The Journal of Analysis (2020) 28:387–413
https://doi.org/10.1007/s41478-019-00175-3

ORIGINAL RESEARCH PAPER

Legendre spectral projection methods for weakly singular Hammerstein integral equations of mixed type

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Received: 10 November 2018 / Accepted: 17 April 2019 / Published online: 26 April 2019

Abstract

This work provides the Legendre spectral projection (Galerkin and collocation), iterated Legendre spectral projection, Legendre spectral multi-projection and iterated Legendre spectral multi-projection methods to approximate the solution of weakly singular Hammerstein integral equations of mixed type. The convergence rates of approximate solutions to the exact solutions are obtained for all the above four methods in both L^2 and infinity norm. The comparison of convergence rates for all these methods have been discussed. We also have shown that iterated Galerkin improves over Galerkin, multi-Galerkin improves over iterated Galerkin and iterated multi-Galerkin improves over multi-Galerkin in L^2 norm using Legendre polynomial bases.

Keywords Hammerstein integral equations of mixed type · Weakly singular kernels · Legendre spectral projection methods · Multi-projection methods

Mathematics Subject Classification 45G05 · 65R20

1 Introduction

We consider the following weakly singular Hammerstein integral equation of mixed type

$$u(s) - \sum_{i=1}^{m} \int_{-1}^{1} k_i(s,t)\, \psi_i(t, u(t))\, \mathrm{d}t = f(s), \quad -1 \le s \le 1, \tag{1}$$

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where the source function f, the kernels $k_i(.,.)$ and the nonlinear functions $\psi_i(.,.)$ for $i = 1, 2, \dots, m$ are known and u is the unknown function to be determined in a Banach space $\mathbb{X}$. We consider the kernel $k_i(.,.)$ as weakly singular type which is of the form

$$k_i(s,t) = m_i(s,t) g_\alpha |s-t|, \tag{2}$$

$m_i(s,t) \in C([-1,1] \times [-1,1])$ and

$$g_\alpha(x) = \begin{cases} x^{\alpha-1}, & \text{if } 0 < \alpha < 1, \\ \log x, & \text{if } \alpha = 1. \end{cases}$$

As a reformulation of boundary value problem, this type of problem (1) arises in nonlinear physical phenomenon such as electromagnetic fluid dynamics [3].

Several authors have used numerical methods such as projection methods (Galerkin and collocation), Petrov-Galerkin method, degenerate kernel method and Nyström method ([6–8, 11, 12, 15]) to solve the various linear and nonlinear integral equations because these integral equations can't solve explicitly. Integral equations of type (1) with smooth and weakly singular kernel were solved numerically in ([2, 5, 9, 10, 13, 14, 18]) using piecewise polynomials as bases. In piecewise polynomial based projection methods the number of partitions should be increased to obtain more accurate approximate solution. So, one has to solve a large system of nonlinear equations, which take lots of time to compute. Therefore, many spectral methods have been developed by using global polynomials in last some years. In the global polynomial based projection methods, if $\mathcal{P}_n$ denotes either orthogonal or interpolatory projection operator, then $\|\mathcal{P}_n\|_\infty$ is unbounded.

We are interested to solve numerically the Hammerstein integral equations of mixed type with weakly singular kernel using Legendre spectral projection, iterated Legendre spectral projection methods. To improve convergence rates further, Legendre multi-projection and iterated Legendre multi-projection method have been used. We evaluate the convergence rates in all the above four methods in both L^2 and infinity norm, even if $\|\mathcal{P}_n\|_\infty$ is unbounded. We have given a comparison of error bounds in all the methods.

We have organized this paper in the following way. We have discussed the abstract framework for the Legendre spectral projection methods for Hammerstein integral equations of mixed type with the weakly singular kernels in Sect. 2. The convergence rates of approximated solution with exact solution have been discussed using spectral projection, iterated spectral projection, spectral multi-projection and iterated spectral multi-projection methods in Sects. 3, 4, 5 and 6, respectively, in both L^2 and infinity norm using Legendre polynomial bases. However, in the end, we have added a remark through which, we have given the comparison of error bounds in all the methods.

Throughout this paper, we assume c is a generic constant which may differ and is independent of n.

2 Hammerstein integral equation of mixed type with weakly singular kernel

In this section, we set up an abstract framework for the Hammerstein integral equation of type (1) with weakly singular kernel of type (2) on the Banach space $\mathbb{X} = C[-1,1]$ or $L^2[-1,1]$. Throughout the paper, the following assumptions are made on f, $k_i(.,.)$ and $\psi_i(.,u(.))$:

(i) $f \in C[-1,1]$,

(ii) $s_i = \sup\limits_{s,t\in[-1,1]} |m_i(s,t)| < \infty$ for $i = 1,2,\dots,m$ and $M = \sum_{i=1}^m s_i$,

(iii) $M_2 = \sup\limits_{s\in[-1,1]} \int_{-1}^1 \left|g_\alpha|s-t|\right|^2 \mathrm{d}t < \infty$, for $\frac{1}{2} < \alpha \le 1$.

(iv) The nonlinear functions $\psi_i(t,u)$ are continuous on $[-1,1]\times\mathbb{R}$ for $i = 1,2,\dots,m$. $\psi_i(t,u)$ are Lipschitz continuous in u, i.e., for any $u_1,\ u_2 \in \mathbb{R}$, $\exists$ constants $c_i > 0,\ i = 1,2,\dots,m$ such that

$$\left|\psi_i(t,u_1) - \psi_i(t,u_2)\right| \le c_i|u_1 - u_2|,\ \forall\, t \in [-1,1],$$

and $l_1 = \sup_{i=1,2,\dots,m} c_i$.

(v) The functions $\psi_i^{(0,1)}(t,u(t))$ exist and are Lipschitz continuous in u, i.e., for any $u_1,\ u_2 \in \mathbb{R}$, $\exists$ constants $q_i > 0,\ i = 1,2,\dots,m$ such that

$$\left|\psi_i^{(0,1)}(t,u_1) - \psi_i^{(0,1)}(t,u_2)\right| \le q_i|u_1 - u_2|,\ \forall\, t \in [-1,1],$$

and $l_2 = \sup_{i=1,2,\dots,m} q_i$. This implies $\psi_i^{(0,1)}(.,.) \in C([-1,1]\times\mathbb{R})$.

Define the integral operator $\mathcal{K}_i : \mathbb{X} \to \mathbb{X},\ i = 1,2,\dots,m$ by

$$(\mathcal{K}_i\psi_i)u(s) = \int_{-1}^1 k_i(s,t)\psi_i(t,u(t))\mathrm{d}t,\ s \in [-1,1].$$

Then the equation (1) can be written in the following operator equation

$$u - \sum_{i=1}^m \mathcal{K}_i\psi_i(u) = f. \tag{3}$$

Next, we define the operator $\mathcal{T}$ on $\mathbb{X}$ by

$$\mathcal{T}u = f + \sum_{i=1}^m \mathcal{K}_i\psi_i(u),\ u \in \mathbb{X}, \tag{4}$$

then the equation (3) can be written as

$$u = \mathcal{T}u. \tag{5}$$

Define the Fréchet derivatives of $\sum_{i=1}^m \mathcal{K}_i\psi_i(u)$ at u_0 by

$$\sum_{i=1}^{m}(\mathcal{K}_i\psi_i)'(u_0)u(s) = \sum_{i=1}^{m}\int_{-1}^{1} k_i(s,t)\psi_i^{(0,1)}(t,u_0(t))u(t)\mathrm{d}t.$$

Lemma 2.1 ([16]) *Let $m_i(s,t) \in C([-1,1]\times[-1,1])$ and $g_\alpha|s-t|$ be the weakly singular part of the kernel $k_i(s,t)$ for $1 \le i \le m$. Then for any $s_1, s_2 \in [-1,1]$, we have the followings:*

(i) $\lim_{s_1\to s_2}\int_{-1}^{1} |m_i(s_1,t)-m_i(s_2,t)|^2\mathrm{d}t \to 0$, for $1 \le i \le m$,

(ii) $\lim_{s_1\to s_2}\int_{-1}^{1} |g_\alpha|s_1-t|-g_\alpha|s_2-t||^2\mathrm{d}t \to 0$, for $\frac{1}{2} < \alpha \le 1$.

The next theorem shows the existence and uniqueness of the solution of equation (5).

Theorem 2.2 *Let $\mathbb{X}=C[-1,1]$, $f\in\mathbb{X}$ and $g_\alpha|s-t|$ satisfies the assumption (iii) with $m_i(.,.)\in C([-1,1]\times[-1,1])$ and $s_i = \sup_{s,t\in[-1,1]}|m_i(s,t)| < \infty$. Let $\psi_i(t,u(t))\in C([-1,1]\times\mathbb{R})$ satisfy the assumption (iv) with $\sqrt{2M_2Ml_1} < 1$, where $M=\sum_{i=1}^{m} s_i$ and $l_1 = \sup_{i=1,2,\ldots,m} c_i$. Then the operator equation $\mathcal{T}u = u$ has a unique isolated solution $u_0\in\mathbb{X}$, i.e., $\mathcal{T}u_0 = u_0$.*

Proof The proof follows exactly by using similar technique given in Theorem-2.4 of [12]. □

We will first approximate the space $\mathbb{X}$ by a finite dimensional space $\mathbb{X}_n$. We consider $\mathbb{X}_n = \mathrm{span}\{\phi_0,\phi_1,\ldots,\phi_n\}$ as the sequence of Legendre polynomial subspaces of $\mathbb{X}$ of degree $\le n$. Define $L_i(s) = \sqrt{\frac{2i+1}{2}}\phi_i(s)$, $i=0,1,\ldots,n$. Since L_i and L_j's are polynomials

$$\langle L_i, L_j\rangle = \int_{-1}^{1} L_i(t)\overline{L_j(t)}\mathrm{d}t = \int_{-1}^{1} L_i(t)L_j(t)\mathrm{d}t = \delta_{i,j} \tag{6}$$

for $i,j = 0,1,\ldots,n$.

Then the Legendre polynomials $\{L_0,L_1,\ldots,L_n\}$ be the orthonormal bases for the subspaces $\mathbb{X}_n$ of $\mathbb{X}$ of degree $\le n$. Now we need to introduce the Legendre orthogonal and Legendre interpolatory projection operator.

Let $\mathcal{P}_n^G : \mathbb{X}\to\mathbb{X}_n$ be the orthogonal projection defined by

$$\mathcal{P}_n^G u = \sum_{j=0}^{n}\langle u, L_j\rangle L_j,\ \ u\in\mathbb{X}, \tag{7}$$

where $\langle u, L_j\rangle = \int_{-1}^{1} u(t)L_j(t)\mathrm{d}t$.

Let $\{\tau_0, \tau_1, \dots, \tau_n\}$ be the zeros of Legendre polynomial of degree $n+1$ and define the interpolatory projection $\mathcal{P}_n^C : \mathbb{X} \to \mathbb{X}_n$ by

$$\mathcal{P}_n^C u \in \mathbb{X}_n,\ \mathcal{P}_n^C u(\tau_i) = u(\tau_i),\ i = 0, 1, \dots, n,\ u \in \mathbb{X}. \tag{8}$$

Now onwards, we assume that the projection operator $\mathcal{P}_n : \mathbb{X} \to \mathbb{X}_n$ is either orthogonal projection $\mathcal{P}_n^G$ or interpolatory projection $\mathcal{P}_n^C$ for notational convenience.

Lemma 2.3 ([4]) *Consider $\mathcal{P}_n = \mathcal{P}_n^G$ or $\mathcal{P}_n^C$ as the projection operator is defined to be Legendre orthogonal projection or Legendre interpolatory projection operator. Then the following conditions hold:*

(i) *For $u \in C[-1, 1]$, $\|\mathcal{P}_n u\|_{L^2} \le p\|u\|_\infty$, where p is a constant independent of n.*
(ii) *There exists a constant $c > 0$ such that for any $u \in \mathbb{X}$ and any $n \in \mathbb{N}$,*

$$\|u - \mathcal{P}_n u\|_{L^2} \le c \inf_{\phi \in \mathbb{X}_n} \|u - \phi\|_{L^2} \to 0 \text{ as } n \to \infty.$$

(iii) *For any $u \in C^r[-1, 1]$, there exists a constant c independent of n such that*

$$\begin{aligned}
&\|\mathcal{P}_n u - u\|_{L^2} \le c n^{-r} \|u^{(r)}\|_{L^2},\\
&\|\mathcal{P}_n^G u - u\|_\infty \le c n^{\frac{3}{4} - r} \|u^{(r)}\|_\infty,\\
&\|\mathcal{P}_n^C u - u\|_\infty \le c n^{\frac{1}{2} - r} \|u^{(r)}\|_\infty.
\end{aligned}$$

(iv) *For any $u \in C^r[-1, 1]$, there exists a constant c independent of n such that*

$$\|u - \mathcal{P}_n^G u\|_\infty \le c n^{\frac{1}{2} - r} V(u^{(r)}),$$

where $V(u^{(r)})$ denotes the total variation of $u^{(r)}$.

Note that $\|u - \mathcal{P}_n u\|_\infty \nrightarrow 0$ as $n \to \infty$ for any $u \in C[-1, 1]$.

Lemma 2.4 ([1]) *Let $\mathbb{X}$ be a Banach space and $\mathcal{T}, \mathcal{T}_n \in \mathbb{BL}(\mathbb{X})$. If $\mathcal{T}_n$ is norm convergent to $\mathcal{T}$ or $\mathcal{T}_n$ is ν-convergent to $\mathcal{T}$ and $(\mathcal{I} - \mathcal{T})^{-1}$ exists and bounded on $\mathbb{X}$, then $(\mathcal{I} - \mathcal{T}_n)^{-1}$ exists and uniformly bounded on $\mathbb{X}$* for *sufficiently large n.*

3 Legendre spectral projection methods

In this section, Legendre spectral projection methods for weakly singular Hammerstein integral equation of mixed type are being discussed. The convergence rates for approximated solution with exact solution have been evaluated in both L^2 and infinity norm.

The Legendre spectral projection methods for the equation (3) is to find an approximate solution $u_n \in \mathbb{X}_n$ such that

$$u_n - \sum_{i=1}^{m} \mathcal{P}_n \mathcal{K}_i \psi_i(u_n) = \mathcal{P}_n f. \tag{9}$$

If $\mathcal{P}_n = \mathcal{P}_n^G$, then the above scheme leads to Legendre Galerkin method, whereas if $\mathcal{P}_n = \mathcal{P}_n^C$, we get the Legendre collocation method.

Let $\mathcal{T}_n$ be the operator defined by

$$\mathcal{T}_n u := \sum_{i=1}^{m} \mathcal{P}_n \mathcal{K}_i \psi_i(u) + \mathcal{P}_n f, \ u \in \mathbb{X}.$$

Then equation (9) can be written as

$$u_n = \mathcal{T}_n u_n.$$

We need the following lemma and theorem for the convergence rates of the approximate solution u_n to the exact solution u_0.

Lemma 3.1 *For any* $u, \ v \in L^2[-1,1]$ *or* $C[-1,1]$, *the followings hold*

$$\left\| \sum_{i=1}^{m} (\mathcal{K}_i \psi_i)(u_0) - \sum_{i=1}^{m} (\mathcal{K}_i \psi_i)(u) \right\|_\infty \leq \sqrt{M_2} M l_1 \|u_0 - u\|_{L^2}, \tag{10}$$

$$\left\| \left[\sum_{i=1}^{m} (\mathcal{K}_i \psi_i)'(u_0) - \sum_{i=1}^{m} (\mathcal{K}_i \psi_i)'(u) \right] v \right\|_\infty \leq \sqrt{M_2} M l_2 \|u_0 - u\|_{L^2} \|v\|_\infty, \tag{11}$$

$$\left\| \left[\sum_{i=1}^{m} (\mathcal{K}_i \psi_i)'(u_0) - \sum_{i=1}^{m} (\mathcal{K}_i \psi_i)'(u) \right] v \right\|_\infty \leq \sqrt{M_2} M l_2 \|u_0 - u\|_\infty \|v\|_{L^2}. \tag{12}$$

Proof Using Lipschitz continuity of $\psi_i(., u(.))$ and Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
&\left\| \sum_{i=1}^{m} (\mathcal{K}_i \psi_i)(u_0) - \sum_{i=1}^{m} (\mathcal{K}_i \psi_i)(u) \right\|_\infty \\
&\quad = \sup_{s \in [-1,1]} \left| \sum_{i=1}^{m} \int_{-1}^{1} m_i(s,t) g_\alpha |s-t| [\psi_i(t, u_0(t)) - \psi_i(t, u(t))] \mathrm{d}t \right| \\
&\quad \leq \sum_{i=1}^{m} \sup_{s,t \in [-1,1]} |m_i(s,t)| \sup_{s \in [-1,1]} \int_{-1}^{1} \big| g_\alpha |s-t| [\psi_i(t, u_0(t)) - \psi_i(t, u(t))] \big| \mathrm{d}t \\
&\quad \leq \sup_{i=1,2,\dots,m} c_i \sum_{i=1}^{m} s_i \sup_{s \in [-1,1]} \|g_\alpha |s-t|\|_{L^2} \left(\int_{-1}^{1} |u_0(t) - u(t)|^2 \mathrm{d}t \right)^{\frac{1}{2}} \\
&\quad \leq \sqrt{M_2} M l_1 \|u_0 - u\|_{L^2},
\end{aligned}$$

which completes the proof of first inequality. Similarly, using Lipschitz continuity and Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
&\left\| \left[\sum_{i=1}^{m} (\mathcal{K}_i \psi_i)'(u_0) - \sum_{i=1}^{m} (\mathcal{K}_i \psi_i)'(u) \right] v \right\|_\infty \\
&\leq \sup_{s\in[-1,1]} \sum_{i=1}^{m} \left| \int_{-1}^{1} m_i(s,t) g_\alpha |s-t| \left[\psi_i^{(0,1)}(t,u_0(t)) - \psi_i^{(0,1)}(t,u(t)) \right] v(t) \mathrm{d}t \right| \\
&\leq \sum_{i=1}^{m} \sup_{s,t\in[-1,1]} |m_i(s,t)| \\
&\quad \int_{-1}^{1} \left| g_\alpha |s-t| \left[\psi_i^{(0,1)}(t,u_0(t)) - \psi_i^{(0,1)}(t,u(t)) \right] v(t) \right| \mathrm{d}t \\
&\leq \sup_{i=1,2,\ldots,m} q_i \sum_{i=1}^{m} s_i \sup_{s\in[-1,1]} \| g_\alpha |s-t| \|_{L^2} \left(\int_{-1}^{1} |u_0(t) - u(t)|^2 |v(t)|^2 \mathrm{d}t \right)^{\frac{1}{2}} \\
&\leq \sqrt{M_2} M l_2 \|u_0 - u\|_{L^2} \|v\|_\infty,
\end{aligned} \tag{13}$$

which completes the proof of (11). From estimate (13), we get

$$\left\| \left[\sum_{i=1}^{m} (\mathcal{K}_i \psi_i)'(u_0) - \sum_{i=1}^{m} (\mathcal{K}_i \psi_i)'(u) \right] v \right\|_\infty \leq \sqrt{M_2} M l_2 \|u_0 - u\|_\infty \|v\|_{L^2}.$$

This completes the proof of the lemma. □

Theorem 3.2 *Let $\sum_{i=1}^{m} (\mathcal{K}_i \psi_i)'(u_0)$ be the Fréchet derivative of $\sum_{i=1}^{m} (\mathcal{K}_i \psi_i)(u)$ at u_0. Then $\left\|(\mathcal{I} - \mathcal{P}_n) \sum_{i=1}^{m} (\mathcal{K}_i \psi_i)'(u_0)\right\|_{L^2} \to 0$ as $n \to \infty$.*

Proof To prove $\sum_{i=1}^{m} (\mathcal{K}_i \psi_i)'(u_0)$ is a compact operator, we have to show that $\sum_{i=1}^{m} (\mathcal{K}_i \psi_i)'(u_0)$ is uniformly bounded and equicontinuous.

Now using Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
\left\| \sum_{i=1}^{m} (\mathcal{K}_i \psi_i)'(u_0) u \right\|_\infty &= \sup_{s\in[-1,1]} \left| \int_{-1}^{1} \sum_{i=1}^{m} m_i(s,t) g_\alpha |s-t| \psi_i^{(0,1)}(t,u_0(t)) u(t) \mathrm{d}t \right| \\
&\leq \sup_{i=1,2,\ldots,m} \sup_{t\in[-1,1]} \left| \psi_i^{(0,1)}(t,u_0(t)) \right| \sum_{i=1}^{m} \sup_{s,t\in[-1,1]} |m_i(s,t)| \\
&\quad \times \int_{-1}^{1} |g_\alpha |s-t| u(t)| \mathrm{d}t \\
&\leq \sup_{i=1,2,\ldots,m} d_i \sum_{i=1}^{m} s_i \| g_\alpha |s-t| \|_{L^2} \|u\|_{L^2} \\
&\leq M d \sqrt{M_2} \|u\|_{L^2} \leq M d \sqrt{2 M_2} \|u\|_\infty,
\end{aligned} \tag{14}$$

where $d_i = \sup_{t\in[-1,1]} |\psi_i^{(0,1)}(t, u_0(t))|$ and $d = \sup_{i=1,2,\ldots,m} d_i$. Hence, $\sum_{i=1}^m (\mathcal{K}_i\psi_i)'(u_0)$ is uniformly bounded. Next to show equicontinuity, consider

$$\begin{aligned}
&\left|\sum_{i=1}^m (\mathcal{K}_i\psi_i)'(u_0)u(s_1) - \sum_{i=1}^m (\mathcal{K}_i\psi_i)'(u_0)u(s_2)\right| \\
&\quad\le \sup_{i=1,2,\ldots,m}\ \sup_{t\in[-1,1]} |\psi_i^{(0,1)}(t, u_0(t))| \sum_{i=1}^m \int_{-1}^1 \Big|[m_i(s_1,t)g_\alpha|s_1-t| \\
&\quad\quad -m_i(s_2,t)g_\alpha|s_2-t|]u(t)\Big|\mathrm{d}t, \\
&\quad\le d\sum_{i=1}^m \sup_{s,t\in[-1,1]} |m_i(s_1,t)| \int_{-1}^1 |[g_\alpha|s_1-t| - g_\alpha|s_2-t|]u(t)|\mathrm{d}t \\
&\quad\quad + d\sum_{i=1}^m \int_{-1}^1 |m_i(s_1,t) - m_i(s_2,t)||g_\alpha|s_2-t|u(t)|\mathrm{d}t.
\end{aligned} \tag{15}$$

Now using Cauchy-Schwarz inequality and Lemma 2.1 in the above estimate, we obtain

$$\begin{aligned}
&\left|\sum_{i=1}^m (\mathcal{K}_i\psi_i)'(u_0)u(s_1) - \sum_{i=1}^m (\mathcal{K}_i\psi_i)'(u_0)u(s_2)\right| \\
&\quad\le dM\left(\int_{-1}^1 |g_\alpha|s_1-t| - g_\alpha|s_2-t||^2\mathrm{d}t\right)^{\frac{1}{2}} \|u\|_{L^2} \\
&\quad\quad + d\sum_{i=1}^m \left(\int_{-1}^1 |m_i(s_1,t) - m_i(s_2,t)|^2\right)^{\frac{1}{2}} \|g_\alpha|s_2-t|\|_{L^2}\|u\|_\infty \\
&\quad\to 0 \ \text{ as } \ s_1 \to s_2.
\end{aligned}$$

This proves that $\sum_{i=1}^m (\mathcal{K}_i\psi_i)'(u_0)$ is equicontinuous. Hence by Arzelá-Ascoli's theorem $\sum_{i=1}^m (\mathcal{K}_i\psi_i)'(u_0)$ is a compact operator.

Let B be a closed unit ball in $\mathbb{X}$. Thus, $S = \left\{\sum_{i=1}^m (\mathcal{K}_i\psi_i)'(u_0)u \,:\, u \in B\right\}$ is relatively compact set in $\mathbb{X}$. By using Lemma 2.3, we get

$$\begin{aligned}
\left\|(\mathcal{I}-\mathcal{P}_n)\sum_{i=1}^m (\mathcal{K}_i\psi_i)'(u_0)\right\|_{L^2} &= \sup\left\{\left\|(\mathcal{I}-\mathcal{P}_n)\sum_{i=1}^m (\mathcal{K}_i\psi_i)'(u_0)u\right\|_{L^2} \,:\, u\in B\right\} \\
&= \sup\left\{\|(\mathcal{I}-\mathcal{P}_n)y\|_{L^2} \,:\, y \in S\right\} \to 0 \ \ as\ n\to\infty.
\end{aligned}$$

Thus, the proof is completed. □

For rest of the paper, we assume that 1 is not an eigenvalue of the operator $\sum_{i=1}^m (\mathcal{K}_i\psi_i)'(u_0)$.

Theorem 3.3 *Let $u_0 \in C^r[-1,1]$, $r \geq 1$. Then for sufficiently large n, the operator $(\mathcal{I} - \mathcal{T}_n'(u_0))$ is invertible, i.e., there exists a constant $A_1 > 0$ such that $\|(\mathcal{I} - \mathcal{T}_n'(u_0))^{-1}\|_{L^2} \leq A_1 < \infty$. Also the equation (9) has a unique solution $u_n \in B(u_0, \delta) = \{u : \|u - u_0\|_{L^2} < \delta\}$ for some $\delta > 0$. Moreover, there exists a constant $0 < q < 1$, independent of n such that*

$$\frac{\alpha_n}{1+q} \leq \|u_n - u_0\|_{L^2} \leq \frac{\alpha_n}{1-q},$$

where $\alpha_n = \|(\mathcal{I} - \mathcal{T}_n'(u_0))^{-1}(\mathcal{T}_n(u_0) - \mathcal{T}(u_0))\|_{L^2}$.

Proof Using Theorem 3.2, we get

$$\begin{aligned}
&\|(\mathcal{T}_n'(u_0) - \mathcal{T}'(u_0))(u)\|_{L^2} \\
&\quad = \left\|(\mathcal{P}_n - \mathcal{I})\left[\sum_{i=1}^{m}(\mathcal{K}_i\psi_i)'(u_0)\right](u)\right\|_{L^2} \\
&\quad \leq \left\|(\mathcal{P}_n - \mathcal{I})\sum_{i=1}^{m}(\mathcal{K}_i\psi_i)'(u_0)\right\|_{L^2}\|u\|_{L^2} \to 0, \text{ as } n \to \infty.
\end{aligned}$$

This shows that $\mathcal{T}_n'(u_0)$ is norm convergent to $\mathcal{T}'(u_0)$ in L^2-norm. Since 1 is not an eigenvalue of $\mathcal{T}'(u_0)$, $(\mathcal{I} - \mathcal{T}'(u_0))$ is invertible. Then by the Lemma 2.4, $(\mathcal{I} - \mathcal{T}_n'(u_0))^{-1}$ exists and is uniformly bounded on $\mathbb{X}$ for sufficiently large n, i.e., there exists a constant $A_1 > 0$ such that $\|(\mathcal{I} - \mathcal{T}_n'(u_0))^{-1}\|_{L^2} \leq A_1 < \infty$.

Using $\|\mathcal{P}_n u\|_{L^2} \leq p\|u\|_\infty$ and estimate (11) for any $u \in B(u_0, \delta)$, we get

$$\begin{aligned}
\|\mathcal{T}_n'(u_0) - \mathcal{T}_n'(u)\|_{L^2} &= \left\|\sum_{i=1}^{m}\mathcal{P}_n(\mathcal{K}_i\psi_i)'(u_0) - \sum_{i=1}^{m}\mathcal{P}_n(\mathcal{K}_i\psi_i)'(u)\right\|_{L^2} \\
&\leq p\left\|\sum_{i=1}^{m}(\mathcal{K}_i\psi_i)'(u_0) - \sum_{i=1}^{m}(\mathcal{K}_i\psi_i)'(u)\right\|_\infty \\
&\leq \sqrt{M_2}Ml_2\|u_0 - u\|_{L^2} \leq \sqrt{M_2}Ml_2\delta.
\end{aligned} \tag{16}$$

Now using the estimate (16) and $\|(\mathcal{I} - \mathcal{T}_n'(u_0))^{-1}\|_{L^2} \leq A_1$, we obtain

$$\begin{aligned}
\sup_{\|u-u_0\|_{L^2} \leq \delta}\|(\mathcal{I} - \mathcal{T}_n'(u_0))^{-1}(\mathcal{T}_n'(u_0) - \mathcal{T}_n'(u))\|_{L^2} &\leq A_1\|\mathcal{T}_n'(u_0) - \mathcal{T}_n'(u)\|_{L^2} \\
&\leq A_1\sqrt{M_2}Ml_2\delta \leq q.
\end{aligned}$$

Choosing δ in such a way that $q \in (0,1)$, this proves the equation (4.4) of Theorem-2 of [19].

Now using Lemma 2.3, we get

$$\begin{aligned}\alpha_n =&\|(\mathcal{I}-\mathcal{T}_n'(u_0))^{-1}(\mathcal{T}_n(u_0)-\mathcal{T}(u_0))\|_{L^2}\\ \le& A_1\|(\mathcal{T}_n(u_0)-\mathcal{T}(u_0))\|_{L^2}\\ =& A_1\left\|(\mathcal{P}_n-\mathcal{I})\sum_{i=1}^{m}[(\mathcal{K}_i\psi_i)(u_0)+f]\right\|_{L^2}\\ =& A_1\|(\mathcal{P}_n-\mathcal{I})u_0\|_{L^2}\le A_1cn^{-r}\|u_0^{(r)}\|_{L^2}\to 0 \text{ as } n\to\infty.\end{aligned} \tag{17}$$

Choose n large enough such that $\alpha_n \le \delta(1-q)$, then equation (4.5) of Theorem-2 of [19] is satisfied. Then by applying Theorem-2 of [19], we get

$$\frac{\alpha_n}{1+q}\le\|u_n-u_0\|_{L^2}\le\frac{\alpha_n}{1-q},$$

where $\alpha_n = \|(\mathcal{I}-\mathcal{T}_n'(u_0))^{-1}(\mathcal{T}_n(u_0)-\mathcal{T}(u_0))\|_{L^2}$. This completes the proof. □

Theorem 3.4 *Let $u_0 \in C^r[-1,1]$. Let u_n^G be the Legendre Galerkin and u_n^C be the Legendre collocation approximation of the equation (9). Then the following hold*

$$\begin{aligned}&\|u_n^G-u_0\|_{L^2}=\mathcal{O}(n^{-r}),\quad \|u_n^C-u_0\|_{L^2}=\mathcal{O}(n^{-r}),\\ &\|u_n^G-u_0\|_{\infty}=\mathcal{O}\left(n^{\frac{1}{2}-r}\right),\quad \|u_n^C-u_0\|_{\infty}=\mathcal{O}\left(n^{\frac{1}{2}-r}\right).\end{aligned}$$

Proof The proof of estimates in L^2 norm follows directly from Theorem-3.3. Using equations (3) and (9), and $\|\mathcal{P}_n\|_\infty \le c\log n$ (cf., Page-147, [3]), we have

$$\begin{aligned}\|u_n-u_0\|_\infty =&\left\|\sum_{i=1}^{m}\mathcal{P}_n(\mathcal{K}_i\psi_i)(u_n)+\mathcal{P}_nf-f-\sum_{i=1}^{m}(\mathcal{K}_i\psi_i)(u_0)\right\|_\infty\\ \le&\left\|\mathcal{P}_n\left[\sum_{i=1}^{m}(\mathcal{K}_i\psi_i)(u_n)-\sum_{i=1}^{m}(\mathcal{K}_i\psi_i)(u_0)\right]\right\|_\infty\\ &+\left\|(\mathcal{P}_n-\mathcal{I})\Big[\sum_{i=1}^{m}(\mathcal{K}_i\psi_i)(u_0)+f\Big]\right\|_\infty\\ \le& c\log n\left\|\sum_{i=1}^{m}(\mathcal{K}_i\psi_i)(u_n)-\sum_{i=1}^{m}(\mathcal{K}_i\psi_i)(u_0)\right\|_\infty\\ &+\|(\mathcal{P}_n-\mathcal{I})u_0\|_\infty.\end{aligned} \tag{18}$$

Using estimate (10) in the first term of the right hand side of estimate (18), we get

$$\begin{aligned}\|u_n-u_0\|_\infty \le&\sqrt{M_2}Ml_1c\log n\|u_n-u_0\|_{L^2}\\ &+\|(\mathcal{P}_n-\mathcal{I})u_0\|_\infty.\end{aligned} \tag{19}$$

Using $u_n = u_n^G$ and $u_n = u_n^C$, then using Lemma 2.3, we obtain the desired results. □

4 Iterated Legendre spectral projection methods

In this section, the iterated Legendre spectral projection methods for Hammerstein integral equations of mixed type with weakly singular kernels have been discussed. The rate of convergence for iterated approximate solution with exact solution have been evaluated for both L^2 and infinity norm.

The iterated approximate solution $\widetilde{u}_n$ corresponding to the approximate solution u_n given by equation (9) is defined as

$$\widetilde{u}_n = \sum_{i=1}^{m} (\mathcal{K}_i \psi_i)(u_n) + f. \tag{20}$$

If $u_n = u_n^G$ in equation (20), we get iterated Legendre Galerkin solution and if $u_n = u_n^C$, then we get iterated Legendre collocation solution. To discuss the convergence of the iterated approximate solution $\widetilde{u}_n$ to the exact solution u_0, we need the following theorem.

Theorem 4.1 *Let $u_0 \in C^r[-1,1]$, $r \geq 1$ and $\widetilde{u}_n$ be the iterated Legendre Galerkin approximation $\tilde{u}_n^G$ or iterated Legendre collocation approximation u_n^C of u_0. Then the* following holds

$$\begin{aligned}\|\widetilde{u}_n - u_0\|_\infty \leq & \sqrt{M_2} M l_2 (1 + M_3 c \log n) \|u_n - u_0\|_{L^2} \|u_n - u_0\|_\infty \\ & + (1 + M_3 c \log n) \sup_{s \in [-1,1]} |\langle h(s,.), (\mathcal{P}_n - \mathcal{I})(u_0)(.)\rangle|,\end{aligned}$$

where $h(s,t) = \sum_{i=1}^{m} k_i(s,t) \psi_i^{(0,1)}(t, u_0(t))$.

Proof The steps of the proof follows similarly as in Theorem-4.1 of [16]. So, we omit it. □

Theorem 4.2 *Let $u_0 \in C^r[-1,1]$, $r \geq 1$ and $\widetilde{u}_n^G$ be the iterated Legendre Galerkin approximation of u_0. Then the following holds*

$$\|\widetilde{u}_n^G - u_0\|_{L^2} \leq \sqrt{2} \|\widetilde{u}_n^G - u_0\|_\infty \leq \begin{cases} c n^{\frac{1}{2} - r - \alpha} (1 + c \log n), & \text{for } \frac{1}{2} < \alpha < 1, \\ c n^{-\frac{1}{2} - r} (1 + c \log n) \log n, & \text{for } \alpha = 1. \end{cases}$$

Proof From Theorem-4.1, we have

$$\begin{aligned}\|\widetilde{u}_n^G - u_0\|_\infty \leq & \sqrt{M_2} M l_2 (1 + M_3 c \log n) \|u_n^G - u_0\|_{L^2} \|u_n^G - u_0\|_\infty \\ & + (1 + M_3 c \log n) \sup_{s \in [-1,1]} |\langle h(s,.), (\mathcal{P}_n^G - \mathcal{I})(u_0)(.)\rangle|.\end{aligned} \tag{21}$$

Since $\mathcal{P}_n^G$ be the orthogonal projection from the space $\mathbb{X}$ into $\mathbb{X}_n$, then we have

$$\langle w, (\mathcal{I} - \mathcal{P}_n^G)u \rangle = 0, \ \ \forall\, w \in \mathbb{X}_n.$$

By using Hölder's inequality, Lemma 2.3, and Theorems 2 and 3 of [17], we obtain

$$\begin{aligned} \sup_{s\in[-1,1]} |\langle h(s,.), (\mathcal{P}_n^G - \mathcal{I})(u_0)(.)\rangle| \leq & \sup_{s\in[-1,1]} |\langle h(s,.) - \phi(s,.), (\mathcal{P}_n^G - \mathcal{I})(u_0)(.)\rangle| \\ \leq & \|h_s - \phi_s\|_{L^1} \|(\mathcal{P}_n^G - \mathcal{I})(u_0)\|_\infty \\ \leq & V(u_0^{(r)}) \begin{cases} cn^{\frac{1}{2}-r-\alpha}, & \text{for } \frac{1}{2} < \alpha < 1, \\ cn^{-\frac{1}{2}-r} \log n, & \text{for } \alpha = 1, \end{cases} \end{aligned} \tag{22}$$

where $\phi_s \in \mathbb{X}_n$. Now using Theorem-3.4 and estimate (22) in estimate (21), we obtain

$$\begin{aligned} \|\widetilde{u}_n^G - u_0\|_\infty \leq & (1 + c\log n)cn^{-r} n^{\frac{1}{2}-r} \\ & + (1 + c\log n) \begin{cases} cn^{\frac{1}{2}-r-\alpha}, & \text{for } \frac{1}{2} < \alpha < 1, \\ cn^{-\frac{1}{2}-r} \log n, & \text{for } \alpha = 1, \end{cases} \\ \leq & \begin{cases} cn^{\frac{1}{2}-r-\alpha}(1 + c\log n), & \text{for } \frac{1}{2} < \alpha < 1, \\ cn^{-\frac{1}{2}-r}(1 + c\log n)\log n, & \text{for } \alpha = 1. \end{cases} \end{aligned}$$

Hence,

$$\|\widetilde{u}_n^G - u_0\|_{L^2} \leq \sqrt{2}\|\widetilde{u}_n^G - u_0\|_\infty \leq \begin{cases} cn^{\frac{1}{2}-r-\alpha}(1 + c\log n), & \text{for } \frac{1}{2} < \alpha < 1, \\ cn^{-\frac{1}{2}-r}(1 + c\log n)\log n, & \text{for } \alpha = 1. \end{cases}$$

Thus, the proof is completed. □

Theorem 4.3 *Let $u_0 \in C^r[-1,1]$, $r \geq 1$ and $\widetilde{u}_n^C$ be the iterated Legendre collocation approximation of u_0. Then the following holds*

$$\|\widetilde{u}_n^C - u_0\|_{L^2} \leq \sqrt{2}\|\widetilde{u}_n^C - u_0\|_\infty \leq (1 + c\log n)cn^{-r}.$$

Proof From Theorem-4.1, we have

$$\begin{aligned} \|\widetilde{u}_n^C - u_0\|_\infty \leq & \sqrt{M_2} M l_2 (1 + M_3 c\log n) \|u_n^C - u_0\|_{L^2} \|u_n^C - u_0\|_\infty \\ & + (1 + M_3 c\log n) \sup_{s\in[-1,1]} |\langle h(s,.), (\mathcal{P}_n^C - \mathcal{I})(u_0)(.)\rangle|. \end{aligned} \tag{23}$$

Using Cauchy-Schwarz inequality and Lemma 2.3, we get

$$\sup_{s\in[-1,1]} |\langle h(s,.), (\mathcal{P}_n^C - \mathcal{I})(u_0)(.)\rangle| \le \sup_{s\in[-1,1]} \|h_s\|_{L^2} \|(\mathcal{P}_n^C - \mathcal{I})(u_0)\|_{L^2}$$
$$\le \sup_{s\in[-1,1]} \|h_s\|_{L^2} cn^{-r} \|u_0^{(r)}\|_\infty. \tag{24}$$

Substituting estimate (24) and Theorem-3.4 in equation (23), we obtain

$$\begin{aligned}\|\widetilde{u}_n^C - u_0\|_\infty \le& c(1 + c\log n)cn^{-r} n^{\frac{1}{2}-r} + c(1 + c\log n)cn^{-r}\\ \le&(1 + c\log n)cn^{-r}.\end{aligned}$$

Hence,

$$\|\widetilde{u}_n^C - u_0\|_{L^2} \le \sqrt{2}\|\widetilde{u}_n^C - u_0\|_\infty \le (1 + c\log n)cn^{-r}.$$

This completes the proof. □

5 Legendre spectral multi-projection methods

To improve the results further, we use now the Legendre spectral multi-projection methods for weakly singular Hammerstein integral equations of mixed type. Define the multi-projection operator $(\mathcal{K}_{n,i}^M \psi_i) : \mathbb{X} \to \mathbb{X}$ for $i = 1, 2, \ldots, m$ by

$$(\mathcal{K}_{n,i}^M \psi_i)(u) := \mathcal{P}_n(\mathcal{K}_i\psi_i)(u) + (\mathcal{K}_i\psi_i)(\mathcal{P}_n u) - \mathcal{P}_n(\mathcal{K}_i\psi_i)(\mathcal{P}_n u). \tag{25}$$

The multi-projection method for equation (3) is to find an approximate solution $u_n^M \in \mathbb{X}$ such that

$$u_n^M - \sum_{i=1}^m (\mathcal{K}_{n,i}^M \psi_i)(u_n^M) = f. \tag{26}$$

If $\mathcal{P}_n = \mathcal{P}_n^G$, then equation (26) leads to multi-Galerkin method and if $\mathcal{P}_n = \mathcal{P}_n^C$, then equation (26) leads to multi-collocation method. Let

$$\mathcal{T}_n^M(u) = \sum_{i=1}^m (\mathcal{K}_{n,i}^M \psi_i)(u) + f, \quad u \in \mathbb{X},$$

then equation (26) can be written as

$$u_n^M = \mathcal{T}_n^M(u_n^M).$$

The Fréchet derivative of $\mathcal{T}_n^M u$ at u_0 is given by

$$
\begin{aligned}
\mathcal{T}_n^{M'}(u_0) &= \sum_{i=1}^{m} (\mathcal{K}_{n,i}^{M}\psi_i)'(u_0) \\
&= \sum_{i=1}^{m} \mathcal{P}_n(\mathcal{K}_i\psi_i)'(u_0) + \sum_{i=1}^{m} (\mathcal{K}_i\psi_i)'(\mathcal{P}_n u_0)\mathcal{P}_n \\
&\quad - \sum_{i=1}^{m} \mathcal{P}_n(\mathcal{K}_i\psi_i)'(\mathcal{P}_n u_0)\mathcal{P}_n.
\end{aligned}
$$

To discuss the convergence rates of u_n^M to u_0, we need the following lemma and theorem.

Lemma 5.1 *For any $x, y \in \mathbb{X}$, we have*

$$
\begin{aligned}
&\left\| \sum_{i=1}^{m} (\mathcal{K}_{n,i}^{M}\psi_i)'(x) - \sum_{i=1}^{m} (\mathcal{K}_{n,i}^{M}\psi_i)'(y) \right\|_{L^2} \\
&\quad \leq \sqrt{M_2} M l_2 p[1 + \sqrt{2}p(1+p)] \|x - y\|_{L^2}.
\end{aligned} \tag{27}
$$

Proof Using estimate (11), we obtain

$$
\begin{aligned}
&\left\| \sum_{i=1}^{m} (\mathcal{K}_{n,i}^{M}\psi_i)'(x) - \sum_{i=1}^{m} (\mathcal{K}_{n,i}^{M}\psi_i)'(y) \right\|_{L^2} \\
&\quad \leq \left\| \mathcal{P}_n \Big[\sum_{i=1}^{m} (\mathcal{K}_i\psi_i)'(x) - \sum_{i=1}^{m} (\mathcal{K}_i\psi_i)'(y) \Big] \right\|_{L^2} \\
&\qquad + \left\| (\mathcal{I} - \mathcal{P}_n) \Big[\sum_{i=1}^{m} (\mathcal{K}_i\psi_i)'(\mathcal{P}_n x) - \sum_{i=1}^{m} (\mathcal{K}_i\psi_i)'(\mathcal{P}_n y) \Big] \mathcal{P}_n \right\|_{L^2} \\
&\quad \leq p \left\| \sum_{i=1}^{m} (\mathcal{K}_i\psi_i)'(x) - \sum_{i=1}^{m} (\mathcal{K}_i\psi_i)'(y) \right\|_{\infty} \\
&\qquad + \sqrt{2}p(1+p) \left\| \sum_{i=1}^{m} (\mathcal{K}_i\psi_i)'(\mathcal{P}_n x) - \sum_{i=1}^{m} (\mathcal{K}_i\psi_i)'(\mathcal{P}_n y) \right\|_{\infty} \\
&\quad \leq p\sqrt{M_2} M l_2 \|x - y\|_{L^2} + \sqrt{2}p(1+p)\sqrt{M_2} M l_2 \|\mathcal{P}_n x - \mathcal{P}_n y\|_{L^2} \\
&\quad = p\sqrt{M_2} M l_2 [1 + \sqrt{2}p(1+p)] \|x - y\|_{L^2}.
\end{aligned}
$$

The proof is completed. □

Theorem 5.2 *Let $u_0 \in C^r[-1, 1]$, $r \geq 1$. Then the operator $(\mathcal{I} - \mathcal{T}_n^{M'}(u_0))$ is invertible on $C[-1, 1]$ for sufficiently large n, and there exists a constant $A_2 > 0$ independent of n such that $\|(\mathcal{I} - \mathcal{T}_n^{M'}(u_0))^{-1}\|_{L^2} \leq A_2 < \infty$.*

Proof For any $u \in \mathbb{X}$, we have

$$
\begin{aligned}
&\|[\mathcal{T}_n^{M\prime}(u_0) - \mathcal{T}'(u_0)]u\|_{L^2}\\
&= \left\| \Big[\sum_{i=1}^{m} (\mathcal{P}_n - \mathcal{I})(\mathcal{K}_i\psi_i)'(u_0) - \sum_{i=1}^{m} (\mathcal{P}_n - \mathcal{I})(\mathcal{K}_i\psi_i)'(\mathcal{P}_n u_0)\mathcal{P}_n \Big] u \right\|_{L^2}\\
&= \left\| (\mathcal{P}_n - \mathcal{I}) \Big[\sum_{i=1}^{m} (\mathcal{K}_i\psi_i)'(u_0) - \sum_{i=1}^{m} (\mathcal{K}_i\psi_i)'(u_0)\mathcal{P}_n \right.\\
&\quad \left. + \sum_{i=1}^{m} (\mathcal{K}_i\psi_i)'(u_0)\mathcal{P}_n - \sum_{i=1}^{m} (\mathcal{K}_i\psi_i)'(\mathcal{P}_n u_0)\mathcal{P}_n \Big] u \right\|_{L^2}\\
&\leq \left\| (\mathcal{P}_n - \mathcal{I}) \sum_{i=1}^{m} (\mathcal{K}_i\psi_i)'(u_0)(\mathcal{I} - \mathcal{P}_n)u \right\|_{L^2}\\
&\quad + \left\| (\mathcal{P}_n - \mathcal{I}) \Big[\sum_{i=1}^{m} (\mathcal{K}_i\psi_i)'(u_0) - \sum_{i=1}^{m} (\mathcal{K}_i\psi_i)'(\mathcal{P}_n u_0) \Big] \mathcal{P}_n u \right\|_{L^2}.
\end{aligned}
\tag{28}
$$

The first term of right hand side of estimate (28) becomes

$$
\begin{aligned}
&\left\| (\mathcal{P}_n - \mathcal{I}) \sum_{i=1}^{m} (\mathcal{K}_i\psi_i)'(u_0)(\mathcal{I} - \mathcal{P}_n)u \right\|_{L^2}\\
&\quad \leq \left\| (\mathcal{P}_n - \mathcal{I}) \sum_{i=1}^{m} (\mathcal{K}_i\psi_i)'(u_0) \right\|_{L^2} \|(\mathcal{I} - \mathcal{P}_n)u\|_{L^2}\\
&\quad \leq (1+p) \left\| (\mathcal{P}_n - \mathcal{I}) \sum_{i=1}^{m} (\mathcal{K}_i\psi_i)'(u_0) \right\|_{L^2} \|u\|_{L^2}.
\end{aligned}
\tag{29}
$$

Using equation (12) and Lemma 2.3 in the second term of right hand side of estimate (28), we obtain

$$
\begin{aligned}
&\left\| (\mathcal{P}_n - \mathcal{I}) \Big[\sum_{i=1}^{m} (\mathcal{K}_i\psi_i)'(u_0) - \sum_{i=1}^{m} (\mathcal{K}_i\psi_i)'(\mathcal{P}_n u_0) \Big] \mathcal{P}_n u \right\|_{L^2}\\
&\quad \leq \sqrt{2}(1+p) \left\| \Big[\sum_{i=1}^{m} (\mathcal{K}_i\psi_i)'(u_0) - \sum_{i=1}^{m} (\mathcal{K}_i\psi_i)'(\mathcal{P}_n u_0) \Big] \mathcal{P}_n u \right\|_{\infty}\\
&\quad \leq \sqrt{2}(1+p)\sqrt{M_2} M l_2 \|u_0 - \mathcal{P}_n u_0\|_\infty \|\mathcal{P}_n u\|_{L^2}\\
&\quad \leq \sqrt{2M_2}\, p(1+p) M l_2 c n^{\beta - r} \|u_0^{(r)}\|_\infty \|u\|_{L^2}.
\end{aligned}
\tag{30}
$$

Substituting estimates (29) and (30) in estimate (28), and then using Theorem-3.2, we obtain

$$\begin{aligned}\|\mathcal{T}_n^{M'}(u_0) - \mathcal{T}'(u_0)\|_{L^2} \leq &(1+p)\left\|(\mathcal{P}_n - \mathcal{I})\sum_{i=1}^{m}(\mathcal{K}_i\psi_i)'(u_0)\right\|_{L^2} \\ &+ \sqrt{2M_2}p(1+p)Ml_2cn^{\beta-r}\|u_0^{(r)}\|_\infty \\ \to &0 \text{ as } n \to \infty \text{ and } \beta < r = 1, 2, \cdots.\end{aligned}$$

Hence, $\mathcal{T}_n^{M'}(u_0)$ is norm convergent to $\mathcal{T}'(u_0)$ in L^2-norm for $r > \beta$. Since 1 is not an eigenvalue of $(\mathcal{K}\psi)'(u_0)$, $(\mathcal{I} - \mathcal{T}'(u_0))$ is invertible on $\mathbb{X}$. Then by Lemma 2.4, $(\mathcal{I} - \mathcal{T}_n^{M'}(u_0))^{-1}$ exists and is uniformly bounded on $\mathbb{X}$, for some sufficiently large n, i.e., there exists some $A_2 > 0$ such that $\|(\mathcal{I} - \mathcal{T}_n^{M'}(u_0))^{-1}\|_{L^2} \leq A_2 < \infty$. Thus, the proof is completed. ☐

Theorem 5.3 *Let $u_0 \in C^r[-1,1]$, $r \geq 1$. Then for sufficiently large n, the equation (26) has a unique solution $u_n^M \in B(u_0, \delta) = \{u : \|u - u_0\|_{L^2} \leq \delta\}$ for some $\delta > 0$. Further, there exists a constant $0 < q < 1$, independent of n such that*

$$\frac{\beta_n}{1+q} \leq \|u_n^M - u_0\|_{L^2} \leq \frac{\beta_n}{1-q},$$

where $\beta_n = \|(\mathcal{I} - \mathcal{T}_n^{M'}(u_0))^{-1}(\mathcal{T}_n^M(u_0) - \mathcal{T}(u_0))\|_{L^2}$.

Proof Using Lemma 5.1, we obtain

$$\begin{aligned}\|\mathcal{T}_n^{M'}(u_0) - \mathcal{T}_n^{M'}(u)\|_{L^2} =&\left\|\sum_{i=1}^{m}(\mathcal{K}_{n,i}^M\psi_i)'(u_0) - \sum_{i=1}^{m}(\mathcal{K}_{n,i}^M\psi_i)'(u)\right\|_{L^2} \\ \leq& p\sqrt{M_2}Ml_2[1 + \sqrt{2}p(1+p)]\|u_0 - u\|_{L^2} \leq c\delta.\end{aligned} \tag{31}$$

Using $\|(\mathcal{I} - \mathcal{T}_n^{M'}(u_0))^{-1}\|_{L^2} \leq A_2$ and estimate (31), we obtain

$$\begin{aligned}&\sup_{\|u-u_0\|_{L^2}\leq\delta} \|(\mathcal{I} - \mathcal{T}_n^{M'}(u_0))^{-1}(\mathcal{T}_n^{M'}(u_0) - \mathcal{T}_n^{M'}(u))\|_{L^2} \\ &\leq A_2\|\mathcal{T}_n^{M'}(u_0) - \mathcal{T}_n^{M'}(u)\|_{L^2} \leq A_2c\delta \leq q.\end{aligned}$$

Choosing δ in such a way that $0 < q < 1$. This proves equation (4.4) of Theorem-2 of [19].

Now using Theorem-5.2 and Lemma 2.3, we obtain

$$
\begin{aligned}
\beta_n \leq & A_2 \|\mathcal{T}_n^M(u_0) - \mathcal{T}(u_0)\|_{L^2} \\
\leq & A_2 \Bigg\| \sum_{i=1}^m \mathcal{P}_n(\mathcal{K}_i\psi_i)(u_0) + \sum_{i=1}^m (\mathcal{K}_i\psi_i)(\mathcal{P}_n u_0) \\
& - \sum_{i=1}^m \mathcal{P}_n(\mathcal{K}_i\psi_i)(\mathcal{P}_n u_0) - \sum_{i=1}^m (\mathcal{K}_i\psi_i)(u_0) \Bigg\|_{L^2} \\
\leq & A_2 \Bigg\| (\mathcal{P}_n - \mathcal{I}) \Big[\sum_{i=1}^m (\mathcal{K}_i\psi_i)(u_0) - \sum_{i=1}^m (\mathcal{K}_i\psi_i)(\mathcal{P}_n u_0) \Big] \Bigg\|_{L^2} \\
\leq & A_2 \sqrt{2}(p+1) \Bigg\| \sum_{i=1}^m (\mathcal{K}_i\psi_i)(u_0) - \sum_{i=1}^m (\mathcal{K}_i\psi_i)(\mathcal{P}_n u_0) \Bigg\|_{\infty} \\
\leq & A_2 \sqrt{2M_2} M l_1 (p+1) \|u_0 - \mathcal{P}_n u_0\|_{L^2} \to 0 \text{ as } n \to \infty.
\end{aligned} \tag{32}
$$

Choosing n sufficiently large such that $\beta_n \leq \delta(1-q)$. Then equation (4.5) of Theorem-2 of [19] is satisfied. Hence, by applying Theorem-2 of [19], we obtain

$$
\frac{\beta_n}{1+q} \leq \|u_n^M - u_0\|_{L^2} \leq \frac{\beta_n}{1-q},
$$

where $\beta_n = \|(\mathcal{I} - \mathcal{T}_n^{M'}(u_0))^{-1}(\mathcal{T}_n^M(u_0) - \mathcal{T}(u_0))\|_{L^2}$. Thus, the proof of the theorem is completed. $\square$

Lemma 5.4 *Let $u_0 \in C[-1,1]$. Then the following holds*

$$
\begin{aligned}
\Bigg\| \sum_{i=1}^m (\mathcal{K}_i\psi_i)(\mathcal{P}_n u_0) - \sum_{i=1}^m (\mathcal{K}_i\psi_i)(u_0) \Bigg\|_{\infty} \leq & M l_2 \sqrt{M_2} \|\mathcal{P}_n u_0 - u_0\|_{L^2} \|\mathcal{P}_n u_0 - u_0\|_{\infty} \\
& + \sup_{s\in[-1,1]} |\langle h(s,.), (\mathcal{P}_n - \mathcal{I})u_0(.)\rangle|,
\end{aligned}
$$

where $h(s,t) = \sum_{i=1}^m k_i(s,t)\psi_i^{(0,1)}(t, u_0(t))$.

Proof Using Mean value Theorem, we obtain

$$
\begin{aligned}
& \Bigg| \Big[\sum_{i=1}^m (\mathcal{K}_i\psi_i)(\mathcal{P}_n u_0) - \sum_{i=1}^m (\mathcal{K}_i\psi_i)(u_0) \Big] v(s) \Bigg| \\
& = \Bigg| \sum_{i=1}^m \int_{-1}^1 k_i(s,t)[\psi_i(t, \mathcal{P}_n u_0(t)) - \psi_i(t, u_0(t))] v(t) \mathrm{d}t \Bigg| \\
& = \Bigg| \sum_{i=1}^m \int_{-1}^1 k_i(s,t)[\psi_i^{(0,1)}(t, u_0 + \theta_3(\mathcal{P}_n u_0 - u_0)(t))(\mathcal{P}_n u_0 - u_0)] v(t) \mathrm{d}t \Bigg|,
\end{aligned}
$$

where $0 < \theta_3 < 1$. Using Lipschitz continuity of $\psi_i^{(0,1)}(t, u_0(.))$ and Cauchy-Schwarz inequality, we have

$$
\begin{aligned}
&\left|\Big[\sum_{i=1}^{m}(\mathcal{K}_i\psi_i)(\mathcal{P}_n u_0)-\sum_{i=1}^{m}(\mathcal{K}_i\psi_i)(u_0)\Big]v(s)\right|\\
&\quad\le\Big|\sum_{i=1}^{m}\int_{-1}^{1} m_i(s,t)g_\alpha|s-t|[\psi_i^{(0,1)}(t,u_0\\
&\qquad+\theta_3(\mathcal{P}_n u_0-u_0)(t))-\psi_i^{(0,1)}(t,u_0(t))]\\
&\qquad\times(\mathcal{P}_n u_0-u_0)v(t)\mathrm{d}t\Big|+\left|\sum_{i=1}^{m}\int_{-1}^{1} k_i(s,t)\psi_i^{(0,1)}(t,u_0(t))(\mathcal{P}_n u_0-u_0)v(t)\mathrm{d}t\right|\\
&\quad\le\sup_{i=1,2,\dots,m} q_i\sum_{i=1}^{m}\sup_{s,t\in[-1,1]}|m_i(s,t)|\|v\|_\infty\int_{-1}^{1}|g_\alpha|s-t|(\mathcal{P}_n u_0-u_0)(\mathcal{P}_n u_0-u_0)|\mathrm{d}t\\
&\qquad+\|v\|_\infty\left|\int_{-1}^{1} h(s,t)(\mathcal{P}_n u_0-u_0)(t)\mathrm{d}t\right|\\
&\quad\le Ml_2\sqrt{M_2}\|\mathcal{P}_n u_0-u_0\|_{L^2}\|\mathcal{P}_n u_0-u_0\|_\infty\|v\|_\infty+\|v\|_\infty\left|\int_{-1}^{1} h(s,t)(\mathcal{P}_n u_0-u_0)(t)\mathrm{d}t\right|,
\end{aligned}
$$

where $h(s,t)=\sum_{i=1}^{m} k_i(s,t)\psi_i^{(0,1)}(t,u_0(t))$. Thus, we obtain the desired result. □

Lemma 5.5 *Let $u_0 \in C^r[-1,1]$, $r \ge 1$. Let u_n^M be the approximation of u_0. Then the following holds*

$$\|u_n^M-u_0\|_\infty \le (c+c\log n)\|u_0-u_n^M\|_{L^2}+n^{-r}(c+c\log n)\|u_0^{(r)}\|_\infty.$$

Proof We have

$$
\begin{aligned}
&\|u_n^M-u_0\|_\infty\\
&\quad=\left\|\sum_{i=1}^{m}(\mathcal{K}_{n,i}^M\psi_i)(u_n^M)-\sum_{i=1}^{m}(\mathcal{K}_i\psi_i)(u_0)\right\|_\infty\\
&\quad=\Bigg\|\sum_{i=1}^{m}\mathcal{P}_n(\mathcal{K}_i\psi_i)(u_n^M)+\sum_{i=1}^{m}(\mathcal{K}_i\psi_i)(\mathcal{P}_n u_n^M)\\
&\qquad-\sum_{i=1}^{m}\mathcal{P}_n(\mathcal{K}_i\psi_i)(\mathcal{P}_n u_n^M)-\sum_{i=1}^{m}(\mathcal{K}_i\psi_i)(u_0)\Bigg\|_\infty\\
&\quad\le\left\|\mathcal{P}_n\Big[\sum_{i=1}^{m}(\mathcal{K}_i\psi_i)(u_n^M)-\sum_{i=1}^{m}(\mathcal{K}_i\psi_i)(u_0)\Big]\right\|_\infty\\
&\qquad+\left\|\sum_{i=1}^{m}(\mathcal{K}_i\psi_i)(\mathcal{P}_n u_n^M)-\sum_{i=1}^{m}(\mathcal{K}_i\psi_i)(\mathcal{P}_n u_0)\right\|_\infty\\
&\qquad+\left\|\mathcal{P}_n\Big[\sum_{i=1}^{m}(\mathcal{K}_i\psi_i)(\mathcal{P}_n u_0)-\sum_{i=1}^{m}(\mathcal{K}_i\psi_i)(\mathcal{P}_n u_n^M)\Big]\right\|_\infty\\
&\qquad+\left\|\sum_{i=1}^{m}(\mathcal{K}_i\psi_i)(\mathcal{P}_n u_0)-\sum_{i=1}^{m}(\mathcal{K}_i\psi_i)(u_0)\right\|_\infty\\
&\qquad+\left\|\mathcal{P}_n\Big[\sum_{i=1}^{m}(\mathcal{K}_i\psi_i)(u_0)-\sum_{i=1}^{m}(\mathcal{K}_i\psi_i)(\mathcal{P}_n u_0)\Big]\right\|_\infty.
\end{aligned}
$$

Using $\|\mathcal{P}_n\|_\infty \leq c\log n$ (cf., Page-147, [3]) and estimate (10) with Lemma 2.3 in the above estimate, we obtain

$$\begin{aligned}
\|u_n^M - u_0\|_\infty \leq & c\log nM\sqrt{M_2}l_1\|u_n^M - u_0\|_{L^2} + M\sqrt{M_2}l_1\|\mathcal{P}_n u_n^M - \mathcal{P}_n u_0\|_{L^2} \\
& + c\log nM\sqrt{M_2}l_1\|\mathcal{P}_n u_0 - \mathcal{P}_n u_n^M\|_{L^2} + M\sqrt{M_2}l_1\|\mathcal{P}_n u_0 - u_0\|_{L^2} \\
& + c\log nM\sqrt{M_2}l_1\|u_0 - \mathcal{P}_n u_0\|_{L^2} \\
\leq & c\log n\|u_n^M - u_0\|_{L^2} + c\|u_n^M - u_0\|_{L^2} + c\log n\|u_n^M - u_0\|_{L^2} \\
& + cn^{-r}\|u_0^{(r)}\|_\infty + cn^{-r}\log n\|u_0^{(r)}\|_\infty \\
= & (c + c\log n)\|u_n^M - u_0\|_{L^2} + n^{-r}(c + c\log n)\|u_0^{(r)}\|_\infty.
\end{aligned}$$

This completes the proof. □

Theorem 5.6 *Let $u_0 \in C^r[-1,1]$, $r \geq 1$. Then the Legendre multi-Galerkin approximation $u_n^{M,G}$ of u_0 satisfies the followings*

$$\|u_n^{M,G} - u_0\|_{L^2} \leq \begin{cases} cn^{\frac{1}{2}-r-\alpha}, & \text{for } \frac{1}{2} < \alpha < 1, \\ cn^{-\frac{1}{2}-r}\log n, & \text{for } \alpha = 1, \end{cases}$$

$$\|u_n^{M,G} - u_0\|_\infty \leq (c + c\log n)cn^{-r}.$$

Proof Using Theorem-5.3, and proceeding similarly as in estimate (32), we obtain

$$\begin{aligned}
\|u_n^{M,G} - u_0\|_{L^2} \leq & \frac{\beta_n}{1-q} \\
\leq & \frac{1}{1-q}\|(\mathcal{I} - \mathcal{T}_n^{M'}(u_0))^{-1}(\mathcal{T}_n^M(u_0) - \mathcal{T}(u_0))\|_{L^2} \\
\leq & cA_2\sqrt{2}(p+1)\left\|\sum_{i=1}^m(\mathcal{K}_i\psi_i)(u_0) - \sum_{i=1}^m(\mathcal{K}_i\psi_i)(\mathcal{P}_n^G u_0)\right\|_\infty. \qquad (33)
\end{aligned}$$

Since $\mathcal{P}_n^G$ be the orthogonal projection from the space $\mathbb{X}$ into $\mathbb{X}_n$, then we have

$$\langle w, (\mathcal{I} - \mathcal{P}_n^G)u\rangle = 0, \ \ \forall\, w \in \mathbb{X}_n.$$

Using Lemma 5.4 with Hölder's inequality and Lemma 2.3, we get

$$\begin{aligned}
& \left\|\sum_{i=1}^m(\mathcal{K}_i\psi_i)(u_0) - \sum_{i=1}^m(\mathcal{K}_i\psi_i)(\mathcal{P}_n^G u_0)\right\|_\infty \\
& \leq Ml_2\sqrt{M_2}\|\mathcal{P}_n^G u_0 - u_0\|_{L^2}\|\mathcal{P}_n^G u_0 - u_0\|_\infty + \sup_{s\in[-1,1]}|\langle h(s,.), (\mathcal{P}_n^G - \mathcal{I})u_0(.)\rangle| \\
& \leq cn^{\frac{1}{2}-2r}\|u_0^{(r)}\|_\infty V(u_0^{(r)}) + \sup_{s\in[-1,1]}|\langle h(s,.) - \phi(s,.), (\mathcal{P}_n^G - \mathcal{I})u_0(.)\rangle| \\
& \leq cn^{\frac{1}{2}-2r}\|u_0^{(r)}\|_\infty V(u_0^{(r)}) + \|h_s - \phi_s\|_{L^1}\|(\mathcal{P}_n^G - \mathcal{I})u_0\|_\infty, \qquad (34)
\end{aligned}$$

where $\phi_s \in \mathbb{X}_n$. Using Theorems 2 and 3 of [17] in estimate (34), we get

$$\begin{aligned}\left\|\sum_{i=1}^{m}(\mathcal{K}_i\psi_i)(u_0)-\sum_{i=1}^{m}(\mathcal{K}_i\psi_i)(\mathcal{P}_n^G u_0)\right\|_\infty &\\ \leq cn^{\frac{1}{2}-2r}\|u_0^{(r)}\|_\infty V(u_0^{(r)}) &\\ + V(u_0^{(r)})\begin{cases} cn^{\frac{1}{2}-r-\alpha}, & \text{for } \frac{1}{2}<\alpha<1,\\ cn^{-\frac{1}{2}-r}\log n, & \text{for } \alpha=1,\end{cases} &\\ \leq \begin{cases} cn^{\frac{1}{2}-r-\alpha}, & \text{for } \frac{1}{2}<\alpha<1,\\ cn^{-\frac{1}{2}-r}\log n, & \text{for } \alpha=1.\end{cases} &\end{aligned} \tag{35}$$

Combining estimates (33) and (35), we obtain

$$\|u_n^{M,G}-u_0\|_{L^2} \leq \begin{cases} cn^{\frac{1}{2}-r-\alpha}, & \text{for } \frac{1}{2}<\alpha<1,\\ cn^{-\frac{1}{2}-r}\log n, & \text{for } \alpha=1.\end{cases} \tag{36}$$

Again, from Lemma 5.5 and using estimate (36), we get

$$\begin{aligned}\|u_n^{M,G}-u_0\|_\infty \leq &(c+c\log n)\begin{cases} cn^{\frac{1}{2}-r-\alpha}, & \text{for } \frac{1}{2}<\alpha<1,\\ cn^{-\frac{1}{2}-r}\log n, & \text{for } \alpha=1,\end{cases}\\ &+ n^{-r}(c+c\log n)\|u_0^{(r)}\|_\infty\\ \leq &(c+c\log n)cn^{-r}.\end{aligned}$$

Hence, the proof is completed. □

Theorem 5.7 *Let* $u_0 \in C^r[-1,1]$, $r \geq 1$. *Then the Legendre multi-collocation approximation* $u_n^{M,C}$ *of* u_0 *satisfies the following*

$$\|u_n^{M,C}-u_0\|_{L^2} \leq cn^{-r}, \quad \|u_n^{M,C}-u_0\|_\infty \leq c(c+c\log n)n^{-r}.$$

Proof As proceeding similarly as in equation (33), we obtain

$$\|u_n^{M,C}-u_0\|_{L^2} \leq c\left\|\sum_{i=1}^{m}(\mathcal{K}_i\psi_i)(u_0)-\sum_{i=1}^{m}(\mathcal{K}_i\psi_i)(\mathcal{P}_n^C u_0)\right\|_\infty. \tag{37}$$

Using Lemmas 5.4 and 2.3 with Cauchy-Schwarz inequality, we get

$$\begin{aligned}\left\|\sum_{i=1}^{m}(\mathcal{K}_i\psi_i)(u_0)-\sum_{i=1}^{m}(\mathcal{K}_i\psi_i)(\mathcal{P}_n^C u_0)\right\|_\infty &\\ \leq Ml_2\sqrt{M_2}cn^{\frac{1}{2}-2r}\|u_0^{(r)}\|_\infty^2 + \sup_{s\in[-1,1]}\|h_s\|_{L^2}\|(\mathcal{P}_n^C-\mathcal{I})u_0\|_{L^2} & \qquad (38)\\ \leq Ml_2\sqrt{M_2}cn^{\frac{1}{2}-2r}\|u_0^{(r)}\|_\infty^2 + \sup_{s\in[-1,1]}\|h_s\|_{L^2}cn^{-r}\|u_0^{(r)}\|_\infty \leq cn^{-r}. &\end{aligned}$$

Now combining estimates (37) and (38), we obtain

$$\|u_n^{M,C} - u_0\|_{L^2} \leq cn^{-r}. \tag{39}$$

Substituting estimate (39) in Lemma 5.5, we get

$$\begin{aligned}\|u_n^{M,C} - u_0\|_\infty \leq &(c + c\log n)cn^{-r} + n^{-r}(c + c\log n)\|u_0^{(r)}\|_\infty \\ \leq &(c + c\log n)cn^{-r}.\end{aligned}$$

This completes the proof. □

6 Iterated Legendre multi-projection methods

In this section, we discuss on the iterated Legendre spectral multi-projection methods for weakly singular Hammerstein integral equations of mixed type in both L^2 and infinity norm.

The iterated approximate solution $\widetilde{u}_n^M$ corresponding to the approximate solution u_n^M given by equation (26) is defined as follows:

$$\widetilde{u}_n^M = \sum_{i=1}^{m} (\mathcal{K}_i\psi_i)(u_n^M) + f. \tag{40}$$

To obtain the supercovergence results for the iterated approximate solution $\widetilde{u}_n^M$ to the exact solution u_0, we need the following lemmas.

Lemma 6.1 *For any $x, y, z \in \mathbb{X}$, the following holds*

$$\begin{aligned}&\left\|\left[\sum_{i=1}^{m} (\mathcal{K}_{n,i}^M\psi_i)'(x) - \sum_{i=1}^{m} (\mathcal{K}_{n,i}^M\psi_i)'(y)\right](z)\right\|_\infty \\ &\leq c\log n(c + c\log n)\|x - y\|_{L^2}\|z\|_\infty.\end{aligned}$$

Proof Using $\|\mathcal{P}_n\|_\infty \leq c\log n$ (cf., Page-147, [3]) and estimate (11), we obtain

$$
\begin{aligned}
&\left\|\Big[\sum_{i=1}^{m}(\mathcal{K}_{n,i}^{M}\psi_i)'(x)-\sum_{i=1}^{m}(\mathcal{K}_{n,i}^{M}\psi_i)'(y)\Big](z)\right\|_{\infty}\\
&\leq \left\|\mathcal{P}_n\Big[\sum_{i=1}^{m}(\mathcal{K}_i\psi_i)'(x)-\sum_{i=1}^{m}(\mathcal{K}_i\psi_i)'(y)\Big]z\right\|_{\infty}\\
&\quad+\left\|(\mathcal{I}-\mathcal{P}_n)\Big[\sum_{i=1}^{m}(\mathcal{K}_i\psi_i)'(\mathcal{P}_n x)-\sum_{i=1}^{m}(\mathcal{K}_i\psi_i)'(\mathcal{P}_n y)\Big]\mathcal{P}_n z\right\|_{\infty}\\
&\leq c\log n\left\|\Big[\sum_{i=1}^{m}(\mathcal{K}_i\psi_i)'(x)-\sum_{i=1}^{m}(\mathcal{K}_i\psi_i)'(y)\Big](z)\right\|_{\infty}\\
&\quad+(1+c\log n)\left\|\Big[\sum_{i=1}^{m}(\mathcal{K}_i\psi_i)'(\mathcal{P}_n x)-\sum_{i=1}^{m}(\mathcal{K}_i\psi_i)'(\mathcal{P}_n y)\Big]\mathcal{P}_n z\right\|_{\infty}\\
&\leq c\log n\sqrt{M_2}Ml_2\|x-y\|_{L^2}\|z\|_{\infty}\\
&\quad+(1+c\log n)\sqrt{M_2}Ml_2\|\mathcal{P}_n x-\mathcal{P}_n y\|_{L^2}\|\mathcal{P}_n z\|_{\infty}\\
&\leq c\log n\sqrt{M_2}Ml_2\|x-y\|_{L^2}\|z\|_{\infty}\\
&\quad+(1+c\log n)\sqrt{M_2}Ml_2 pc\log n\|x-y\|_{L^2}\|z\|_{\infty}\\
&= c\log n(c+c\log n)\|x-y\|_{L^2}\|z\|_{\infty}.
\end{aligned}
\tag{41}
$$

The proof is completed. □

Theorem 6.2 *Let $u_0 \in C[-1,1]$. Let $\widetilde{u}_n^M$ be the iterated approximation of u_0. Then the following holds*

$$
\begin{aligned}
&\|\widetilde{u}_n^M-u_0\|_{\infty}\\
&\quad\leq [c+c\log n(c+c\log n)]\|u_n^M-u_0\|_{L^2}\|u_n^M-u_0\|_{\infty}+(1+c\log n)\\
&\qquad\times\sup_{s\in[-1,1]}\left|\langle h(s,.),(\mathcal{P}_n-\mathcal{I})\Big[\sum_{i=1}^{m}(\mathcal{K}_i\psi_i)(\mathcal{P}_n u_0)-\sum_{i=1}^{m}(\mathcal{K}_i\psi_i)(u_0)\Big](.)\rangle\right|.
\end{aligned}
$$

Proof The steps of the proof follows similarly as in Theorem 4.1 of [16]. So, we omit it. □

Theorem 6.3 *Let $u_0 \in C^r[-1,1],\ r \geq 1$. Then the iterated Legendre multi-Galerkin approximation $\widetilde{u}_n^{M,G}$ of u_0 satisfies the followings*

$$
\|\widetilde{u}_n^{M,G}-u_0\|_{L^2}\leq\sqrt{2}\|\widetilde{u}_n^{M,G}-u_0\|_{\infty}\leq(1+c\log n)^2\begin{cases} cn^{\frac{1}{2}-r-2\alpha}, & \text{for } \frac{1}{2}<\alpha<1,\\ cn^{-\frac{3}{2}-r}(\log n)^2, & \text{for } \alpha=1.\end{cases}
$$

Proof From Theorem 6.2, we have

$$\begin{aligned}
\|\widetilde{u}_n^{M,G} - u_0\|_\infty \leq & [c + c\log n(c + c\log n)]\|u_n^{M,G} - u_0\|_{L^2}\|u_n^{M,G} \\
& - u_0\|_\infty + (1 + c\log n) \\
& \times \sup_{s\in[-1,1]} \Big|\langle h(s,.), (\mathcal{P}_n^G - \mathcal{I}) \\
& \Big[\sum_{i=1}^m (\mathcal{K}_i\psi_i)(u_0) - \sum_{i=1}^m (\mathcal{K}_i\psi_i)(\mathcal{P}_n^G u_0)\Big](.)\rangle\Big|. \qquad (42)
\end{aligned}$$

Since $\mathcal{P}_n^G$ be the orthogonal projection from the space $\mathbb{X}$ into $\mathbb{X}_n$, then we have

$$\langle w, (\mathcal{I} - \mathcal{P}_n^G)u\rangle = 0, \ \ \forall\, w \in \mathbb{X}_n.$$

By applying Hölder's inequality, estimate (35), and Theorems 2 and 3 of [17], we obtain

$$\begin{aligned}
& \sup_{s\in[-1,1]} \Big|\langle h(s,.), (\mathcal{P}_n^G - \mathcal{I})\Big[\sum_{i=1}^m (\mathcal{K}_i\psi_i)(u_0) - \sum_{i=1}^m (\mathcal{K}_i\psi_i)(\mathcal{P}_n^G u_0)\Big](.)\rangle\Big| \\
& \leq \Big|\langle h_s - \phi_s, (\mathcal{P}_n^G - \mathcal{I})\Big[\sum_{i=1}^m (\mathcal{K}_i\psi_i)(u_0) - \sum_{i=1}^m (\mathcal{K}_i\psi_i)(\mathcal{P}_n^G u_0)\Big](.)\rangle\Big| \\
& \leq \|h_s - \phi_s\|_{L^1}\Big\|(\mathcal{P}_n^G - \mathcal{I})\Big[\sum_{i=1}^m (\mathcal{K}_i\psi_i)(u_0) - \sum_{i=1}^m (\mathcal{K}_i\psi_i)(\mathcal{P}_n^G u_0)\Big]\Big\|_\infty \\
& \leq (1 + c\log n)\Big\|\sum_{i=1}^m (\mathcal{K}_i\psi_i)(u_0) \\
& \quad - \sum_{i=1}^m (\mathcal{K}_i\psi_i)(\mathcal{P}_n^G u_0)\Big\|_\infty \begin{cases} cn^{-\alpha}, & \text{for } \frac{1}{2} < \alpha < 1, \\ cn^{-1}\log n, & \text{for } \alpha = 1, \end{cases} \\
& \leq (1 + c\log n)\begin{cases} cn^{\frac{1}{2}-r-2\alpha}, & \text{for } \frac{1}{2} < \alpha < 1, \\ cn^{-\frac{3}{2}-r}(\log n)^2, & \text{for } \alpha = 1, \end{cases} \qquad (43)
\end{aligned}$$

where $\phi_s \in \mathbb{X}_n$. Substituting estimate (43) and Theorem 5.6 in equation (42), we obtain

$$\begin{aligned}
&\|\widetilde{u}_n^{M,G} - u_0\|_{L^2} \\
&\quad \le \sqrt{2}\|\widetilde{u}_n^{M,G} - u_0\|_\infty \\
&\quad \le [c + c\log n(c + c\log n)]\begin{cases} cn^{\frac{1}{2}-r-\alpha}, & \text{for } \frac{1}{2} < \alpha < 1, \\ cn^{-\frac{1}{2}-r}\log n, & \text{for } \alpha = 1. \end{cases} \\
&\qquad \times (c + c\log n)cn^{-r} + (1 + c\log n)^2 \begin{cases} cn^{\frac{1}{2}-r-2\alpha}, & \text{for } \frac{1}{2} < \alpha < 1, \\ cn^{-\frac{3}{2}-r}(\log n)^2, & \text{for } \alpha = 1, \end{cases} \\
&\quad \le (1 + c\log n)^2 \begin{cases} cn^{\frac{1}{2}-r-2\alpha}, & \text{for } \frac{1}{2} < \alpha < 1, \\ cn^{-\frac{3}{2}-r}(\log n)^2, & \text{for } \alpha = 1. \end{cases}
\end{aligned}$$

This completes the proof. □

Theorem 6.4 *Let $u_0 \in C^r[-1,1]$, $r \ge 1$. Then the iterated Legendre multi-collocation approximation $\widetilde{u}_n^{M,C}$ of u_0 satisfies the following*

$$\|\widetilde{u}_n^{M,C} - u_0\|_\infty \le c(1 + c\log n)n^{-r}.$$

Proof From Theorem 6.2, we have

$$\begin{aligned}
\|\widetilde{u}_n^{M,C} - u_0\|_\infty \le & [c + c\log n(c + c\log n)]\|u_n^{M,C} - u_0\|_{L^2}\|u_n^{M,C} \\
& - u_0\|_\infty + (1 + c\log n) \\
& \times \sup_{s\in[-1,1]} \Big|\langle h(s,.), (\mathcal{P}_n^C - \mathcal{I})\Big[\sum_{i=1}^m (\mathcal{K}_i\psi_i)(u_0) \\
& - \sum_{i=1}^m (\mathcal{K}_i\psi_i)(\mathcal{P}_n^C u_0)\Big](.)\rangle\Big|.
\end{aligned} \tag{44}$$

Using Cauchy-Schwarz inequality, Lemmas 2.3 and 5.4, we obtain

$$\begin{aligned}
& \sup_{s\in[-1,1]} \Big|\langle h(s,.), (\mathcal{P}_n^C - \mathcal{I})\Big[\sum_{i=1}^m (\mathcal{K}_i\psi_i)(u_0) - \sum_{i=1}^m (\mathcal{K}_i\psi_i)(\mathcal{P}_n^C u_0)\Big](.)\rangle\Big| \\
& \quad \le \sup_{s\in[-1,1]} \|h_s\|_{L^2}\Big\|(\mathcal{P}_n^C - \mathcal{I})\Big[\sum_{i=1}^m (\mathcal{K}_i\psi_i)(u_0) - \sum_{i=1}^m (\mathcal{K}_i\psi_i)(\mathcal{P}_n^C u_0)\Big]\Big\|_{L^2} \\
& \quad \le \sqrt{2}(1+p)\sup_{s\in[-1,1]} \|h_s\|_{L^2}\Big\|\sum_{i=1}^m (\mathcal{K}_i\psi_i)(u_0) - \sum_{i=1}^m (\mathcal{K}_i\psi_i)(\mathcal{P}_n^C u_0)\Big\|_\infty \\
& \quad \le \sqrt{2}(1+p)\sup_{s\in[-1,1]} \|h_s\|_{L^2}\Big[Ml_2\sqrt{M_2}\|\mathcal{P}_n^C u_0 - u_0\|_{L^2}\|\mathcal{P}_n^C u_0 - u_0\|_\infty \\
& \qquad + \sup_{s\in[-1,1]} |\langle h(s,.), (\mathcal{P}_n^C - \mathcal{I})u_0(.)\rangle|\Big] \\
& \quad \le cn^{\frac{1}{2}-2r}\|u_0^{(r)}\|_\infty V(u_0^{(r)}) + \sup_{s\in[-1,1]} \|h_s\|_{L^2}\|(\mathcal{P}_n^C - \mathcal{I})u_0\|_{L^2} \\
& \quad \le cn^{\frac{1}{2}-2r}\|u_0^{(r)}\|_\infty V(u_0^{(r)}) + \sup_{s\in[-1,1]} \|h_s\|_{L^2} cn^{-r}\|u_0^{(r)}\|_\infty.
\end{aligned} \tag{45}$$

Substituting estimate (45) and Theorem 5.7 in equation (44), we obtain

$$\begin{aligned}\|\widetilde{u}_n^{M,C} - u_0\|_\infty \leq & [c + c\log n(c + c\log n)]cn^{-r}cn^{-r}(c + c\log n) + (1 + c\log n) \\ & \times \left[cn^{\frac{1}{2}-2r}\|u_0^{(r)}\|_\infty V(u_0^{(r)}) + \sup_{s\in[-1,1]} \|h_s\|_{L^2}cn^{-r}\|u_0^{(r)}\|_\infty\right] \\ \leq & c(1 + c\log n)n^{-r}.\end{aligned}$$

Hence, the result. □

Remark 6.5 Let u_n^G, u_n^C, $\widetilde{u}_n^G$, $\widetilde{u}_n^C$ be the Legendre Galerkin, Legendre collocation, iterated Legendre Galerkin and iterated Legendre collocation approximations of u, respectively. Let $u_n^{M,G}$, $u_n^{M,C}$, $\widetilde{u}_n^{M,G}$, $\widetilde{u}_n^{M,C}$ be the Legendre multi-Galerkin, Legendre multi-collocation, iterated Legendre multi-Galerkin, iterated Legendre multi-collocation approximations of u.

From Theorems-3.4, 4.2, 4.3, 5.6, 5.7, 6.3 and 6.4, we observe the following convergence rates in the respective methods.

Legendre projection methods:

$$\|u_n^G - u_0\|_{L^2} \leq cn^{-r}, \qquad \|u_n^C - u_0\|_{L^2} \leq cn^{-r},$$
$$\|u_n^G - u_0\|_\infty \leq cn^{\frac{1}{2}-r}, \qquad \|u_n^C - u_0\|_\infty \leq cn^{\frac{1}{2}-r}.$$

Iterated Legendre projection methods:

$$\|\widetilde{u}_n^G - u_0\|_{L^2} \leq \begin{cases} cn^{\frac{1}{2}-r-\alpha}(1 + c\log n), & \text{for } \frac{1}{2} < \alpha < 1, \\ cn^{-\frac{1}{2}-r}(1 + c\log n)\log n, & \text{for } \alpha = 1, \end{cases}$$
$$\|\widetilde{u}_n^G - u_0\|_\infty \leq \begin{cases} cn^{\frac{1}{2}-r-\alpha}(1 + c\log n), & \text{for } \frac{1}{2} < \alpha < 1, \\ cn^{-\frac{1}{2}-r}(1 + c\log n)\log n, & \text{for } \alpha = 1, \end{cases}$$
$$\|\widetilde{u}_n^C - u_0\|_{L^2} \leq cn^{-r}(1 + c\log n),$$
$$\|\widetilde{u}_n^C - u_0\|_\infty \leq cn^{-r}(1 + c\log n).$$

Legendre multi-projection methods:

$$\|u_n^{M,G} - u_0\|_{L^2} \leq \begin{cases} cn^{\frac{1}{2}-r-\alpha}, & \text{for } \frac{1}{2} < \alpha < 1, \\ cn^{-\frac{1}{2}-r}\log n, & \text{for } \alpha = 1, \end{cases}$$
$$\|u_n^{M,G} - u_0\|_\infty \leq (c + c\log n)cn^{-r}.$$
$$\|u_n^{M,C} - u_0\|_{L^2} \leq cn^{-r},$$
$$\|u_n^{M,C} - u_0\|_\infty \leq c(c + c\log n)n^{-r}.$$

Iterated Legendre multi-projection methods:

$$\begin{aligned}\|\widetilde{u}_n^{M,G} - u_0\|_{L^2} &\leq \sqrt{2}\|\widetilde{u}_n^{M,G} - u_0\|_\infty \\ &\leq (1 + c\log n)^2 \begin{cases} cn^{\frac{1}{2}-r-2\alpha}, & \text{for } \frac{1}{2} < \alpha < 1, \\ cn^{-\frac{3}{2}-r}(\log n)^2, & \text{for } \alpha = 1, \end{cases} \\ \|\widetilde{u}_n^{M,C} - u_0\|_{L^2} &\leq \sqrt{2}\|\widetilde{u}_n^{M,C} - u_0\|_\infty \leq c(1 + c\log n)n^{-r}.\end{aligned}$$

We observe that

1. Iterated Galerkin improves over Galerkin, multi-Galerkin improves over Iterated Galerkin and Iterated multi-Galerkin improves over multi-Galerkin in L^2 norm using Legendre polynomial bases.
2. However, in infinity norm, multi-Galerkin improves over Galerkin and iterated multi-Galerkin improves over iterated Galerkin using Legendre polynomial bases.
3. In collocation method, no improvement recorded from collocation to multi-collocation and iterated collocation to iterated multi-collocation in L^2 norm.
4. In infinity norm, multi-collocation improves over collocation method. However, there is no improvement from iterated collocation to iterated multi-collocation method.

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