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ORIGINAL RESEARCH PAPER

Some Riemann–Liouville fractional integral inequalities for convex functions

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Abstract We are pleased to investigate some Riemann–Liouville fractional integral inequalities in a very simple and novel way. By using convexity of a function f and a simple inequality over the domain of f we establish some interesting results.

Keywords Convex function · Riemann–Liouville fractional integral

Mathematics Subject Classification 26A51 · 26A33 · 26D15

1 Introduction

The study on the fractional calculus continued with contributions from Fourier, Abel, Lacroix, Leibniz, Grunwald and Letnikov for detail (see, [2, 4]). Riemann–Liouville fractional integral operator is the first formulation of an integral operator of non-integral order.

Definition 1 Let $f \in L_1[a,b]$. Then the Riemann–Liouville fractional integrals of f of order $\alpha > 0$ with $a \geq 0$ are defined by

$$I_{a^+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a$$

and

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$$I_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \quad x<b.$$

In fact these formulations of fractional integral operators have been established due to Letnikov [5], Sonin [6] and then by Laurent [3].

Since the inequalities always have been proved worthy in establishing the mathematical models and their solutions in almost all branches of applied sciences. Especially the convexity takes very important role in the optimization theory. The aim of this paper is to introduce some fractional inequalities for the Riemann–Liouville fractional integral operators via the convexity property of the functions.

2 Main results

First we give the following estimate of the sum of left and right handed Riemann–Liouville fractional integrals.

Theorem 1 *Let $f : I \longrightarrow \mathbb{R}$ be a positive convex function. Then for $a, b \in I; a<b$ and $\alpha, \beta \geq 1$, the following inequality for Riemann–Liouville fractional Integrals holds*

$$\begin{aligned} &\Gamma(\alpha) I_{a+}^{\alpha} f(x) + \Gamma(\beta) I_{b-}^{\beta} f(x) \\ &\quad \leq \frac{(x-a)^{\alpha} f(a) + (b-x)^{\beta} f(b)}{2} + f(x)\left[\frac{(x-a)^{\alpha} + (b-x)^{\beta}}{2}\right]. \end{aligned} \tag{1}$$

Proof Let us consider the function f on the interval $[a, x], x \in [a, b]$. Then for $t \in [a, x]$ and $\alpha \geq 1$ the following inequality holds

$$(x-t)^{\alpha-1} \leq (x-a)^{\alpha-1}. \tag{2}$$

Since f is convex therefore for $t \in [a, x]$ we have

$$f(t) \leq \frac{x-t}{x-a} f(a) + \frac{t-a}{x-a} f(x). \tag{3}$$

Multiplying inequalities (2) and (3), then integrating with respect to t over $[a, x]$ we have

$$\int_a^x (x-t)^{\alpha-1} f(t)dt \leq \frac{(x-a)^{\alpha-1}}{x-a}\left[f(a)\int_a^x (x-t)dt + f(x)\int_a^x (t-a)dt\right]$$

$$\Gamma(\alpha)I_{a^+}^{\alpha}f(x) \leq \frac{(x-a)^{\alpha}}{2}[f(a)+f(x)]. \tag{4}$$

Now we consider the function f on the interval $[x,b], x \in [a,b]$. Then for $t \in [x,b]$ and $\beta \geq 1$ the following inequality holds

$$(t-x)^{\beta-1} \leq (b-x)^{\beta-1}. \tag{5}$$

Since f is convex on $[a, b]$, therefore for $t \in [x,b]$ we have

$$f(t) \leq \frac{t-x}{b-x}f(b) + \frac{b-t}{b-x}f(x). \tag{6}$$

Multiplying inequalities (5) and (6), then integrating with respect to t over $[x, b]$ we have

$$\int_x^b (t-x)^{\beta-1}f(t)dt \leq \frac{(b-x)^{\beta-1}}{b-x}\left[f(b)\int_x^b (t-x)dt + f(x)\int_x^b (b-t)dt\right]$$

$$\Gamma(\beta)I_{b^-}^{\beta}f(x) \leq \frac{(b-x)^{\beta}}{2}[f(b)+f(x)]. \tag{7}$$

Adding (4) and (7) we get the required inequality in (1). □

It is nice to see that the following implications hold.

Corollary 1 *By setting $\alpha = \beta$ in (1) we get the following fractional integral inequality*

$$\begin{aligned}&\Gamma(\alpha)\left(I_{a^+}^{\alpha}f(x) + I_{b^-}^{\alpha}f(x)\right)\\ &\quad \leq \frac{(x-a)^{\alpha}f(a) + (b-x)^{\alpha}f(b)}{2} + f(x)\left[\frac{(x-a)^{\alpha} + (b-x)^{\alpha}}{2}\right].\end{aligned} \tag{8}$$

Corollary 2 *By setting $\alpha = \beta = 1$ and taking $x = b$ or $x = a$ in (1) we get*

$$\frac{1}{b-a}\int_a^b f(t)dt \leq \frac{f(a)+f(b)}{2}. \tag{9}$$

Corollary 3 *By setting $\alpha = \beta = 1$ and taking $x = \frac{a+b}{2}$ in (1) we get*

$$0 \leq \frac{1}{b-a}\int_a^b f(t)dt - f\left(\frac{a+b}{2}\right) \leq \frac{f(a)+f(b)}{2}. \tag{10}$$

Remark 1 It is interesting to see that if in Theorem 1 we consider f is concave function and $0 < \alpha, \beta < 1$, then reverse of inequalities (1) holds.

In the following result we investigate the fractional integral inequality that appears as the generalization and refinement of a well known inequality for functions whose derivative in absolute value is convex.

Theorem 2 *Let* $f : I \longrightarrow \mathbb{R}$ *be a differentiable function. If* $|f'|$ is *convex, then for* $a, b \in I$, $a<b$ *and* $\alpha, \beta > 0$ *the following inequality for the Riemann–Liouville fractional integrals holds*

$$\begin{aligned}&|\Gamma(\alpha+1)I^{\alpha}_{a^+}f(x) + \Gamma(\beta+1)I^{\beta}_{b^-}f(x) - ((x-a)^{\alpha}f(a) + (b-x)^{\beta}f(b))|\\&\quad\leq \frac{(x-a)^{\alpha+1}|f'(a)| + (b-x)^{\beta+1}|f'(b)|}{2} + |f'(x)|((x-a)^{\alpha+1} + (b-x)^{\beta+1}).\end{aligned} \tag{11}$$

Proof Since $|f'|$ is convex, therefore for $t \in [a, x]$ we have

$$|f'(t)| \leq \frac{x-t}{x-a}|f'(a)| + \frac{t-a}{x-a}|f'(x)|$$

from which we can write

$$-\left(\frac{x-t}{x-a}|f'(a)| + \frac{t-a}{x-a}|f'(x)|\right) \leq f'(t) \leq \frac{x-t}{x-a}|f'(a)| + \frac{t-a}{x-a}|f'(x)| \tag{12}$$

We consider the right hand side of inequality (12)

$$f'(t) \leq \frac{x-t}{x-a}|f'(a)| + \frac{t-a}{x-a}|f'(x)|. \tag{13}$$

Now for $\alpha > 0$ we have the following inequality

$$(x-t)^{\alpha} \leq (x-a)^{\alpha}, \quad t \in [a, x]. \tag{14}$$

The product of last two inequalities give

$$(x-t)^{\alpha}f'(t) \leq (x-a)^{\alpha-1}((x-t)|f'(a)| + (t-a)|f'(x)|).$$

Integrating with respect to t over $[a, x]$ we have

$$\begin{aligned}&\int_a^x (x-t)^{\alpha}f'(t)dt\\&\quad\leq (x-a)^{\alpha-1}\left[|f'(a)|\int_a^x (x-t)dt + |f'(x)|\int_a^x (t-a)dt\right]\\&\quad= (x-a)^{\alpha+1}\left[\frac{|f'(a)| + |f'(x)|}{2}\right],\end{aligned} \tag{15}$$

and

$$\int_a^x (x-t)^\alpha f'(t)dt = f(t)(x-t)^\alpha|_a^x + \alpha\int_a^x (x-t)^{\alpha-1}f(t)dt$$
$$= -f(a)(x-a)^\alpha + \Gamma(\alpha+1)I_{a^+}^\alpha f(x).$$

Therefore (15) takes the form

$$\Gamma(\alpha+1)I_{a^+}^\alpha f(x) - f(a)(x-a)^\alpha \leq (x-a)^{\alpha+1}\left[\frac{|f'(a)|+|f'(x)|}{2}\right]. \qquad (16)$$

If we consider from (12) the left hand side inequality and proceeding as we did for the right side inequality we get

$$f(a)(x-a)^\alpha - \Gamma(\alpha+1)I_{a^+}^\alpha f(x) \leq (x-a)^{\alpha+1}\left[\frac{|f'(a)|+|f'(x)|}{2}\right]. \qquad (17)$$

From (16) and (17) we get

$$|\Gamma(\alpha+1)I_{a^+}^\alpha f(x) - f(a)(x-a)^\alpha| \leq (x-a)^{\alpha+1}\left[\frac{|f'(a)|+|f'(x)|}{2}\right]. \qquad (18)$$

On the other hand for $t \in [x,b]$ using convexity of $|f'|$ we have

$$|f'(t)| \leq \frac{t-x}{b-x}|f'(b)| + \frac{b-t}{b-x}|f'(x)|. \qquad (19)$$

Also for $t \in [x,b]$ and $\beta > 0$ we have

$$(t-x)^\beta \leq (b-x)^\beta. \qquad (20)$$

By adopting the same treatment as we have done for (12) and (14) one can obtain from (19) and (20) the following inequality

$$\left|\Gamma(\beta+1)I_{b^-}^\beta f(a) - f(b)(b-x)^\beta\right| \leq (b-x)^{\beta+1}\left[\frac{|f'(b)|+|f'(x)|}{2}\right]. \qquad (21)$$

By combining the inequalities (18) and (21) via triangular inequality we get the required inequality. □

It is interesting to see the following inequalities as special cases.

Corollary 4 *By setting* $\alpha = \beta$ in (11) *we get the following fractional integral inequality*

$$|\Gamma(\alpha+1)[I_{a^+}^\alpha f(x) + I_{b^-}^\alpha f(x)] - ((x-a)^\alpha f(a) + (b-x)^\alpha f(b))|$$
$$\leq \frac{(x-a)^{\alpha+1}|f'(a)| + (b-x)^{\alpha+1}|f'(b)|}{2} + |f'(x)|[(x-a)^{\alpha+1} + (b-x)^{\alpha+1}].$$

Corollary 5 *By setting* $\alpha = \beta = 1$ and $x = \frac{a+b}{2}$ in (11) *we get the following inequality*

$$\left|\frac{1}{b-a}\int_a^b f(t)dt - \frac{f(a)+f(b)}{2}\right| \leq \frac{b-a}{8}\left[|f'(a)| + |f'(b)| + 2f'\left(\frac{a+b}{2}\right)\right]. \tag{22}$$

Remark 2 If f' passes through $x = \frac{a+b}{2}$, then from (22) we get [1], Theorem 2.2]. If $f'(x) \leq 0$, then (22) gives the refinement of [1], Theorem 2.2].

Before going to the next theorem we observe the following result.

Lemma 1 *Let $f : [a,b] \longrightarrow \mathbb{R}$, be a convex function. If f is symmetric about $\frac{a+b}{2}$, then the following inequality holds*

$$f\left(\frac{a+b}{2}\right) \leq f(x) \quad x \in [a,b]. \tag{23}$$

Proof We have

$$\frac{a+b}{2} = \frac{1}{2}\left(\frac{x-a}{b-a}b + \frac{b-x}{b-a}x\right) + \frac{1}{2}\left(\frac{x-a}{b-a}a + \frac{b-x}{b-a}b\right). \tag{24}$$

Since f is convex, therefore we have

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \frac{1}{2}\left[f\left(\frac{x-a}{b-a}b + \frac{b-x}{b-a}x\right) + f\left(\frac{x-a}{b-a}a + \frac{b-x}{b-a}b\right)\right] \\ &= \frac{1}{2}(f(x) + f(a+b-x)). \end{aligned} \tag{25}$$

Also f is symmetric about $\frac{a+b}{2}$, therefore we have $f(a+b-x) = f(x)$ and inequality in (23) holds. □

Theorem 3 *Let $f : I \longrightarrow \mathbb{R}$ be a positive convex function. If f is symmetric about $\frac{a+b}{2}$, then the following inequality for Riemann–Liouville fractional integrals holds*

$$\begin{aligned} &\frac{1}{2}\left(\frac{1}{\alpha+1} + \frac{1}{\beta+1}\right)f\left(\frac{a+b}{2}\right) \\ &\leq \frac{\Gamma(\beta+1)I_{b^-}^{\beta+1}f(a)}{2(b-a)^{\beta+1}} + \frac{\Gamma(\alpha+1)I_{a^+}^{\alpha+1}f(b)}{2(b-a)^{\alpha+1}} \\ &\leq \frac{f(a)+f(b)}{2}. \end{aligned} \tag{26}$$

Proof For $x \in [a,b]$ we have

$$(x-a)^\beta \leq (b-a)^\beta, \quad \beta > 0. \tag{27}$$

Also f is convex function we have

$$f(x) \leq \frac{x-a}{b-a} f(b) + \frac{b-x}{b-a} f(a). \tag{28}$$

Multiplying (27) and (28) and then integrating with respect to x over $[a, b]$ we have

$$\int_a^b (x-a)^\beta f(x)dx \leq \frac{(b-a)^\beta}{b-a} f(b) \int_a^b (x-a)dx + \frac{(b-a)^\beta}{b-a} f(a) \int_a^b (b-x)dx.$$

From which we have

$$\frac{\Gamma(\beta+1) I_{b^-}^{\beta+1} f(a)}{(b-a)^{\beta+1}} \leq \frac{f(a)+f(b)}{2}. \tag{29}$$

On the other hand for $x \in [a, b]$ we have

$$(b-x)^\alpha \leq (b-a)^\alpha, \quad \alpha > 0. \tag{30}$$

Multiplying (28) and (30) and then integrating with respect to x over $[a, b]$ we get

$$\int_a^b (b-x)^\alpha f(x)dx \leq (b-a)^{\alpha+1} \frac{f(a)+f(b)}{2}.$$

From which we have

$$\frac{\Gamma(\alpha+1) I_{a^+}^{\alpha+1} f(b)}{(b-a)^{\alpha+1}} \leq \frac{f(a)+f(b)}{2}. \tag{31}$$

Adding (29) and (31) we get

$$\frac{\Gamma(\beta+1) I_{b^-}^{\beta+1} f(a)}{2(b-a)^{\beta+1}} + \frac{\Gamma(\alpha+1) I_{a^+}^{\alpha+1} f(b)}{2(b-a)^{\alpha+1}} \leq \frac{f(a)+f(b)}{2}.$$

Using Lemma 1 and multiplying (23) with $(x-a)^\beta$, then integrating over $[a, b]$ we have

$$f\left(\frac{a+b}{2}\right) \int_a^b (x-a)^\beta dx \leq \int_a^b (x-a)^\beta f(x)dx \tag{32}$$

$$f\left(\frac{a+b}{2}\right) \frac{(b-a)^{\beta+1}}{\beta+1} \leq \Gamma(\beta+1) I_{b^-}^{\beta} f(a) \tag{33}$$

$$f\left(\frac{a+b}{2}\right) \frac{1}{2(\beta+1)} \leq \frac{\Gamma(\beta+1) I_{b^-}^{\beta} f(a)}{2(b-a)^{\beta+1}}. \tag{34}$$

Using Lemma 1 and multiplying (23) with $(b-x)^\alpha$, then integrating over $[a, b]$ one can get

$$f\left(\frac{a+b}{2}\right)\frac{1}{2(\alpha+1)} \leq \frac{\Gamma(\alpha+1)I_{a^+}^{\alpha}f(b)}{2(b-a)^{\alpha+1}}. \tag{35}$$

Adding (34) and (35) we get the required inequality. □

Corollary 6 *If we put* $\alpha = \beta$ *in* (26), *then we get*

$$f\left(\frac{a+b}{2}\right)\frac{1}{(\alpha+1)} \leq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha+1}}\left[I_{b^-}^{\alpha+1}f(a) + I_{a^+}^{\alpha+1}f(b)\right] \leq \frac{f(a)+f(b)}{2}.$$

Remark 3 If $\alpha \to 0$, then from above inequality we get the Hadamard inequality.

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Compliance with ethical standards

Research involving human participants and/or animals In this paper none of the authors have performed any experiment and study with human participants or animals.

Conflict of interest Author has not any conflict of interest.

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