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ORIGINAL RESEARCH PAPER

An n-dimensional cubic functional equation and its Hyers–Ulam stability

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Abstract

In this article, we introduce and discuss the general solution of a new n-dimensional cubic functional equation. Furthermore, we establish the generalized Hyers–Ulam stability of this functional equation in fuzzy normed spaces using direct and fixed point approach.

Keywords Cubic functional equation · Fixed point · Hyers–Ulam stability · Fuzzy normed space

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1 Introduction

The notion of fuzzy sets which is a powerful hand set for modeling uncertainty and vagueness in various problems arising in the field of science and engineering has been introduced by Zadeh [30] in 1965. After that, fuzzy theory has become a very active area of research and a lot of developments have been made in the theory of fuzzy sets to find the fuzzy analogues of the classical set theory, for example, see [23]. Some of them are fuzzy linear systems which have many applications in science, such as control problems, information, physics, statistics, engineering, economics, finance and even social sciences; for instance, refer to [1, 4] and [5].

The study of stability problems for functional equations is related to a question of Ulam [28] concerning the stability of group homomorphisms and affirmatively answered for Banach spaces by Hyers [16]. Since then, the stability problems have been extensively investigated for a variety of functional equations and spaces. Later on, various generalizations and extension of Hyers' result were ascertained by Rassias [26], Aoki [2] and Rassias [25] in different versions. The generalized Hyers–Ulam stability of different functional equations in various fuzzy normed spaces has been studied by a number of authors, see [3, 6, 11, 12, 20, 21]) and references therein.

The famous functional equation in the field of stability of functional equation is the cubic functional equation

$$f(x+2y)-3f(x+y)+3f(x)-f(x-y)=6f(y), \tag{1}$$

which is introduced by Rassias in [24]. He established the solution of the Ulam–Hyers stability problem for these cubic mappings. It is easy to see that the function $f(x)=ax^3$ satisfies (1). Thus, every solution of the cubic functional equation (1) is said to be a cubic function. The following alternative cubic functional equation

$$f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+12f(x), \tag{2}$$

has been introduced by Jun and Kim in [18]. They found out the general solution and established the Hyers–Ulam stability for the functional equation (2). They also [17] introduced the cubic functional equation

$$f(x+2y)+f(x-2y)=4f(x+y)+4f(x-y)-6f(x), \tag{3}$$

and proved the Hyers–Ulam stability problem for (3). Since $f(2x)=8f(x)$, the last functional equation is equal to the following:

$$f(x+2y)+f(x-2y)=2f(x)-f(2x)+4f(x+y)+4f(x-y). \tag{4}$$

Some generalized cubic functional equations of (2) and (7) have been introduced in [19] and [22], respectively. In [10], Bodaghi et al. introduced a generalization of the cubic functional equation (4) as follows:

$$\begin{aligned} f(x+ny)+f(x-ny)=2\Big(2\cos\Big(\frac{n\pi}{2}\Big)+n^2-1\Big)f(x)\\ -\frac{1}{2}\Big(\cos\Big(\frac{n\pi}{2}\Big)+n^2-1\Big)f(2x)+n^2[f(x+y)+f(x-y)], \end{aligned} \tag{5}$$

for an integer $n \geq 1$. They determined the general solution and proved the Hyers–Ulam stability problem for the functional equation (5) related to cubic Jordan $*$-derivations (for stability of cubic derivations on Banach algebras and cubic $*$-derivations on Banach $*$-algebras see [9] and [29], respectively). The stability of the Eq. (5) in various spaces was studied in [8]. For the other version of a cubic functional equation, refer to [7] and [15].

In this paper, we investigate and introduce the general solution and generalized Hyers–Ulam stability of cubic functional equation

$$\begin{aligned} \sum_{1\leq i<j<k<l\leq n} f\left(x_i + x_j + x_k + x_l\right) &= (n-3) \sum_{1\leq i<j<k\leq n} f\left(x_i + x_j + x_k\right) \\ &\quad - \left(\frac{(n-2)(n-3)}{2}\right) \sum_{1\leq i<j\leq n}^{n} f\left(x_i + x_j\right) \\ &\quad + \left(\frac{n^3 - 6n^2 + 11n - 54}{6}\right) \sum_{i=1}^{n} f(x_i) + \sum_{j=1}^{n} f(2x_j), \end{aligned} \tag{6}$$

where n is a positive integer in $\mathbb{N}\backslash\{0,1,2,3\}$ in fuzzy normed space by applying direct and fixed point methods.

2 Preliminary notations

In this section, we restate the usual terminology, notations and conventions of the theory of fuzzy normed space.

Definition 2.1 Let X be a real linear space. A function $N : X \times \mathbb{R} \longrightarrow [0,1]$ is said to be fuzzy norm on X if, for all $x, y \in X$ and $a, b \in \mathbb{R}$,

(N1) $N(x,a) = 0$, for $a \leq 0$;
(N2) $N(x,a) = 1$, for all $a > 0$ if and only if $x = 0$;
(N3) $N(ax,b) = N\left(x, \frac{b}{|a|}\right)$ if $a \neq 0$;
(N4) $N(x+y, a+b) \geq min\{N(x,a), N(y,b)\}$;
(N5) $N(x,\cdot)$ is a non-decreasing function on $\mathbb{R}$ and $lim_{a\to\infty} N(x,a) = 1$;
(N6) For $x \neq 0$ and $N(x,\cdot)$ is continuous on $\mathbb{R}$.

The pair (X, N) is called a fuzzy normed linear space. One may regard $N(x, a)$ as the truth value of the statement the norm of x is less than or equal to the real number a.

Definition 2.2 Let (X, N) be a fuzzy normed linear space.

(i) Let x_n be a sequence in X. Then x_n is said to be convergent if there exists $x \in X$ such that $lim_{n\to\infty}N\big(x_n - x, t\big) = 1$ for all $t > 0$. In that case, x is called the limit of the sequence x_n and we denote it by $N - lim_{n\to\infty}x_n = x$;

(ii) A sequence x_n in X is called Cauchy if for each $\epsilon > 0$ and each $t > 0$ there exists n_0 such that for all $n \geq n_0$ and all $p > 0$, we have $N\big(x_{n+p} - x_n, t\big) > 1 - \epsilon$;

(iii) If each Cauchy sequence is convergent, then the fuzzy norm is said to be complete and the fuzzy normed space is called a fuzzy Banach space.

Definition 2.3 A mapping $f : X \to Y$ between fuzzy normed spaces X and Y is continuous at a point x_0 if for each sequence $\{x_n\}$ converging to x_0 in X, the sequence $f\{x_n\}$ converges to $f\{x_0\}$. If f is continuous at each point of $x_0 \in X$, then f is said to be continuous on X.

We bring the following theorems, of which some result in fixed point theory [13]. These results play a fundamental role to arrive our purpose in this paper (an extension of the result was given in [27]).

Theorem 2.4 [Banach's contraction principle] *Let* (X, d) *be a complete metric space and consider a mapping* $T : X \longrightarrow X$ *which is strictly contractive mapping, that is,*

(A1) $d(Tx, Ty) \leq Ld(x, y)$ *for some (Lipschitz constant)* $L < 1$, *then*

- (i) *The mapping* T *has one and only fixed point* $x^* = T(x^*)$;
- (ii) *The fixed point for each given element* x^* *is globally attractive, that is,*

(A2) $lim_{n\longrightarrow\infty}T^n x = x^*$, *for any starting point* $x \in X$;

- (iii) *One has the following estimation inequalities:*

(A3) $(T^n x, x^*) \leq \frac{1}{1-L}d(T^n x, T^{n+1}x)$, *for all* $n \geq 0, x \in X$.
(A4) $(x, x^*) \leq \frac{1}{1-L}d(x, x^*), \forall x \in X$.

Theorem 2.5 [The alternative of fixed point] *Suppose that for a complete generalized metric space* (X, d) *and a strictly contractive mapping* $T : X \longrightarrow X$ *with Lipschitz constant L. Then for each given element* $x \in X$ *either*

(B1) $(T^n x, T^{n+1}x) = +\infty$, *for all* $n \geq 0$, *or*
(B2) *there exists natural number* n_0 *such that*

- (i) $d(T^n x, T^{n+1}x) < \infty$, *for all* $n \geq n_0$;
- (ii) *The sequence* $(T^n x)$ *is convergent to a fixed point* y^* *of T;*
- (iii) y^* *is the unique fixed point of* T *in the set* $Y = \{y \in X; d(T^{n_0}x, y) < \infty\}$;
- (iv) $d(y^*, y) \leq \frac{1}{1-L}d(y, Ty)$, *for all* $y \in L$.

3 Solution of the functional equation (6)

In this part, we discuss the general solution of the functional equation (6). Checking that the mapping $f(x) = cx^3$ satisfies the functional equation (6) is long and tedious. Hence, in the next result, we show that if the mapping f satisfies (6), then it satisfies the functional equation (2). Therefore, (6) is a cubic functional equation.

Lemma 3.1 *Let X and Y be real vector spaces. If the mapping $f : X \longrightarrow Y$ satisfies the functional equation* (6), *then $f : X \longrightarrow Y$ satisfies the functional equation* (2).

Proof Let $f : X \longrightarrow Y$ satisfies the functional equation (6). Replacing $(x_1, \dots, x_n)$ by $(\overbrace{0, \dots, 0}^{n-times})$ in (6), we get

$$\begin{aligned}(n(n-1)(n-2)(n-3))f(0) = {} & (n(n-1)(n-2)(n-3))f(0) \\ & - \left(\frac{n(n-1)(n-2)(n-3)}{2}\right)f(0) \\ & + \left(\frac{n^3-6n^2+11n-54}{6}\right)nf(0) + nf(0),\end{aligned} \tag{7}$$

for all $x \in X$. The relation (7) implies that $2(n^3 - 6n^2 + 11n + 15)f(0) = 0$, and hence $f(0) = 0$. Interchanging $(x_1, x_2, \dots, x_n)$ by $(x, -x, \overbrace{0, \dots, 0}^{(n-2)-times})$ in (6), we find

$$\begin{aligned}\left(\frac{(n-2)(n-3)(n-4)}{6}\right)f(x+(-x)) = {} & \left(\frac{(n-3)^2(n-4)}{2}\right)f(x+(-x)) \\ & - \left(\frac{(n-2)(n-3)^2}{2}\right)f(x+(-x)) \\ & - \left(\frac{n^3-6n^2+11n-54}{6}\right)(f(x)+f(-x)) \\ & + 8(f(x)+f(-x)),\end{aligned} \tag{8}$$

for all $x \in X$. The equality $f(0) = 0$ and the relation (8) shows that $\left(\frac{(n-1)(n-2)(n-3)}{6}\right)(f(x)+f(-x)) = 0$. Since $n \neq 1, 2, 3$, we have $f(-x) = -f(x)$. This means that f is a odd mapping. Now, setting $(x_1, x_2, \dots, x_n)$ by $\left(x, \overbrace{0, \dots, 0}^{(n-1)-times}\right)$ in (6), we get

$$\left(\frac{(n-1)(n-2)(n-3)}{6}\right)f(x) = \left(\frac{(n-1)(n-2)(n-3)}{2}\right)f(x) - \left(\frac{(n-1)(n-2)(n-3)}{2}\right)f(x) + \left(\frac{n^3-6n^2+11n-54}{6}\right)f(x) + f(2x), \tag{9}$$

for all $x \in X$. It follows from (9) that $f(2x) = 8f(x)$. Switching $(x_1, x_2, \dots, x_n)$ by $\left(2x, y, -y, y, \overbrace{0, \dots, 0}^{(n-4)-times}\right)$ in (6), we obtain

$$\begin{aligned} f(2x+y) &+ (n-4)f(2x) + (n-4)f(2x+2y) + \left(\frac{(n-4)(n-5)}{2}\right)f(2x+y) \\ &+ (n-4)f(2x) + \left(\frac{(n-4)(n-5)}{2}\right)f(2x-y) + \left(\frac{(n-4)(n-5)}{2}\right)f(2x+y) \\ &+ \left(\frac{(n-4)(n-5)(n-6)}{6}\right)f(2x) + (n-4)f(y) + \left(\frac{(n-4)(n-5)}{2}\right)f(2y) \\ &+ \left(\frac{(n-4)(n-5)(n-6)}{6}\right)f(y) + \left(\frac{(n-4)(n-5)(n-6)}{6}\right)f(-y) \\ &+ \left(\frac{(n-4)(n-5)(n-6)}{6}\right)f(y) \\ = (n-3)&[f(2x) + f(2x+2y) + (n-4)f(2x+y) + f(2x) \\ &+ (n-4)f(2x-y) + (n-4)f(2x+y) + \frac{(n-4)(n-5)}{2}f(2x)] \\ &- \frac{(n-2)(n-3)}{2}[f(2x+y) + f(2x-y) + f(2x+y) \\ &+ (n-4)f(2x) + f(2y) + (n-4)f(y) + (n-4)f(-y) + (n-4)f(y)] \\ &+ \left(\frac{n^3-6n^2+11n-54}{6}\right)[f(2x) + f(y) + f(-y) + f(y)], \end{aligned} \tag{10}$$

for all $x, y \in X$. Using the property $f(2x) = 8f(x)$, the oddness of f and a simple computation one show that the relation (10) can be converted to

$$3f(2x+y) + f(2x-y) = 8f(x+y) + 24f(x), \tag{11}$$

for all $x, y \in X$. Substituting y by $-y$ in (11), we arrive at

$$3f(2x-y) + f(2x+y) = 8f(x-y) + 24f(x), \tag{12}$$

for all $x, y \in X$. Plugging (11) into (12), we have

$$f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+12f(x),$$

for all $x, y \in X$. This finishes the proof. □

4 Stability results for (6)-direct method

In this section, we establish the stability of (6) in fuzzy normed space using direct method. Throughout this paper, we denote $\overbrace{X \times X \times \cdots \times X}^{n-times}$ by X^n. In addition, we assume that X, (Z, F) and (Y, F') are linear space, fuzzy normed space and fuzzy Banach space, respectively. Given $f : X \longrightarrow Y$, for notational handiness, we define the difference operators $Df : X \times X \longrightarrow Y$ by

$$\begin{aligned} Df(x_1, x_2, \dots, x_n) = &\sum_{1\le i<j<k<l\le n} f\left(x_i+x_j+x_k+x_l\right) - (n-3)\sum_{1\le i<j<k\le n} f\left(x_i+x_j+x_k\right) \\ &+\left(\frac{(n-2)(n-3)}{2}\right)\sum_{1\le i<j\le n}^{n} f\left(x_i+x_j\right) \\ &-\left(\frac{n^3-6n^2+11n-54}{6}\right)\sum_{i=1}^{n} f(x_i) - \sum_{j=1}^{n} f(2x_j), \end{aligned}$$

for all $x_1, x_2, \dots, x_n \in X$.

Theorem 4.1 *Let $\beta \in \{-1, 1\}$ and $d > 0$. Let $\chi : X^n \longrightarrow Z$ be a multi linear mapping with $0 < \left(\frac{d}{2}\right)^3 < 1$ and*

$$F'\left(\chi\left(2^{\beta k}x, 0, \dots, 0\right), r\right) \ge F'(d^{\beta}\chi(x, 0, \dots, 0), r), \tag{13}$$

for all $x \in X$ and

$$\lim_{k\to\infty} F'\left(\chi\left(2^{\beta k}x_1, 2^{\beta k}x_2, \dots, 2^{\beta k}x_n\right), 2^{3\beta k}r\right) = 1, \tag{14}$$

for all $x_1, x_2, \dots, x_n \in X$ and all $r > 0$. Suppose that a mapping $f : X \longrightarrow Y$ satisfies the inequality

$$F\left(Df\left(x_1, x_2, \dots, x_n\right), r\right) \ge F'\left(\chi\left(x_1, x_2, \dots, x_n\right), r\right), \tag{15}$$

for all $r > 0$ and all $x_1, x_2, \dots, x_n \in X$. Then, the limit

$$C(x) = F - \lim_{k\to\infty} \frac{f(2^{\beta k}x)}{2^{3\beta k}}, \tag{16}$$

exists for all $x \in X$ and the mapping $C : X \longrightarrow Y$ is a unique cubic mapping such that

$$F(f(x) - C(x), r) \geq F'\left(\chi(x, 0, \dots, 0), r \mid 2^3 - d \mid\right), \tag{17}$$

for all $x \in X$ and all $r > 0$.

Proof Replacing $\left(x_1, x_2, \dots, x_n\right)$ by $(0, \dots, 0)$ in (15), we have

$$F\left(2(n^3 - 6n^2 + 11n + 15)f(0), r\right) \geq F'(\chi(0, 0, \dots, 0), r), \tag{18}$$

for all $r > 0$. The relation (18) implies that

$$F\left(f(0), \frac{r}{2(n^3 - 6n^2 + 11n + 15)}\right) \geq F'\left(\chi(0, 0, \dots, 0), \frac{r}{2(n^3 - 6n^2 + 11n + 15)}\right) \tag{19}$$

for all $r > 0$. Since $\chi(0, 0, \dots, 0) = 0$, it follows from (19) that $f(0) = 0$. Interchanging $\left(x_1, x_2, \dots, x_n\right)$ by $\left(x, \overbrace{0, \dots, 0}^{(n-1)-times}\right)$ in (15), we get

$$F\left(\frac{(n-1)(n-2)(n-3)}{6}f(x) - \left(\frac{n^3 - 6n^2 + 11n - 54}{6}\right)f(x) - f(2x), r\right) \geq F'(\chi(x, 0, \dots, 0), r), \tag{20}$$

for all $x \in X$ and all $r > 0$. Hence,

$$F(8f(x) - f(2x), r) \geq F'(\chi(x, 0, \dots, 0), r), \tag{21}$$

for all $x \in X$ and all $r > 0$. For $\beta = 1$, we have

$$F\left(f(x) - \frac{f(2x)}{8}, \frac{r}{8}\right) \geq F'(\chi(x, 0, \dots, 0), r), \tag{22}$$

and for $\beta = -1$, we have

$$F\left(8f\left(\frac{x}{2}\right) - f(x), r\right) \geq F'(\chi\left(\frac{x}{2}, 0, \dots, 0\right), r), \tag{23}$$

for all $x \in X$ and all $r > 0$. Replacing x by $2^n x$ in (22), we obtain

$$F\left(\frac{f(2^{n+1}x)}{8} - f(2^n x), \frac{r}{8}\right) \geq F'(\chi(2^n x, 0, \ldots, 0), r), \tag{24}$$

for all $x \in X$ and all $r > 0$. Using (13), the property (N3) in (24), we arrive

$$F\left(\frac{f(2^{n+1}x)}{8} - f(2^n x), \frac{r}{8}\right) \geq F'\left(\chi(x, 0, \ldots, 0), \frac{r}{d^n}\right), \tag{25}$$

for all $x \in X$ and all $r > 0$. It is easy to verify from (25) that

$$F\left(\frac{f(2^{n+1}x)}{2^{3(n+1)}} - \frac{f(2^n x)}{2^{3n}}, \frac{r}{2^{3(n+1)}}\right) \geq F'\left(\chi(x, 0, \ldots, 0), \frac{r}{d^n}\right), \tag{26}$$

holds for all $x \in X$ and all $r > 0$. Replacing r by $d^n r$ in (26), we get

$$F\left(\frac{f(2^{n+1}x)}{2^{3(n+1)}} - \frac{f(2^n x)}{2^{3n}}, \frac{d^n r}{2^{3(n+1)}}\right) \geq F'(\chi(x, 0, \ldots, 0), r), \tag{27}$$

for all $x \in X$ and all $r > 0$. It is easy to see that

$$\frac{f(2^n x)}{2^{3n}} - f(x) = \sum_{i=0}^{n-1} \left[\frac{f(2^{i+1}x)}{2^{3(i+1)}} - \frac{f(2^i x)}{2^{3i}}\right], \tag{28}$$

for all $x \in X$. From Eqs. (27) and (28), we have

$$\begin{aligned} F\left(\frac{f(2^n x)}{2^{3n}} - f(x), \sum_{i=0}^{n-1} \frac{d^i r}{2^3 . 2^{3n}}\right) &\geq \min \cup_{i=0}^{n-1} \left\{\frac{f(2^{i+1}x)}{2^{3(i+1)}} - \frac{f(2^i x)}{2^{3i}}, \frac{d^i r}{2^3 . 2^{3n}}\right\} \\ &\geq \min \cup_{i=0}^{n-1} \{F'(\chi(x, 0, \ldots, 0), r)\} \\ &\geq F'(\chi(x, 0, \ldots, 0), r) \\ &\geq F'(\chi(x, 0, \ldots, 0), r) \end{aligned} \tag{29}$$

for all $x \in X$ and all $r > 0$. Switching x by $2^m x$ in (29) and using (13), (N3), we obtain

$$F\left(\frac{f(2^{n+m}x)}{2^{3(n+m)}} - \frac{f(2^m x)}{2^{3m}}, \sum_{i=0}^{n-1} \frac{d^i r}{2^3 . 2^{3(i+m)}}\right) \geq F'(\chi(x, 0, \ldots, 0), \frac{r}{d^m}), \tag{30}$$

for all $x \in X$ and all $r > 0$ and all $m, n \geq 0$. Substituting r by $d^m r$ in (30), we get

$$F\left(\frac{f(2^{n+m}x)}{2^{3(n+m)}} - \frac{f(2^m x)}{2^{3m}}, \sum_{i=m}^{m+n-1} \frac{d^i r}{2^3.2^{3i}}\right) \geq F'(\chi(x,0,\ldots,0),r), \tag{31}$$

for all $x \in X$ and all $r > 0$ and all $m, n \geq 0$. Using the property (N3) in (30), we find

$$F\left(\frac{f(2^{n+m}x)}{2^{3(n+m)}} - \frac{f(2^m x)}{2^{3m}}, r\right) \geq F'\left(\chi(x,0,\ldots,0), \frac{r}{\sum_{i=m}^{m+n-1} \frac{d^i}{2^3.2^{3i}}}\right), \tag{32}$$

for all $x \in X$ and all $r > 0$ and all $m, n \geq 0$. Since $0 < d < 2^3$ and $\sum_{i=0}^{n}\left(\frac{d}{2^{3i}}\right) < \infty$, the Cauchy criterion for convergence and the property (N5) implies that $\left\{\frac{f(2^n x)}{2^{3n}}\right\}$ is a Cauchy sequence in (Y, N). Since (Y, N) is a fuzzy Banach space, this sequence converges to some point $C(x) \in Y$. Hence, one can define the mapping $C : X \longrightarrow Y$ by

$$C(x) = F - \lim_{n\to\infty}\left\{\frac{f(2^n x)}{2^{3n}}\right\},$$

for all $x \in X$. Letting $m = 0$ in (32), we get

$$F\left(\frac{f(2^n x)}{2^{3n}} - f(x), r\right) \geq F'\left(\chi(x,0,\ldots,0), \frac{r}{\sum_{i=0}^{n-1} \frac{d^i}{2^3.2^{3i}}}\right), \tag{33}$$

for all $x \in X$ and all $r > 0$. Letting $n \to \infty$ in (33) and using (N_6), we arrive

$$F(f(x) - C(x), r) \geq F'(\chi(x,0,\ldots,0), r(2^3 - d)),$$

for all $x \in X$ and all $r > 0$. Here, we claim that C is cubic. Replacing $(x_1, x_2, \ldots, x_n)$ by $(2^n x_1, 2^n x_2, \ldots, 2^n x_n)$ in (15), respectively, we get

$$F\left(\frac{1}{2^{3n}} Df(2^n x_1, 2^n x_2, \ldots, 2^n x_n), r\right) \geq F'\left(\chi(2^n x_1, 2^n x_2, \ldots, 2^n x_n), 2^{3n} r\right),$$

for all $x_1, x_2, \ldots, x_n \in X$ and all $r > 0$. Since

$$\lim_{n\to\infty} F'\left(\chi(2^n x_1, 2^n x_2, \ldots, 2^n x_n), 2^{3n} r\right) = 1,$$

C satisfies the functional equation (6). Hence, the mapping $C : X \longrightarrow T$ is cubic. To prove the uniqueness of C, let $D : X \longrightarrow Y$ be another cubic mapping satisfying

(17). Fix $x \in X$. Clearly, $C(2^n x) = 2^{3n}C(x)$ and $D(2^n x) = 2^{3n}D(x)$ for all $x \in X$ for all $x \in X$ and all $n \in \mathbb{N}$. It follows from (17) that

$$\begin{aligned}
F(C(x) - D(x), r) &= N\left(\frac{C(2^n x)}{2^{3n}} - \frac{D(2^n x)}{2^{3n}}, r\right) \\
&\geq \min\left\{F\left(\frac{C(2^n x)}{2^{3n}} - \frac{f(2^n x)}{2^{3n}}, \frac{r}{2}\right), F\left(\frac{f(2^n x)}{2^{3n}} - \frac{D(2^n x)}{2^{3n}}, \frac{r}{2}\right)\right\} \\
&\geq F'\left(\chi(2^n x, 0, \ldots, 0), \frac{2^{3n} r(2^3 - d)}{2}\right) \\
&\geq F'\left(\chi(x, 0, \ldots, 0), \frac{2^{3n} r(2^3 - d)}{2d^n}\right)
\end{aligned}$$

for all $x \in X$ and all $r > 0$. Since $\lim_{n\to\infty} \frac{2^{3n} r(2^3-d)}{2d^n} = \infty$, we obtain

$$\lim_{n\to\infty} F'\left(\chi(x, 0, \ldots, 0), \frac{2^{3n} r(2^3 - d)}{2d^n}\right) = 1.$$

Thus, $F(C(x) - D(x), r) = 1$ for all $x \in X$ and all $r > 0$, and so $C(x) = D(x)$. For $\beta = -1$, one can prove the other part of the proof in a similar method. Therefore, the proof is completed. □

The following corollary is a direct consequence of Theorem 4.1 concerning the stability of (6). We include it without the proof.

Corollary 4.2 *Let ϵ, s be constants with $\epsilon > 0$. Suppose that the mapping $f : X \longrightarrow Y$ satisfies the inequality*

$$F(Df(x_1, x_2, \ldots, x_n), r) \geq \begin{cases} F'(\epsilon, r) \\ F'(\epsilon \sum_{i=1}^n ||x_i||^s, r) \\ F'(\epsilon(\sum_{i=1}^n ||x_i||^{ns} + \Pi_{i=1}^n ||x_i||^s), r) \end{cases},$$

for all $x_1, x_2, \ldots, x_n \in X$ and all $r > 0$. Then there exists a unique cubic function $C : X \longrightarrow Y$ such that

$$F(f(x) - C(x), r) \geq \begin{cases} F'(\epsilon, r \mid 7 \mid) \\ F'\left(\epsilon\|x\|^s, r \mid 2^3 - 2^s \mid\right); \quad s \neq 3 \\ F'\left((\epsilon\|x\|^{ns}, r \mid 2^3 - 2^{ns} \mid\right); \quad s \neq \frac{3}{n} \end{cases}$$

for all $x \in X$ and $r > 0$.

5 Stability results for (6)-fixed point method

In this section, in analogy with Theorem 4.1, we bring the following generalized Hyers–Ulam stability results for the functional equations (6) in fuzzy normed space using fixed point method. To prove the stability result, we define the following ψ_i is a constant such that

$$\psi_i = \begin{cases} 2 & if \quad i = 0 \\ \frac{1}{2} & if \quad i = 1 \end{cases},$$

and Ω is the set such that $\Omega = \{p | p : X \longrightarrow Y, P(0) = 0\}$.

Theorem 5.1 *Let $f : X \longrightarrow Y$ be a mapping for which there exists a mapping $\chi : X^n \longrightarrow Z$ with condition*

$$lim_{k\to\infty} F'\left(\chi\left(\psi_i^k x_1, \psi_i^k x_2, \dots, \psi_i^k x_n\right), \psi_i^{3k} r\right) = 1, \tag{34}$$

and satisfying the inequality

$$F\left(Df\left(x_1, x_2, \dots, x_n\right), r\right) \geq F'\left(\chi\left(x_1, x_2, \dots, x_n\right), r\right), \tag{35}$$

for all $x_1, x_2, \dots, x_n \in X$ and $r > 0$. If there exists $L = L(i)$ such that the mapping $x \longrightarrow \beta(x) = \chi\left(\frac{x}{2}, 0, \dots, 0\right)$ has the property

$$F'\left(L \frac{1}{\psi_i^3} \beta(\psi_i x), r\right) = F'(\beta(x), r), \tag{36}$$

for all $x \in X$ and $r > 0$, then there exists a unique cubic function $C : X \longrightarrow Y$ satisfying the functional equation (6) and

$$F(f(x) - C(x), r) \geq F'\left(\frac{L^{1-i}}{1 - L} \beta(x), r\right), \tag{37}$$

for all $x \in X$ and $r > 0$.

Proof Let d be a general metric on Ω such that

$$d(t, u) = inf\left\{k \in (0, \infty) | F(t(x) - u(x), r) \geq F'(\beta(x), kr), x \in X, r > 0\right\}.$$

It is easy to see that (Ω, d) is complete. Define $T : \Omega \longrightarrow \Omega$ by $Tt(x) = \frac{1}{\eta_i^3} t(\eta_i x)$ for all $x \in X$. Assume that for $t, u \in \Omega$, $d(t, u) = k$. Then $F(t(x) - u(x), r) \geq F'(\beta(x), kr)$. So, $F\left(\frac{t(\eta_i x)}{\eta_i^3} - \frac{u(\eta_i x)}{\eta_i^3}, r\right) \geq F'(\beta(\eta_i x), k\eta_i^3 r)$. This implies that $F(Tt(x) - Tu(x), r) \geq F'(\beta(\eta_i x), k\eta_i^3 r)$. Thus, $F(Tt(x) - Tu(x), r) \geq F'(\beta(x), kLr)$ and hence $d(Tt(x), Tu(x)) \leq kL$. The last inequality shows that $d(Tt, Tu) \leq Ld(t, u)$ for all

$t, u \in \Omega$. Therefore, T is strictly contractive mapping on Ω with Lipschitz constant L. Replacing $(x_1, x_2, \dots, x_n)$ by $(x, 0, \dots, 0)$ in (35), we get

$$F(f(2x) - 8f(x), r) \geq F'(\psi(x, 0, \dots, 0), r), \tag{38}$$

for all $x \in X$ and $r > 0$. Using the property (N3) in (38), we arrive at

$$F\left(\frac{f(2x)}{2^3} - f(x), r\right) \geq F'\left(\frac{1}{2^3}\psi(x, 0, \dots, 0), r\right), \tag{39}$$

for all $x \in X$ and $r > 0$. Applying (36) when $i = 0$, it follows from (39) that

$$F\left(\frac{f(2x)}{2^3} - f(x), r\right) \geq F'(L\beta(x), r), \tag{40}$$

for all $x \in X$ and $r > 0$. Interchanging x by $\frac{x}{2}$ in (38), we obtain

$$F\left(f(x) - 8f\left(\frac{x}{2}\right), r\right) \geq F'\left(\frac{1}{2^3}\psi\left(\frac{x}{2}, 0, \dots, 0\right), r\right), \tag{41}$$

for all $x \in X$ and $r > 0$. When $i = 1$, it follows from (41) that

$$F\left(f(x) - 8f\left(\frac{x}{2}\right), r\right) \geq F'(\beta(x), r), \tag{42}$$

for all $x \in X$ and $r > 0$. From (40) and (42), we can conclude $T(f, Tf) \leq L^{1-i} < \infty$ for $i \in \{0, 1\}$. Now, Theorem 2.5 in both cases necessitates that there exists a fixed point C of T in Ω such that

$$C(x) = F - \lim_{k\to\infty} \frac{f(\eta^k x)}{\eta^{3k}},$$

for all $x \in X$ and $r > 0$. Letting $(x_1, x_2, \dots, x_n)$ by $(\eta_i^k x_1, \eta_i^k x_2, \dots, \eta_i^k x_n)$ in (35), we find

$$F\left(\frac{1}{\eta_i^{3k}} Df(\eta_i^k x_1, \eta_i^k x_2, \dots, \eta_i^k x_n), r\right) \geq F'(\psi(\eta_i^k x_1, \eta_i^k x_2, \dots, \eta_i^k x_n), \eta_i^{3k} r),$$

for all $r > 0$ and all $x_1, x_2, \dots, x_n \in X$. By proceeding with the same procedure of Theorem 4.1, we can prove the function $C : X \longrightarrow Y$ is cubic and its satisfies the functional equation (6). By Theorem 2.5, C is a unique fixed point of T in the set $\Delta = \{f \in \Omega |\, d(f, C) < \infty\}$. In other words, C is a unique function such that

$$F(f(x) - C(x), r) \geq F'(\beta(x), kr),$$

for all $x \in X$ and $r > 0$. Again using Theorem 2.5, we have $d(f, C) \leq \frac{1}{1-L} d(f, Tf)$ and so $d(f, C) \leq \frac{L^{1-i}}{1-L}$. Thus, $F(f(x) - C(x), r) \geq F'\left(\beta(x)\frac{L^{1-i}}{1-L}, r\right)$ for all $x \in X$ and $r > 0$. This completes the proof. □

The upcoming corollary shows that the functional equation (6) can be stable if we apply Theorem 5.1.

Corollary 5.2 *Let ϵ, s be real numbers with $\epsilon > 0$. Suppose that the mapping $f : X \longrightarrow Y$ satisfies the inequality*

$$F(Df(x_1, x_2, \dots, x_n), r) \geq \begin{cases} F'(\epsilon, r) \\ F'(\epsilon \sum_{i=1}^{n} ||x_i||^s, r) \\ F'(\epsilon(\sum_{i=1}^{n} ||x_i||^{ns} + \Pi_{i=1}^{n} ||x_i||^s), r), \end{cases},$$

for all $x_1, x_2, \dots, x_n \in X$ and all $r > 0$. Then there exists a unique cubic mapping $C : X \longrightarrow Y$ such that

$$F(f(x) - C(x), r) \geq \begin{cases} F'(\epsilon, 7r) \\ F'\left(\epsilon\|x\|^s, r \mid 2^3 - 2^s \mid\right); \quad s \neq 3 \\ F'\left((\epsilon\|x\|^{ns}, r \mid 2^3 - 2^{ns} \mid\right); \quad s \neq \frac{3}{n} \end{cases},$$

for all $x \in X$ and $r > 0$.

Proof Set

$$\chi(x_1, x_2, \dots, x_n) = \begin{cases} \epsilon, \\ \epsilon \sum_{i=1}^{n} ||x_i||^s \\ \epsilon(\sum_{i=1}^{n} ||x_i||^{ns} + \Pi_{i=1}^{n} ||x_i||^s) \end{cases},$$

for all $x_1, x_2, \dots, x_n \in X$. Then

$$F'\left(\chi\left(\psi_i^k x_1, \psi_i^k x_2, \dots, \psi_i^k x_n\right), \psi_i^{3k} r\right) = \begin{cases} F'(\epsilon, \psi_i^{3k} r) \\ F'(\epsilon \sum_{i=1}^{n} ||x_i||^s, \psi_i^{(3-s)k} r) \\ F'(\epsilon(\sum_{i=1}^{n} ||x_i||^{ns} + \Pi_{i=1}^{n} ||x_i||^s), \psi_i^{(3-ns)k} r) \end{cases}$$

$$= \begin{cases} \to 1 \;\; as \;\; k \to \infty \\ \to 1 \;\; as \;\; k \to \infty \\ \to 1 \;\; as \;\; k \to \infty. \end{cases}$$

This means that (34) holds. We also have

$$F'\left(L\frac{1}{\psi_i^3}\beta(\psi_i x), r\right) = F'(\beta(x), r),$$

for all $x \in X$ and $r > 0$ where $\beta(x)$ is defined in Theorem 5.1. Hence,

$$F'(\beta(x), r) = F'\left(\chi\left(\frac{x}{2}, 0, \ldots, 0\right), r\right) = \begin{cases} F'(\epsilon, r) \\ F'(n\epsilon\|x\|^s, r2^s) \\ F'(\epsilon\|x\|^{ns}, r2^{ns}). \end{cases}$$

On the other hand,

$$F'\left(\frac{1}{\psi_i^3}\beta\left(\psi_i x\right), r\right) = \begin{cases} F'\left(\frac{\epsilon}{\psi_i^3}, r\right) \\ F'\left(\frac{\epsilon n\|x\|^s\psi_i^s}{\psi_i^3}, r2^s\right) \\ F'\left(\frac{\epsilon\|x\|^{ns}\psi_i^{ns}}{\psi_i^3}, r2^{ns}\right) \end{cases} = \begin{cases} \psi_i^{-3}\beta(x) \\ \psi_i^{s-3}\beta(x) \\ \psi_i^{ns-3}\beta(x) \end{cases}$$

for all $x \in X$. To finish the proof we consider the following cases:
Case 1. $L = 2^{-3}$ if $i = 0$.

$$F(f(x) - C(x), r) \geq F'\left(\frac{L^{1-i}}{1-L}\beta(x), r\right) = F'\left(\frac{2^{-3}}{1-2^{-3}}\epsilon, r\right) \geq F'(\epsilon, 7r).$$

Case 2. $L = 2^3$ if $i = 1$.

$$F(f(x) - C(x), r) \geq F'\left(\frac{L^{1-i}}{1-L}\beta(x), r\right) = F'\left(\frac{1}{1-2^3}\epsilon, r\right) \geq F'(\epsilon, -7r).$$

Case 3. $L = 2^{s-3}$ for $s > 3$ if $i = 0$.

$$\begin{aligned} F(f(x) - C(x), r) &\geq F'\left(\frac{L^{1-i}1-L}{\beta}(x), r\right) \\ &= F'\left(\frac{2^{s-3}}{1-2^{s-3}}\epsilon\|x\|^s 2^s, r\right) \geq F'\left(\epsilon\|x\|^s, r(2^3 - 2^s)\right). \end{aligned}$$

Case 4. $L = 2^{3-s}$ for $s < 3$ if $i = 1$.

$$F(f(x) - C(x), r) \geq F'\left(\frac{L^{1-i}}{1-L}\beta(x), r\right) = F'\left(\frac{1}{1-2^{3-s}}\epsilon\|x\|^s 2^s, r\right) \geq F'\left(\epsilon\|x\|^s, r(2^s - 2^3)\right).$$

Case 5. $L = 2^{ns-3}$ for $s > \frac{3}{n}$ if $i = 0$.

$$\begin{aligned} F(f(x) - C(x), r) &\geq F'\left(\frac{L^{1-i}}{1-L}\beta(x), r\right) \\ &= F'\left(\frac{2^{ns-3}}{1-2^{ns-3}}\epsilon\|x\|^{ns} 2^{ns}, r\right) \geq F'\left(\epsilon\|x\|^{ns}, r(2^3 - 2^{ns})\right). \end{aligned}$$

Case 6. $L = 2^{3-ns}$ for $s < \frac{3}{n}$ if $i = 1$.

$$F(f(x) - C(x), r) \geq F'\left(\frac{L^{1-i}}{1-L}\beta(x), r\right) = F'\left(\frac{1}{1-2^{3-ns}}\epsilon\|x\|^{ns} 2^{ns}, r\right) \geq F'\left(\epsilon\|x\|^{ns}, r(2^{ns} - 2^3)\right)$$

Now, the proof is completed. □

The idea of the following example is taken from [14]. In fact, we illustrate that the functional equation (6) can be non-stable on the space of real numbers.

***Example*5.3** Let $\chi : \mathbb{R} \to \mathbb{R}$ be a function defined by

$$\chi(x) = \begin{cases} rx^3, & \mathit{if}\ |x| < 1 \\ r, & \mathit{otherwise}, \end{cases}$$

where $r > 0$ is a constant. Define the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ by

$$f(x) = \sum_{n=0}^{\infty} \frac{\chi(2^n x)}{2^{3n}},$$

for all $x \in \mathbb{R}$. We first show that f satisfies the functional inequality

$$\begin{aligned} &|Df(x_1, x_2, \dots, x_n)| \\ &\leq \left(\frac{(n^3 - 3n^2 + 2n)}{2} + \frac{(n^3 - 6n^2 + 11n - 6)}{6}\left[\frac{n(n+1)}{2}\right]^2\right)\left(\sum_{i=0}^{n} |x_i|^3\right) r, \end{aligned} \tag{43}$$

for all $x, y \in \mathbb{R}$. Note that f is bounded. In other words,

$$| f(x) | \leq \sum_{n=0}^{\infty} \frac{| \chi(2^n x) |}{2^{3n}} = \sum_{n=0}^{\infty} \frac{r}{2^{3n}} = \frac{8r}{7},$$

for all $x \in \mathbb{R}$. If $x_1 = x_2 = \cdots = x_n = 0$, then (24) is trivial. If $\sum_{i=0}^{n} | x_i |^3 \geq \frac{1}{8}$, then the left-hand side of (43) is less than $\left(\frac{(n^3-3n^2+2n)}{2} + \frac{(n^3-6n^2+11n-6)}{6} \left[\frac{n(n+1)}{2} \right]^2 \right) r$. Now suppose that $0 < \sum_{i=0}^{n} | x_i |^3 < \frac{1}{8}$. Thus, there exists a positive integer k such that

$$\frac{1}{8^{k+1}} \leq \sum_{i=0}^{n} | x_i |^3 < \frac{1}{8^k}, \tag{44}$$

such that $8^{k-1} x_1^3 < \frac{1}{8}, 8^{k-1} x_2^3 < \frac{1}{8}, \ldots, 8^{k-1} x_n^3 < \frac{1}{8}$, and consequently

$$\sum_{1 \leq i<j<k<l \leq n} 2^{k-1} \left(x_i + x_j + x_k + x_l \right), \; 2^{k-1} \sum_{1 \leq i<j<k \leq n} \left(x_i + x_j + x_k \right),$$
$$\sum_{1 \leq i<j \leq n}^{n} 2^{k-1} \left(x_i + x_j \right), \; \sum_{i=1}^{n} 2^{k-1} (x_i), \; \sum_{j=1}^{n} 2^{k-1} (2x_j) \in (-1, 1).$$

Therefore, for each $n \in \{0, 1, \ldots, k-1\}$, we have

$$\sum_{1 \leq i<j<k<l \leq n} 2^{n} \left(x_i + x_j + x_k + x_l \right), \; 2^{n} \sum_{1 \leq i<j<k \leq n} \left(x_i + x_j + x_k \right),$$
$$\sum_{1 \leq i<j \leq n}^{n} 2^{n} \left(x_i + x_j \right), \; \sum_{i=1}^{n} 2^{n} (x_i), \; \sum_{j=1}^{n} 2^{n} (2x_j) \in (-1, 1)$$

and

$$\sum_{1 \leq i<j<k<l \leq n} \chi\left(2^n (x_i + x_j + x_k + x_l)\right) - (n-3) \sum_{1 \leq i<j<k \leq n} \chi\left(2^n (x_i + x_j + x_k)\right)$$
$$+ \left(\frac{(n-2)(n-3)}{2} \right) \sum_{1 \leq i<j \leq n}^{n} \chi\left(2^n (x_i + x_j)\right) - \left(\frac{n^3 - 6n^2 + 11n - 54}{6} \right) \sum_{i=1}^{n} \chi(2^n (x_i))$$
$$- \sum_{j=1}^{n} \chi(2^n (2x_j)) = 0.$$

From the definition of *f* and (26), we obtain

$$\begin{aligned}
&| Df(x_1, x_2, \dots, x_n) | \\
&\leq \sum_{n=0}^{\infty} \frac{1}{8^n} \mid \sum_{1\leq i<j<k<l\leq n} \chi\left(2^n(x_i + x_j + x_k + x_l)\right) \\
&\quad - (n-3) \sum_{1\leq i<j<k\leq n} \chi\left(2^n(x_i + x_j + x_k)\right) + \left(\frac{(n-2)(n-3)}{2}\right) \sum_{1\leq i<j\leq n}^{n} \chi\left(2^n(x_i + x_j)\right) \\
&\quad - \left(\frac{n^3 - 6n^2 + 11n - 54}{6}\right) \sum_{i=1}^{n} \chi(2^n(x_i)) - \sum_{j=1}^{n} \chi(2^n(2x_j)) \mid \\
&\leq \sum_{n=k}^{\infty} \frac{1}{8^n} \mid \sum_{1\leq i<j<k<l\leq n} \chi\left(2^n(x_i + x_j + x_k + x_l)\right) \\
&\quad - (n-3) \sum_{1\leq i<j<k\leq n} \chi\left(2^n(x_i + x_j + x_k)\right) + \left(\frac{(n-2)(n-3)}{2}\right) \sum_{1\leq i<j\leq n}^{n} \chi\left(2^n(x_i + x_j)\right) \\
&\quad - \left(\frac{n^3 - 6n^2 + 11n - 54}{6}\right) \sum_{i=1}^{n} \chi(2^n(x_i)) - \sum_{j=1}^{n} \chi(2^n(2x_j)) \mid \\
&\leq \left(\frac{(n^3 - 3n^2 + 2n)}{2} + \frac{(n^3 - 6n^2 + 11n - 6)}{6}\left[\frac{n(n+1)}{2}\right]^2\right)\left(\sum_{i=0}^{n} \mid x_i \mid^3\right) r.
\end{aligned}$$

Thus, f satisfies (43) for all $x_1, x_2, \dots, x_n \in \mathbb{R}$ with $0 < \sum_{i=0}^{n} \mid x_i \mid^3 < \frac{1}{8}$. We claim that there is not any cubic mapping $C : \mathbb{R} \longrightarrow \mathbb{R}$ and a constant $\beta > 0$ such that

$$| f(x) - C(x) | \leq \beta \mid x \mid^3, \tag{45}$$

for all $x \in \mathbb{R}$. Suppose the contrary to our claim. Hence, there exists a constant $\alpha \in \mathbb{R}$ such that $C(x) = \alpha x^3$ for all $x \in \mathbb{R}$. So

$$|f(x)| \leq (|\alpha| + \beta)|x|^3, \tag{46}$$

for all $x \in \mathbb{R}$. On the other hand, consider $m \in \mathbb{N}$ such that $(m+1)r > |\alpha| + \beta$. If x is a real number in $\left(0, \frac{1}{2^{k-1}}\right)$, then $2^n x \in (0, 1)$ for all $n = 0, 1, \dots, k-1$. Therefore, for such x, we get

$$f(x) = \sum_{n=0}^{\infty} \frac{\chi(2^n x)}{2^{3n}} \geq \sum_{n=0}^{m} \frac{2^{3n} r x^3}{2^{3n}} = (m+1) r x^3 > (|\alpha| + \beta) x^3.$$

The above relation contradicts (46).

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