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ORIGINAL RESEARCH PAPER

Complement of the generalized total graph of commutative rings

Tamizh Chelvam Thirugnanam[1] · Balamurugan Mariappan[1]

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Abstract Let R be a commutative ring with identity, $Z(R)$ its set of zero-divisors, and H a nonempty proper multiplicative prime subset of R. *The generalized total graph* of R is the simple undirected graph $GT_H(R)$ with the vertex set R and two distinct vertices x and y are adjacent if and only if $x + y \in H$. In this paper, we investigate several graph theoretical properties of the complement $\overline{GT_H(R)}$. In particular, we obtain a characterization for $\overline{GT_P(R)}$ to be claw-free or unicyclic or pancyclic. Also, we obtain the clique number and chromatic number of $\overline{GT_P(R)}$ and discuss the perfect, planar and outer planarity nature for $\overline{GT_P(R)}$. Further, we discuss various domination parameters for $\overline{GT_P(R)}$ where P is a prime ideal of R.

Keywords Commutative rings · Total graph · Complement · Domination · Gamma sets

Mathematics Subject Classification 05C75 · 05C25 · 13A15 · 13M05

1 Introduction

Throughout this paper R denotes a commutative ring with identity, $Z(R)$ its set of all zero-divisors, $Z^*(R) = Z(R)\backslash\{0\}$ and $U(R)$ its units. Anderson and Livingston [5] introduced the *zero-divisor graph* of R, denoted by $\Gamma(R)$, as the undirected simple graph with vertex set $Z^*(R)$ and two distinct vertices $x, y \in Z^*(R)$ are adjacent if and

✉ Tamizh Chelvam Thirugnanam
tamche59@gmail.com; tamizhchelvam@msuniv.ac.in

Balamurugan Mariappan
bm4050@gmail.com

[1] Department of Mathematics, Manonmaniam Sundaranar University, Tirunelveli, Tamil Nadu 627 012, India

only if $xy = 0$. Subsequently, Anderson and Badawi [4] introduced the concept of *total graph* of commutative rings. The *total graph* $T_\Gamma(R)$ of R is the undirected graph with vertex set R and for distinct elements $x, y \in R$ are adjacent if and only if $x + y \in Z(R)$. Tamizh Chelvam and Asir [7, 15–18] have extensively studied about total graphs of commutative rings.

Recently, Anderson and Badawi [4] introduced the concept of generalized total graph of commutative rings. A nonempty proper subset H of R to be a multiplicative prime subset of R if the following two conditions hold: (1) $ab \in H$ for every $a \in H$ and $b \in R$; (2) if $ab \in H$ for $a, b \in R$, then either $a \in H$ or $b \in H$. For example, H is a multiplicative prime subset of R if H is a prime ideal of R or H is a union of prime ideals of R or $H = Z(R)$, or $H = R \backslash U(R)$. For a multiplicative prime subset H of R, the *generalized total graph* $GT_H(R)$ of R is the simple undirected graph with vertex set R and two distinct vertices x and y are adjacent if and only if $x + y \in H$. The *unit graph* $G(R)$ of R is the simple undirected graph with vertex set R in which two distinct vertices x and y are adjacent if and only if $x + y \in U(R)$. Tamizh Chelvam and Balamurugan [19] studied some graph theoretical properties of the generalized total graph of finite fields and its complement.

Let $G = (V, E)$ be a graph with vertex set V and edge set E. The complement $\overline{G}$ of the graph G is the simple graph with vertex set $V(G)$ and two distinct vertices x and y are adjacent in $\overline{G}$ if and only if they are not adjacent in G. We say that G is connected if there is a path between any two distinct vertices of G. For a vertex $v \in V(G)$, $deg(v)$ is the degree of v. For any graph G, $\delta(G)$ and $\Delta(G)$ denote the minimum and maximum degrees of vertices in G respectively. K_n denotes the complete graph of order n and $K_{m,n}$ denotes the complete bipartite graph. For basic definitions in graph theory, we refer the reader to [10] and for the terms regarding algebra one can refer [13]. Note that if R is finite, then $\overline{GT_{Z(R)}(R)}$ is the unit graph [14].

Throughout this paper, we assume that P is a prime ideal of R with $|P| = \lambda$ and $|R/P| = \mu$. In this paper, we study about the complement $\overline{GT_P(R)}$ of the generalized total graph $GT_P(R)$. In Sect. 2, we study the graph theoretical properties namely girth, clique number, chromatic number and Eulerian of $\overline{GT_P(R)}$. In Sect. 3, we obtain a characterization for $\overline{GT_P(R)}$ to be claw-free or unicylic or pancyclic or planar or outerplanar. In Sect. 4, we study about various domination parameters of $\overline{GT_P(R)}$ and further obtain domatic number of $\overline{GT_P(R)}$.

2 Graph theoretical properties of $\overline{GT_P(R)}$

In this section, we discuss about some graph theoretical properties of $\overline{GT_P(R)}$. More specifically, we discuss about girth and Eulerian of $\overline{GT_P(R)}$. Also, we obtain the independence number, clique number and chromatic number of $\overline{GT_P(R)}$. Let us start with some basic properties of $\overline{GT_P(R)}$.

For a prime ideal P of a commutative ring R which is not an integral domain with $|P| = \lambda$ and $|R/P| = \mu$, we have $|P| = |a_i + P| \geq 2$ for $a_i + P \in R/P$ for $1 \leq i \leq \mu$.

Since R contains identity 1, we denote $1+1$ by 2. First we recall the following structure theorem for $GT_P(R\backslash P)$.

Theorem 1 [4, Theorem 2.2] *Let P be a prime ideal of a commutative ring R, and let* $|P| = \lambda$ *and* $|R/P| = \mu$.

(i) *If* $2 \in P$, *then* $GT_P(R\backslash P)$ *is the union of* $\mu - 1$ disjoint K_λ's;
(ii) *If* $2 \notin P$, *then* $GT_P(R\backslash P)$ *is the union of* $\frac{\mu-1}{2}$ *disjoint* $K_{\lambda,\lambda}$'s.

Assume that R is finite and $2 \in P$. Since $|R|$ is finite, $R \cong R_1 \times \cdots \times R_q$ where each R_i is a local ring. Note that $P = P_1 \times \cdots \times P_q$ where $P_i \subseteq R_i$ for $1 \le i \le q$. Since P is a prime ideal in $R, P_i = Z(R_i)$ for exactly one $i, 1 \le i \le q$ and $P_j = R_j$ for $1 \le j \ne i \le q$. Since $2 \in P \subseteq Z(R), 2 \in P_i \subseteq Z(R_i)$ for some $i, 1 \le i \le q$. By [14, Corollary 2.3] $|R_i| = 2^{\alpha_i}$ where α_i is a positive integer. This implies that μ is even. Hence we get that μ is even if $2 \in P$ and μ is odd if $2 \notin P$. Throughout this paper, we consider the following partition of R into distinct cosets $a_0 + P, a_1 + P, \ldots, a_{\mu-1} + P$ of P with $a_0 \in P$.

(i) If $2 \in P$, then $R = P \cup \left(\bigcup_{i=1}^{\mu-1}(a_i + P)\right)$;
(ii) If $2 \notin P$, then $R = P \cup \bigcup_{i=1}^{\frac{\mu-1}{2}}(a_i + P) \bigcup_{i=1}^{\frac{\mu-1}{2}}(-a_i + P)$.

Now, using Theorem 1, we obtain the degrees of vertices in $\overline{GT_P(R)}$.

Lemma 1 *Let R be a finite commutative ring which is not an integral domain and P be a prime ideal in R. Assume that* $|P| = \lambda$ *and* $|R/P| = \mu$. *Then the following are true in* $\overline{GT_P(R)}$.

(i) *If* $2 \in P$, *then* $deg(v) = (\mu - 1)\lambda$ *for every* $v \in R$;
(ii) *If* $2 \notin P$, *then* $deg(v) = \begin{cases} (\mu-1)\lambda & \text{for } v \in P; \\ (\mu-1)\lambda - 1 & \text{for } v \in R\backslash P. \end{cases}$

Now, we observe some of the immediate consequences of Lemma 1.

Lemma 2 *Let R be a finite commutative ring which is not an integral domain and P be a prime ideal in R. Assume that* $|P| = \lambda$ *and* $|R/P| = \mu$. *Then the following hold*:

(i) $\overline{GT_P(R)}$ *contains no isolated vertex*;
(ii) $\overline{GT_P(R)}$ *contains no vertex of degree* $|R| - 1$;
(iii) $\overline{GT_P(R)}$ *is complete* μ*-partite if and only if* $2 \in P$;
(iv) $\overline{GT_P(R)}$ *is connected bi-regular if and only if* $2 \notin P$. *Moreover*, $\Delta(\overline{GT_P(R)}) = \delta(\overline{GT_P(R)}) + 1$;
(v) $\overline{GT_P(R)}$ *is connected.*

The following Lemma follows from Theorem 1 and is useful throughout this paper.

Lemma 3 *Let R be a finite commutative ring which is not an integral domain and P be a prime ideal in R. Assume that $|P| = \lambda$ and $|R/P| = \mu$. Then the following are true in $\overline{GT_P(R)}$.*

(i) *Let $2 \in P$. Two distinct vertices x and y are adjacent in $\overline{GT_P(R)}$ if and only if x and y are not in the same coset of P;*

(ii) *Let $2 \notin P$. Two distinct vertices x and y are adjacent in $\overline{GT_P(R)}$ if and only if $x \in a_i + P$ and $y \in R \backslash (-a_i + P)$ for some $i, 0 \leq i \leq \frac{\mu-1}{2}$.*

Now we obtain the girth of $\overline{GT_P(R)}$.

Lemma 4 *Let R be a finite commutative ring which is not an integral domain and P be a prime ideal in R. Assume that $|P| = \lambda$ and $|R/P| = \mu$. Then*

$$gr(\overline{GT_P(R)}) = \begin{cases} 4 & \text{if } 2 \in P \text{ and } \mu = 2; \\ 3 & \text{if } 2 \in P \text{ and } \mu \geq 3; \\ 3 & \text{if } 2 \notin P. \end{cases}$$

Proof Assume that $2 \in P$. If $\mu = 2$, then by Lemma 2(iii), $\overline{GT_P(R)} = K_{\frac{|R|}{2},\frac{|R|}{2}}$ and so $gr(\overline{GT_P(R)}) = 4$.

If $\mu \geq 3$, then by Lemma 2(iii), $\overline{GT_P(R)} = K_{\underbrace{\lambda, \lambda, \ldots, \lambda}_{\mu times}}$. Since $\mu \geq 3$, choose $x \in P, y \in a_1 + P, z \in a_2 + P$ and $S = \{x, y, z\}$. Then the subgraph induced by S is C_3 in $\overline{GT_P(R)}$ and so $gr(\overline{GT_P(R)}) = 3$.

Assume that $2 \notin P$. Note that, for $1 \leq i \leq \frac{\mu-1}{2}$, $|P| = |a_i + P| = |-a_i + P| \geq 2$. Let $S = \{x, a, b\}$ where $x \in P$ and $a, b \in a_i + P$ for some i. Then the subgraph induced by S is C_3 in $\overline{GT_P(R)}$ and so $gr(\overline{GT_P(R)}) = 3$.

Note that, a *clique* in a graph G is a complete subgraph of G. The order of the largest clique in a graph G is its *clique number*, which is denoted by $\omega(G)$. Now, we obtain the clique number of $\overline{GT_P(R)}$.

Lemma 5 *Let R be a finite commutative ring which is not an integral domain and P be a prime ideal in R. Assume that $|P| = \lambda$ and $|R/P| = \mu$. Then*

$$\omega(\overline{GT_P(R)}) = \begin{cases} \mu & \text{if } 2 \in P; \\ \dfrac{(\mu-1)}{2}\lambda + 1 & \text{if } 2 \notin P. \end{cases}$$

Proof Suppose $2 \in P$. By Lemma 2(iii), $\overline{GT_P(R)}$ is complete μ-partite. Consider the set $S = \{x_0, x_1, \ldots, x_{\mu-1}\}$ where $x_0 \in P$ and $x_i \in a_i + P$, for $1 \leq i \leq \mu - 1$. Clearly $<S> = K_\mu$ is a maximum clique in $\overline{GT_P(R)}$ and so $\omega(\overline{GT_P(R)}) = \mu$.

If $2 \notin P$, then $R = P \cup (\bigcup_{i=1}^{\frac{\mu-1}{2}}((a_i + P) \cup (-a_i + P)))$. Let $S = \{a_0\} \cup \bigcup_{i=1}^{\frac{\mu-1}{2}}(a_i + P)$ where $a_0 \in P$. By Lemma 3(ii), $<S>$ induces a complete subgraph of order $\frac{(\mu-1)}{2}\lambda + 1$ in $\overline{GT_P(R)}$. Note that, any two elements in P are non-adjacent in $\overline{GT_P(R)}$. Also, the elements of $a_i + P$ and $-a_i + P$ are not adjacent. Thus S is a maximum clique in $\overline{GT_P(R)}$ and so $\omega(\overline{GT_P(R)}) = \frac{(\mu-1)}{2}\lambda + 1$.

An assignment of colors to the vertices of a graph G so that adjacent vertices are assigned different colors is called a *proper coloring* of G. The smallest number of colors in any proper coloring of a graph G is called the *chromatic number* of G and is denoted by $\chi(G)$. A set of vertices in a graph G is an *independent* if no two vertices in the set are adjacent. In the following lemma, we obtain the chromatic number of $\overline{GT_P(R)}$.

Lemma 6 *Let R be a finite commutative ring which is not an integral domain and P be a prime ideal in R. Assume that $|P| = \lambda$ and $|R/P| = \mu$. Then*

$$\chi(\overline{GT_P(R)}) = \begin{cases} \mu & \text{if } 2 \in P; \\ \left(\frac{\mu-1}{2}\right)\lambda + 1 & \text{if } 2 \notin P. \end{cases}$$

Proof If $2 \in P$, then by Lemma 2(iii), $\overline{GT_P(R)}$ is complete μ-partite and so $\chi(\overline{GT_P(R)}) = \mu$.

If $2 \notin P$, then $R = P \cup \bigcup_{i=1}^{\frac{\mu-1}{2}}(a_i + P) \bigcup_{i=1}^{\frac{\mu-1}{2}}(-a_i + P)$. Note that , for $1 \leq i \leq \frac{\mu-1}{2}$, $<a_i + P> = <-a_i + P> = K_\mu$ in $\overline{GT_P(R)}$. By Lemma 3(ii), the elements of $a_i + P$ and $-a_i + P$ are not adjacent in $\overline{GT_P(R)}$. Hence one can use λ colours to the elements of $a_i + P$ and $-a_i + P$. Let $S = \bigcup_{i=1}^{\frac{\mu-1}{2}} a_i + P$ and $T = \bigcup_{i=1}^{\frac{\mu-1}{2}}(-a_i + P)$. Then $<S> = <T> = \bigcup_{\frac{\mu-1}{2}} K_\lambda$ in $\overline{GT_P(R)}$. Thus we require $\frac{(\mu-1)}{2}\lambda$ colours for the union $S \cup T$. Note that $<P>$ is an independent set and each element of P is adjacent to every element of $S \cup T$ in $\overline{GT_P(R)}$. Assign a different color to all the elements of P. This implies $\chi(\overline{GT_P(R)}) = (\frac{\mu-1}{2})\lambda + 1$.

Corollary 1 ([10]) *A nontrivial connected graph G is Eulerian if and only if every vertex of G has even degree.*

Using Corollary 1, we state below a characterization for $\overline{GT_P(R)}$ to be Eulerian.

Lemma 7 *Let R be a finite commutative ring which is not an integral domain and P be a prime ideal in R. Assume that $|P| = \lambda$ and $|R/P| = \mu$. Then the following are true*:

(i) *If* $2 \in P$, *then* $\overline{GT_P(R)}$ *is Eulerian if and only if* λ *is even*;
(ii) *If* $2 \notin P$, *then* $\overline{GT_P(R)}$ *is not Eulerian.*

Proof Proof of (i) follows from Lemma 2(iii), Lemma 2(v) and Lemma 1(i).
Proof of (ii) follows from Lemma 2(iv).

The *vertex independence number* (or the *independence number*) $\beta(G)$ of a graph G is the maximum cardinality of an independent set of vertices in G. In the following lemma, we obtain the independence number of $\overline{GT_P(R)}$.

Lemma 8 *Let* R *be a finite commutative ring which is not an integral domain and* P *be a prime ideal in* R. *Assume that* $|P| = \lambda$ *and* $|R/P| = \mu$. *Then the independence number* $\beta(\overline{GT_P(R)}) = \lambda$.

Proof Suppose $2 \in P$. Then by Lemma 2(iii), $\beta(\overline{GT_P(R)}) = \lambda$.

Suppose $2 \notin P$. Then P is an independent set in $\overline{GT_P(R)}$ and $|P| = \lambda \geq 2$. Assume that $S \subseteq V(\overline{GT_P(R)}) \backslash P$ is an independent subset.

Claim : $|S| \leq 2$.

Suppose $|S| \geq 3$. Let x, y and z be three distinct vertices in S such that $x \in a_i + P$, $y \in a_j + P$ and $z \in a_k + P$ for $1 \leq i, j, k \leq \mu - 1$.

Assume that any two of i, j and k are equal. Without loss of generality, let us take $i = j$. Then x, y are adjacent in $\overline{GT_P(R)}$ and so S is not an independent in $\overline{GT_P(R)}$.

Assume that i, j and k are all distinct. Suppose sum of at least any two of a_i, a_j and a_k is 0. Without loss of generality, let us assume that $a_i + a_j = 0$. Then z is adjacent with both x and y. Hence S is not an independent set in $\overline{GT_P(R)}$.

Suppose sum of any two of a_i, a_j and a_k is not equal to 0. Then $<S>$ contains a K_3, a contradiction to S is an independent set in $\overline{GT_P(R)}$. Hence $|S| \leq 2$. Since each element in P is adjacent to every element in $R \backslash P$ in $\overline{GT_P(R)}, P$ is a maximal independent set in $\overline{GT_P(R)}$ having order $\lambda \geq 2$ and so $\beta(\overline{GT_P(R)}) = \lambda$.

The *edge independence number* $\beta_1(G)$ of a graph G is the maximum cardinality of an independent set of edges. In the following lemma, we obtain the edge independence number of $\overline{GT_P(R)}$.

Lemma 9 *Let* R *be a finite commutative ring which is not an integral domain and* P *be a prime ideal in* R. *Assume that* $|P| = \lambda$ *and* $|R/P| = \mu$. *Then the edge independence number*

$$\beta_1(\overline{GT_P(R)}) = \begin{cases} \dfrac{|R|}{2} & \text{if } 2 \in P; \\ \left(\dfrac{\mu - 1}{2}\right)\lambda + \dfrac{\lfloor \lambda \rfloor}{2} & \text{if } 2 \notin P. \end{cases}$$

Proof **Case 1.** If $2 \in P$, then μ is even. Consider the partition $R = \bigcup_{i=0}^{\mu-1}(a_i + P)$ where $P = a_0 + P$. Let $S = \bigcup_{i=0}^{\frac{\mu}{2}-1}(a_i + P)$ and $T = \bigcup_{i=\frac{\mu}{2}}^{\mu-1}(a_i + P)$. Consider $E = \{\{x_j y_j\} : x_j \in S \text{ and } y_j \in T\}$ for $1 \leq j \leq \frac{|R|}{2}$. Clearly E is a maximal edge independent set in $\overline{GT_P(R)}$ and hence $\beta_1(\overline{GT_P(R)}) = \frac{|R|}{2}$.

Case 2. Assume that $2 \notin P$. Let $S_i = (a_i + P) \cup (-a_{i+1} + P)$ for $i \in \{1, \ldots, \frac{\mu-3}{2}\}$ and $S_{\frac{\mu-1}{2}} = P \cup (-a_1 + P)$. Then $K_{\lambda,\lambda} \subseteq <S_i>$ for each i in $\overline{GT_P(R)}$ and so $\beta_1(<S_i>) = \lambda$. Since $<a_{\frac{\mu-1}{2}} + P> = K_\lambda$, we have $\beta_1(<a_{\frac{\mu-1}{2}} + P>) = \frac{\lfloor \lambda \rfloor}{2}$. Thus we have $\beta_1(\overline{GT_P(R)}) = (\frac{\mu-1}{2})\lambda + \frac{\lfloor \lambda \rfloor}{2}$.

3 Some characterizations of $\overline{GT_P(R)}$

In this section, we characterize when $\overline{GT_P(R)}$ to be claw-free or unicylic or pancyclic or perfect. A graph G is said to be *unicyclic* if G contains exactly one cycle. A graph G is a *claw-free* if G does not have $K_{1,3}$(a claw) as the induced subgraph.

Theorem 2 *Let R be a finite commutative ring which is not an integral domain and P be a prime ideal in R with $|P| = \lambda$ and $|R/P| = \mu$. Then the following hold*:

(i) *$\overline{GT_P(R)}$ is claw-free if and only if $|P| = 2$;*

(ii) *$\overline{GT_P(R)}$ is unicyclic if and only if R is isomorphic to either of the rings $\mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_4, \frac{\mathbb{Z}_2[x]}{<x^2>}$.*

Proof

(i) Assume that $|P| = 2$ and $S \subseteq V(\overline{GT_P(R)})$ with $|S| = 4$. Then $<S>$ contains a C_4 as a subgraph and so $\overline{GT_P(R)}$ is claw-free. Conversely, assume that $\overline{GT_P(R)}$ is claw-free. Suppose $|P| \geq 3$. If $2 \in P$, then by Lemma 2(iii), $\overline{GT_P(R)}$ is $K_{\underbrace{\lambda, \lambda, \cdots, \lambda}_{\mu times}}$ where $\mu \geq 2$ and so $\overline{GT_P(R)}$ contains a $K_{1,3}$ as an induced subgraph, a contradiction. Suppose $2 \notin P$. Let $S = \{u, v, w, x\}$ where $u, v, w \in P$, $x \in a_i + P$ for some $i, 1 \leq i \leq \frac{\mu-1}{2}$. Then $<S> = K_{1,3}$ in $\overline{GT_P(R)}$, a contradiction.

(ii) Assume that R is isomorphic to any one of $\mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_4$ or $\frac{\mathbb{Z}_2[x]}{<x^2>}$. Then $\overline{GT_P(R)} = C_4$ and so $\overline{GT_P(R)}$ is unicyclic. Conversely, assume that $\overline{GT_P(R)}$ is unicyclic. Since R is not an integral domain, |R| is not a prime number. Then $|R| \geq 4$. Suppose $|R| \geq 6$.

Case 1. Let $2 \in P$. Then R satisfies any one of the following:
(a) $\lambda = 2$ and $\mu \geq 4$; (b) $\lambda \geq 3$ and $\mu \geq 2$.

In both the situations, $\overline{GT_P(R)}$ contains $K_{3,3}$ as a subgraph, a contradiction. Hence $|R| = 4$ and so R is isomorphic to either $\mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_4$ or $\frac{\mathbb{Z}_2[x]}{<x^2>}$.

Case 2. Let $2 \notin P$. Since R is not an integral domain, $|P| = |a_i + P| = \lambda \geq 2$, for all $i, 1 \leq i \leq \frac{\mu-1}{2}$. Let $x_1, x_2 \in P$ and $y_1, y_2 \in a_i + P$ for some i. Let $S_1 = \{x_1, y_1, y_2\}$ and $S_2 = \{x_2, y_1, y_2\}$. Then $<S_1> = <S_2>$ are two different cycles in $\overline{GT_P(R)}$ and so $\overline{GT_P(R)}$ is not unicylic and hence no other R exists.

Note that a graph G is perfect if and only if both G and $\overline{G}$ have no induced subgraph that is an odd cycle of length at least 5 [20, 8.1.2]. Using this, now we prove that $\overline{GT_P(R)}$ is perfect.

Theorem 3 *Let R be a finite commutative ring which is not an integral domain and P be a prime ideal in R with $|P| = \lambda$ and $|R/P| = \mu$. Then $\overline{GT_P(R)}$ is a perfect graph.*

Proof By Theorem 1, $GT_P(R)$ does not contain an odd cycle of length ≥ 5 as an induced subgraph. It remains to prove that $\overline{GT_P(R)}$ does not contain an odd cycle of length ≥ 5 as an induced subgraph.

Case 1. Assume that $2 \in P$. By Lemma 2(iii), $\overline{GT_P(R)}$ is complete μ-partite. If $\mu = 2$, then $\overline{GT_P(R)}$ is complete bipartite and so $\overline{GT_P(R)}$ does not contain any odd cycle.

Assume that $\mu \geq 3$ and $S \subseteq V(\overline{GT_P(R)})$ with $|S| \geq 5$. Consider the partition of $R = \bigcup_{i=0}^{\mu-1}(a_i + P)$ with $a_0 \in P$. If $S \subseteq a_i + P$ for some i, then $<S>$ is totally disconnected in $\overline{GT_P(R)}$. If S is not contained in $(a_i + P)$ for some i, let $n = \max\{|S \cap (a_i + P)|\}$ for all i. Then $n \geq 1$ and $K_{n,|S|-n} \subseteq <S> \subseteq \overline{GT_P(R)}$. Hence $<S>$ contains a vertex of degree at least 3. From this, any cycle of length at least 5 cannot be an induced subgraph of $\overline{GT_P(R)}$.

Case 2. Assume that $2 \notin P$. Consider $R = P \cup \bigcup_{i=1}^{\frac{\mu-1}{2}}(a_i + P \cup -a_i + P)$. Suppose $S \subseteq V(\overline{GT_P(R)})$ with $|S| \geq 5$.

Case 2.1. Assume that $S \cap P \neq \phi$. If $S \subseteq P$, then $<S> = \overline{K}_{|S|}$ in $\overline{GT_P(R)}$. If S is not contained in P, then the induced subgraph $<S>$ of $\overline{GT_P(R)}$ contains a $K_{n,|S \setminus P|}$ where $n = |S \cap P|$. Hence $<S>$ contains a vertex of degree at least 3 and so any cycle of length at least 5 cannot be an induced subgraph of $\overline{GT_P(R)}$.

Case 2.2. Assume that $S \cap P = \phi$. Then $S \subseteq \bigcup_{i=1}^{\frac{\mu-1}{2}}(a_i + P \cup -a_i + P)$. Let $A_i = (a_i + P) \cup (-a_i + P)$ for all $i, 1 \leq i \leq \frac{\mu-1}{2}$. Suppose $S \subseteq A_i$ for some i. If either $S \subseteq (a_i + P)$ or $S \subseteq (-a_i + P)$, then $<S> = K_{|S|}$ in $\overline{GT_P(R)}$. If $S \cap (a_i + P) \neq \phi$ and $S \cap (-a_i + P) \neq \phi$, then either $|S \cap (a_i + P)| \geq 3$ or $|S \cap (-a_i + P)| \geq 3$. Then K_3 is a subgraph of $<S>$ and in turn an induced subgraph of $\overline{GT_P(R)}$.

Assume that S is not a subset of A_i for some i. Then $S \cap A_i \neq \phi$ and $S \cap A_j \neq \phi$ for some $1 \leq i \neq j \leq \frac{\mu-1}{2}$. If $S \cap A_i \neq \phi$ for $1 \leq i \leq \frac{\mu-1}{2}$ and $|S \cap A_i| = 1$, then $<S>$ contains a K_5 as a subgraph in $\overline{GT_P(R)}$. Suppose $|S \cap A_i| \geq 2$ for some i. Let

$A_i = A_1, |S \cap A_1| \geq 2, n = |S \cap A_1|$ and $m = |S \cap \bigcup_{i=2}^{\frac{\mu-1}{2}} A_i|$. Since $|S| \geq 5$, either $n \geq 3$ or $m \geq 3$. Thus $K_{n,m}$ is a subgraph of $<S>$. Hence, in this case also, $\overline{GT_P(R)}$ has no induced subgraph that is an odd cycle of length at least 5.

A graph G of order $m \geq 3$ is *pancyclic* [8, Definition 6.3.1] if G contains cycles of all lengths from 3 to m.

Theorem 4 *Let R be a finite commutative ring which is not an integral domain and P be a prime ideal in R, where $|P| = \lambda$ and $|R/P| = \mu$. Then $\overline{GT_P(R)}$ is pancyclic if and only if either $2 \notin P$ or $2 \in P$ with $\mu > 2$.*

Proof Assume that $2 \notin P$. Consider the partition $R = P \cup \bigcup_{i=1}^{\frac{\mu-1}{2}} (a_i + P \cup -a_i + P)$. Let $P = \{x_{01}, x_{02}, \ldots, x_{0\lambda}\}, a_i + P = \{x_{i1}, x_{i2}, \ldots, x_{i\lambda}\}$ and $-a_i + P = \{y_{i1}, y_{i2}, \ldots, y_{i\lambda}\}$ for $1 \leq i \leq \frac{\mu-1}{2}$.

Note that $<P> = \overline{K}_\lambda$ and $<a_i + P> = <-a_i + P> = K_\lambda$ in $\overline{GT_P(R)}$. Let P_i and P'_i be spanning paths of length $\lambda - 1$ in $<a_i + P>$ and $<-a_i + P>$ respectively for $i, 1 \leq i \leq \frac{\mu-1}{2}$.

If $\mu = 3$, then $C: x_{01} - x_{11} - x_{02} - x_{12} - x_{03} - \cdots - x_{0(\lambda-1)} - x_{1(\lambda-1)} - x_{1\lambda} - x_{0\lambda} - {P_1}' - x_{01}$ is a spanning cycle of length $|R|$ in $\overline{GT_P(R)}$. By removing vertices one by one from ${P_1}' \backslash \{x_{11}\}, P \backslash \{x_{01}, x_{0\lambda}\}, P_1 \backslash \{x_{1\lambda}\}$, we get cycles of lengths $|R| - 1, |R| - 2, \ldots, 4$ as subgraphs in $\overline{GT_P(R)}$. Also $x_{01} - x_{11} - x_{1\lambda} - x_{01}$ is a 3-cycle in $\overline{GT_P(R)}$. From this, we get cycles of length from 3 to $|R|$ as subgraphs in $\overline{GT_P(R)}$.

If $\mu > 3$, then $C: x_{01} - x_{11} - x_{02} - x_{12} - x_{03} - \cdots - x_{0(\lambda-1)} - x_{1(\lambda-1)} - x_{0\lambda} - x_{1\lambda} - P_2 - P_3 - \cdots - P_{\frac{\mu-1}{2}} - P'_1 - P'_2 - \cdots - P'_{\frac{\mu-1}{2}} - x_{01}$ is a spanning cycle of length $|R|$ in $\overline{GT_P(R)}$. By removing vertices one by one from the spanning paths $P'_{\frac{\mu-1}{2}}, P'_{\frac{\mu-3}{2}}, \ldots, P'_1, P_{\frac{\mu-1}{2}}, P_{\frac{\mu-3}{2}}, \ldots, P_2, P \backslash \{x_{01}\}, P_1 \backslash \{x_{11}, x_{1\lambda}\}$, we get cycles of lengths $|R| - 1, |R| - 2, \ldots, 4, 3$ as subgraphs in $\overline{GT_P(R)}$. From this, we get cycles of length from 3 to $|R|$ as subgraphs in $\overline{GT_P(R)}$. Hence $\overline{GT_P(R)}$ is pancyclic.

Assume that $2 \in P$ and $\mu > 2$. Note that $R = P \cup \bigcup_{i=1}^{\mu-1} (a_i + P)$ and $\overline{GT_P(R)}$ is complete μ-partite graph. Let $P = P_0 = \{x_{01}, \ldots, x_{0\lambda}\}$ and $a_i + P = \{x_{i1}, \ldots, x_{i\lambda}\}$ for $1 \leq i \leq \mu - 1$. Consider $C = x_{01} - x_{11} \cdots - x_{(\mu-1)1} - x_{02} - x_{12} - \cdots - x_{(\mu-1)2} - \cdots - x_{0\lambda} - \cdots - x_{(\mu-2)\lambda} - x_{(\mu-1)\lambda} - x_{01}$ in $\overline{GT_P(R)}$. Clearly C is a spanning cycle of length $|R|$ in $\overline{GT_P(R)}$. By removing the vertices one by one from the sets $(a_{\mu-1} + P) \backslash \{x_{(\mu-1)\lambda}\}, a_{\mu-2} + P, \ldots, a_2 + P$ and alternatively removing the vertices one by one from the sets $(a_1 + P) \backslash \{x_{11}\}$ and $P \backslash \{x_{01}\}$, we get the cycles of lengths $|R|, |R| - 1, \ldots, 4, 3$ as subgraphs in $\overline{GT_P(R)}$. Thus $\overline{GT_P(R)}$ is pancyclic.

Conversely, assume that $2 \in P$. If $\mu = 2$, then $\overline{GT_P(R)} = K_{\lambda,\lambda}$ and so $\overline{GT_P(R)}$ contains no odd cycle, which is a contradiction to our assumption.

Now, we discuss about planarity and outerplanarity of $\overline{GT_P(R)}$. For this purpose, we recall the following results.

Theorem 5 ([10, Theorem 9.7]) *A graph is planar if and only if it does not contain a subdivision of* K_5 *or* $K_{3,3}$.

Theorem 6 ([11, Theorem 11.10]) *A graph G is outerplanar if and only if it does not contain a subdivision of* K_4 *or* $K_{2,3}$.

Using the above theorems, we determine all finite commutative rings R for which $\overline{GT_P(R)}$ to be planar or outerplanar.

Theorem 7 *Let R be a finite commutative ring which is not an integral domain and P be a prime ideal in R, where* $|P| = \lambda$ *and* $|R/P| = \mu$. *Then the following hold*:

(i) *If* $2 \in P$, *then* $\overline{GT_P(R)}$ *is planar if and only if R is isomorphic to any one of* $\mathbb{Z}_4, \mathbb{Z}_2 \times \mathbb{Z}_2, \frac{\mathbb{Z}_2[x]}{<x^2>}$;

(ii) *If* $2 \notin P$, *then* $\overline{GT_P(R)}$ *is planar if and only if* $R = \mathbb{Z}_6$;

(iii) $\overline{GT_P(R)}$ *is outerplanar if and only if R is either* $\mathbb{Z}_4, \mathbb{Z}_2 \times \mathbb{Z}_2 or \frac{\mathbb{Z}_2[x]}{<x^2>}$.

Proof

(i) Assume that R is isomorphic to either $\mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$ or $\frac{\mathbb{Z}_2[x]}{<x^2>}$. Then $\overline{GT_P(R)} = C_4$ and so $\overline{GT_P(R)}$ is planar. Conversely assume that $\overline{GT_P(R)}$ is planar. If $|R| \geq 8$, then any one of the following is true: (a) $|P| = 2$ with $\mu \geq 4$; (b) $|P| \geq 3$.

  (a) Assume that $|P| = 2$ with $\mu \geq 4$. Then $\overline{GT_P(R)} = K_{2,2,\ldots,2}$ and so $\overline{GT_P(R)}$ contains a $K_{3,3}$, as a subgraph. From this $\overline{GT_P(R)}$ is not planar.

  (b) Assume that $|P| \geq 3$. Note that $\mu \geq 2$. Then once again $\overline{GT_P(R)}$ contains a $K_{3,3}$, as a subgraph and hence $\overline{GT_P(R)}$ is not planar.

If $|R| = 6$, by assumption on R, $R \cong \mathbb{Z}_6$ and $P = <2>$. From this one can see that the graph $\overline{GT_P(R)}$ is $K_{3,3}$, which is not planar. From the above arguments, $|R| = 4$ and so R is isomorphic to either $\mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$ or $\frac{\mathbb{Z}_2[x]}{<x^2>}$.

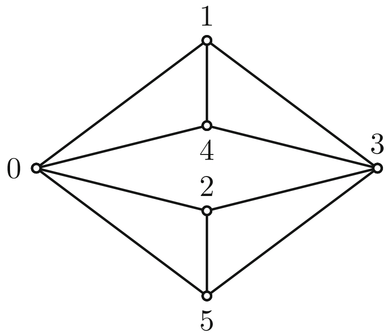
(ii) Assume that $R \cong \mathbb{Z}_6$. Since $2 \notin P, P = <3>$. The planar embedding of $\overline{GT_P(\mathbb{Z}_6)}$ is given in Fig. 1 and so it is planar.

Conversely, assume that $\overline{GT_P(R)}$ is planar and $2 \notin P$. If $|R| \geq 9$, then any one of the following is true:

(a) $|P| = 2$ with $\mu \geq 5$;

(b) $|P| \geq 3$.

Fig. 1 $\overline{GT_{\langle 3\rangle}(\mathbb{Z}_6)}$



(a) Assume that $|P| = 2$ with $\mu \geq 5$. Then $GT_P(R) = K_2 \cup \bigcup_{\frac{\mu-1}{2}} K_{2,2}$ and so $\overline{GT_P(R)}$ contains a $K_{3,3}$, a contradiction.

(b) Proof is trivial.

(iii) By above (i) and (ii), it is enough to show that $\overline{GT_P(R)}$ is not outerplanar for $R = \mathbb{Z}_6$ and $2 \notin P$. For $R = \mathbb{Z}_6$ and $2 \notin P$, from the Fig. 1, $\overline{GT_P(R)}$ contains a $K_{2,3}$ and so $\overline{GT_P(R)}$ is not outerplanar.

4 Domination parameters of $\overline{GT_P(R)}$

In this section, we discuss about various domination parameters of $\overline{GT_P(R)}$. More specifically, we discuss about $\gamma_t, \gamma_c, \gamma_{cl}, \gamma_p, \gamma_s, \gamma_w$ and γ_i of $\overline{GT_P(R)}$. A nonempty subset S of V is called a *dominating set* if every vertex in $V \backslash S$ is adjacent to at least one vertex in S. A subset S of V is called a *total dominating set* if every vertex in V is adjacent to some vertex in S. A dominating set S is called a *connected (or clique) dominating* set if the subgraph induced by S is connected (or complete). A dominating set S is called an *independent dominating set* if no two vertices of S are adjacent. A dominating set S is called a *perfect dominating set* if every vertex in $V \backslash S$ is adjacent to exactly one vertex in S. A dominating set S is called a *strong (or weak) dominating set*, if for every vertex $u \in V \backslash S$ there is a vertex $v \in S$ with $deg(v) \geq deg(u) (ordeg(v) \leq deg(u))$ and u is adjacent to v. The *domination number* γ of G is defined to be the minimum cardinality of a dominating set in G and the corresponding dominating set is called as a γ-set of G. Similar definition is applicable for the *total domination number* γ_t, *connected domination number* γ_c,*clique domination number* γ_{cl}, *independent domination number* γ_i, *perfect domination number* γ_p, *strong domination number* γ_s *and the weak domination number* γ_w. For all these definitions, one can refer Haynes et al. [12]. In the following Lemma, we obtain the domination number of $\overline{GT_P(R)}$.

Lemma 10 *Let R be a finite commutative ring which is not an integral domain and P be a prime ideal in R. Assume that* $|P| = \lambda$ *and* $|R/P| = \mu$. *Then* $\gamma(\overline{GT_P(R)}) = 2$.

Proof By Lemma 2(ii), $\gamma(\overline{GT_P(R)}) > 1$. Let x and y be two distinct elements in R such that $x \in P$ and $y \notin P$.

If $2 \in P$, then by Lemma 3(i), x covers all the elements in $V(\overline{GT_P(R)}) \backslash P$ and y covers all the elements in P. Hence $\gamma(\overline{GT_P(R)}) = 2$.

Assume that $2 \notin P$. Let $z \in V(\overline{GT_P(R)}) \backslash \{x, y\}$. If $z \in P$, then by Lemma 3(ii) z, y are adjacent in $\overline{GT_P(R)}$. Suppose $z \notin P$. Then by Lemma 3(ii) z, x are adjacent in $\overline{GT_P(R)}$. Therefore $\{x, y\}$ is a dominating set in $\overline{GT_P(R)}$. By Lemma 2(ii), $\{x, y\}$ is a minimal dominating set in $\overline{GT_P(R)}$. From this we get that $\gamma(\overline{GT_P(R)}) = 2$.

In view of Lemma 10, we have the following characterization of γ-sets in $\overline{GT_P(R)}$.

Lemma 11 *Let R be a finite commutative ring which is not an integral domain and P be a prime ideal in R. Then* $S = \{x, y\} \subseteq V(\overline{GT_P(R)})$ *is a* γ*-set in* $\overline{GT_P(R)}$ *if and only if x, y are in two distinct cosets of P in R.*

Corollary 2 *Let R be a finite commutative ring which is not an integral domain and P be a prime ideal in R. Assume that* $|P| = \lambda$ *and* $|R/P| = \mu$. *Then* $\gamma_t(\overline{GT_P(R)}) = \gamma_c(\overline{GT_P(R)}) = \gamma_{cl}(\overline{GT_P(R)}) = 2$.

Proof Let $x \in P$ and $y \in V(\overline{GT_P(R)}) \backslash P$. By Lemma 11, $S = \{x, y\}$ is a dominating set of $\overline{GT_P(R)}$. By Lemma 3, x, y are adjacent in $\overline{GT_P(R)}$. Therefore S is a total dominating set of $\overline{GT_P(R)}$ and so $\gamma_t(\overline{GT_P(R)}) = \gamma_c(\overline{GT_P(R)}) = \gamma_{cl}(\overline{GT_P(R)}) = 2$.

A graph G is called *excellent* if, for every vertex $v \in V(G)$, there is a γ-set S containing v. Using Lemma 11, we have the following result.

Corollary 3 *Let R be a finite commutative ring which is not an integral domain and P be a prime ideal in R. Assume that* $|P| = \lambda$ *and* $|R/P| = \mu$. *Then* $\overline{GT_P(R)}$ *is excellent.*

Lemma 12 *Let R be a finite commutative ring which is not an integral domain and P be a prime ideal in R. Assume that* $|P| = \lambda$ *and* $|R/P| = \mu$. *Then* $\overline{GT_P(R)}$ *has a perfect dominating set if and only if* $2 \in P$ *and* $\mu = 2$.

Proof Assume that $2 \in P$ and $\mu = 2$. Then $\overline{GT_P(R)}$ is a complete bi-partite graph. Let $S = \{x, y\}$ where $x \in P$ and $y \in V(\overline{GT_P(R)}) \backslash P$. Then $S = \{x, y\}$ is a perfect dominating set of $\overline{GT_P(R)}$.

Conversely, assume that S is a perfect dominating set in $\overline{GT_P(R)}$.

Suppose $2 \notin P$. Since R is not an integral domain, $|P| = |a_i + P| = |-a_i + P| \geq 2, 1 \leq i \leq \frac{\mu-1}{2}$. Let x, y be two distinct vertices in $V(\overline{GT_P(R)})$. If $x, y \in P$, then every $z \in V(\overline{GT_P(R)}) \backslash P$ is adjacent to both x and y and hence both x and y cannot be in S.

Suppose $x \in P \cap S$ and $y \in P$ with $y \notin S$. Since P is an independent set, x and y are not adjacent in $\overline{GT_P(R)}$. Since S is dominating set, $\exists z_1 \in S$ such that z_1 and y are adjacent in $\overline{GT_P(R)}$. By Lemma 3(ii), $z_1 \in (a_i + P) \cup (-a_i + P)$ for some i where $1 \leq i \leq \frac{\mu-1}{2}$. Without loss of generality, assume that $z_1 \in (a_1 + P)$. Since $|a_1 + P| \geq 2$, $\exists z_2 \in a_1 + P$ with $z_2 \neq z_1$. If $z_2 \in S$, then, by Lemma 3(ii) and $2 \notin P$, y is adjacent to z_1, z_2 which is a contradiction. Hence $z_2 \in V(\overline{GT_P(R)}) \backslash S$. By Lemma 3(ii), x, z_2 are adjacent and z_1, z_2 are also adjacent in $\overline{GT_P(R)}$, which is a contradiction to S is a perfect dominating set.

Suppose $x, y \in S \backslash P$. Since every element in P is adjacent to both x, y again a contradiction. Hence $\overline{GT_P(R)}$ has no perfect dominating set.

The above argument implies that $2 \in P$.

Suppose $2 \in P$ and $\mu \geq 3$. By Lemma 2(iii), $\overline{GT_P(R)}$ is a complete μ-partite graph. Clearly $\overline{GT_P(R)}$ does not contain any perfect dominating set, which is a contradiction to S is a perfect dominating set. Hence $2 \in P$ and $\mu = 2$.

Lemma 13 *Let R be a finite commutative ring which is not an integral domain and P be a prime ideal in R. Assume that* $|P| = \lambda$ *and* $|R/P| = \mu$. *Then the following are true*:

(i) *If* $2 \in P$, *then* $\gamma_s(\overline{GT_P(R)}) = \gamma_w(\overline{GT_P(R)}) = 2$;
(ii) *If* $2 \notin P$, *then* $\gamma_s(\overline{GT_P(R)}) = \lambda$ *and* $\gamma_w(\overline{GT_P(R)}) = 2$.

Proof

(i) Assume that $2 \in P$. Then $\overline{GT_P(R)}$ is a complete μ-partite graph. Let $x \in P$ and $y \in a_i + P$ for some i, $1 \leq i \leq \mu - 1$. Then $S = \{x, y\}$ is a dominating set of $\overline{GT_P(R)}$ and so by Lemma 1(i), $deg(v) = (\mu - 1)\lambda \, \forall \, v \in \overline{GT_P(R)}$. Therefore $\gamma_s(\overline{GT_P(R)}) = \gamma_w(\overline{GT_P(R)}) = 2$.
(ii) Assume that $2 \notin P$. By the definition of $\overline{GT_P(R)}$, each vertex $x \in P$ is adjacent with each vertex $y \in V(\overline{GT_P(R)}) \backslash P$. By Lemma 1(ii), we have $deg(x) = (\mu - 1)\lambda \, \forall \, x \in P$ and $deg(y) = (\mu - 1)\lambda - 1 \, \forall \, y \in V(\overline{GT_P(R)}) \backslash P$. Since P dominates $\overline{GT_P(R)}$ and $deg(x) > deg(y)$, P is a strong dominating set in $\overline{GT_P(R)}$. Then $\gamma_s(\overline{GT_P(R)}) = \lambda$. Let $x \in a_i + P$ and $y \in -a_i + P$ where $1 \leq i \leq \frac{\mu-1}{2}$. By Lemma 11, the set $S = \{x, y\}$ is a dominating set of $\overline{GT_P(R)}$ and by Lemma 1(ii), $deg(x) = deg(y) = \lambda(\mu - 1) = \delta(\overline{GT_P(R)})$. Then $\{x, y\}$ is a weak dominating set in $\overline{GT_P(R)}$ and so $\gamma_w(\overline{GT_P(R)}) = 2$.

In following lemma, we obtain the independent domination number of $\overline{GT_P(R)}$.

Lemma 14 *Let R be a finite commutative ring which is not an integral domain and P be a prime ideal in R. Assume that* $|P| = \lambda$ *and* $|R/P| = \mu$. *Then*

$$\gamma_i(\overline{GT_P(R)}) = \begin{cases} \lambda & \text{if } 2 \in P; \\ 2 & \text{if } 2 \notin P. \end{cases}$$

Proof If $2 \in P$, then by Lemma 2(iii), $\overline{GT_P(R)}$ is $K_{\underbrace{\lambda, \ldots, \lambda}_{\mu times}}$ and so $\gamma_i(\overline{GT_P(R)}) = \lambda$.

Suppose $2 \notin P$. Let $x \in a_i + P$ and $y \in -a_i + P$ where $1 \leq i \leq \frac{\mu-1}{2}$. By Lemma 11, $\{x, y\}$ is a dominating set in $\overline{GT_P(R)}$. By Lemma 3(ii), x, y are not adjacent in $\overline{GT_P(R)}$ and so $\gamma_i(\overline{GT_P(R)}) = 2$.

Note that, a graph G is said to be *well-covered* if $\beta(G) = \gamma_i(G)$. The following lemma follows from Lemmas 8 and 14.

Lemma 15 *Let R be a finite commutative ring which is not an integral domain and P be a prime ideal in R. Assume that* $|P| = \lambda$ *and* $|R/P| = \mu$. *Then*

(i) *If* $2 \in P$, *then* $\overline{GT_P(R)}$ *is well-covered*;

(ii) *If* $2 \notin P$, *then* $\overline{GT_P(R)}$ is *well-covered if and only if* $|P| = 2$.

A *domatic partition* of G is a partition of $V(G)$ into dominating sets of G. The maximum number of sets in a domatic partition of G is called the *domatic number* of G and the same is denoted by $d(G)$.

Lemma 16 *Let R be a finite commutative ring which is not an integral domain and P be a prime ideal in R. Assume that* $|P| = \lambda$ *and* $|R/P| = \mu$. *Then*

$$d(\overline{GT_P(R)}) = \begin{cases} \dfrac{|R|}{2} & \text{if} 2 \in P; \\ \dfrac{|R| - \lambda}{2} + 1 & \text{if} 2 \notin P. \end{cases}$$

Proof If $2 \in P$, then μ is even. Let $S = \bigcup_{i=1}^{\frac{\mu}{2}} (a_i + P)$ and $T = \bigcup_{i=\frac{\mu}{2}+1}^{\mu} (a_i + P)$. Let $X_j = \{\{x_j, y_j\} : x_j \in S \, and \, y_j \in T\}$ for $1 \leq j \leq \frac{|R|}{2}$. Clearly X_j is a dominating set in $\overline{GT_P(R)}$ and hence $d(\overline{GT_P(R)}) = \frac{|R|}{2}$.

Suppose $2 \notin P$. Then μ is odd and $\mu \geq 3$. Let $S = \bigcup_{i=1}^{\frac{\mu-1}{2}} (a_i + P)$ and $T = \bigcup_{i=1}^{\frac{\mu-1}{2}} (-a_i + P)$ where $a_i + (-a_i) = 0$. Let $X_j = \{\{x_j, y_j\} : x_j \in S \, and \, y_j \in T\}$ for $1 \leq j \leq \frac{|R| - \lambda}{2}$. Clearly, each X_j is a dominating set in $\overline{GT_P(R)}$. Also, P is a dominating set in $\overline{GT_P(R)}$. Hence $d(\overline{GT_P(R)}) = \frac{|R| - \lambda}{2} + 1$.

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