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ORIGINAL RESEARCH PAPER

Generalized Ulam–Hyers stability of (*a*, *b*; *k* > 0)-cubic functional equation in intuitionistic fuzzy normed spaces

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Abstract In this paper, we proved the general solution in vector space and established the generalized Ulam–Hyers stability of $(a, b; k > 0)-$ cubic functional equation

$$\frac{a+\sqrt{k}b}{2}f\left(ax+\sqrt{k}by\right)+\frac{a-\sqrt{k}b}{2}f\left(ax-\sqrt{k}by\right)+k(a^2-kb^2)b^2f(y)$$
$$=k(ab)^2f(x+y)+(a^2-kb^2)a^2f(x)$$

where $a \neq \pm 1, 0; b \neq \pm 1, 0; k > 0$ in Banach space and Intuitionistic fuzzy normed spaces using both direct and fixed point methods.

Keywords Cubic functional equation · Generalized Ulam–Hyers stability · Banach space · Intuitionistic fuzzy normed space · Fixed point

Mathematics Subject Classification 39B52 · 32B72 · Banach space · 32B82

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1 Introduction

One of the most interesting questions in the theory of functional equations concerning the famous Ulam stability problem is, as follows: when is it true that a mapping satisfying a functional equation approximately, must be close to an exact solution of the given functional equation?

The first stability problem was raised by Ulam [13] during his talk at the University of Wisconsin in 1940. In fact, we are given a group $(G_1, \cdot)$ and let $(G_2, *)$ be a metric group with the metric $d(\cdot,\cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$, such that if a mapping $h : G_1 \to G_2$ satisfies the inequality $d(h(x \cdot y), h(x)*h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$?

Hyers [3] gave the first affirmative partial answer to the question of Ulam for Banach spaces. It was further generalized via excellent results obtained by a number of authors [1, 2, 8, 10, 11]. The solution and stability of the following cubic functional equations

$$C(x+2y) + 3C(x) = 3C(x+y) + C(x-y) + 6C(y), \tag{1.1}$$

$$f(2x+y) + f(2x-y) = 2f(x+y) + 2f(x-y) + 12f(x), \tag{1.2}$$

$$\begin{aligned} &f(x+y+2z) + f(x+y-2z) + f(2x) + f(2y) \\ &\quad = 2[f(x+y) + 2f(x+z) + 2f(y+z) + 2f(x-z) + 2f(y-z)], \end{aligned} \tag{1.3}$$

$$\begin{aligned} kf(x+ky) - f(kx+y) = &\frac{k(k^2-1)}{2}[f(x+y) + f(x-y)] + (k^4-1)f(y) \\ &- 2k(k^2-1)f(x), k \geq 2 \end{aligned} \tag{1.4}$$

were investigated by Rassias [9], Jun et al. [4], Jung and Chang [6], Rassias et. al. [5].

Now, we will recall the fundamental results in fixed point theory.

Theorem 1.1 (Banach's contraction principle) *Let* (X, d) *be a complete metric space and consider a mapping* $T : X \to X$ *which is strictly contractive mapping, that is*

(A_1) $d(Tx, Ty) \leq Ld(x, y)$ *for some (Lipschitz constant)* $L < 1$. *Then,*

(i) *The mapping* T *has one and only one fixed point* $x^* = T(x^*)$;
(ii) *The fixed point for each given element* x^* *is globally attractive; that is*

(A_2) $lim_{n\to\infty} T^n x = x^*$, *for any starting point* $x \in X$;

(iii) *The following estimation inequalities hold:*

(A_3) $d(T^n x, x^*) \leq \frac{1}{1-L} d(T^n x, T^{n+1}x), \forall n \geq 0, \forall x \in X$;
(A_4) $d(x, x^*) \leq \frac{1}{1-L} d(x, x^*), \forall x \in X$.

Theorem 1.2 [7] (The alternative of fixed point) *Suppose that for a complete generalized metric space* (X, d) *and a strictly contractive mapping* $T : X \to X$ *with Lipschitz constant L. Then, for each given element* $x \in X$, either

(B_1) $d(T^n x, T^{n+1}x) = \infty \quad \forall n \geq 0$, or
(B_2) *there exists a natural number* n_0 *such that:*

(i) $d(T^n x, T^{n+1}x) < \infty$ for all $n \geq n_0$;
(ii) *The sequence* $(T^n x)$ *is convergent to a fixed point* y^* *of* T
(iii) y^* *is the unique fixed point of T in the set* $Y = \{y \in X : d(T^{n_0}x, y) < \infty\}$;
(iv) $d(y^*, y) \leq \frac{1}{1-L} d(y, Ty)$ *for all* $y \in Y$.

There are many situations where the norm of a vector is not possible to find and the concept of intuitionistic fuzzy norm seems to be more suitable in such cases, that is, we can deal with such situations by modelling the inexactness by intuitionistic fuzzy norm.

In this paper, we established the generalized Ulam–Hyers stability of $(a, b; k > 0)-$ cubic functional equation

$$\frac{a+\sqrt{k}b}{2} f\left(ax+\sqrt{k}by\right) + \frac{a-\sqrt{k}b}{2} f\left(ax-\sqrt{k}by\right) + k(a^2-kb^2)b^2 f(y) = k(ab)^2 f(x+y) + (a^2-kb^2)a^2 f(x) \tag{1.5}$$

where $a \neq \pm 1, 0; b \neq \pm 1, 0; k > 0$ in intuitionistic fuzzy normed space using direct and fixed point methods.

2 Solution of the cubic functional Eq. (1.5)

In this section, the general solution of the functional Eq. (1.5) is given. Throughout this section, assume that $\mathcal{A}_1$ and $\mathcal{A}_2$ are vector spaces.

Lemma 2.1 *If a mapping* $f : \mathcal{A}_1 \to \mathcal{A}_2$ *satisfies the functional* Eq. (1.5), *then the following properties hold*

(i) $f(0) = 0$,
(ii) $f(ax) = a^3 f(x)$, *for all* $x \in \mathcal{A}_1$.
(iii) $f(-y) = -f(y)$, *for all* $y \in \mathcal{A}_1$; *that is, f is an odd function.*

Proof Letting (x, y) by $(0, 0)$ in (1.5), we get

$$f(0)\left(\frac{a+\sqrt{k}b}{2}+\frac{a-\sqrt{k}b}{2}+k(a^2-kb^2)b^2-k(ab)^2-(a^2-kb^2)a^2\right)=0, or$$
$$f(0)\left(a+ka^2b^2-k^2b^4-k(ab)^2-a^4+ka^2b^2\right)=0, or$$
$$f(0)\left(a(1-a^3)+b^2(ka^2-k^2b^2)\right)=0.$$

Since $a \neq \pm 1, 0; b \neq \pm 1, 0; k > 0$, we arrive (i).

Replacing (*x*, *y*) by (*x*, 0) in (1.5), we obtain

$$\frac{a+\sqrt{k}b}{2}f(ax)+\frac{a-\sqrt{k}b}{2}f(ax)=k(ab)^2f(x)+(a^2-kb^2)a^2f(x), or$$
$$af(ax)=(k(ab)^2+a^4-ka^2b^2)f(x), or$$
$$af(ax)=a^4f(x), a\neq 0, or$$
$$f(ax)=a^3f(x)$$

for all $x \in \mathcal{A}_1$. Thus, (ii) holds.

With the help of (ii) and setting *x* by 0 in (1.5), we get

$$\frac{a+\sqrt{k}b}{2}f\left(\sqrt{k}by\right)+\frac{a-\sqrt{k}b}{2}f\left(-\sqrt{k}by\right)+k(a^2-kb^2)b^2f(y)=k(ab)^2f(y)$$
$$\frac{(a+\sqrt{k}b)b^3k^{\frac{3}{2}}}{2}f(y)+\frac{(a-\sqrt{k}b)b^3k^{\frac{3}{2}}}{2}f(-y)-k^2b^4f(y)=0$$
$$(a+\sqrt{k}b)b^3k^{\frac{3}{2}}f(y)+(a-\sqrt{k}b)b^3k^{\frac{3}{2}}f(-y)-2k^2b^4f(y)=0$$
$$b^3k^{\frac{3}{2}}f(y)\left[(a-\sqrt{k}b)\right]+(a-\sqrt{k}b)b^3k^{\frac{3}{2}}f(-y)=0$$
$$(a-\sqrt{k}b)b^3k^{\frac{3}{2}}[f(y)+f(-y)]=0$$

for all $y \in \mathcal{A}_1$. Finally, (iii) holds, since $a \neq \pm 1, 0; b \neq \pm 1, 0; k > 0$. Thus f is an odd function. Hence the proof is complete. □

3 Stability results in Banach space

In this section, we investigate the generalized Ulam–Hyers stability of the functional Eq. (1.5) in Banach space using direct and fixed point methods.

Throughout this section, let us consider $\mathcal{Z}_1$ be a normed space and $\mathcal{Z}_2$ be a Banach space. Define a mapping $Df_{(a,b;k)} : \mathcal{Z}_1 \longrightarrow \mathcal{Z}_2$ by

$$Df_{(a,b;k)}(x,y)=\frac{a+\sqrt{k}b}{2}f\left(ax+\sqrt{k}by\right)+\frac{a-\sqrt{k}b}{2}f\left(ax-\sqrt{k}by\right)$$
$$+k(a^2-kb^2)b^2f(y)-k(ab)^2f(x+y)-(a^2-kb^2)a^2f(x)$$

where $a \neq \pm 1, 0; b \neq \pm 1, 0; k > 0$ for all $x, y \in \mathcal{Z}_1$.

3.1 Banach space: direct method

Theorem 3.1 *Let $q = \pm 1$ and $K : \mathcal{Z}_1^2 \longrightarrow [0,\infty)$ be a function such that*

$$\sum_{s=0}^{\infty} \frac{K(a^{qs}x, a^{qs}y)}{a^{3qs}} \quad \text{convergesin} \quad \mathbb{R} \quad \text{and} \quad \lim_{s\to\infty} \frac{K(a^{qs}x, a^{qs}y)}{a^{3qs}} = 0 \tag{3.1}$$

for all $x, y \in \mathcal{Z}_1$. Let $Df_{(a,b;k)} : \mathcal{Z}_1 \longrightarrow \mathcal{Z}_2$ be a function fulfilling the inequality

$$\left\|Df_{(a,b;k)}(x,y)\right\| \leq K(x,y) \tag{3.2}$$

for all $x, y \in \mathcal{Z}_1$. Then there exists a unique cubic function $\mathcal{C} : \mathcal{Z}_1 \longrightarrow \mathcal{Z}_2$ which satisfies (1.5) and

$$\|f(x) - \mathcal{C}(x)\| \leq \frac{1}{a^4} \sum_{s=\frac{1-q}{2}}^{\infty} \frac{K(a^{qs}x, a^{qs}x)}{a^{3qs}} \tag{3.3}$$

where $\mathcal{C}(x)$ are defined by

$$\mathcal{C}(x) = \lim_{p\to\infty} \frac{f(a^{pq}x)}{a^{3pq}} \tag{3.4}$$

for all $x \in \mathcal{Z}_1$, respectively.

Proof **Case (i):** Assume $q = 1$.
Changing (x, y) by $(x, 0)$ in (3.2), we get

$$\left\|af(ax) - a^4 f(x)\right\| \leq K(x,0) \tag{3.5}$$

for all $x \in \mathcal{Z}_1$. It follows from (3.5) that

$$\left\|\frac{f(ax)}{a^3} - f(x)\right\| \leq \frac{K(x,0)}{a^4} \tag{3.6}$$

for all $x \in \mathcal{Z}_1$. Now replacing x by ax and dividing by a^3 in (3.6), we have

$$\left\|\frac{f(a^2x)}{a^6} - \frac{f(ax)}{a^3}\right\| \leq \frac{K(ax,0)}{a^7} \tag{3.7}$$

for all $x \in \mathcal{Z}_1$. From (3.6) and (3.7), we obtain

$$\left\|\frac{f(a^2x)}{a^6} - f(x)\right\| \leq \left\|\frac{f(a^2x)}{a^6} - \frac{f(ax)}{a^3}\right\| + \left\|\frac{f(ax)}{a^3} - f(x)\right\| \leq \frac{1}{a^4}\left[K(x,0) + \frac{K(ax,0)}{a^3}\right] \tag{3.8}$$

for all $x \in \mathcal{Z}_1$. Generalizing, for a positive integer p, we obtain

$$\left\|\frac{f(a^px)}{a^{3p}} - f(x)\right\| \leq \frac{1}{a^4} \sum_{s=0}^{p-1} \frac{K(a^sx,0)}{a^{3s}} \tag{3.9}$$

for all $x \in \mathcal{Z}_1$. To prove the convergence of the sequence $\left\{\frac{f(a^p x)}{a^{3p}}\right\}$, replacing x by $a^s x$ and dividing by a^{3s} in (3.9), for any $p, s > 0$, we get

$$\left\|\frac{f(a^{p+s}x)}{a^{3(p+s)}} - \frac{f(a^s x)}{a^{3s}}\right\| = \frac{1}{a^{3s}}\left\|\frac{f(a^p \cdot a^s x)}{a^{3p}} - f(a^s x)\right\| \leq \frac{1}{a^{3s}}\frac{1}{a^4}\sum_{s=0}^{p-1}\frac{K(a^p \cdot a^s x, 0)}{a^{3p}}$$
$$\leq \frac{1}{a^4}\sum_{s=0}^{\infty}\frac{K(a^{p+s}x, 0)}{a^{3(p+s)}} \to 0 \quad as \;\; s \to \infty$$

for all $x \in \mathcal{Z}_1$. Thus it follows that the sequence $\left\{\frac{f(a^p x)}{a^{3p}}\right\}$ is a Cauchy in $\mathcal{Z}_2$ and so it converges. Define a mapping $\mathcal{C}(x) : \mathcal{Z}_1 \to \mathcal{Z}_2$ by

$$\mathcal{C}(x) = \lim_{p\to\infty}\frac{f(2^p x)}{2^{3p}}$$

for all $x \in \mathcal{Z}_1$. In order to show that $\mathcal{C}$ satisfies (1.5), replacing (x, y) by $(a^p x, a^p y)$ and dividing by a^{3p} in (3.2), we have

$$\left\|D\mathcal{C}(x,y)_{(a,b;k)}\right\| = \lim_{p\to\infty}\frac{1}{a^{3p}}\left\|Df_{(a,b;k)}(a^p x, a^p y)\right\| \leq \lim_{p\to\infty}\frac{1}{a^{3p}}K(a^p x, a^p y)$$

for all $x, y \in \mathcal{Z}_1$ and hence the mapping $\mathcal{C}$ is cubic. Taking the limit as p approaches to infinity in (3.9), we find that the mapping $\mathcal{C}$ is a cubic mapping satisfying the inequality (3.3) near the approximate mapping $f : \mathcal{Z}_1 \to \mathcal{Z}_2$ of Eq. (1.5). Hence, $\mathcal{C}$ satisfies (1.5), for all $x, y \in \mathcal{Z}_1$.

To prove that $\mathcal{C}$ is unique, we assume now that there is $\mathcal{C}'$ as another cubic mapping satisfying (1.5) and the inequality (3.3). Then it is easily proved that

$$\mathcal{C}(a^s x) = a^{3s}\mathcal{C}(x), \quad \mathcal{C}'(a^s x) = a^{3s}\mathcal{C}'(x)$$

for all $x \in \mathcal{Z}_1$ and all $s \in \mathbb{N}$. Thus

$$\|\mathcal{C}(x) - \mathcal{C}'(x)\| = \frac{1}{a^{3s}}\|\mathcal{C}(a^s x) - \mathcal{C}'(a^s x)\|$$
$$\leq \frac{1}{a^{3s}}\{\|\mathcal{C}(a^s x) - f(a^s x)\| + \|f(a^s x) - \mathcal{C}'(a^s x)\|\}$$
$$\leq \frac{2}{a^4}\sum_{s=0}^{\infty}\frac{\kappa(a^{p+s}x, a^{p+s}x)}{a^{3(p+s)}}$$

for all $x \in \mathcal{Z}_1$. Therefore, as $p \to \infty$ in the above inequality, we get the uniqueness of $\mathcal{C}$. Hence the theorem holds for $q = 1$.

Case (ii): Assume $q = -1$. Now replacing x by $\frac{x}{a}$ in (3.5), we get

$$\left\|f(x) - a^3 f\left(\frac{x}{a}\right)\right\| \leq \frac{1}{a}K\left(\frac{x}{a}, 0\right) \tag{3.10}$$

for all $x \in \mathcal{Z}_1$. The rest of the proof is similar to that of case $q = 1$. Hence for $q = -1$ also the theorem holds. This completes the proof. □

The following corollary is an immediate consequence of Theorem 3.1 concerning the stabilities of (1.5).

Corollary 3.2 *Let $Df_{(a,b;k)} : \mathcal{Z}_1 \longrightarrow \mathcal{Z}_2$ be a mapping. If there exist real numbers β and t such that*

$$\left\|Df_{(a,b;k)}(x,y)\right\| \leq \begin{cases} \beta, \\ \beta\{||x||^t + ||y||^t\}, \\ \beta\left\{||x||^t||y||^t + \left\{||x||^{2t} + ||y||^{2t}\right\}\right\}, \end{cases} \tag{3.11}$$

for all $x, y \in \mathcal{Z}_1$, then there exists a unique cubic function $\mathcal{C} : \mathcal{Z}_1 \longrightarrow \mathcal{Z}_2$ such that

$$\|f(x) - \mathcal{C}(x)\| \leq \begin{cases} \dfrac{a\beta}{a^3 - 1}, \\ \dfrac{a\beta||x||^t}{|a^3 - a^t|}, & t \neq 3, \\ \dfrac{a\beta||x||^{at}}{|a^3 - a^{2t}|}, & 2t \neq 3, \end{cases} \tag{3.12}$$

for all $x \in \mathcal{Z}_1$.

Using Theorem 1.2, we obtain the generalized Ulam–Hyers stability of (1.5).

3.2 Banach space: fixed point method

Theorem 3.3 *Let $Df_{(a,b;k)} : \mathcal{Z}_1 \longrightarrow \mathcal{Z}_2$ be a mapping for which there exists a function $K : \mathcal{Z}_1^2 \longrightarrow [0, \infty)$ with the condition*

$$\lim_{n\to\infty} \frac{1}{\delta_i^{3n}} K(\delta_i^n x, \delta_i^n y) = 0 \tag{3.13}$$

where

$$\delta_i = \begin{cases} a & if \quad i = 0, \\ \dfrac{1}{a} & if \quad i = 1 \end{cases} \tag{3.14}$$

such that the functional inequality

$$\left\|Df_{(a,b;k)}(x,y)\right\| \leq K(x,y) \tag{3.15}$$

holds for all $x, y \in \mathcal{Z}_1$. Assume that there exists $L = L(i)$ such that the function

$$D(x,0) = \frac{1}{a} K\left(\frac{x}{a}, 0\right)$$

with the property

$$\frac{1}{\delta_i^3} D(\delta_i x, 0) = LD(x, 0) \tag{3.16}$$

for all $x \in \mathcal{Z}_1$. *Then there exists a unique cubic mapping* $\mathcal{C} : \mathcal{Z}_1 \longrightarrow \mathcal{Z}_2$ *satisfying the functional* Eq. (1.5) *and*

$$\|f(x) - \mathcal{C}(x)\| \; \| \leq \left(\frac{L^{1-i}}{1-L} \right) D(x, 0) \tag{3.17}$$

for all $x \in \mathcal{Z}_1$.

Proof Consider the set

$$\mathcal{A} = \{h/h : \mathcal{Z}_1 \longrightarrow \mathcal{Z}_2, h(0) = 0\}$$

and introduce the generalized metric on $\mathcal{A}$,

$$d(h,f) = \inf\{\omega \in (0, \infty) :\| \; h(x) - f(x) \; \| \; \leq \omega D(x, x), x \in \mathcal{Z}_1\}. \tag{3.18}$$

It is easy to see that (3.18) is complete with respect to the defined metric. Define $J : \mathcal{A} \longrightarrow \mathcal{A}$ by

$$Jh(x) = \frac{1}{\delta_i^3} h(\delta_i x),$$

for all $x \in \mathcal{Z}_1$. Now, from (3.18) and $h,f \in \mathcal{A}$

$$\begin{aligned}
&\inf\{\omega \in (0, \infty) :\| \; h(x) - f(x) \; \| \; \leq \omega D(x, x), x \in \mathcal{Z}_1\} or \\
&\inf\left\{ \omega \in (0, \infty) : \left\| \frac{1}{\delta_i^3} h(\delta_i x) - \frac{1}{\delta_i^3} f(\delta_i x) \right\| \leq \frac{\omega}{\delta_i^3} D(\delta_i x, \delta_i x), x \in \mathcal{Z}_1 \right\} or \\
&\inf\left\{ L\omega \in (0, \infty) : \left\| \frac{1}{\delta_i^3} h(\delta_i x) - \frac{1}{\delta_i^3} f(\delta_i x) \right\| \leq L\omega D(x, x), x \in \mathcal{Z}_1 \right\} or \\
&\inf\{L\omega \in (0, \infty) :\| \; Jh(x) - Jf(x) \; \| \; \leq L\omega D(x, x), x \in \mathcal{Z}_1\}
\end{aligned}$$

This implies J is a strictly contractive mapping on $\mathcal{A}$ with Lipschitz constant L. It follows from (3.18), (3.6) and (3.16) for the case $i = 0$, we reach

$$\inf\left\{1 \in (0, \infty) : \left\| af(ax) - a^4 f(x) \right\| \leq K(x, 0), x \in \mathcal{Z}_1 \right\} or \tag{3.19}$$

$$\begin{aligned}
&\inf\left\{ 1 \in (0, \infty) : \left\| \frac{f(ax)}{a^3} - f(x) \right\| \leq \frac{1}{a^4} K(x, 0), x \in \mathcal{Z}_1 \right\} or \\
&\inf\{L \in (0, \infty) : \|Jf(x) - f(x)\| \leq LD(x, 0), x \in \mathcal{Z}_1\} or \\
&\inf\left\{L^1 \in (0, \infty) : \|Jf(x) - f(x)\| \leq LD(x, 0), x \in \mathcal{Z}_1 \right\} or \\
&\inf\left\{L^{1-0} \in (0, \infty) : \|Jf(x) - f(x)\| \leq LD(x, 0), x \in \mathcal{Z}_1 \right\}
\end{aligned} \tag{3.20}$$

Again replacing $x = \frac{x}{a}$ in (3.19) and (3.16) for the case $i = 1$, we get

$$\begin{aligned}
&\inf\left\{1 \in (0,\infty) : \left\| af(x) - a^4 f\left(\frac{x}{a}\right) \right\| \le K\left(\frac{x}{a}, 0\right), x \in \mathcal{Z}_1 \right\} \; or \\
&\inf\left\{1 \in (0,\infty) : \left\| f(x) - a^3 f\left(\frac{x}{a}\right) \right\| \le \frac{1}{a} K\left(\frac{x}{a}, 0\right), x \in \mathcal{Z}_1 \right\} \; or \\
&\inf\{1 \in (0,\infty) : \|f(x) - Jf(x)\| \le D(x,0), x \in \mathcal{Z}_1\} \; or \\
&\inf\{L^0 \in (0,\infty) : \|f(x) - Jf(x)\| \le D(x,0), x \in \mathcal{Z}_1\} \; or \\
&\inf\{L^{1-1} \in (0,\infty) : \|f(x) - Jf(x)\| \le D(x,0), x \in \mathcal{Z}_1\}.
\end{aligned} \tag{3.21}$$

Thus, from (3.20) and (3.21), we arrive

$$\inf\{L^{1-i} \in (0,\infty) : \|f(x) - Jf(x)\| \le L^{1-i} D(x,0), x \in \mathcal{Z}_1\}. \tag{3.22}$$

Hence property (FP1) holds. It follows from property (FP2) that there exists a fixed point $\mathcal{C}$ of J in $\mathcal{A}$ such that

$$\mathcal{C}(x) = \lim_{n\to\infty} \frac{1}{\delta_i^{3n}} f(\delta_i^n x) \tag{3.23}$$

for all $x \in \mathcal{Z}_1$. In order to show that $\mathcal{C}$ satisfies (1.5), replacing (x, y) by $(\delta_i^n x, \delta_i^n y)$ and dividing by δ_i^{3n} in (3.15), we have

$$\left\| D\mathcal{C}(x,y)_{(a,b;k)} \right\| = \lim_{n\to\infty} \frac{1}{\delta_i^{3n}} \left\| Df_{(a,b;k)}(\delta_i^n x, \delta_i^n y) \right\| \le \lim_{n\to\infty} \frac{1}{\delta_i^{3n}} K\left(\delta_i^n x, \delta_i^n y\right) = 0$$

for all $x, y \in \mathcal{Z}_1$, and so the mapping $\mathcal{C}$ is cubic. i.e., $\mathcal{C}$ satisfies the functional Eq. (1.5). By property (FP3), $\mathcal{C}$ is the unique fixed point of J in the set

$$\Delta = \{\mathcal{C} \in \mathcal{A} : d(f, \mathcal{C}) < \infty\},$$

such that

$$\inf\{\omega \in (0,\infty) : \|f(x) - \mathcal{C}(x)\| \le \omega D(x,0), x \in \mathcal{Z}_1\}.$$

Finally by property (FP4), we obtain

$$\|f(x) - \mathcal{C}(x)\| \le \|f(x) - Jf(x)\|.$$

This implies

$$\|f(x) - \mathcal{C}(x)\| \le \frac{L^{1-i}}{1-L}$$

which yields

$$\inf\left\{\frac{L^{1-i}}{1-L} \in (0,\infty) : \|f(x) - \mathcal{C}(x)\| \le \left(\frac{L^{1-i}}{1-L}\right) D(x,0), x \in \mathcal{Z}_1 \right\}.$$

So, the proof is completed. □

Using Theorem 3.3, we prove the following corollary concerning the stabilities of (1.5).

Corollary 3.4 *Let $Df_{(a,b;k)} : \mathcal{Z}_1 \longrightarrow \mathcal{Z}_2$ be a mapping. If there exist real numbers Θ and ρ such that*

$$\left\|Df_{(a,b;k)}(x,y)\right\| \leq \begin{cases} \Theta, \\ \Theta\{||x||^{\rho} + ||y||^{\rho}\}, \\ \Theta\left\{||x||^{\rho}||y||^{\rho} + \left\{||x||^{2\rho} + ||y||^{2\rho}\right\}\right\}, \end{cases} \tag{3.24}$$

for all $x, y \in \mathcal{Z}_1$, then there exists a unique cubic function $\mathcal{C} : \mathcal{Z}_1 \to \mathcal{Z}_2$ such that

$$\|f(x) - \mathcal{C}(x)\| \leq \begin{cases} \dfrac{\Theta}{a|a^3 - 1|}, & \\ \dfrac{a^{\rho}\Theta}{a|a^3 - a^{\rho}|}, & \rho \neq 3, \\ \dfrac{a^{2\rho}\Theta}{a|a^3 - a^{2\rho}|} & 2\rho \neq 3, \end{cases} \tag{3.25}$$

for all $x \in \mathcal{Z}_1$.

Proof Let

$$K(x,y) = \begin{cases} \Theta, \\ \Theta\{||x||^{\rho} + ||y||^{\rho}\} \\ \Theta\left\{||x||^{\rho}||y||^{\rho} + \left\{||x||^{2\rho} + ||y||^{2\rho}\right\}\right\} \end{cases}$$

for all $x, y \in \mathcal{Z}_1$. Now

$$\frac{1}{\delta_i^{3n}} K(\delta_i^n x, \delta_i^n y) = \begin{cases} \dfrac{\Theta}{\delta_i^{3n}}, \\ \dfrac{\Theta}{\delta_i^{3n}}\{||\delta_i^n x||^{\rho} + ||\delta_i^n y||^{\rho}\}, \\ \dfrac{\Theta}{\delta_i^{3n}}\left\{||\delta_i^n x||^{\rho}||\delta_i^n y||^{\rho} + \left\{||\delta_i^n x||^{2\rho} + ||\delta_i^n y||^{2\rho}\right\}\right\} \end{cases} = \begin{cases} \to 0 \text{ as } n \to \infty, \\ \to 0 \text{ as } n \to \infty, \\ \to 0 \text{ as } n \to \infty. \end{cases}$$

Thus, (3.13) holds. But, we have

$$D(x,0) = \frac{1}{a} K\left(\frac{x}{a}, 0\right)$$

has the property

$$\frac{1}{\delta_i^3} D(\delta_i x, 0) = LD(x,0)$$

for all $x \in \mathcal{Z}_1$. Hence,

$$D(x,0) = \frac{1}{a}K\left(\frac{x}{a},0\right) = \begin{cases} \frac{\Theta}{a}, \\ \frac{\Theta}{a \cdot a^{\rho}}||x||^{\rho}, \\ \frac{\Theta}{a \cdot a^{2\rho}}||x||^{2\rho} \end{cases} \tag{3.26}$$

for all $x \in \mathcal{Z}_1$. It follows from (3.26),

$$\frac{1}{\delta_i^3}D(\delta_i x,0) = \begin{cases} \delta_i^{-3}\frac{\Theta}{a}, \\ \delta_i^{\rho-3}\frac{\Theta}{a}||x||^{\rho} \\ \delta_i^{2\rho-3}\frac{\Theta}{a}||x||^{2\rho}. \end{cases}$$

Now, from (3.17), we prove the following cases for condition (i).

$$L = \delta_i^{-3}, i = 0 \qquad L = \frac{1}{\delta_i^{-3}}, i = 1$$

$$L = a^{-3}, i = 0 \qquad L = \frac{1}{a^{-3}}, i = 1$$

$$L = a^{-3}, i = 0 \qquad L = a^{3}, i = 1$$

$$\begin{aligned} &\| f(x) - \mathcal{C}(x) \| \\ &\leq \left(\frac{L^{1-i}}{1-L}\right)D(x,0) \\ &= \left(\frac{(a^{-3})^{1-0}}{1-a^{-3}}\right)\cdot\frac{\Theta}{a} \\ &= \left(\frac{a^{-3}}{1-a^{-3}}\right)\cdot\frac{\Theta}{a} \\ &= \left(\frac{\Theta}{a(a^3-1)}\right) \end{aligned} \qquad \begin{aligned} &\| f(x) - \mathcal{C}(x) \| \\ &\leq \left(\frac{L^{1-i}}{1-L}\right)D(x,0) \\ &= \left(\frac{(a^{3})^{1-1}}{1-a^{3}}\right)\cdot\frac{\Theta}{a} \\ &= \left(\frac{1}{1-a^{3}}\right)\cdot\frac{\Theta}{a} \\ &= \left(\frac{\Theta}{a(1-a^3)}\right) \end{aligned}$$

Also, from (3.17), we prove the following cases for condition (ii).

$$
\begin{array}{ll}
L = \delta_i^{\rho-3}, \rho < 3, i = 0 & L = \dfrac{1}{\delta_i^{\rho-3}}, \rho > 3, i = 1 \\
L = a^{\rho-3}, \rho < 3, i = 0 & L = \dfrac{1}{a^{\rho-3}}, \rho < 3, i = 1 \\
L = a^{\rho-3}, \rho < 3, i = 0 & L = a^{3-\rho}, \rho > 3, i = 1 \\
\| f(x) - \mathcal{C}(x) \| & \| f(x) - \mathcal{C}(x) \| \\
\leq \left(\dfrac{L^{1-i}}{1-L} \right) D(x, 0) & \leq \left(\dfrac{L^{1-i}}{1-L} \right) D(x, 0) \\
= \left(\dfrac{(a^{\rho-3})^{1-0}}{1-a^{\rho-3}} \right) \cdot \dfrac{\Theta}{a} & = \left(\dfrac{(a^{3-\rho})^{1-1}}{1-a^{3-\rho}} \right) \cdot \dfrac{\Theta}{a} \\
= \left(\dfrac{a^{\rho-3}}{1-a^{\rho-3}} \right) \cdot \dfrac{\Theta}{a} & = \left(\dfrac{1}{1-a^{3-\rho}} \right) \cdot \dfrac{\Theta}{a} \\
= \left(\dfrac{a^{\rho}}{a^3 - a^{\rho}} \right) \cdot \dfrac{\Theta}{a} & = \left(\dfrac{a^{\rho}}{a^{\rho} - a^3} \right) \cdot \dfrac{\Theta}{a}
\end{array}
$$

Finally, the proof of (3.17) for condition (iii) is similar to that of condition (ii). Hence the proof is complete. □

4 Intuitionistic fuzzy normed space stability results

In this section, we investigate the generalized Ulam–Hyers stability of the functional Eq. (1.5) in Intuitionistic Fuzzy Normed Space using direct and fixed point methods.

Now, we recall the basic definitions and notations in Intuitionistic Fuzzy Normed Space.

Definition 4.1 A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is said to be continuous $t-$ norm if $*$ satisfies the following conditions:

(1) $*$ is commutative and associative;
(2) $*$ is continuous;
(3) $a*1 = a$ for all $a \in [0,1]$;
(4) $a*b \leq c*d$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0,1]$.

Definition 4.2 A binary operation $\diamond : [0,1] \times [0,1] \to [0,1]$ is said to be continuous $t-$ conorm if $\diamond$ satisfies the following conditions:

(1') $\diamond$ is commutative and associative;
(2') $\diamond$ is continuous;
(3') $a \diamond 0 = a$ for all $a \in [0,1]$;
(4') $a \diamond b \leq c \diamond d$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0,1]$.

Using the notions of continuous $t-$ norm and $t-$ conorm, Saadati and Park [12] introduced the concept of intuitionistic fuzzy normed space as follows:

Definition 4.3 The five-tuple $(X, \mu, v, *, \diamond)$ is said to be an Intuitionistic Fuzzy Normed Space (for short, IFNS) if X is a vector space, $*1$ is a continuous $t-$ norm, $\diamond$ is a continuous $t-$ conorm, and μ, v are fuzzy sets on $X \times (0, \infty)$ satisfying the following conditions. For every $x, y \in X$ and $s, t > 0$

(IFN1) $\mu(x, t) + v(x, t) \leq 1$,
(IFN2) $\mu(x, t) > 0$,
(IFN3) $\mu(x, t) = 1$, if and only if $x = 0$.
(IFN4) $\mu(\alpha x, t) = \mu\left(x, \frac{t}{\alpha}\right)$ for each $\alpha \neq 0$,
(IFN5) $\mu(x, t)*\mu(y, s) \leq \mu(x + y, t + s)$,
(IFN6) $\mu(x, \cdot) : (0, \infty) \to [0, 1]$ is continuous,
(IFN7) $\lim_{t\to\infty} \mu(x, t) = 1$ and$\lim_{t\to 0} \mu(x, t) = 0$,
(IFN8) $v(x, t) < 1$,
(IFN9) $v(x, t) = 0$, if and only if $x = 0$.
(IFN10) $v(\alpha x, t) = v\left(x, \frac{t}{\alpha}\right)$ for each $\alpha \neq 0$,
(IFN11) $v(x, t)\diamond v(y, s) \geq v(x + y, t + s)$,
(IFN12) $v(x, \cdot) : (0, \infty) \to [0, 1]$ is continuous,
(IFN13) $\lim_{t\to\infty} v(x, t) = 0$ and $\lim_{t\to 0} v(x, t) = 1$.

In this case, (μ, v) is called an intuitionistic fuzzy norm.

Example 4.4 Let $(X, \|\cdot\|)$ be a normed space. Let $a*b = ab$ and $a\diamond b = \min\{a + b, 1\}$ for all $a, b \in [0, 1]$. For all $x \in X$ and every $t > 0$, consider

$$\mu(x,t) = \begin{cases} \dfrac{t}{t + \|x\|} & if \quad t > 0; \\ 0 & if \quad t \leq 0; \end{cases} \quad and \quad v(x,t) = \begin{cases} \dfrac{\|x\|}{t + \|x\|} & if \quad t > 0; \\ 0 & if \quad t \leq 0. \end{cases}$$

Then $(X, \mu, v, *, \diamond)$ is an IFN-space.

The concepts of convergence and Cauchy sequences in an intuitionistic fuzzy normed space are studied in [12].

Definition 4.5 Let $(X, \mu, v, *, \diamond)$ be an IFNS. Then, a sequence $x = \{x_k\}$ is said to be *intuitionistic fuzzy convergent* to a point $L \in X$ if

$$\lim \mu(x_k - L, t) = 1 \quad and \quad \lim v(x_k - L, t) = 0$$

for all $t > 0$. In this case, we write

$$x_k \xrightarrow{IF} L \quad as \quad k \to \infty.$$

Definition 4.6 Let $(X, \mu, v, *, \diamond)$ be an IFN-space. Then, $x = \{x_k\}$ is said to be *intuitionistic fuzzy Cauchy sequence* if

$$\mu\left(x_{k+p} - x_k, t\right) = 1 \quad and \quad \nu\left(x_{k+p} - x_k, t\right) = 0$$

for all $t > 0$, and $p = 1, 2, \ldots$.

Definition 4.7 Let $(X, \mu, \nu, *, \diamond)$ be an IFN-space. Then $(X, \mu, \nu, *, \diamond)$ is said to be *complete* if every intuitionistic fuzzy Cauchy sequence in $(X, \mu, \nu, *, \diamond)$ is intuitionistic fuzzy convergent $(X, \mu, \nu, *, \diamond)$.

Hereafter, throughout this section, assume that X is a linear space, (Z, μ', ν') is an intuitionistic fuzzy normed space and (Y, μ, ν) an intuitionistic fuzzy Banach space. Now, we use the following notation for a given mapping $Df_{(a,b;k)} : X \to Y$ such that

$$\begin{aligned} Df_{(a,b;k)}(x,y) = & \frac{a+\sqrt{k}b}{2} f\left(ax + \sqrt{k}by\right) + \frac{a-\sqrt{k}b}{2} f\left(ax - \sqrt{k}by\right) \\ & + k(a^2 - kb^2)b^2 f(y) - k(ab)^2 f(x+y) - (a^2 - kb^2)a^2 f(x) \end{aligned}$$

where $a \neq \pm 1, 0; b \neq \pm 1, 0; k > 0$ for all $x, y \in X$.

4.1 IFNS: direct method

Theorem 4.8 *Let $\tau \in \{1, -1\}$. Let $K : X \times X \to Z$ be a function such that for some $0 < \left(\frac{p}{a}\right)^\tau < 1$,*

$$\left.\begin{aligned} \mu'\left(K_\mu(a^{n\tau}x, a^{n\tau}y), r\right) \geq \mu'\left(p^{n\tau}K_\mu(x,y), r\right) \\ \nu'(K_\nu(a^{n\tau}x, a^{n\tau}y), r) \leq \nu'(p^{n\tau}K_\nu(x,y), r) \end{aligned}\right\} \tag{4.1}$$

for all $x \in X$ and all $r > 0$ and

$$\left.\begin{aligned} \lim_{n\to\infty} \mu'\left(K_\mu(a^{\tau n}x, a^{\tau n}y), a^{\tau n}r\right) = 1 \\ \lim_{n\to\infty} \nu'(K_\nu(a^{\tau n}x, a^{\tau n}y), a^{\tau n}r) = 0 \end{aligned}\right\} \tag{4.2}$$

for all $x, y \in X$ and all $r > 0$. Let $Df_{(a,b;k)} : X \to Y$ be a function satisfying the inequality

$$\left.\begin{aligned} \mu\left(Df_{(a,b;k)}(x,y), r\right) \geq \mu'\left(K_\mu(x,y), r\right) \\ \nu\left(Df_{(a,b;k)}(x,y), r\right) \leq \nu'(K_\nu(x,y), r) \end{aligned}\right\} \tag{4.3}$$

for all $x, y \in X$ and all $r > 0$. Then there exists a unique cubic mapping $N : X \to Y$ satisfying (1.5) and

$$\left.\begin{aligned} \mu(f(x) - C(x), r) \geq \mu'\left(K_\mu(x,0), a^4|a^3 - p|r\right) \\ \nu(f(x) - C(x), r) \leq \nu'(K_\nu(x,0), a^4|a^3 - p|r) \end{aligned}\right\} \tag{4.4}$$

for all $x \in X$ and all $r > 0$.

Proof Let $\tau = 1$. Replacing (x, y) by $(x, 0)$ in (4.3), we get

$$\left.\begin{array}{l}\mu(af(ax)-a^4f(x),r)\geq\mu'\left(K_\mu(x,0),r\right)\\ \nu(af(ax)-a^4f(x),r)\leq\nu'(K_\nu(x,0),r)\end{array}\right\}\tag{4.5}$$

for all $x\in X$ and all $r>0$. Using (IFN4) in (4.5), we arrive

$$\left.\begin{array}{l}\mu\left(\frac{f(ax)}{a^3}-f(x),\frac{r}{a^4}\right)\geq\mu'\left(K_\mu(x,0),r\right)\\ \nu\left(\frac{f(ax)}{a^3}-f(x),\frac{r}{a^4}\right)\leq\nu'(K_\nu(x,0),r)\end{array}\right\}\tag{4.6}$$

for all $x\in X$ and all $r>0$. Replacing x by a^nx in (4.6), we have

$$\left.\begin{array}{l}\mu\left(\frac{f(a^{n+1}x)}{a^3}-f(a^nx),\frac{r}{a^4}\right)\geq\mu'\left(K_\mu(a^nx,0),r\right)\\ \nu\left(\frac{f(a^{n+1}x)}{a^3}-f(a^nx),\frac{r}{a^4}\right)\leq\nu'(K_\nu(a^nx,0),r)\end{array}\right\}\tag{4.7}$$

for all $x\in X$ and all $r>0$. It is easy to verify from (4.7) and using (4.1), (IFN4) that

$$\left.\begin{array}{l}\mu\left(\frac{f(a^{n+1}x)}{a^{3(n+1)}}-\frac{f(a^nx)}{a^{3n}},\frac{r}{a^4\cdot a^{3n}}\right)\geq\mu'\left(K_\mu(x,0),\frac{r}{p^n}\right)\\ \nu\left(\frac{f(a^{n+1}x)}{a^{3(n+1)}}-\frac{f(a^nx)}{a^{3n}},\frac{r}{a^4\cdot a^{3n}}\right)\leq\nu'\left(K_\nu(x,0),\frac{r}{p^n}\right)\end{array}\right\}\tag{4.8}$$

for all $x\in X$ and all $r>0$. Replacing r by p^nr in (4.8), we have

$$\left.\begin{array}{l}\mu\left(\frac{f(a^{n+1}x)}{a^{3(n+1)}}-\frac{f(a^nx)}{a^{3n}},\frac{r\cdot p^n}{a^4\cdot a^{3n}}\right)\geq\mu'\left(K_\mu(x,0),r\right)\\ \nu\left(\frac{f(a^{n+1}x)}{a^{3(n+1)}}-\frac{f(a^nx)}{a^{3n}},\frac{r\cdot p^n}{a^4\cdot a^{3n}}\right)\leq\nu'(K_\nu(x,0),r)\end{array}\right\}\tag{4.9}$$

for all $x\in X$ and all $r>0$. It is easy to see that

$$\frac{f(a^nx)}{a^{3n}}-f(x)=\sum_{i=0}^{n-1}\frac{f(a^{i+1}x)}{a^{3(i+1)}}-\frac{f(a^ix)}{a^{3i}}\tag{4.10}$$

for all $x\in X$. From Eqs. (4.8) and (4.9), we have

$$\left.\begin{array}{l}\mu\left(\frac{f(a^nx)}{a^{3n}}-f(x),\sum_{i=0}^{n-1}\frac{p^ir}{a^4\cdot a^{3i}}\right)=\mu\left(\sum_{i=0}^{n-1}\frac{f(a^{i+1}x)}{a^{3(i+1)}}-\frac{f(a^ix)}{a^{3i}},\sum_{i=0}^{n-1}\frac{p^ir}{a^4\cdot a^{3i}}\right)\\ \nu\left(\frac{f(a^nx)}{a^{3n}}-f(x),\sum_{i=0}^{n-1}\frac{p^ir}{a^4\cdot a^{3i}}\right)=\nu\left(\sum_{i=0}^{n-1}\frac{f(a^{i+1}x)}{a^{3(i+1)}}-\frac{f(a^ix)}{a^{3i}},\sum_{i=0}^{n-1}\frac{p^ir}{a^4\cdot a^{3i}}\right)\end{array}\right\}\tag{4.11}$$

for all $x\in X$ and all $r>0$. From Eqs. (4.10) and (4.11), we have

$$\left.\begin{aligned} \mu\left(\frac{f(a^n x)}{a^{3n}} - f(x), \sum_{i=0}^{n-1} \frac{p^i r}{a^4 \cdot a^{3i}}\right) &\geq \prod_{i=0}^{n-1} \mu\left(\frac{f(a^{i+1}x)}{a^{3(i+1)}} - \frac{f(a^i x)}{a^{3i}}, \frac{p^i r}{a^4 \cdot a^{3i}}\right) \\ \nu\left(\frac{f(a^n x)}{a^{3n}} - f(x), \sum_{i=0}^{n-1} \frac{p^i r}{a^4 \cdot a^{3i}}\right) &\leq \coprod_{i=0}^{n-1} \nu\left(\frac{f(a^{i+1}x)}{a^{3(i+1)}} - \frac{f(a^i x)}{a^{3i}}, \frac{p^i r}{a^4 \cdot a^{3i}}\right) \end{aligned}\right\} \quad (4.12)$$

where

$$\prod_{i=0}^{n-1} c_j = c_1 * c_2 * \cdots * c_n \quad \text{and} \quad \coprod_{i=0}^{n-1} d_j = d_1 \diamond d_2 \diamond \cdots \diamond d_n$$

for all $x \in X$ and all $r > 0$. Hence

$$\left.\begin{aligned} \mu\left(\frac{f(a^n x)}{a^{3n}} - f(x), \sum_{i=0}^{n-1} \frac{p^i r}{a^4 \cdot a^{3i}}\right) &\geq \prod_{i=0}^{n-1} \mu'\big(K_\mu(x,0), r\big) = \mu'\big(K_\mu(x,0), r\big) \\ \nu\left(\frac{f(a^n x)}{a^{3n}} - f(x), \sum_{i=0}^{n-1} \frac{p^i r}{a^4 \cdot a^{3i}}\right) &\leq \coprod_{i=0}^{n-1} \nu'(K_\nu(x,0), r) = \nu'(K_\nu(x,0), r) \end{aligned}\right\} \quad (4.13)$$

for all $x \in X$ and all $r > 0$. Replacing x by $a^m x$ in (4.13) and using (4.2), ($IFN4$), we obtain

$$\left.\begin{aligned} \mu\left(\frac{f(a^{n+m} x)}{a^{3(n+m)}} - \frac{f(a^m x)}{a^{3m}}, \sum_{i=0}^{n-1} \frac{p^i r}{a^4 \cdot a^{3(i+m)}}\right) &\geq \mu'\big(K_\mu(a^m x,0), r\big) = \mu'\left(K_\mu(x,0), \frac{r}{p^m}\right) \\ \nu\left(\frac{f(a^{n+m} x)}{a^{3(n+m)}} - \frac{f(a^m x)}{a^{3m}}, \sum_{i=0}^{n-1} \frac{p^i r}{a^4 \cdot a^{3(i+m)}}\right) &\leq \nu'(K_\nu(a^m x,0), r) = \nu'\left(K_\nu(x,0), \frac{r}{p^m}\right) \end{aligned}\right\} \quad (4.14)$$

for all $x \in X$ and all $r > 0$ and all $m, n \geq 0$. Replacing r by $p^m r$ in (4.14), we get

$$\left.\begin{aligned} \mu\left(\frac{f(a^{n+m} x)}{a^{3(n+m)}} - \frac{f(a^m x)}{a^{3m}}, \sum_{i=0}^{n-1} \frac{p^{i+m} r}{a^4 \cdot a^{3(i+m)}}\right) &\geq \mu'\big(K_\mu(x,0), r\big) \\ \nu\left(\frac{f(a^{n+m} x)}{a^{3(n+m)}} - \frac{f(a^m x)}{a^{3m}}, \sum_{i=0}^{n-1} \frac{p^{i+m} r}{a^4 \cdot a^{3(i+m)}}\right) &\leq \nu'(K_\nu(x,0), r) \end{aligned}\right\} \quad (4.15)$$

for all $x \in X$ and all $r > 0$ and all $m, n \geq 0$. It follows from (4.16), that

$$\left.\begin{aligned}\mu\left(\frac{f(a^{n+m}x)}{a^{3(n+m)}}-\frac{f(a^m x)}{a^{3m}},r\right)&\geq\mu'\left(K_\mu(x,0),\frac{r}{\sum_{i=m}^{n-1}\frac{p^i}{a^4\cdot a^{3i}}}\right)\\ v\left(\frac{f(a^{n+m}x)}{a^{3(n+m)}}-\frac{f(a^m x)}{a^{3m}},r\right)&\leq v'\left(K_v(x,0),\frac{r}{\sum_{i=m}^{n-1}\frac{p^i}{a^4\cdot a^{3i}}}\right)\end{aligned}\right\}\tag{4.16}$$

holds for all $x\in X$ and all $r>0$ and all $m,n\geq 0$. Since $0<p<3$ and $\sum_{i=0}^{n}\left(\frac{p}{3}\right)^i<\infty$. The Cauchy criterion for convergence in IFNS shows that $\left\{\frac{f(a^n x)}{a^{3n}}\right\}$ is a Cauchy sequence in (Y,μ,v). Since (Y,μ,v) is a complete IFN-space this sequence converges to some point $C(x)\in Y$. So, one can define the mapping $C:X\to Y$ by $\left\{\frac{f(a^n x)}{a^{3n}}\right\}$

$$\lim_{n\to\infty}\mu\left(\frac{f(a^n x)}{a^{3n}}-C(x),r\right)=1,\ \lim_{n\to\infty}v\left(\frac{f(a^n x)}{a^{3n}}-C(x),r\right)=0$$

for all $x\in X$ and all $r>0$. Hence

$$\frac{f(a^n x)}{a^{3n}}\xrightarrow{IF}C(x),\quad as\quad n\to\infty.$$

Letting $m=0$ in (4.16), we arrive

$$\left.\begin{aligned}\mu\left(\frac{f(a^n x)}{a^{3n}}-f(x),r\right)&\geq\mu'\left(K_\mu(x,0),\frac{r}{\sum_{i=0}^{n-1}\frac{p^i}{a^4\cdot a^{3i}}}\right)\\ v\left(\frac{f(a^n x)}{a^{3n}}-f(x),r\right)&\leq v'\left(K_v(x,0),\frac{r}{\sum_{i=0}^{n-1}\frac{p^i}{a^4\cdot a^{3i}}}\right)\end{aligned}\right\}\tag{4.17}$$

for all $x\in X$ and all $r>0$. Letting n tend to infinity in (4.17), we have

$$\left.\begin{aligned}\mu(C(x)-f(x),r)&\geq\mu'\left(K_\mu(x,0),ar|a^3-p|\right)\\ v(C(x)-f(x),r)&\leq v'(K_v(x,0),ar|a^3-p|)\end{aligned}\right\}\tag{4.18}$$

for all $x\in X$ and all $r>0$. To prove C satisfies (1.5), replacing $(x,\ y)$ by $(a^n x,a^n y)$ in (4.3) respectively, we obtain

$$\left.\begin{aligned}\mu\left(\frac{1}{a^{3n}}Df_{(a,b;k)}(a^n x,a^n y),r\right)&\geq\mu'(K(a^n x,a^n y),a^{3n}r)\\ v\left(\frac{1}{a^{3n}}Df_{(a,b;k)}(a^n x,a^n y),r\right)&\leq v'(K(a^n x,a^n y),a^{3n}r)\end{aligned}\right\}\tag{4.19}$$

for all $x \in X$ and all $r > 0$. Now,

$$\begin{aligned}
&\mu\Big(\frac{a+\sqrt{k}b}{2}C\Big(ax+\sqrt{k}by\Big)+\frac{a-\sqrt{k}b}{2}C\Big(ax-\sqrt{k}by\Big)\\
&\qquad +k(a^2-kb^2)b^2C(y)-k(ab)^2C(x+y)-(a^2-kb^2)a^2C(x),r\Big)\\
&\geq \mu\Big(\frac{a+\sqrt{k}b}{2}C\Big(ax+\sqrt{k}by\Big)-\frac{1}{a^{3n}}\frac{a+\sqrt{k}b}{2}f\Big(a^n\Big(ax+\sqrt{k}by\Big)\Big),\frac{r}{6}\Big)*\\
&\quad \mu\Big(\frac{a-\sqrt{k}b}{2}C\Big(ax-\sqrt{k}by\Big)-\frac{1}{a^{3n}}\frac{a-\sqrt{k}b}{2}f\Big(a^n\Big(ax-\sqrt{k}by\Big)\Big),\frac{r}{6}\Big)*\\
&\quad \mu\Big(k(a^2-kb^2)b^2C(y)-\frac{1}{a^{3n}}k(a^2-kb^2)b^2f(a^ny),\frac{r}{6}\Big)*\\
&\quad \mu\Big(-k(ab)^2C(x+y)+\frac{1}{a^{3n}}k(ab)^2f(a^n(x+y)),\frac{r}{6}\Big)*\\
&\quad \mu\Big(-(a^2-kb^2)a^2C(x)+\frac{1}{a^{3n}}(a^2-kb^2)a^2f(a^nx),\frac{r}{6}\Big)*\\
&\quad \mu\Big(\frac{1}{a^{3n}}\frac{a+\sqrt{k}b}{2}f\Big(a^n\Big(ax+\sqrt{k}by\Big)\Big)+\frac{1}{a^{3n}}\frac{a-\sqrt{k}b}{2}f\Big(a^n\Big(ax-\sqrt{k}by\Big)\Big)\\
&\quad +\frac{1}{a^{3n}}k(a^2-kb^2)b^2f(a^ny)+\frac{1}{a^{3n}}k(ab)^2f(a^n(x+y))+\frac{1}{a^{3n}}(a^2-kb^2)a^2f(a^nx),\frac{r}{6}\Big)
\end{aligned} \tag{4.20}$$

and

$$\begin{aligned}
&\nu\Big(\frac{a+\sqrt{k}b}{2}C\Big(ax+\sqrt{k}by\Big)+\frac{a-\sqrt{k}b}{2}C\Big(ax-\sqrt{k}by\Big)\\
&\qquad +k(a^2-kb^2)b^2C(y)-k(ab)^2C(x+y)-(a^2-kb^2)a^2C(x),r\Big)\\
&\leq \nu\Big(\frac{a+\sqrt{k}b}{2}C\Big(ax+\sqrt{k}by\Big)-\frac{1}{a^{3n}}\frac{a+\sqrt{k}b}{2}f\Big(a^n\Big(ax+\sqrt{k}by\Big)\Big),\frac{r}{6}\Big)\diamond\\
&\quad \nu\Big(\frac{a-\sqrt{k}b}{2}C\Big(ax-\sqrt{k}by\Big)-\frac{1}{a^{3n}}\frac{a-\sqrt{k}b}{2}f\Big(a^n\Big(ax-\sqrt{k}by\Big)\Big),\frac{r}{6}\Big)\diamond\\
&\quad \nu\Big(k(a^2-kb^2)b^2C(y)-\frac{1}{a^{3n}}k(a^2-kb^2)b^2f(a^ny),\frac{r}{6}\Big)\diamond\\
&\quad \nu\Big(-k(ab)^2C(x+y)+\frac{1}{a^{3n}}k(ab)^2f(a^n(x+y)),\frac{r}{6}\Big)\diamond\\
&\quad \nu\Big(-(a^2-kb^2)a^2C(x)+\frac{1}{a^{3n}}(a^2-kb^2)a^2f(a^nx),\frac{r}{6}\Big)\diamond\\
&\quad \nu\Big(\frac{1}{a^{3n}}\frac{a+\sqrt{k}b}{2}f\Big(a^n\Big(ax+\sqrt{k}by\Big)\Big)+\frac{1}{a^{3n}}\frac{a-\sqrt{k}b}{2}f\Big(a^n\Big(ax-\sqrt{k}by\Big)\Big)\\
&\quad +\frac{1}{a^{3n}}k(a^2-kb^2)b^2f(a^ny)+\frac{1}{a^{3n}}k(ab)^2f(a^n(x+y))\\
&\quad +\frac{1}{a^{3n}}(a^2-kb^2)a^2f(a^nx),\frac{r}{6}\Big)
\end{aligned} \tag{4.21}$$

for all $x, y \in X$ and all $r > 0$. Since

$$\left.\begin{aligned}\lim_{n\to\infty}\mu\left(\frac{1}{a^{3n}}Df_{(a,b;k)}(a^nx,a^ny),\frac{r}{6}\right)&=1\\ \lim_{n\to\infty}\nu\left(\frac{1}{a^{3n}}Df_{(a,b;k)}(a^nx,a^ny),\frac{r}{6}\right)&=0\end{aligned}\right\}\tag{4.22}$$

for all $x \in X$ and all $r > 0$. Letting $n \to \infty$ in (4.20), (4.21) and using (4.22), we observe that C fulfills (1.5). Therefore C is a cubic mapping.

In order to prove $C(x)$ is unique, let $C'(x)$ be another cubic functional equation satisfying (1.5) and (4.4). Hence,

$$\begin{aligned}\mu(C(x)-C'(x),r)&=\mu\left(\frac{C(a^nx)}{a^{3n}}-\frac{C'(a^nx)}{a^{3n}},r\right)\\ &\geq\mu\left(C(a^nx)-f(a^nx),\frac{r.a^{3n}}{2}\right)*\mu\left(f(a^nx)-C'(a^nx),\frac{r.a^{3n}}{2}\right)\\ &\geq\mu'\left(K_\mu(a^nx,0),\frac{a^4ra^{3n}|a^3-p|}{2}\right)\geq\mu'\left(K_\mu(x,0),\frac{a^4ra^{3n}|a^3-p|}{2\cdot p^n}\right)\\ \nu(C(x)-C'(x),r)&=\nu\left(\frac{C(a^nx)}{a^{3n}}-\frac{C'(a^nx)}{a^{3n}},r\right)\\ &\leq\nu\left(C(a^nx)-f(a^nx),\frac{r.a^{3n}}{2}\right)\diamond\nu\left(f(a^nx)-C'(a^nx),\frac{r.a^{3n}}{2}\right)\\ &\leq\nu'\left(K_\nu(a^nx,0),\frac{a^4ra^{3n}|a^3-p|}{2}\right)\leq\nu'\left(K_\nu(x,0),\frac{a^4ra^{3n}|a^3-p|}{2\cdot p^n}\right)\end{aligned}$$

for all $x \in X$ and all $r > 0$. Since

$$\lim_{n\to\infty}\frac{a^4ra^{3n}|a^3-p|}{2p^n}=\infty,$$

we obtain

$$\left.\begin{aligned}\lim_{n\to\infty}\mu'\left(K_\mu(x,0),\frac{a^4ra^{3n}|a^3-p|}{2\cdot p^n}\right)&=1\\ \lim_{n\to\infty}\nu'\left(K_\nu(x,0),\frac{a^4ra^{3n}|a^3-p|}{2\cdot p^n}\right)&=0\end{aligned}\right\}$$

for all $x \in X$ and all $r > 0$. Thus

$$\left.\begin{aligned}\mu(C(x)-C'(x),r)&=1\\ \nu(C(x)-C'(x),r)&=0\end{aligned}\right\}$$

for all $x \in X$ and all $r > 0$. Hence $C(x) = C'(x)$. Therefore $C(x)$ is unique.

For $\tau = -1$, we can prove the similar stability result. This completes the proof of the theorem. □

The following corollary is an immediate consequence of Theorem 4.8, regarding the stability of (1.5)

Corollary 4.9 *Suppose that a function $Df_{(a,b;k)} : X \to Y$ satisfies the double inequality*

$$\left.\begin{aligned} \mu\big(Df_{(a,b;k)}(x,y), r\big) &\geq \begin{cases} \mu(\lambda, r), \\ \mu(\lambda(||x||^s + ||y||^s), r), \\ \mu\Big(\lambda\Big\{||x||^s||y||^s + \Big(||x||^{2s} + ||y||^{2s}\Big)\Big\}, r\Big), \end{cases} \\ v\big(Df_{(a,b;k)}(x,y), r\big) &\leq \begin{cases} v(\lambda, r), \\ v(\lambda(||x||^s + ||y||^s), r), \\ v\Big(\lambda\Big\{||x||^s||y||^s + \Big(||x||^{2s} + ||y||^{2s}\Big)\Big\}, r\Big), \end{cases} \end{aligned}\right\} \tag{4.23}$$

for all $x, y \in X$ and all $r > 0$, where λ, s are constants with $\lambda > 0$. Then there exists a unique cubic mapping $C : X \to Y$ such that

$$\left.\begin{aligned} \mu(f(x) - C(x), r) &\geq \begin{cases} \mu(\lambda, a|a^3 - 1|r), \\ \mu(\lambda||x||^s, a|a^3 - a^s|r), \\ \mu\Big(\lambda||x||^{2s}, a|a^3 - a^{2s}|r\Big), \end{cases} \\ v(f(x) - C(x), r) &\leq \begin{cases} v(\lambda, a|a^3 - 1|r), \\ v(\lambda||x||^s, a|a^3 - a^s|r), \\ v\Big(\lambda||x||^{2s}, a|a^3 - a^{2s}|r\Big), \end{cases} \end{aligned}\right\} \tag{4.24}$$

for all $x \in X$ and all $r > 0$.

4.2 IFNS: fixed point method

Theorem 4.10 *Let $Df_{(a,b;k)} : X \to Y$ be a mapping for which there exists a function $K : X \times X \to Z$ with the double condition*

$$\left.\begin{aligned} \lim_{n\to\infty} \mu'\big(K\big(\chi_i^n x, \chi_i^n y\big), \chi^{3n} r\big) &= 1 \\ \lim_{n\to\infty} v'\big(K\big(\chi_i^n x, \chi_i^n y\big), \chi^{3n} r\big) &= 0 \end{aligned}\right\} \tag{4.25}$$

for all $x, y \in X$ and all $r > 0$ where

$$\chi_i = \begin{cases} a & if \quad i = 0 \\ \dfrac{1}{a} & if \quad i = 1 \end{cases} \tag{4.26}$$

and satisfying the double functional inequality

$$\left.\begin{aligned} \mu\big(Df_{(a,b;k)}(x,y), r\big) &\geq \mu'(K(x,y), r) \\ v\big(Df_{(a,b;k)}(x,y), r\big) &\leq v'(K(x,y), r) \end{aligned}\right\} \tag{4.27}$$

for all $x, y \in X$ and all $r > 0$. If there exists $L = L(i)$ such that the function

$$x \to \rho(x) = \frac{1}{a} K\Big(\frac{x}{a}, 0\Big),$$

has the property

$$\left.\begin{aligned} \mu\left(L\frac{\rho(\chi_i x)}{\chi_i^3}, r\right) &= \mu(\rho(x), r) \\ \nu\left(L\frac{\rho(\chi_i x)}{\chi_i^3}, r\right) &= \nu(\rho(x), r) \end{aligned}\right\} \tag{4.28}$$

for all $x \in X$ *and all* $r > 0$, *then there exists a unique cubic function* $C : X \to Y$ *satisfying the functional* Eq. (1.5) *and*

$$\left.\begin{aligned} \mu(f(x) - C(x), r) &\geq \mu'\left(\rho(x), \frac{L^{1-i}}{1-L} r\right) \\ \nu(f(x) - C(x), r) &\leq \nu'\left(\rho(x), \frac{L^{1-i}}{1-L} r\right) \end{aligned}\right\} \tag{4.29}$$

for all $x \in X$ *and all* $r > 0$.

Proof Consider the set

$$\Lambda = \{h/h : X \to Y, h(0) = 0\}$$

and introduce the generalized metric on Λ,

$$d(h,f) = \inf\left\{L \in (0,\infty) : \left\{\begin{array}{l} \mu(h(x) - f(x), r) \geq \mu'(\rho(x), Lr), x \in X \\ \nu(h(x) - f(x), r) \leq \nu'(\rho(x), Lr), x \in X \end{array}\right\}\right\} \tag{4.30}$$

It is easy to see that (4.30) is complete with respect to the defined metric. Define $J : \Lambda \to \Lambda$ by

$$Jh(x) = \frac{1}{\chi_i^3} h(\chi_i x),$$

for all $x \in \mathcal{X}$. Now, from (4.30) and $h,f \in \Lambda$

$$\left.\begin{array}{l} \left\{\begin{array}{l} \inf\{1 \in (0,\infty) : \mu(h(x) - f(x), r) \geq \mu'(\rho(x), r), x \in X\} \\ \inf\{1 \in (0,\infty) : \mu(\frac{1}{\chi_i^3} h(\chi_i x) - \frac{1}{\chi_i^3} f(\chi_i x), r) \geq \mu'(\rho(\chi_i x), \chi_i^3 r), x \in X\} \\ \inf\{L \in (0,\infty) : \mu(\frac{1}{\chi_i^3} h(\chi_i x) - \frac{1}{\chi_i^3} f(\chi_i x), r) \geq \mu'(\rho(x), Lr), x \in X\} \\ \inf\{L \in (0,\infty) : \mu(Jh(x) - Jf(x), r) \geq \mu'(\rho(x), Lr), x \in X\} \end{array}\right. \\ \left\{\begin{array}{l} \inf\{1 \in (0,\infty) : \nu(h(x) - f(x), r) \leq \nu'(\rho(x), r), x \in X\} \\ \inf\{1 \in (0,\infty) : \nu(\frac{1}{\chi_i^3} h(\chi_i x) - \frac{1}{\chi_i^3} f(\chi_i x), r) \leq \nu'(\rho(\chi_i x), \chi_i^3 r), x \in X\} \\ \inf\{L \in (0,\infty) : \nu(\frac{1}{\chi_i^3} h(\chi_i x) - \frac{1}{\chi_i^3} f(\chi_i x), r) \leq \nu'(\rho(x), Lr), x \in X\} \\ \inf\{L \in (0,\infty) : \nu(Jh(x) - Jf(x), r) \leq \nu'(\rho(x), Lr), x \in X\} \end{array}\right. \end{array}\right\}$$

This implies J is a strictly contractive mapping on Λ with Lipschitz constant L.

It follows from (4.30), (4.5), we reach

$$\inf\left\{1 \in (0,\infty) : \left\{ \begin{array}{l} \mu(af(ax) - a^4 f(x), r) \geq \mu'(K(x,0), r) \\ \nu(af(ax) - a^4 f(x), r) \leq \nu'(K(x,0), r) \end{array} \right\}\right\} \tag{4.31}$$

for all $x \in X$ and all $r > 0$. Now, from (4.31) and (4.28) for the case $i = 0$, we reach

$$\left.\begin{array}{l} \left\{\begin{array}{l} \inf\{1 \in (0,\infty) : \mu(af(ax) - a^4 f(x), r) \geq \mu'(K(x,0), r)\} \\ \inf\left\{1 \in (0,\infty) : \mu\left(\frac{f(ax)}{a^3} - f(x), r\right) \geq \mu'(K(x,0), a^4 r)\right\} \\ \inf\{L \in (0,\infty) : \mu(Jf(x) - f(x), r) \geq \mu'(\rho(x), Lr)\} \\ \inf\{L^1 \in (0,\infty) : \mu(Jf(x) - f(x), r) \geq \mu'(\rho(x), Lr)\} \\ \inf\{L^{1-0} \in (0,\infty) : \mu(Jf(x) - f(x), r) \geq \mu'(\rho(x), Lr)\} \end{array}\right. \\ \left\{\begin{array}{l} \inf\{1 \in (0,\infty) : \nu(af(ax) - a^4 f(x), r) \leq \nu'(K(x,0), r)\} \\ \inf\left\{1 \in (0,\infty) : \nu\left(\frac{f(ax)}{a^3} - f(x), r\right) \leq \nu'(K(x,0), a^4 r)\right\} \\ \inf\{L \in (0,\infty) : \nu(Jf(x) - f(x), r) \leq \nu'(\rho(x), Lr)\} \\ \inf\{L^1 \in (0,\infty) : \nu(Jf(x) - f(x), r) \leq \nu'(\rho(x), Lr)\} \\ \inf\{L^{1-0} \in (0,\infty) : \nu(Jf(x) - f(x), r) \leq \nu'(\rho(x), Lr)\} \end{array}\right. \end{array}\right\} \tag{4.32}$$

for all $x \in X$ and all $r > 0$. Again replacing x by $\frac{x}{a}$ in (4.33) and (4.28) for the case $i = 1$, we get

$$\left.\begin{array}{l} \left\{\begin{array}{l} \inf\{1 \in (0,\infty) : \mu(f(ax) - a^3 f(x), r) \geq \mu'(K(x,0), ar)\} \\ \inf\{1 \in (0,\infty) : \mu(f(x) - Jf(x), r) \geq \mu'(\rho(x), r)\} \\ \inf\{L^0 \in (0,\infty) : \mu(f(x) - Jf(x), r) \geq \mu'(\rho(x), r)\} \\ \inf\{L^{1-1} \in (0,\infty) : \mu(f(x) - Jf(x), r) \geq \mu'(\rho(x), r)\} \end{array}\right. \\ \left\{\begin{array}{l} \inf\{1 \in (0,\infty) : \nu(f(ax) - a^3 f(x), r) \leq \nu'(K(x,0), ar)\} \\ \inf\{1 \in (0,\infty) : \nu(f(x) - Jf(x), r) \leq \nu'(\rho(x), r)\} \\ \inf\{L^0 \in (0,\infty) : \nu(f(x) - Jf(x), r) \leq \nu'(\rho(x), r)\} \\ \inf\{L^{1-1} \in (0,\infty) : \nu(f(x) - Jf(x), r) \leq \nu'(\rho(x), r)\} \end{array}\right. \end{array}\right\} \tag{4.33}$$

Thus, from (4.31) and (4.33), we arrive

$$\inf\left\{L^{1-i} \in (0,\infty) : \left\{ \begin{array}{l} \mu(h(x) - f(x), r) \geq \mu'(\rho(x), L^{1-i} r), x \in X \\ \nu(h(x) - f(x), r) \leq \nu'(\rho(x), L^{1-i} r), x \in X \end{array} \right\}\right\} \tag{4.34}$$

Hence property (FP1) holds. The rest of the proof is similar to that of Theorem 3.3. □

The following corollary is an immediate consequence of Theorem 4.10, regarding the stability of (1.5).

Corollary 4.11 *Suppose that a function* $Df_{(a,b;k)} : X \to Y$ *satisfies the double inequality*

$$\left.\begin{array}{l} \mu\left(Df_{(a,b;k)}(x,y),r\right) \geq \begin{cases} \mu(\lambda,r), \\ \mu(\lambda(||x||^s+||y||^s),r), \\ \mu\left(\lambda\left\{||x||^s||y||^s+\left(||x||^{2s}+||y||^{2s}\right)\right\},r\right), \end{cases} \\ v\left(Df_{(a,b;k)}(x,y),r\right) \leq \begin{cases} v(\lambda,r), \\ v(\lambda(||x||^s+||y||^s),r), \\ v\left(\lambda\left\{||x||^s||y||^s+\left(||x||^{2s}+||y||^{2s}\right)\right\},r\right), \end{cases} \end{array}\right\} \quad (4.35)$$

for all $x, y \in X$ *and all* $r > 0$, *where* λ, s *are constants with* $\lambda > 0$. *Then there exists a unique cubic mapping* $C : X \to Y$ *such that the double inequality*

$$\left.\begin{array}{l} \mu(f(x)-C(x),r) \geq \begin{cases} \mu'(\lambda, a|a^3-1|r), \\ \mu'(\lambda||x||^s, a|a^3-a^s|r), \\ \mu'\left(\lambda||x||^{2s}, a|a^3-a^{2s}|r\right), \end{cases} \\ v(f(x)-C(x),r) \leq \begin{cases} v'\left(\lambda, a|a^3-1|r\right), \\ v'\left(\lambda||x||^s, a|a^3-a^s|r\right), \\ v'\left(\lambda||x||^{2s}, a|a^3-a^{2s}|r\right), \end{cases} \end{array}\right\} \quad (4.36)$$

holds for all $x \in X$ *and all* $r > 0$.

Proof Set

$$\mu'\left(K(\chi_i^n x, \chi_i^n y), \chi_i^{3n} r\right) = \begin{cases} \mu'\left(\lambda, \chi_i^{3n} r\right), \\ \mu'\left(\lambda(||x||^s+||y||^s), \chi_i^{(3-s)n} r\right), \\ \mu'(\lambda\{||x||^s||y||^s \\ \quad +\left(||x||^{2s}+||y||^{2s}\right)\}, \chi_i^{(3-2s)n} r\right), \end{cases} = \begin{cases} \to 1 \text{ as k} \to \infty \\ \to 1 \text{ as k} \to \infty \\ \to 1 \text{ as k} \to \infty \end{cases}$$

$$v'\left(K(\chi_i^n x, \chi_i^n y), \chi_i^{3n} r\right) = \begin{cases} v'\left(\lambda, \chi_i^{3n} r\right), \\ v'\left(\lambda(||x||^s+||y||^s), \chi_i^{(3-s)n} r\right), \\ v'(\lambda\{||x||^s||y||^s \\ \quad +\left(||x||^{2s}+||y||^{2s}\right)\}, \chi_i^{(3-2s)n} r\right), \end{cases} = \begin{cases} \to 0 \text{ as k} \to \infty \\ \to 0 \text{ as k} \to \infty \\ \to 0 \text{ as k} \to \infty \end{cases}$$

for all $x \in X$ and all $r > 0$. Thus, the relation (4.25) holds. It follows from (4.28), (4.29) and (4.35)

$$\left.\begin{array}{l} \mu'\left(\frac{1}{a}K\left(\frac{x}{a},0\right),r\right) = \begin{cases} \mu'(\lambda, at) \\ \mu'(\lambda||x||^s, a^{1-s}r) \\ \mu'\left(\lambda||x||^{2s}, a^{1-2s}r\right) \end{cases} \\ \nu'\left(\frac{1}{a}K\left(\frac{x}{a},0\right),r\right) = \begin{cases} \nu'(\lambda, at) \\ \nu'(\lambda||x||^s, a^{1-s}r) \\ \nu'\left(\lambda||x||^{2s}, a^{1-2s}r\right) \end{cases} \end{array}\right\}$$

for all $x, y \in X$ and all $r > 0$. Also from (4.28), we have

$$\left.\begin{array}{l} \mu'\left(\frac{\rho(\chi_i x)}{\chi_i^3},r\right) = \begin{cases} \mu'(\lambda, {\chi_i}^3 at) \\ \mu'(\lambda||x||^s, {\chi_i}^{3-s}at) \\ \mu'\left(\lambda||x||^{2s}, {\chi_i}^{3-2s}at\right) \end{cases} \\ \nu'\left(\frac{\rho(\chi_i x)}{\chi_i^3},r\right) = \begin{cases} \nu'(\lambda, {\chi_i}^3 at) \\ \nu'(\lambda||x||^s, {\chi_i}^{3-s}at) \\ \nu'\left(\lambda||x||^{2s}, {\chi_i}^{3-2s}at\right) \end{cases} \end{array}\right\}$$

for all $x \in X$ and all $r > 0$. Hence, the inequality (4.29) is true for

	L	$\chi_i^{-3},\ i = 0$	L	$\chi_i^{-3},\ i = 1$
1.	a^{-3}	0	a^3	0
2.	a^{s-3}	$s<3$	a^{s+3}	$s > 3$
3.	a^{2s-3}	$2s<3$	a^{2s+3}	$2s > 3$

Now, for condition 1. and $i = 0$, we have

$$\left.\begin{array}{l} \mu(f(x) - C(x), r) \geq \mu'\left(\rho(x), \frac{a^{-3}}{1 - a^{-3}}r\right) = \mu'(\lambda, a|a^3 - 1|r) \\ \nu(f(x) - C(x), r) \leq \nu'\left(\rho(x), \frac{a^{-3}}{1 - a^{-3}}r\right) = \nu'(\lambda, a|a^3 - 1|r) \end{array}\right\}$$

for all $x \in X$ and all $r > 0$. Also, for condition 1. and $i = 1$, we get

$$\left.\begin{array}{l} \mu(f(x) - C(x), r) \geq \mu'\left(\rho(x), \frac{(a^3)^{1-1}}{1 - a^3}r\right) = \mu'(\lambda, a|1 - a^3|r) \\ \nu(f(x) - C(x), r) \leq \nu'\left(\rho(x), \frac{(a^3)^{1-1}}{1 - a^3}r\right) = \nu'(\lambda, a|1 - a^3|r) \end{array}\right\}$$

for all $x \in X$ and all $r > 0$. Again, for condition 2. and $i = 0$, we obtain

$$\left.\begin{aligned} \mu(f(x) - C(x), r) &\geq \mu'\left(\rho(x), \frac{(a^{s-3})^{1-0}}{1 - a^{s-3}} r\right) = \mu'(\lambda||x||^s, a|a^3 - a^s|r) \\ v(f(x) - C(x), r) &\leq v'\left(\rho(x), \frac{(a^{s-3})^{1-0}}{1 - a^{s-3}} r\right) = v'(\lambda||x||^s, a|a^3 - a^s|r) \end{aligned}\right\}$$

for all $x \in X$ and all $r > 0$. Also, for condition 2. and $i = 1$, we arrive

$$\left.\begin{aligned} \mu(f(x) - C(x), r) &\geq \mu'\left(\rho(x), \frac{(a^{3-s})^{1-1}}{1 - (a^{3-s})} r\right) = \mu'(\lambda||x||^s, a|a^s - a^3|r) \\ v(f(x) - C(x), r) &\leq v'\left(\rho(x), \frac{(a^{3-s})^{1-1}}{1 - (a^{3-s})} r\right) = v'(\lambda||x||^s, a|a^s - a^3|r) \end{aligned}\right\}$$

for all $x \in X$ and all $r > 0$. The rest of the proof is similar to that of previous cases. This finishes the proof. □

Compliance with ethical standards

Conflict of interest No conflicts of interest to declare.

References

1. Aoki, T. 1950. On the stability of the linear transformation in Banach spaces. *Journal of the Mathematical Society of Japan* 2: 64–66.
2. Gavruta, P. 1994. A generalization of the Hyers–Ulam–Rassias stability of approximately additive mappings. *Journal of Mathematical Analysis and Applications* 184: 431–436.
3. Hyers, D.H. 1941. On the stability of the linear functional equation. *Proceedings of the National Academy of Sciences USA* 27: 222–224.
4. Jun, K.W., and H.M. Kim. 2002. The generalized Hyers–Ulam–Rassias stability of a cubic functional equation. *Journal of Mathematical Analysis and Applications* 274: 867–878.
5. Rassias, M.J., M. Arunkumar, and E. Sathya. 2015. Stability of a k- cubic functional equation in Quasi-beta normed spaces: Direct and Fixed point methods. *British Journal of Mathematics and Computer Science* 8 (5): 346–360.
6. Jung, Y.S., and I.S. Chang. 2005. The stability of a cubic type functional equation with the fixed point alternative. *Journal of Mathematical Analysis and Applications* 306: 752–760.
7. Margolis, B., and J.B. Diaz. 1968. A fixed point theorem of the alternative for contractions on a generalized complete metric space. *Bulletin of the American Mathematical Society* 126: 305–309.
8. Rassias, J.M. 1982. On approximately of approximately linear mappings by linear mappings. *Journal of Functional Analysis* 46: 126–130.
9. Rassias, J.M. 2000. Solution of the Ulam problem for cubic mappings. *Analele Universitatii din Timisoara. Seria Matematica-Informatica* 38: 121–132.
10. Rassias, ThM. 1978. On the stability of the linear mapping in Banach spaces. *Proceedings of the American Mathematical Society* 72: 297–300.
11. Ravi, K., M. Arunkumar, and J.M. Rassias. 2008. On the Ulam stability for the orthogonally general Euler–Lagrange type functional equation. *International Journal of Mathematical Sciences* 3: 36–47.
12. Saadati, R., and J.H. Park. 2006. On the intuitionistic fuzzy topological spaces. *Chaos, Solitons and Fractals* 27: 331–344.
13. Ulam, S.M. 1964. *Problems in Modern Mathematics*. Wiley, New York: Science Editions.