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ORIGINAL RESEARCH PAPER

A note on Ankeny–Rivlin theorem

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Abstract In this paper, we consider the class of polynomials not vanishing in the unit disk and obtain a result that improves the results of Dubinin, Aziz and Dawood and the classical result of Ankeny and Rivlin.

Keywords Polynomial · Maximum modulus principle · Zeros

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1 Introduction and statement of results

For an arbitrary entire function $f(z)$, let $M(f,r) := \max_{|z|=r} |f(z)|$. For a polynomial $P(z)$ of degree n, it is known that

$$M(P,\rho) \leq \rho^n M(P,1), \quad \rho \geq 1. \tag{1.1}$$

Inequality (1.1) is a simple consequence of Maximum Modulus Principle (see [4]). It was shown by Ankeny and Rivlin [1] that if $P(z) \neq 0$ in $|z| < 1$, then (1.1) can be replaced by

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$$M(P,\rho) \le \frac{\rho^n + 1}{2} M(P,1), \quad \rho \ge 1. \tag{1.2}$$

In 1988, Aziz and Dawood further improved the bound in (1.2) and proved under the same hypothesis that

$$M(P,\rho) \le \frac{\rho^n + 1}{2} M(P,1) - \frac{\rho^n - 1}{2} \min_{|z|=1} |P(z)|, \quad \rho \ge 1. \tag{1.3}$$

Recently, Dubinin [3] obtained the following refinement of (1.2) by using the classical Schwarz lemma.

Theorem 1 *If $P(z) = \sum_{j=0}^{n} c_j z^j$ is a polynomial of degree $n \ge 2$ with no zeros in $|z| < 1$, then for any $\rho > 1$,*

$$M(P,\rho) \le \frac{(1+\rho^n)(|c_0| + \rho|c_n|)}{(1+\rho)(|c_0| + |c_n|)} M(P,1). \tag{1.4}$$

The result is best possible and equality holds in (1.4) for $P(z) = \frac{\mu + \nu z^n}{2}$, $|\mu| = |\nu| = 1$.

In this note, we prove the following generalization of (1.4) which sharpens the bounds in (1.2) and (1.3) as well.

Theorem 2 *If $P(z) = \sum_{j=0}^{n} c_j z^j$ is a polynomial of degree $n \ge 2$ with no zeros in $|z| < 1$, then for any $\rho > 1$ and $0 \le t \le 1$,*

$$\begin{aligned} M(P,\rho) \le & \left(\frac{(1+\rho^n)(|c_0| + \rho|c_n| - tm)}{(1+\rho)(|c_0| + |c_n| - tm)} \right) M(P,1) \\ & - \left(\frac{(1+\rho^n)(|c_0| + \rho|c_n| - tm)}{(1+\rho)(|c_0| + |c_n| - tm)} - 1 \right) tm, \end{aligned} \tag{1.5}$$

where $m = min_{|z|=1} |P(z)|$.

The result is best possible and equality holds in (1.5) for $P(z) = \frac{\mu + \nu z^n}{2}$, $|\mu| = |\nu| = 1$.

Remark 1 Since if $P(z) = \sum_{j=0}^{n} c_j z^j \ne 0$ in $|z| < 1$, then $|c_0| \ge |c_n|$. Also, as in the proof of the Theorem 2 (given in the next section), we have for every λ with $|\lambda| \le 1$, the polynomial $P(z) - \lambda m$ does not vanish in $|z| < 1$, hence

$$|c_0 - \lambda m| \ge |c_n|. \tag{1.6}$$

If in (1.6), we choose the argument of λ suitably and note that $|c_0| > m$ (from (2.2), proof of Theorem 2), we get

$$|c_0| - |\lambda| m \ge |c_n|. \tag{1.7}$$

If we take $|\lambda| = t$ in (1.7) so that $0 \le t \le 1$, we get $tm + |c_n| \le |c_0|$.

Remark 2 Here, we show that for $\rho > 1$,

$$\frac{|c_0| + \rho|c_n| - tm}{|c_0| + |c_n| - tm} \leq \frac{1+\rho}{2}, \tag{1.8}$$

which is equivalent to showing

$$|c_0| + \rho|c_n| - tm \leq |c_n| + \rho|c_0| - \rho tm,$$

that is

$$|c_n| + tm \leq |c_0|,$$

which clearly holds by Remark 1. Also, the function $xM(P,1) - (x-1)tm$ is a non-decreasing function of x. If we combine this fact with (1.8) according to which

$$\frac{(1+\rho^n)(|c_0| + \rho|c_n| - tm)}{(1+\rho)(|c_0| + |c_n| - tm)} \leq \frac{1+\rho^n}{2},$$

it follows that the right hand side of (1.5) does not exceed $(\frac{1+\rho^n}{2})M(P,1) - (\frac{\rho^n-1}{2})tm$, we have a refinement of (1.3).

Remark 3 For $t = 0$, (1.5) reduces to (1.4).

2 Proof of Theorem

Proof of Theorem 2. Since $P(z) = \sum_{j=0}^{n} c_j z^j$ has all its zeros in $|z| \geq 1$ and $m = \min_{|z|=1} |P(z)|$, therefore

$$m \leq |P(z)|, for |z| = 1. \tag{2.1}$$

It follows by the Maximum and Minimum Modulus Principles that the strict inequality

$$m < |P(z)| < M(P,1), \tag{2.2}$$

holds for $|z| < 1$.

We show that for every complex α with $|\alpha| \leq 1$, the polynomial $F(z) = P(z) - \alpha m$ does not vanish in $|z| < 1$. For if $F(z) = P(z) - \alpha m$ has a zero in $|z| < 1$, say at $z = z_1$ with $|z_1| < 1$, then

$$F(z_1) = P(z_1) - \alpha m = 0.$$

This gives,

$$|P(z_1)| = |\alpha| m \leq m,$$

where $|z_1| < 1$, which contradicts (2.2).

Hence, we conclude that the polynomial $F(z)$ does not vanish in $|z| < 1$. Applying Theorem 1 to the polynomial $F(z) = P(z) - \alpha m = (c_0 - \alpha m) + \sum_{j=1}^{n} c_j z^j$, we get for every complex α with $|\alpha| \leq 1$ and $\rho > 1$,

$$\max_{|z|=\rho}\left|P(z)-\alpha m\right| \leq \left(\frac{\rho^n+1}{\rho+1}\right)\left(\frac{|c_0-\alpha m|+\rho|c_n|}{|c_0-\alpha m|+|c_n|}\right)\max_{|z|=1}\left|P(z)-\alpha m\right|. \tag{2.3}$$

For every α with $|\alpha|\leq 1$, we have

$$|c_0-\alpha m| \geq \left||c_0|-|\alpha|m\right| = |c_0|-|\alpha|m,$$

since $|\alpha|m \leq m < |P(0)| = |c_0|$, by (2.2).

Further, the function $\left(\frac{x+\rho|c_n|}{x+|c_n|}\right)$ is decreasing on $\{x : x > -|c_n|\} \cup \{x : x < -|c_n|\}$ for every $\rho > 1$, it follows from (2.3) that for every α with $|\alpha| \leq 1$ and for every $\rho > 1$,

$$M(P,\rho)-|\alpha|m \leq \left(\frac{\rho^n+1}{\rho+1}\right)\left(\frac{|c_0|-|\alpha|m+\rho|c_n|}{|c_0|-|\alpha|m+|c_n|}\right)\left|P(z_0)-\alpha m\right|, \tag{2.4}$$

where z_0 is a point on $|z|=1$ such that $|P(z_0)| = M(P,1)$. Also by (2.1) and (2.2), we have

$$m \leq |P(z)| \; for \; |z| \leq 1, \tag{2.5}$$

we take in particular $z = z_0$ in (2.5) and get

$$m \leq |P(z_0)|. \tag{2.6}$$

Choosing the argument of α with $|\alpha| \leq 1$ on the right hand side of (2.4) such that

$$|P(z_0)-\alpha m| = |P(z_0)|-|\alpha|m,$$

which is possible by (2.6), we obtain from (2.4) that

$$M(P,\rho)-|\alpha|m \leq \left(\frac{\rho^n+1}{\rho+1}\right)\left(\frac{|c_0|-|\alpha|m+\rho|c_n|}{|c_0|-|\alpha|m+|c_n|}\right)\left(|P(z_0)|-|\alpha|m\right),$$

for every α with $|\alpha| \leq 1$ and for every $\rho > 1$.

The above inequality is equivalent to (1.5) and this completes the proof of Theorem 2.

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Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

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