



Markov chains and generalized wavelet multiresolutions

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Abstract

We develop some new results for a general class of transfer operators, as they are used in a construction of multi-resolutions. We then proceed to give explicit and concrete applications. We further discuss the need for such a constructive harmonic analysis/dynamical systems approach to fractals.

Keywords Markov chains · Multiresolution · Fractals · Cuntz relations · Optimization

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1 Introduction and setting

While there are already a number of approaches to harmonic analysis of fractals, “non-smooth” settings, we propose below a focus on a certain family of positive operators. They will serve as transfer operators.

Our paper is divided into two parts: in the first we develop the needed results on transfer operators, and the second part will be concrete applications. There are many justification for the need of a constructive harmonic analysis of fractals; one is the discovery of Jorgensen–Pedersen that certain fractal L^2 spaces admit Fourier bases; while others do not. However the lack of available Fourier bases in many example suggests a need for alternative approaches.

The Cantor fractals are special cases of more general IFS systems. Our present paper will deal with this more general framework. In addition to fractal Fourier

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analyses (fractals in the large), we shall also study multiresolution and wavelet techniques. In work of Dutkay–Jorgensen, it was shown that the general affine IFS-systems, even if not amenable to Fourier analysis, in fact do admit wavelet bases, and so in particular can be analyzed with the use of multiresolutions; reflecting the inherent self-similarity to the fractal under consideration. But this approach in fact depends on the use of certain transfer operators. The latter in turn ties in with intriguing new work on cascade algorithms, with an analysis of representations of non-commutative generators and relations (especially the Cuntz relations), as well as with certain stochastic processes; and we shall make connections to recent research on Markov processes, and to reproducing kernel theory.

2 Generalized multi-resolution measures on solenoids

In Sect. 2 below we introduce a certain multi-resolution approach to problems that arise in analysis of fractals and more generally in stochastic analysis. Examples will be given in Sect. 4, and a wavelet representation approach in Sect. 5.

Definition 2.1 Let X be a compact Hausdorff space. Consider a linear operator

$$R : C(X) \rightarrow L^\infty(X) \tag{2.1}$$

and assume it is positive. i.e., $f \geq 0 \Rightarrow Rf \geq 0$.

Proof Riesz. We also consider positive Borel measures λ on X . □

Definition 2.2 Let λ be a finite positive Borel measure on X . Then the following are equivalent:

- (1) $\lambda R \ll \lambda$ with Radon-Nikodym derivative w .
- (2) $\int wfd\lambda = \int (Rf)d\lambda$, for all $f \in C(X)$

Proof This is immediate from the definition of the Radon-Nikodym derivative. To understand it in more detail, it helps to have a generalized Perron-Frobenius theorem. Recall, R satisfies an additional condition thus as follows: In general, it may be difficult, but recall $R : C(X) \rightarrow L^\infty(X, \lambda)$ also satisfies the following property (i.e., transition operator has the pull-out property.) Let σ be an embedding in the measure space X , and assume that

$$R[(f \circ \sigma)g] = fRg, \quad \forall f, g \in C(X). \tag{2.2}$$

□

Remark 2.3 The axiom (2.2) is a generalized conditional expectation property. But, in general, as an operator in $L^2(\lambda)$, R may still be unbounded. Nonetheless, if

$$\mu = \lambda \circ R \quad \text{and} \quad \frac{d\mu}{d\lambda} = w \quad \text{Radon-Nikodym derivative.}$$

Note w depends on both R and on λ .

□

We now study domains of the unbounded operators in $L^2(\lambda)$.

Theorem 2.4 *Suppose (2) and (2.2) hold, then*

$$C(X) \subset \text{domain}(R^*), \quad \text{the domain of the adjoint operator } R^*; \text{ in general possibly unbounded.} \tag{2.3}$$

Generally, $R^*f = wf \circ \sigma$, for all $f \in C(X)$. Moreover, $R : L^2(\lambda) \rightarrow L^2(\lambda)$ is bounded if and only if

$$w \in L^\infty(\lambda), \quad \text{and then} \quad \|R\|_{2 \rightarrow 2} = \|R(w)\|_\infty^{1/2}. \tag{2.4}$$

But in general, R is an unbounded operator in $L^2(\lambda)$. As noted, we have:

- (1) $\lambda R \ll \lambda$
- (2) $\frac{d\lambda R}{d\lambda} = w$

Lemma 2.5 *Let $C(X) \subset \text{domain}(R^*)$, and*

$$R^*f = wf \circ \sigma, \quad \forall f \in C(X) \quad \text{i.e., } (R^*f)(x) = w(x)f(\sigma(x)), \quad \forall x \in X. \tag{2.5}$$

*implies $RR^*f = fR(w)$.*

Proof (of the Lemma) Assume (2) and (2.2), then

$$\int w(f \circ \sigma)gd\lambda \stackrel{\text{by(2.2)}}{=} \int f(Rg)d\lambda \quad \forall g \in C(X). \tag{2.6}$$

Indeed, the right hand side of (2.6),

$$\begin{aligned} \text{RHS of (2.6)} &= \int fRgd\lambda \stackrel{\text{by(2.2)}}{=} \int R((f \circ \sigma)g)d\lambda \\ &\stackrel{\text{by(2)}}{=} \int w(f \circ \sigma)gd\lambda, \quad \text{where we use that } w = \frac{d\mu}{d\lambda} = R^*(\mathbf{1}). \end{aligned}$$

Hence,

$$\begin{aligned} \text{RHS of (2.6)} &\stackrel{\text{Schwarz}}{\leq} \int |wf \circ \sigma|^2 d\lambda \int |g|^2 d\lambda \\ &\Rightarrow f \in \text{dom}(R^*), \quad \text{and} \\ &R^*f = wf \circ \sigma. \end{aligned}$$

□

this is a weighted composition operator.

2.1 The case of unbounded w

Even if $w = \frac{d\mu}{d\lambda}$ is only in $L^1(\lambda)$, then the following two operators are well defined as $L^2(\lambda) \rightarrow L^2(\lambda)$ operators; each with $C(X) \subset L^2(\lambda)$ as dense domain, and $R \subset S^*$, $S \subset R^*$; (containments of operators) see (2.9) below.

$$L^2(\lambda) \supset \begin{cases} C(X) \ni f \xrightarrow{R} R(f) \text{ or} \\ C(X) \ni f \xrightarrow{S} w(f \circ \sigma) \in L^2(\lambda), \quad R : C(X) \rightarrow L^\infty(\lambda); \quad Rf \in L^\infty(\lambda), \quad \text{and we have} \end{cases} \tag{2.7}$$

$$\langle Sf, g \rangle_{L^2(\lambda)} = \langle f, Rg \rangle_{L^2(\lambda)}, \quad \forall f, g \in C(X). \tag{2.8}$$

Proof of (2.8)

$$\int w(f \circ \sigma)gd\lambda = \int fR(g)d\lambda, \tag{2.9}$$

and we verified (2.9) above.

2.2 The bounded case

Moreover, assuming $w \in L^\infty(\lambda)$, we get

$$RR^*f = R(wf \circ \sigma) \stackrel{\text{by (2.9)}}{=} f(Rw)$$

so RR^* is a multiplication operator on $L^2(\lambda)$, i.e., multiplication by the function $R(w) \in L^\infty(\lambda)$. So if $\|RR^*\|_{2 \rightarrow 2} = \|Rw\|_\infty < \infty$. Recall, by Riesz $\|Rw\|_\infty \leq \|w\|_\infty$. If and only if RR^* is bounded: $L^2(\lambda) \rightarrow L^2(\lambda)$. If and only if R is bounded: $L^2(\lambda) \rightarrow L^2(\lambda)$. So by the Hilbert space $\|RR^*\|_{2 \rightarrow 2} = \|R\|_{2 \rightarrow 2}^2 = \|R^*\|_{2 \rightarrow 2}^2$. (The L^2 -operator norms.)

Proof

$$\int R((f \circ \sigma)g) = \int fRGd\lambda$$

$$\int (f \circ \sigma)gwd\lambda = \int (R^*f)gd\lambda$$

$$R^*f = wf \circ \sigma$$

$$RR^*f = R(wf \circ \sigma) = fRw$$

$$f : L^2(\lambda) \rightarrow L^2(\lambda)$$

$$RR^* \text{ in } L^2(\lambda) \text{ is bounded} \iff Rw \in L^\infty \iff w \in RR^* \iff R \text{ in } L^2(\lambda) \text{ is bounded}$$

□

The converse also holds: If $C(X) \ni f \xrightarrow{S} w(f \circ \sigma)$ extends to a bounded operator in $L^2(\lambda)$, then $w \in L^\infty(\lambda)$. Then adjoint operator $R := S^*$ (adjoint with respect to the $L^2(\lambda)$ -Hilbert space), then

$$R((f \circ \sigma)g) = fR(g), \quad \forall f, g \in C(X).$$

Moreover, set $\mu = \lambda R$, i.e., $\int f d\mu = \int Rf d\lambda$, then $\mu \ll \lambda$, and $w = \frac{d\mu}{d\lambda}$ is the Radon-Nikodym derivative.

Corollary 2.6 S^*S is a multiplication operator in $L^2(\lambda)$.

Proof $S^*Sf = R(wf \circ \sigma) = fR(w)$ so $S^*S = RS$ is a multiplicative operator, $m = R(w)$. Then set $f \rightarrow mf(\lambda \circ \sigma^{-1})(E) = \lambda(\sigma^{-1}(E))$, $\sigma^{-1}(E) = \{x \in X : \sigma(x) \in E\}$. □

Corollary 2.7 $\lambda \circ \sigma^{-1} \ll \lambda$, and $\frac{d\lambda \circ \sigma^{-1}}{d\lambda} = R\left(\frac{1}{w}\right)$.

Proof

$$\begin{aligned} \int f d\lambda \circ \sigma^{-1} &= \int f \circ \sigma d\lambda \\ &= \int \frac{1}{w} wf \circ \sigma d\lambda \quad wf \circ \sigma = R^*f \\ &= \int R\left(\frac{1}{w}\right) f d\lambda; \end{aligned}$$

and the corollary follows. □

Assume in addition that $w = \frac{d\mu}{d\lambda}$, $\mu = \lambda \circ R$. Let $w \in L^\infty(\lambda)$, and $\|w\|_\infty \leq 1$, then $\|R\|_{2 \rightarrow 2} \leq 1$ by the theorem. Then by Hilbert space theorem

$$Rh = h \Rightarrow R^*h = h \tag{2.10}$$

and so

$$\begin{aligned} h &= R^*h = wh \circ \sigma \\ h &= Rh = RR^*h = hR(w), \\ RR^*h(x) &= h(x)R(w)(x), \end{aligned}$$

so

$$h(x) > 0 \Rightarrow R(w)(x) = 1. \tag{2.11}$$

Corollary 2.8 Suppose $w^{-1} = \frac{1}{w}$ is well defined then X, λ, R, σ especially $R((f \circ \sigma)g) = fRg$, then $\lambda \circ \sigma^{-1} \ll \lambda$, and $\frac{d\lambda \circ \sigma^{-1}}{d\lambda} = R\left(\frac{1}{w}\right)$; and so λ is σ -invariant if and only if $R\left(\frac{1}{w}\right) = 1$.

Proof Let $f \in C(X)$, then

$$\begin{aligned} \int f d\lambda \circ \sigma^{-1} &= \int (f \circ \sigma) d\lambda \\ &= \int \frac{1}{w} w(f \circ \sigma) d\lambda \quad \text{where } w(f \circ \sigma) = R^*f \\ &= \int R\left(\frac{1}{w}\right) f d\lambda \quad R\left(\frac{1}{w}\right) = \text{Radon-Nikodym} \Rightarrow R\left(\frac{1}{w}\right) d\lambda = \lambda \circ \sigma^{-1}. \end{aligned}$$

the desired conclusion. □

Corollary 2.9

$$\begin{aligned} \lambda \circ \sigma^{-1} &= \lambda \\ R\left(\frac{1}{w}\right) &= 1 \end{aligned}$$

There is a generalized family of multi-resolution measures on solenoids: The solenoid may be defined for any endomorphism $\sigma : X \rightarrow X$ where X is compact, and σ is assumed to be onto. In addition, we fix a positive operator $R : C(X) \rightarrow L^\infty(X)$, and $h \geq 0$ function on X such that $Rh = h$. Also, given a finite positive measure λ on X such that $\mu(f) = \int Rf d\mu$ satisfies $\mu \ll \lambda$.

Theorem 2.10 Set $w := \frac{d\mu}{d\lambda}$. From this, we define \mathbb{P} on $Sol_\sigma(X)$ such that $\frac{d\mathbb{P} \circ \tilde{\sigma}}{d\mathbb{P}} = w \circ \pi_0$, where $\tilde{\sigma}$ is then indeed automorphism on $Sol_\sigma(X)$.

Remark 2.11 $\tilde{\sigma}(x_0 x_1 x_2, \dots) = (\sigma(x_0) x_0 x_1 x_2 \dots)$ for all $x \in Sol_\sigma(X)$. R, h, λ or $R_i, h_i, i \in \mathbb{N}$ (properties as stated in the theorem) but it is or item to measure \mathbb{P} on $Sol_\sigma(X)$. Given

$$\begin{cases} R_1, R_2, R_3, \dots \\ h_1, h_2, h_3, \dots, \quad h_i \geq 0 \quad \text{Axiom: } R_{i+1}(h_{i+1}) = h_i \end{cases}$$

in sense of that (R_i, h_i) governs the transition: $\pi_{i-1} \rightarrow \pi_i$, for all i . But if $R_{i+1}(h_{i+1}) = h_i$ and each R_i satisfy $R_i((f \circ \sigma)g) = fR_i(g)$, for all $f, g \in C(X)$, then there exists a unique \mathbb{P}_x such that we get consisting a cylinder, and so \mathbb{P}_x is well-defined. \mathbb{P}_x on a cylinder function is: Conditions C_n for cylinder functions over n .

Proof $E_x(\text{cyl}^{(n)}) = \int_{\pi_0^{-1}(x)} \text{cyl}^{(n)} d\mathbb{P}_x = C_n : f_0(x)R_1(f_1R_2(f_2 \dots R_n(f_n h_n) \dots))(x)$ The following holds

$$C_{n+1}(f_0, \dots, f_n, \mathbf{1}) = C_n(f_0, \dots, f_n)$$

holds if $R_{n+1}(h_{n+1}) = h_n$. □

2.2.1 Proof preliminaries

General setting: In detail, recall: $(\pi_i)_{i=0,1,\dots}$, $\pi_i(x) := x_i$ and $Sol_\sigma(X) := \{(x_i)_\sigma^\infty\} \subset \Pi_\sigma^\infty X \supseteq Sol_\sigma(X)$, $\sigma(x_{i+1}) = x_i$.

Definition 2.12 $x = (x_i)_\sigma^\infty$ such that $\sigma(x_{i+1}) = x_i$, for all $i = 0, 1, 2, \dots$, and we set $\pi_j(x) = x_j$, $j = 0, 1, 2, \dots$, coordinate functions; $\tilde{\sigma}(x_0 x_1 x_2 \dots) = (\sigma(x_0) x_0 x_1, x_2 \dots)$; note that $\tilde{\sigma}$ is an automorphism with inverse $\tilde{\sigma}^{-1}(x_0 x_1 x_2 \dots) = (x_1, x_2 \dots)$.

Lemma 2.13 For all $x \in X$, there exists a unique positive measure \mathbb{P}_x on $\pi_0^{-1}(x) \subset Sol_\sigma(x)$ such that $\int (f_0 \pi_0 f_1 \pi_1 \dots f_n \pi_n) d\mathbb{P}_x = f(x)R(f_1 F(f_2 \dots R(f_n h) \dots))(x)$ or $f_0(x)R_1(f_1 R_2(f_2 \dots R_n(f_n h_n) \dots))(x)$

2.2.2 General setting and assumptions

X compact, $\sigma : X \rightarrow X$ endomorphism, onto; λ finite positive measure on X , $R_i : C(X) \rightarrow L^\infty(X)$ positive such that

$$R_i((f \circ \sigma)g) = fR_i g, \quad \forall f, g \in C(X) \tag{2.12}$$

or same conditions on $R_i, i \in \mathbb{N}$. Assume there exists

$$h \geq 0, \quad \int h d\lambda = 1, \quad Rh = h, \tag{2.13}$$

i.e. a spcf. More generally $h_1, h_2, \dots R_{i+1}(h_{i+1}) = h_i$, for all i . Also, $w \in L^\infty(\lambda)$, there exists a finite constant such that

$$\left| \int (Rf) d\lambda \right|^2 \leq \text{const} \int |f|^2 d\lambda, \quad \forall f \in C(X). \tag{2.14}$$

Let $\mathbf{1}$ denote constant finite one on X . Set $w = R^* \mathbf{1} \in L^2(\lambda)$, by Riesz. i.e.,

$$\int w f d\lambda = \int (Rf) d\lambda \quad \Rightarrow \quad R^* f = w f \circ \sigma. \tag{2.15}$$

Condition (2.14) is really a Radon-Nikodym derivative as follows: Since R is positive, we have:

$$\mu(f) = \lambda(R(f)) = \int_X (Rf)(x)d\lambda(x) = \int f(wd\lambda)$$

is a measure on X by the Riesz theorem; and (2.14) is the case that $d\mu \ll d\lambda$ (absolute continuous) so the Radon-Nikodym derivative $\frac{d\mu}{d\lambda} = w \in L^1_+(X, \lambda)$ is well defined

$$\int f w d\lambda = \int f d\mu = \int (Rf) d\lambda. \quad \mu = \lambda R \tag{2.16}$$

Conversely, suppose (2.16) holds, then

$$\left| \int f w d\lambda \right|^2 \leq \int |f|^2 d\lambda \int w^2 d\lambda$$

Given $\frac{d\mathbb{P} \circ \tilde{\sigma}}{d\mathbb{P}}$ where $\mathbb{P} \leftrightarrow (R_1, R_2, R_3, \dots)$.

2.2.3 Alternative representation for \mathbb{P}_x

By Riesz, there exists $\{\mu_x\}_x$ such that $\{\pi_i\} (R_i f)(x) = \int_T f(y) d\mu_x^{(i)}(y) = \int f(y) P_i(dy|x)$

$$\begin{aligned} R_1(f_1 R_2(f_2 h))(x) &= \int \int f_1(y_1) f_2(y_2) h_2(y_2) d\mu_{y_1}(y_2) d\mu_x(y_1) \\ &= \int \int f_1(y_1) f_2(y_2) h_2(y_2) P_2(dy_2|y_1) P_1(dy_1|x) \end{aligned}$$

$$R(f_1 R(f_2 \dots R(f_n h) \dots))(x) = \int \int \dots \int f_1(y_1) f_2(y_2) \dots f_n(y_n) h_n(y_n) P(dy_n|y_{n-1}) \dots P(dy_1|x)$$

cylinder set $\sigma(E_{i+1}) \subset E_i$,

$$\mathbb{P}_x(\text{cyl}^n) = \int_{E_1} \dots \int_{E_n} h_n(y_n) P_n(dy_n|y_{n-1}) P_{n-1}(dy_{n-1}|y_{n-2}) \dots P_1(dy_1|x)$$

where $\int_{E_1} \dots \int_{E_n} h_n(y_n) P_n(dy_n|y_{n-1}) P = \mathbb{P}_x(\text{cyl})$.

3 A transfer operator

A popular tool for deciding if a candidate for a wavelet basis is in fact an ONB uses a certain transfer operator. Variants of this operator is used in diverse areas of applied mathematics. It is an operator which involves a weighted average over a finite set of possibilities. Hence it is natural for understanding random walk algorithms. As remarked in for example [6, 11–13], it was also studied in physics, for

example by David Ruelle who used to prove results on phase transition for infinite spin systems in quantum statistical mechanics. In fact the transfer operator has many incarnations (many of them known as Ruelle operators), and all of them based on N -fold branching laws.

In our wavelet application, the Ruelle operator weights in input over the N branch possibilities, and the weighting is assigned by a chosen scalar function w . the and the w -Ruelle operator is denoted R_w . In the wavelet setting there is in addition a low-pass filter function m_0 which in its frequency response formulation is a function on the d -torus $\mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d$.

Since the scaling matrix A has integer entries A passes to the quotient $\mathbb{R}^d / \mathbb{Z}^d$, and the induced transformation $r_A : \mathbb{T}^d \rightarrow \mathbb{T}^d$ is an N -fold cover, where $N = |\det A|$, i.e., for every x in \mathbb{T}^d there are N distinct points y in \mathbb{T}^d solving $r_A(y) = x$.

In the wavelet case, the weight function w is $w = |m_0|^2$. Then with this choice of w , the ONB problem for a candidate for a wavelet basis in the Hilbert space $L^2(\mathbb{R}^d)$ as it turns out may be decided by the dimension of a distinguished eigenspace for R_w , by the so called Perron-Frobenius problem.

This has worked well for years for the wavelets which have an especially simple algorithm, the wavelets that are initialized by a single function, called the scaling function. These are called the multiresolution analysis (MRA) wavelets, or for short the MRA-wavelets. But there are instances, for example if a problem must be localized in frequency domain, when the MRA-wavelets do not suffice, where it will by necessity include more than one scaling function. And we are then back to trying to decide if the output from the discrete algorithm, and the \mathcal{O}_N representation is an ONB, or if it has some stability property which will serve the same purpose, in case where asking for an ONB is not feasible.

4 Future directions

The idea of a scientific analysis by subdividing a fixed picture or object into its finer parts is not unique to wavelets. It works best for structures with an inherent self-similarity; this self-similarity can arise from numerical scaling of distances. But there are more subtle non-linear self-similarities. The Julia sets in the complex plane are a case in point [2–5, 14, 15]. The simplest Julia set come from a one parameter family of quadratic polynomials $\varphi_c(z) = z^2 + c$, where z is a complex variable and where c is a fixed parameter. The corresponding Julia sets J_c have a surprisingly rich structure. A simple way to understand them is the following: Consider the two branches of the inverse $\beta_{\pm} = z \mapsto \pm\sqrt{z-c}$. Then J_c is the unique minimal non-empty compact subset of \mathbb{C} , which is invariant under $\{\beta_{\pm}\}$. (There are alternative ways of presenting J_c but this one fits our purpose. The Julia set J of a holomorphic function, in this case $z \mapsto z^2 + c$, informally consists of those points whose long-time behavior under repeated iteration, or rather iteration of substitutions, can change drastically under arbitrarily small perturbations.) Here “long-time” refers to largen n , where $\varphi^{(n+1)}(z) = \varphi(\varphi^{(n)}(z))$, $n = 0, 1, \dots$, and $\varphi^{(0)}(z) = z$. Please see Figs. 1 and 2 for examples of Julia set graphs.

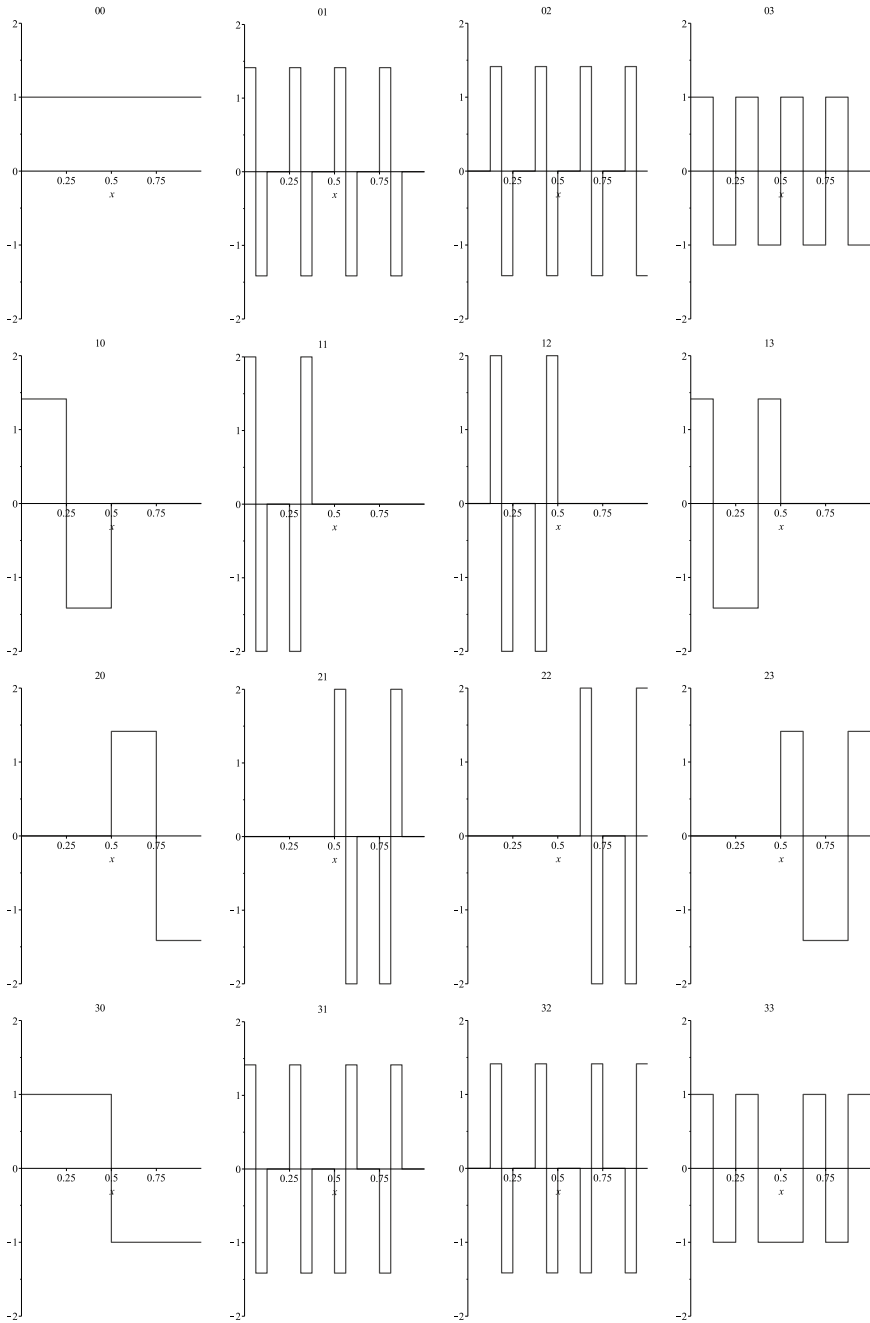
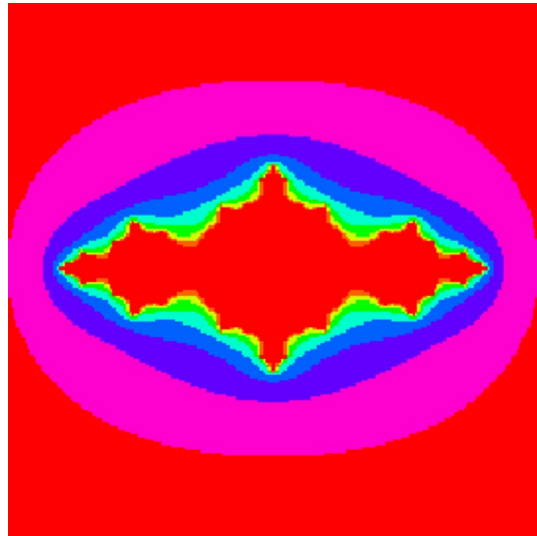


Fig. 1 Walsh functions $S_w,1$ for words w of length two [9]

Fig. 2 Julia set with $c = -1$



It would be interesting to adapt and modify the Haar wavelet, and the other wavelet algorithms to the Julia sets. The two papers [7, 8] initiated such a development. Then an attempt to adapt and modify the Haar wavelet to the Julia sets was made, [9] however, there were some limitations in finding the filters. Perhaps trying another fractal set such as tent map or others may work.

4.1 Orthonormal bases generated by Cuntz algebras

We present new results from [9] by borrowing Sect. 3 and part of Sect. 2 from [9] in the rest of this Sect. 4.1. It gives a general criterion for a family generated by the Cuntz isometries to be an orthonormal basis.

Theorem 4.1 [9] *Let \mathcal{H} be a Hilbert space and $(S_i)_{i=0}^{N-1}$ be a representation of the Cuntz algebra \mathcal{O}_N . Let \mathcal{E} be an orthonormal set in \mathcal{H} and $f : X \rightarrow \mathcal{H}$ a norm continuous function on a topological space X with the following properties:*

- (i) $\mathcal{E} = \cup_{i=0}^{N-1} S_i \mathcal{E}$.
- (ii) $\overline{\text{span}}\{f(t) : t \in X\} = \mathcal{H}$ and $\|f(t)\| = 1$, for all $t \in X$.
- (iii) *There exist functions $\mathbf{m}_i : X \rightarrow \mathbb{C}$, $g_i : X \rightarrow X$, $i = 0, \dots, N - 1$ such that*

$$S_i^* f(t) = \mathbf{m}_i(t) f(g_i(t)), \quad t \in X. \tag{4.1}$$

- (iv) *There exist $c_0 \in X$ such that $f(c_0) \in \overline{\text{span}}\mathcal{E}$.*
- (v) *The only function $h \in C(X)$ with $h \geq 0$, $h(c) = 1, \forall c \in \{x \in X : f(x) \in \overline{\text{span}}\mathcal{E}\}$, and*

$$h(t) = \sum_{i=0}^{N-1} |\mathbf{m}_i(t)|^2 h(g_i(t)), \quad t \in X \iff Rh = h \tag{4.2}$$

are the constant functions.

Then \mathcal{E} is an orthonormal basis for \mathcal{H} .

Proof Define

$$h(t) := \sum_{e \in \mathcal{E}} |\langle f(t), e \rangle|^2 = \|Pf(t)\|^2, \quad t \in X$$

where P is the orthogonal projection onto the closed linear span of \mathcal{E} .

Since $t \mapsto f(t)$ is norm continuous we get that h is continuous. Clearly $h \geq 0$. Also, if $f(c) \in \overline{\text{span}}\mathcal{E}$, then $\|Pf(c)\| = \|f(c)\| = 1$ so $h(c) = 1$. In particular, from (ii) and (iv), $h(c_0) = 1$. We check (4.2). Since the sets $S_i\mathcal{E}$, $i = 0, \dots, N - 1$ are mutually orthogonal, the union in (i) is disjoint. Therefore for all $t \in X$:

$$\begin{aligned} h(t) &= \sum_{i=0}^{N-1} \sum_{e \in \mathcal{E}} |\langle f(t), S_i e \rangle|^2 = \sum_{i=0}^{N-1} \sum_{e \in \mathcal{E}} |\langle S_i^* f(t), e \rangle|^2 = \sum_{i=0}^{N-1} |\mathbf{m}_i(t)|^2 \sum_{e \in \mathcal{E}} |\langle f(g_i(t)), e \rangle|^2 \\ &= \sum_{i=0}^{N-1} |\mathbf{m}_i(t)|^2 h(g_i(t)) \end{aligned}$$

By (v), h is constant and, since $h(c_0) = 1$, $h(t) = 1$ for all $t \in X$. Then $\|Pf(t)\| = 1$ for all $t \in X$. Since $\|f(t)\| = 1$ it follows that $f(t) \in \text{span } \mathcal{E}$ for all $t \in X$. But the vectors $f(t)$ span \mathcal{H} so $\overline{\text{span}}\mathcal{E} = \mathcal{H}$ and \mathcal{E} is an orthonormal basis. □

Remark 4.2 [9] The operators of the form

$$Rh(t) = \sum_{i=0}^{N-1} |\mathbf{m}_i(t)|^2 h(g_i(t)), \quad t \in X, h \in C(X),$$

that appear in (4.2), are sometimes called Ruelle operators or transfer operators, see e.g. [1].

4.1.1 Piecewise exponential bases on fractals

Example 4.3 [9] We consider affine iterated function systems with no overlap. Let R be a $d \times d$ expansive real matrix, i.e., all the eigenvalues of R have absolute value strictly greater than 1. Let $B \subset \mathbb{R}^d$ a finite set such that $N = |B|$. Define the affine iterated function system

$$\tau_b(x) = R^{-1}(x + b) \quad (x \in \mathbb{R}^d, b \in B) \tag{4.3}$$

By [10] there exists a unique compact subset X_B of \mathbb{R}^d which satisfies the invariance equation

$$X_B = \cup_{b \in B} \tau_b(X_B) \tag{4.4}$$

X_B is called the attractor of the iterated function system $(\tau_b)_{b \in B}$. Moreover X_B is given by

$$X_B = \left\{ \sum_{k=1}^{\infty} R^{-k} b_k : b_k \in B \text{ for all } k \geq 1 \right\} \tag{4.5}$$

Also from [10] there is a unique probability measure μ_B on \mathbb{R}^d satisfying the invariance equation

$$\int f d\mu_B = \frac{1}{N} \sum_{b \in B} \int f \circ \tau_b d\mu_B \tag{4.6}$$

for all continuous compactly supported functions f on \mathbb{R} . We call μ_B the invariant measure for the iterated function system (IFS) $(\tau_b)_{b \in B}$. By [10], μ_B is supported on the attractor X_B . We say that the IFS has no overlap if $\mu_B(\tau_b(X_B) \cap \tau_{b'}(X_B)) = \emptyset$ for all $b \neq b'$ in B .

Assume that the IFS $(\tau_b)_{b \in B}$ has no overlap. Define the map $r : X_B \rightarrow X_B$

$$r(x) = \tau_b^{-1}(x), \text{ if } x \in \tau_b(X_B) \tag{4.7}$$

Then r is an N -to-1 onto map and μ_B is strongly invariant for r . Note that $r^{-1}(x) = \{\tau_b(x) : b \in B\}$ for μ_B -a.e. $x \in X_B$.

We apply Theorem 4.1 to the setting of Example 4.3, in dimension $d = 1$ for affine iterated function systems, when the set $\frac{1}{R}B$ has a spectrum L [9].

Definition 4.4 [9] Let L in \mathbb{R} , $|L| = N$, $R > 1$ such that L is a spectrum for the set $\frac{1}{R}B$.

We say that $c \in \mathbb{R}$ is an *extreme cycle point* for (B, L) if there exists l_0, l_1, \dots, l_{p-1} in L such that, if $c_0 = c$, $c_1 = \frac{c_0 + l_0}{R}$, $c_2 = \frac{c_1 + l_1}{R} \dots c_{p-1} = \frac{c_{p-2} + l_{p-2}}{R}$ then $\frac{c_{p-1} + l_{p-1}}{R} = c_0$, and $|m_B(c_i)| = 1$ for $i = 0, \dots, p - 1$ where

$$m_B(x) = \frac{1}{N} \sum_{b \in B} e^{2\pi i b x} \quad x \in \mathbb{R}.$$

Proposition 4.5 [9] Let $(m_i)_{i=0}^{N-1}$ be a QMF basis. Define the operators on $L^2(X, \mu)$

$$S_i(f) = m_i f \circ r, \quad i = 0, \dots, N - 1 \tag{4.8}$$

Then the operators S_i are isometries and they form a representation of the Cuntz algebra \mathcal{O}_N , i.e.

$$S_i^* S_j = \delta_{ij}, \quad i, j = 0, \dots, N - 1, \quad \sum_{i=0}^{N-1} S_i^* S_i = I \tag{4.9}$$

The adjoint of S_i is given by the formula

$$S_i^*(f)(z) = \frac{1}{N} \sum_{r(w)=z} \overline{m_i(w)} f(w) \tag{4.10}$$

Proof We compute the adjoint: take f, g in $L^2(X, \mu)$. We use the strong invariance of μ .

$$\langle S_i^* f, g \rangle = \int f \overline{m_i g \circ r} d\mu = \int \frac{1}{N} \sum_{r(w)=z} \overline{m_i(w)} f(w) \overline{g(z)} d\mu(z)$$

Then (4.10) follows. The Cuntz relations in (4.9) are then easily checked with Proposition ?? □

Definition 4.6 [9] We denote by L^* the set of all finite words with digits in L , including the empty word. For $l \in L$ let S_l be given as in (4.8) where m_l is replaced by the exponential e_l . If $w = l_1 l_2 \dots l_n \in L^*$ then by S_w we denote the composition $S_{l_1} S_{l_2} \dots S_{l_n}$.

Theorem 4.7 [9] Let $B \subset \mathbb{R}, 0 \in B, |B| = N, R > 1$ and let μ_B be the invariant measure associated to the IFS $\tau_b(x) = R^{-1}(x + b), b \in B$. Assume that the IFS has no overlap and that the set $\frac{1}{R}B$ has a spectrum $L \subset \mathbb{R}, 0 \in L$. Then the set

$$\mathcal{E}(L) = \{S_w e_{-c} : c \text{ is an extreme cycle point for } (B, L), w \in L^*\}$$

is an orthonormal basis in $L^2(\mu_B)$. Some of the vectors in $\mathcal{E}(L)$ are repeated but we count them only once.

Proof Let c be an extreme cycle point. Then $|m_B(c)| = 1$. Using the fact that we have equality in the triangle inequality ($1 = |m_B(c)| \leq \frac{1}{N} \sum_{b \in B} |e^{2\pi i b c}| = 1$), and since $0 \in B$, we get that $e^{2\pi i b c} = 1$ so $bc \in \mathbb{Z}$ for all $b \in B$. Also there exists another extreme cycle point d and $l \in L$ such that $\frac{d+l}{R} = c$. Then we have: $S_l e_{-c}(x) = e^{2\pi i l x} e^{2\pi i (R x - b)(-c)}$, if $x \in \tau_b(X_B)$. Since $bc \in \mathbb{Z}$ and $R(-c) + l = -d$, we obtain

$$S_l e_{-c} = e_{-d} \tag{4.11}$$

We use this property to show that the vectors $S_w e_{-c}, S_{w'} e_{-c'}$ are either equal or orthogonal for w, w' in L^* and c, c' extreme cycle points for (B, L) . Using (4.11), we can append some letters at the end of w and w' such that the new words have the same length:

$$S_w e_{-c} = S_{w\alpha} e_{-d}, \quad S_{w'} e_{-c'} = S_{w'\beta} e_{-d'}, \quad |w\alpha| = |w'\beta| \quad \text{where } d, d' \text{ are cycle points.}$$

Moreover, repeating the letters for the cycle points d and d' as many times as we want, we can assume that α ends in a repetition of the letters associated to d and similarly for β and d' . But, since $|w\alpha| = |w'\beta|$, the Cuntz relations imply that $S_{w\alpha} e_{-d} \perp S_{w'\beta} e_{-d'}$ or $w\alpha = w'\beta$. Assume $|w| \leq |w'|$. Then $\alpha = w''\beta$ for some word w'' . Then $S_{w\alpha} e_{-d} \perp S_{w''\beta} e_{-d}$ iff $S_\alpha e_{-d} \perp S_{w''\beta} e_{-d}$. Also, α consists of repetitions of the digits of the cycle associated to d and similarly for d' . So $S_\alpha e_{-d} = e_{-f}$, $S_{w''\beta} e_{-d} = e_{-f'}$, and all points d, d', f, f', c, c' all belong to the same cycle. So the only case when $S_w e_{-c}$ is not orthogonal to $S_{w'} e_{-c'}$ is when they are equal.

Next we check that the hypotheses of Theorem 4.1 are satisfied. We let $f(t) = e_{-t} \in L^2(\mu_B)$. To check (i) we just have to see that $e_{-c} \in \cup_{l \in L} S_l \mathcal{E}(L)$. But this follows from (4.11). Requirement (ii) is clear. For (iii) we compute

$$\begin{aligned} S_l^* e_{-l}(x) &= \frac{1}{N} \sum_{b \in B} e^{-2\pi i l \cdot \frac{1}{R}(x+b)} e^{-2\pi i t \cdot \frac{1}{R}(x+b)} = e^{-2\pi i x \cdot \frac{1}{R}(t+l)} \frac{1}{N} \sum_{b \in B} e^{-2\pi i b(\frac{t+l}{R})} \\ &= \overline{m_B}\left(\frac{t+l}{R}\right) e_{-\frac{t+l}{R}}(x) \end{aligned}$$

So (iii) is satisfied with $m_l(t) = \overline{m_B}\left(\frac{t+l}{R}\right)$, $g_l(t) = \frac{t+l}{R}$.

For (iv) take $c_0 = -c$ for any extreme cycle point (0 is always one). For (v), take h continuous on \mathbb{R} , $0 \leq h \leq 1$, $h(c) = 1$ for all c with $e_{-c} \in \overline{\text{span}}\mathcal{E}(L)$, and

$$h(t) = \sum_{l \in L} \left| m_B\left(\frac{t+l}{R}\right) \right|^2 h\left(\frac{t+l}{R}\right) := Rh(t)$$

In particular, we have $h(c) = 1$ for every extreme cycle point c . Assume $h \not\equiv 1$. First we will restrict our attention to $t \in I := [a, b]$ with $a \leq \frac{\min L}{R-1}$, $b \geq \frac{\max L}{R-1}$, and note that $g_l(I) \subset I$ for all $l \in L$. Let $m = \min_{l \in L} h(t)$. Then let $h' = h - m$, assume $m < 1$. Then $Rh'(t) = h'(t)$ for all $t \in \mathbb{R}$, h' has a zero in I and $h \geq 0$ on I , $h'(z_0) = 0$. But this implies that $|m_B(g_l(z_0))|^2 h'(g_l(z_0)) = 0$ for all $l \in L$. Since $\sum_{l \in L} |m_B(g_l(z_0))|^2 = 1$, it follows that for one of the $l_0 \in L$ we have $h'(g_{l_0}(z_0)) = 0$. By induction, we can find $z_n = g_{l_{n-1}} \dots g_{l_0} z_0$ such that $h'(z_n) = 0$. We prove that z_0 is a cycle point. Suppose not. Since m_B has finitely many zeros, for n large enough $g_{\alpha_k} \dots g_{\alpha_1} z_n$ is not a zero for m_B , for any choice of digits $\alpha_1, \dots, \alpha_k$ in L . But then, by using the same argument as above we get that $h'(g_{\alpha_k} \dots g_{\alpha_1} z_n) = 0$ for any $\alpha_1, \dots, \alpha_k \in L$. The points $\{g_{\alpha_k} \dots g_{\alpha_1} z_n : \alpha_1, \dots, \alpha_k \in L, k \in \mathbb{N}\}$ are dense in the attractor X_L of the IFS $\{g_l\}_{l \in L}$, thus h' is constant zero on X_L . But the extreme cycle points c are in X_L and since $h(c) = 1$ we have $0 = h'(c) = 1 - m$, so $m = 1$. Thus $h = 1$ on I . Since we can let $a \rightarrow -\infty$ and $b \rightarrow \infty$ we obtain that $h \equiv 1$. □

Remark 4.8 [9] The functions in $\mathcal{E}(L)$ are piecewise exponential. The formula for $S_{l_1 \dots l_n} e_{-c}$ is

$$S_{l_1 \dots l_n} e_{-c}(x) = e^{\alpha^{(b,l,c)}} \cdot e_{l_1 + Rl_2 + \dots + R^{n-1}l_n + R^n(-c)}(x) \tag{4.12}$$

where $\alpha(b, l, c) = -[b_1 l_2 + (Rb_1 + b_2)l_3 + \dots + (R^{n-2}b_1 + \dots + b_{n-1})l_n] + (R^{n-1}b_1 + \dots + b_n) \cdot c$ if $x \in \tau_{b_1} \dots \tau_{b_n} X_B$. We have

$$S_{l_1} \dots S_{l_n} e_{-c}(x) = e_{l_1}(x)e_{l_2}(rx) \dots e_{l_n}(r^{n-1}x)e_c(r^n x)$$

If $x \in \tau_{b_1} \dots \tau_{b_n} X_B$ then $rx \in \tau_{b_2} \dots \tau_{b_n} X_B, r^{n-1}x \in \tau_{b_n} X_B$. So

$$\begin{aligned} rx &= Rx - b_1 \\ r^2x &= Rrx - b_2 = R^2x - Rb_1 - b_2 \\ &\vdots \\ r^{n-1}x &= R^{n-1}x - R^{n-2}b_1 - \dots - Rb_{n-2} - b_{n-1} \\ r^n x &= R^n x - R^{n-1}b_1 - R^{n-2}b_2 - \dots - Rb_{n-1} - b_n. \end{aligned}$$

The rest follows from a direct computation.

Corollary 4.9 [9] *In the hypothesis of Theorem 4.1, if in addition $B, L \subset \mathbb{Z}$ and $R \in \mathbb{Z}$, then there exists a set Λ such that $\{e_\lambda : \lambda \in \Lambda\}$ is an orthonormal basis for $L^2(\mu_B)$.*

Proof If everything is an integer then, it follows from Remark 4.8 that $S_w e_{-c}$ is an exponential function for all w and extreme cycle points c . Note that, as in the proof of Theorem 4.1, $bc \in \mathbb{Z}$ for all $b \in B$. □

Example 4.10 [9] We consider the IFS that generates the middle third Cantor set: $R = 3, B = \{0, 2\}$. The set $\frac{1}{3}\{0, 2\}$ has spectrum $L = \{0, 3/4\}$. We look for the extreme cycle points for (B, L) . We need $|m_B(-c)| = 1$ so $|\frac{1+e^{2\pi i 2c}}{2}| = 1$, therefore $c \in \frac{1}{2}\mathbb{Z}$. Also c has to be a cycle for the IFS $g_0(x) = x/3, g_{3/4}(x) = \frac{x+3/4}{3}$ so $0 \leq c \leq \frac{3/4}{3-1} = 3/8$. Thus, the only extreme cycle is $\{0\}$. By Theorem 4.1 $\mathcal{E} = \{S_w 1 : w \in \{0, 3/4\}^*\}$ is an orthonormal basis for $L^2(\mu_B)$. Note also that the numbers $e^{2\pi i \alpha(b, l, c)}$ in formula (4.12) are ± 1 because $2\pi i B \cdot L \subset \pi i \mathbb{Z}$.

4.1.2 Walsh bases

In the following, we will focus on the unit interval, which can be regarded as the attractor of a simple IFS and we use step functions for the QMF basis to generate Walsh-type bases for $L^2[0, 1]$ [9].

Example 4.11 [9] The interval $[0, 1]$ is the attractor of the IFS $\tau_0 x = \frac{x}{2}, \tau_1 x = \frac{x+1}{2}$, and the invariant measure is the Lebesgue measure on $[0, 1]$. The map r defined in

Example 4.3 is $rx = 2x \pmod 1$. Let $m_0 = 1, m_1 = \chi_{[0,1/2)} - \chi_{[1/2,1)}$. It is easy to see that $\{m_0, m_1\}$ is a QMF basis. Therefore S_0, S_1 defined as in Proposition 4.5 form a representation of the Cuntz algebra \mathcal{O}_2 .

Proposition 4.12 [9] *The set $\mathcal{E} := \{S_w 1 : w \in \{0, 1\}^*\}$ is an orthonormal basis for $L^2[0, 1]$, the Walsh basis.*

Proof We check the conditions in Theorem 4.1. To see that (i) holds note that $S_0 1 = 1$. Define $f(t) = e_t, t \in \mathbb{R}$. (ii) is clear. For (iii) we compute

$$S_1^* e_t(x) = \frac{1}{2}(e^{2\pi i t \cdot x/2} + e^{2\pi i t \cdot (x+1)/2}) = e^{2\pi i t \cdot x/2} \frac{1}{2}(1 + e^{2\pi i t/2})$$

$$S_1^* e_t(x) = \frac{1}{2}(e^{2\pi i t \cdot x/2} - e^{2\pi i t \cdot (x+1)/2}) = e^{2\pi i t \cdot x/2} \frac{1}{2}(1 - e^{2\pi i t/2})$$

Thus (iii) holds with $\mathbf{m}_0(t) = \frac{1}{2}(1 + e^{2\pi i t/2}), \mathbf{m}_1(t) = \frac{1}{2}(1 - e^{2\pi i t/2}), g_0(t) = g_1(t) = \frac{t}{2}$. Since $e_0 = 1$ it follows that (iv) holds.

For (v) take h continuous on $\mathbb{R}, 0 \leq h \leq 1, h(c) = 1$ for all $c \in \mathbb{R}$ with $e_t \in \overline{\text{span}} \mathcal{E}$, in particular $h(0) = 1$ and

$$h(t) = \left| \frac{1}{2}(1 + e^{2\pi i t/2}) \right|^2 h(t/2) + \left| \frac{1}{2}(1 - e^{2\pi i t/2}) \right|^2 h(t/2) = h(t/2)$$

Then $h(t) = h(t/2^n)$ for all $t \in \mathbb{R}, n \in \mathbb{N}$. Letting $n \rightarrow \infty$ and using the continuity of h , we get $h(t) = h(0) = 1$ for all $t \in \mathbb{R}$. Since all conditions hold, we get that \mathcal{E} is an orthonormal basis. That \mathcal{E} is actually the Walsh basis follows from the following calculations: for $|w| = n$ in $\{0, 1\}^*$ let $n = \sum_i x_i 2^i$ be the base two expansion of n . Because $S_0 f = f$ or, $S_1 f = m_1 f$ or and $m_0 \equiv 1$ we obtain the following decomposition:

$S_w 1(x) = m_1(r^{i_1} x) \cdot m_1(r^{i_2} x) \dots m_1(r^{i_k} x),$ where i_1, i_2, \dots, i_k correspond to those i with $x_i = 1$
 Also $m_1(r^i x) = m_1(2^i x \pmod 1)$ are the Rademacher functions and thus we obtain the Walsh basis (see e.g. [16]). □

The Walsh bases can be easily generalized by replacing the matrix

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

which appears in the definition of the filters m_0, m_1 , with an arbitrary unitary matrix A with constant first row and by changing the scale from two to N .

Theorem 4.13 Let $N \in \mathbb{N}$, $N \geq 2$. Let $A = [a_{ij}]$ be an $N \times N$ unitary matrix whose first row is constant $\frac{1}{\sqrt{N}}$. Consider the IFS $\tau_j x = \frac{x+j}{N}$, $x \in \mathbb{R}$, $j = 0, \dots, N - 1$ with the attractor $[0, 1]$ and invariant measure the Lebesgue measure on $[0, 1]$. Define

$$m_i(x) = \sqrt{N} \sum_{j=0}^{N-1} a_{ij} \chi_{[j/N, (j+1)/N]}(x)$$

Then $\{m_i\}_{i=0}^{N-1}$ is a QMF basis. Consider the associated representation of the Cuntz algebra \mathcal{O}_N . Then the set $\mathcal{E} := \{S_w 1 : w \in \{0, \dots, N - 1\}^*\}$ is an orthonormal basis for $L^2[0, 1]$.

Proof We check the conditions in Theorem 4.1. Let $f(t) = e_t$, $t \in \mathbb{R}$.

To check (i) note that $S_0 1 \equiv 1$. (ii) is clear. For (iii) we compute:

$$S_k^* e_t = \frac{1}{N} \sum_{j=0}^{N-1} \overline{m_k}(\tau_j x) e_t(\tau_j x) = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} \overline{a_{kj}} e^{2\pi i t \cdot (x+j)/N} = e^{2\pi i t \cdot x/N} \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} \overline{a_{kj}} e^{2\pi i t \cdot j/N}$$

So (iii) is true with $m_k(t) = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} \overline{a_{kj}} e^{2\pi i t \cdot j/N}$ and $g_k(t) = \frac{t}{N}$.

(iv) is true with $c_0 = 0$. For (v) take $h \in C(\mathbb{R})$, $0 \leq h \leq 1$, $h(c) = 1$ for all $c \in \mathbb{R}$ with $e_c \in \overline{\text{span} \mathcal{E}}$ (in particular $h(0) = 1$), and

$$h(t) = \sum_{k=0}^{N-1} |m_k(t)|^2 h(t/N) = h(t/N) \sum_{k=0}^{N-1} \frac{1}{N} \left| \sum_{j=0}^{N-1} a_{kj} e^{-2\pi i t \cdot j/N} \right|^2 = h(t/N) \cdot \frac{1}{N} \|Av\|^2$$

where $v = (e^{-2\pi i t \cdot j/N})_{j=0}^{N-1}$. Since A is unitary, $\|Av\|^2 = \|v\|^2 = N$. Then $h(t) = h(t/N^n)$. Letting $n \rightarrow \infty$ and using the continuity of h we obtain that $h(t) = 1$ for all $t \in \mathbb{R}$. Thus, Theorem 4.1 implies that \mathcal{E} is an orthonormal basis. □

Remark 4.14 [9] We can read the constants that appear in the step function $S_w 1$ from the tensor of A with itself n times, where n is the length of the word w .

Let A be an $N \times N$ matrix, B an $M \times M$ matrix. Then $A \otimes B$ has entries:

$$(A \otimes B)_{i_1+Mi_2, j_1+Mj_2} = a_{i_1, j_1} b_{i_2, j_2}, \quad i_1, j_1 = 0, \dots, N - 1, i_2, j_2 = 0, \dots, M - 1$$

$$A \otimes B = \begin{pmatrix} Ab_{0,0} & Ab_{0,1} & \dots & Ab_{0,M-1} \\ Ab_{1,0} & Ab_{1,1} & \dots & Ab_{1,M-1} \\ \vdots & \vdots & \ddots & \vdots \\ Ab_{M-1,0} & Ab_{M-1,1} & \dots & Ab_{M-1,M-1} \end{pmatrix}$$

The matrix $A^{\otimes n}$ is obtained by induction, tensoring to the left: $A^{\otimes n} = A \otimes A^{\otimes(n-1)}$.

Thus $A \otimes A \otimes A \otimes \dots \otimes A$, n times, has entries

$$A^{\otimes n}_{i_0+N_1+N^2i_2+\dots+N^{n-1}i_{n-1}j_0+Nj_1+\dots+N^{n-1}j_{n-1}} = a_{i_0j_0}a_{i_1j_1} \dots a_{i_{n-1}j_{n-1}}$$

Now compute for $i_0, \dots, i_{n-1} \in \{0, \dots, N - 1\}$:

$$S_{i_0 \dots i_{n-1}} 1(x) = m_{i_0}(x)m_{i_1}(rx) \dots m_{i_{n-1}}(r^{n-1}x)$$

Suppose $x \in [\frac{k}{N^n}, \frac{k+1}{N^n})$, $0 \leq k < N^n$ and $k = N^{n-1}j_0 + N^{n-2}j_1 + \dots + Nj_{n-2} + j_{n-1}$, where $0 \leq j_0, \dots, j_{n-1} < N$.

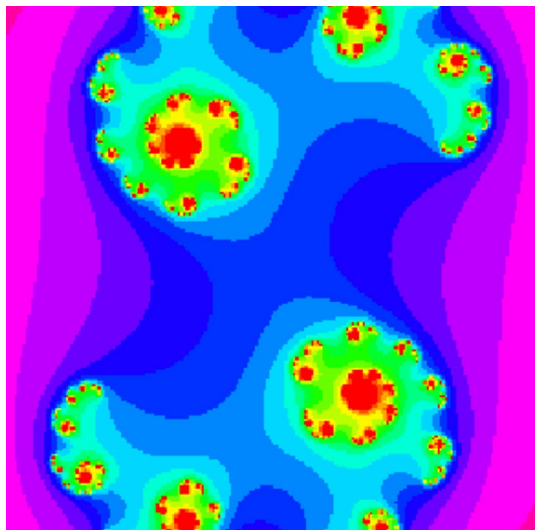
Then $x \in [\frac{j_0}{N}, \frac{j_0+1}{N})$, $rx = (Nx) \bmod 1 \in [\frac{j_1}{N}, \frac{j_1+1}{N})$, ..., $r^{n-1}x = (N^{n-1}x) \bmod 1 \in [\frac{j_{n-1}}{N}, \frac{j_{n-1}+1}{N})$, so $m_{i_0}(x) = \sqrt{N}a_{i_0j_0}$, $m_{i_1}(rx) = \sqrt{N}a_{i_1j_1}$, ..., $m_{i_{n-1}}(r^{n-1}x) = \sqrt{N}a_{i_{n-1}j_{n-1}}$ hence

$$S_{i_0 \dots i_{n-1}} 1(x) = \sqrt{N^n}a_{i_0j_0} \dots a_{i_{n-1}j_{n-1}} = \sqrt{N^n}A^{\otimes n}_{i_0+N_1+N^2i_2+\dots+N^{n-1}i_{n-1}j_0+Nj_1+\dots+N^{n-1}j_{n-1}}$$

Example 4.15 [9] The pictures in Fig. 3 show the Walsh functions that correspond to the scale $N = 4$ and the matrix

$$A = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 & 0 \\ 0 & 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

Fig. 3 Julia set with $c = 0.45 - 0.1428i$



for the words of length two, indicated at the top.

5 Multi-resolutions and generalized wavelet representations

As is illustrated in [12], and the references given there; as well as in the papers [6–9], there is a host of problems from analysis of fractals and more generally in stochastic analysis which lend themselves to the present multi-resolution approach. Below we discuss related wavelet representations.

Lemma 5.1 *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probabilistic space, and let $\mathcal{A} : \Omega \rightarrow X$ be a random variable with values in a fixed measure space (X, \mathcal{B}_X) , then $V_{\mathcal{A}}f := f \circ \mathcal{A}$ defines an isometry. $L^2(X, \mu_{\mathcal{A}}) \rightarrow L^2(\Omega, \mathbb{P})$ where $\mu_{\mathcal{A}}$ is the law of \mathcal{A} , i.e., $\mu_{\mathcal{A}}(\Delta) := \mathbb{P}(\mathcal{A}^{-1}(\Delta))$, for all $\Delta \in \mathbb{B}_X$; and $V_{\mathcal{A}}^*(x) = \mathbb{E}_{\mathcal{A}=x}(\psi | \mathcal{F}_{\mathcal{A}})$ for all $\psi \in L^2(\Omega, \mathbb{P})$, and all $x \in X$.*

If $(\Omega, \mathcal{F}, \mathbb{P})$ is a solenoid probability span on $\Omega_X = \prod_{n=0}^{\infty} X$, we shall apply the lemma to the each vertices

$\pi_n : \Omega_X \rightarrow X$ given by $\pi_n(x_0x_1x_2 \dots) = x_n$, for all $n \in \mathbb{N}_0$, and the isometry can span to π_n will simply be denoted V_n . The sigma-algebra given by π_n will be denoted \mathcal{F}_n . Let (X, \mathbb{B}) be fixed, let $Rf(x) = \int f(y)\mu(dy|x)$, $f \in \mathcal{F}(X, \mathcal{B})$ and $Rh = h$ i.e., $\mu(\dots |x)$ is a probability space and (X, \mathcal{B}) a.e. $x \in X$. Suppose there exists an X and w such that $\int \mu(B|x)d\lambda(x) = \int_B w d\lambda$, for all $B \in \mathcal{B}_X$ then there exists a probability space (Ω, \mathbb{P}) which is the all paths on (X, \mathcal{B}) such that

$$\int_{\Omega} (f_0 \circ \pi_0)(f_1 \circ \pi_1) \dots (f_n \circ \pi_n) d\mathbb{P} = \int_X f_0(x)R(f, R(f_2 \dots R(f_n) \dots))(x)d\lambda(x)$$

and $\mathbb{P} \circ \pi_1^{-1} = ((W \circ \pi_0)d\mathbb{P}) \circ \pi_0^{-1}$. Moreover,

$$\begin{aligned} \text{suppt}(\mathbb{P}) &= \text{Sol}_{\sigma}(X) \iff \\ R[(f \circ \sigma)g] &= fR(g), \quad \forall f, g \in \mathcal{F}(X, \mathcal{B}). \end{aligned}$$

Suppose (R, λ) has the representation $(Rf)(x) = \int_X f(x)\mu(dy, x)$ where $\mu(x)$ is a measure of (X, \mathcal{B}) for all $x \in X$, and each function $X \mu(B, x)$ is measurable for all $B \in \mathcal{B}$. This is only a mild restriction.

Note that a definition by application Riesz if X is locally compact Hausdorff and \mathcal{B}_X is the Borel-sigma algebra. Suppose $R(1) = 1$, then the following representation of \mathbb{P} on $(\text{Sol}(X), \text{cylindersets}, \mathbb{P})$ are equivalent: The following are equivalent

- (i) $\int_{\text{Sol}} (f \circ \pi_0)(g \circ \pi_n) d\mathbb{P} = \int_X f(x)(R^n g)(x) d\lambda(x)$
- (ii) $\text{Prob}_{w.r.t} \mathbb{P}(\pi_0 = x, \pi_1 \in B_1, \dots, \pi_k \in B_k) = \int_{B_1} \int_{B_2} \dots \int_{B_k} \mu(dy_1|x)\mu(dy_2|y_1) \dots \mu(dy_k|y_{k-1})$
for all k and for all $B_i \in \mathcal{B}_X$;
- (iii) $\text{Prob}_{w.r.t} \mathbb{P}(\pi_0 = x, \pi_1 \in B_1, \dots, \pi_k \in B_k) = R(\chi_{B_1} R(\chi_{B_2} R \dots R(\chi_{B_k}) \dots)(x)) = \mathbb{E}_x(\dots) = \int_{\text{Sol}} \dots d\mathbb{P}_x$

In the case $R\mathbf{1} = 1$. But if not, then pick h such that $Rh = h$, and set $R'(f) = \frac{R(fh)}{h}$ will $R'(\mathbf{1}) = \mathbf{1}$.

$$\begin{aligned}
 & Prob(\pi_0 = x, \pi_1 = y_1, \dots, \pi_k = y_k) \\
 & \frac{1}{N^k} W(y_1)W(y_2) \dots W(y_k) \\
 & Pr(x \rightarrow y_1)Pr(y_1 \rightarrow y_2) \dots Pr(y_{k-1} \rightarrow y_k) \\
 Rf(x) &= \int_X f(y)\mu(dy|x) \quad \text{represent R as an...} \\
 \mu(B|x) &:= R(\chi_B)(x), \forall B \in \mathcal{B} \\
 &= Prob(\pi_1 \in B | \pi_0 = x).
 \end{aligned}$$

More generally:

$$\begin{aligned}
 & Prob(\pi_0 = x, \pi_1 \in B_1, \dots, \pi_k \in B_k) \\
 &= \int_{B_1} \int_{B_2} \dots \int_{B_k} \mu(dy_1|x)\mu(dy_2|y_1) \dots \mu(dy_k|y_{k-1}) \\
 &= R(\chi_{B_1}R(\chi_{B_2}R \dots R(\chi_{B_k}) \dots (x))
 \end{aligned}$$

Same manner prop of $\{\mu(- | x)\}_{x \in X}$.

Lemma 5.2 *If $B \in \mathcal{B}_X$ then*

$$\int_X \mu(B|x)d\lambda(x) = \int_B W(x)d\lambda(x), \quad \text{where } W = \frac{d\lambda(R)}{d\lambda}.$$

Proof Let $\{\mu(B|x)\}_{x \in X}$ be a Markov process by $x \in X$ and (X, \mathcal{B}) is a fixed measure space and let \mathbb{P} be the corresponding path space measure $\mathbb{P}(\pi_0 = x, \pi_1 \in B_1, \dots, \pi_k \in B_k)$. Let

$$\sigma \in \text{End}(XB) = \int_{B_1} \int_{B_2} \dots \int_{B_k} \mu(dy_1|x)\mu(dy_2|y_1) \dots \mu(dy_k|y_{k-1})$$

then \mathbb{P} is ...as equation σ -solenoid $Sol_\sigma(X)$ if and only if

$$\mathbb{P}(\pi_{k-1} \in B \cap \sigma^{-1}(A) | \pi_k = x) = \chi_A(x)\mathbb{P}(\pi_k \in B | \pi_k = x)$$

if and only if $\text{supp}(\mathbb{P}) \subset Sol_\sigma(X)$. □

Lemma 5.3 *Suppose R has a representation*

$$R(\chi_B)(x) = \mu(B|x), \quad B \in \mathcal{B}_X, \quad x \in X. \tag{5.1}$$

The following are equivalent: as described, i.e., $(Rf)(x) = \int_X f(x)\mu(dy|x)$; then

$$R[(f \circ \sigma)g](x) = f(x)R(g)(x), \quad \forall x, \quad \forall f, g. \tag{5.2}$$

If and only if

$$\mu(\sigma^{-1}(A) \cap B|x) = \chi_A(x)\mu(B|x) \quad \forall A, B \in \mathcal{B} \quad \forall x \in X. \tag{5.3}$$

Notation: Let $\{\mu(\cdot|x)\}_{x \in X}$ be the family of measures on (x, B) define R as in (5.1); and set $\mu_x(\cdot) := \mu(\cdot|x)$ then (5.2) if and only if (conditional measures):

$$\mu_x(\sigma^{-1}(A)|B) = \chi_A(x), \quad \text{where } \mu_x(\cdot|B)$$

denote the conditional probability i.e.,

$$\mu_x(\sigma^{-1}(A)|B) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

$$\begin{aligned} &\mu_x(\sigma^{-1}(A))\mu_x(B|\sigma^{-1}(A)) \\ &= \mu_x(\sigma^{-1}(A) \cap B) = \chi_A(x)\mu_x(B) \\ &\mu_x(\sigma^{-1}(A)|B) = \chi_A(x) \end{aligned}$$

Proof

$$\begin{aligned} \chi_A(x)\mu_x(B|\sigma^{-1}(A)) &= \mu(B)\mu_x(\sigma^{-1}(A)|B) \\ \chi_A(x)\mu_x(B|\sigma^{-1}(A)) &= \chi_A(x)\mu_x(B) = \mu(B)\mu_x(\sigma^{-1}(A)|B)\mu_x(B) \end{aligned}$$

□

(i) (LP) $\frac{m_0(0)}{\sqrt{N}} = 1$ Low-Pass. $W = |r_0|(Rf)(x) = \frac{1}{N} \sum_{\sigma(y)=x} (Wf)(y)$

(ii) $R(h) = h$
 $\prod_{k=1}^{\infty} \frac{m_0(t/N^k)}{\sqrt{N}} \rightarrow |\hat{\varphi}(t)|^2$
 $\prod_{k=1}^{\infty} \frac{m_0(t/N^k)}{\sqrt{N}} \in L^2(\mathbb{R})$

Theorem 5.4 $(X, \sigma, R, \lambda) \rightarrow Sol_{\sigma}(X)$. We have

- (i) $\exists!$ \mathbb{P} such that $dist(\pi_k) = \mu_k, \int_X f d\mu_R = \int_X R^k(f) d\lambda$, and
- (ii) \mathbb{P} has the property: $\frac{d\mathbb{P} \cdot \hat{\sigma}}{d\mathbb{P}} = w$

Given

$$\begin{aligned}
 X &\xrightarrow{f} \mathbb{R} \quad \text{we get} \\
 X &\xrightarrow{f \circ \pi_n} \text{Sol} \\
 V_n f &:= f \circ \pi_n, \quad \text{where} \\
 \mu_n &:= \text{dist}(\pi_n) \\
 V_n &: L^2(X, \mu_n) \xrightarrow{\text{isometry}} \text{Sol}(X) \\
 V_n &: f \rightarrow f \circ \pi_n
 \end{aligned}$$

Here “dist” is short for distribution.

The prove follows from the above discussion.

$\mathcal{H} : (\Omega, \mathcal{F}, \mathbb{P})$. Let λ be on X and $\lambda R \ll \lambda$ Radon-Nikodym on W . Let U be on $F \in L^2(\Omega, \mathbb{P})$, $R(f)d\lambda = \int f w d\lambda$, $m_0 = \sqrt{W \circ \pi_0}$, $UF = \sqrt{W \circ \pi_0} F \circ \hat{\sigma}$.

Theorem 5.5 *Let $L^2(\Omega, \mathbb{P})$, $\mathbf{1}$, U , ρ . Then there exists a Hilbert space \mathcal{H} , a representation ρ of $L^2(\Omega, \mathbb{P})$ on \mathcal{H} , a unitary operator U on \mathcal{H} and a vector φ in \mathcal{H} such that*

$$\begin{aligned}
 \rho(f) &= m(f \circ \pi_0) \\
 f \in L^\infty(X) & \quad F \rightarrow (f \circ \pi_0)F
 \end{aligned}$$

$(\mathcal{H}, \varphi, U, \rho) \rightarrow$ path space measure

(i) (Covariance)

$$U\rho(f) = \rho(f \circ r)U \quad f \in L^\infty(X). \tag{5.4}$$

(ii) (Scaling equation)

$$F \rightarrow (f \circ \pi_0)F \rightarrow (f \circ \sigma) \circ \pi_0 \sqrt{W \circ \pi_0} F \circ \hat{\sigma} \tag{5.5}$$

(iii) (Orthogonality)

$$\int_{\Omega} \rho(f) 1 d\mathbb{P} = \int (f \circ \pi_0) d\lambda = \int f d\lambda \tag{5.6}$$

(iv)

$$U^{-1}F = \frac{1}{W \circ \pi_1} F \circ \hat{\sigma}^{-1} \quad \pi_n(w) = x_n \tag{5.7}$$

Example 5.6

$$UF = \sqrt{W \circ \pi_0} F \circ \hat{\sigma} \quad \text{where } \sigma : X \rightarrow X$$

$$U^* = U^{-1}$$

$$U = \frac{1}{\sqrt{2}} f\left(\frac{x}{2}\right)$$

$$U\hat{\varphi} = m\hat{\varphi} \quad \text{where } \hat{\varphi} \in L^2(\mathbb{R})$$

We are interested in finding the filter function analogous to Dutkay-Jorgensen Haar-Cantor filter. $m_0 = \frac{1+z}{\sqrt{2}}$.

An attempt to find filter function for Julia set: Let $m_i : X_c \rightarrow \mathbb{C}$. For Julia set $r(z) = z^N$ on \mathbb{T} . μ is Haar measure on \mathbb{T} . Strong invariance of μ with respect to $r(z)$.

$$\int \frac{1}{\#r^{-1}(z)} \sum_{r(w)=z} f(w)dz = \int f(z)d\mu(z)$$

By Brolin’s theorem there exists a unique μ strongly invariant for r .

Quadrature mirror filter

$$\frac{1}{\#r^{-1}(z)} \sum_{r(w)=1} |m_0(w)|^2 = 1 \quad z \in J_c.$$

We want to find nice m_0 and m_1 .
 $w^2 + c = z \Rightarrow w_{\pm} = \pm\sqrt{z - c}$

$$\begin{aligned} \frac{1}{2}(|m_0(w_1)|^2 + |m_0(w_2)|^2) &= 1 \\ \frac{1}{2}(|m_1(w_1)|^2 + |m_1(w_2)|^2) &= 1 \\ \frac{1}{2}(m_0(w_1)m_1(w_1) + m_0(w_2)m_1(w_2)) &= 1 \end{aligned}$$

Can we solve the following matrix over polynomial

$$\frac{1}{\sqrt{2}} \begin{bmatrix} m_0(w_1) & m_0(w_2) \\ m_1(w_1) & m_1(w_2) \end{bmatrix}$$

where it is unitary and $m_0 = 1$?

$$\begin{aligned} \frac{1}{2}(m_1(\sqrt{z - c}) + m_1(-\sqrt{z - c})) &= 0 \\ \frac{1}{2}(|m_1(\sqrt{z - c})|^2 + |m_1(-\sqrt{z - c})|^2) &= 1 \end{aligned}$$

$$m_1(w) = -m(-w)?$$

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