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The fractional Hankel-type integral wavelet packet transformation

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Abstract The wavelet packet transformation involving the fractional powers of Hankel-type integral transformation is defined and discussed on its some basic properties. An inversion formula of this transformation is also obtained. Some examples are given.

Keywords Hankel-type integral transformation · Hankel-type integral wavelet transformation · Fractional Hankel-type integral convolution

Mathematics Subject Classification 46F12 · 65T60

1 Introduction

Let $L^p_{\mu,\alpha,\nu}(I)$ be the space of all real valued measurable functions ψ defined on $I=(0,\infty)$ for which $\int_0^\infty |\psi(t)|t^{\nu\mu-\alpha+2\nu-1}dt$ exist. Also the space $L^\infty(I)$ be the collection of almost everywhere bounded integrable functions. Hence the norm is defined as

$$\|\psi\|_{L^p_{\mu,\alpha,\nu}(I)} = \begin{cases} \left(\int_0^\infty \mid \psi(t) \mid^p t^{\nu\mu-\alpha+2\nu-1}dt\right)\frac{1}{p}, & 1<p<\infty, \quad \mu,\alpha,\nu\in\mathbb{R} \\ ess\sup_{t\in I} \mid \psi(t) \mid, & p=\infty. \end{cases} \tag{1}$$

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The fractional powers of $\theta(0<\theta<\pi)$ of Hankel-type integral transformation $H^{\theta}_{\mu,\alpha,\beta,\nu}$ of functions $f \in L^{p}_{\mu,\alpha,\nu}(I)$ depending upon three real parameters (α, β, ν) is defined as [6, 7]

$$(H^{\theta}_{\mu,\alpha,\beta,\nu}f)(w) = \hat{f}^{\theta}_{\mu,\alpha,\beta,\nu}(w) = \int_0^{\infty} K^{\theta}(t,w)f(t)dt \tag{2}$$

where

$$K^{\theta}(t,w) = \begin{cases} \nu\beta C^{\theta}_{\mu,\alpha}(tw)^{\alpha}J_{\mu}\big(\beta(tw)^{\nu}\csc\theta\big)e^{\frac{i\beta}{2}(t^{2\nu}+w^{2\nu})\cot\theta}t^{-1-2\alpha+2\nu}, & \theta \neq n\pi \\ \nu\beta(tw)^{\alpha}J_{\mu}\big(\beta(tw)^{\nu}\big)t^{-1-2\alpha+2\nu}, & \theta = \dfrac{\pi}{2} \\ \delta(t-w), & \theta = n\pi \end{cases}$$

$\forall n \in \mathbb{Z}$, $\nu\mu + 2\nu - \alpha \geq 1$, $C^{\theta}_{\mu,\alpha} = \frac{e^{i(1+\mu)(\theta-\frac{\pi}{2})}}{\sin\theta}$ and J_{μ} is the Bessel function of first kind of order μ.

The inverse of (1) is defined as follows:

$$f(t) = \left(H^{-\theta}_{\mu,\alpha,\beta,\nu}\hat{f}^{\theta}_{\mu,\alpha,\beta,\nu}\right)(t) = \int_0^{\infty} K^{-\theta}(w,t)\hat{f}^{\theta}_{\mu,\alpha,\beta,\nu}(w)dw, \tag{3}$$

where δ is Dirac-delta function and it is given by

$$\delta(t-w) = \nu^2\beta^2 t^{\alpha} w^{-1-\alpha+2\nu}\int_o^{\infty} y^{-1+2\nu}J_{\mu}(\beta(ty)^{\nu})J_{\mu}(\beta(wy)^{\nu})dy. \tag{4}$$

The Parseval's identity becomes

$$\int_0^{\infty} t^{-1-2\alpha+2\nu}f(t)\overline{g(t)}dt = \int_0^{\infty} w^{-1-2\alpha+2\nu}\hat{f}^{\theta}_{\mu,\alpha,\beta,\nu}(w)\overline{\hat{g}^{\theta}}_{\mu,\alpha,\beta,\nu}(w)dw. \tag{5}$$

Let $\varphi, \psi \in L^{1}_{\mu,\alpha,\nu}(I)$. Then fractional Hankel-type integral convolution $(\varphi\#_{\theta}\psi)(t)$ and Hankel-type integral translation $(\tau^{\theta}_{x}\varphi)(w)$ are respectively defined as:

$$(\varphi\#_{\theta}\psi)(t) = \nu\beta C^{\theta}_{\mu,\alpha}\int_0^{\infty}\varphi(w)e^{\frac{i\beta}{2}w^{2\nu}\cot\theta}(\tau^{\theta}_{t}\psi)(w)w^{\nu\mu-\alpha+2\nu-1}dw, \tag{6}$$

where $0<t, w<\infty$, and

$$(\tau^{\theta}_{t}\psi)(w) = \psi^{\theta}(t,w) = \nu\beta C^{\theta}_{\mu,\alpha}\int_0^{\infty}\varphi(z)D^{\theta}_{\mu,\alpha,\beta,\nu}(t,w,z)e^{\frac{i\beta}{2}z^{2\nu}\cot\theta}z^{\nu\mu-\alpha+2\nu-1}dz, \tag{7}$$

provided that the above integrals exist, and being

$$\begin{aligned} D^{\theta}_{\mu,\alpha,\beta,\nu}(t,w,z) &= \nu\beta(\csc\theta)^{2\alpha}C^{-\theta}_{\mu,\alpha}\int_0^{\infty}(ts)^{\alpha}(ws)^{\alpha}(ts)^{\alpha}J_{\mu}(\beta(ts)^{\nu}\csc\theta) \\ &\times J_{\mu}(\beta(ws)^{\nu}\csc\theta)J_{\mu}(\beta(zs)^{\nu}\csc\theta)e^{-\frac{i\beta}{2}(t^{2\nu}+w^{2\nu}+z^{2\mu})\cot\theta}s^{-1-3\alpha+2\nu-\nu\mu}ds. \end{aligned} \tag{8}$$

Moreover for $\nu\mu + 2\nu - \alpha \geq 1$,

$$\begin{aligned}&\nu\beta C^{\theta}_{\mu,\alpha}\int_0^\infty z^{-1-2\alpha+2\nu}(zs)^\alpha J_\mu(\beta(zs)^\nu\csc\theta)e^{\frac{i\beta}{2}(z^{2\nu}+s^{2\nu})\cot\theta}\\&\quad\times D^{\theta}_{\mu,\alpha,\beta,\nu}(t,w,z)dz=(\csc\theta)^{2\alpha}(ts)^\alpha J_\mu(\beta(ts)^\nu\csc\theta))(ws)^\alpha J_\mu(\beta(ws)^\nu\csc\theta)\\&\quad\times e^{-\frac{i\beta}{2}(t^{2\nu}+w^{2\nu})\cot\theta}e^{\frac{i\beta}{2}s^{2\nu}\cot\theta}s^{-\alpha-\nu\mu},\end{aligned}\tag{9}$$

and

$$\begin{aligned}&\nu\beta C^{\theta}_{\mu,\alpha}\int_0^\infty z^{\nu\mu-\alpha+2\nu-1}D^{\theta}_{\mu,\alpha,\beta,\nu}(t,w,z)e^{\frac{i\beta}{2}z^{2\nu}\cot\theta}dz\\&\quad=\frac{\beta^\mu(tw)^{\alpha+\nu\mu}(\csc\theta)^{2\alpha+\mu}e^{-\frac{i\beta}{2}(t^{2\nu}+w^{2\nu})\cot\theta}}{2^\mu\Gamma(\mu+1)}.\end{aligned}\tag{10}$$

The mathematical theory of wave packet analysis on $\mathbb{R}$ is originated from dyadic dilations, integral translations and crips modulation of a particular signal [2, 3]. Theory of this paper depending on the ideas of fractional wavelets given by [4, 8, 9], the fractional Hankel-type integral wavelet $\psi_{b,a,\theta}$ as the dilation and translation of function $\psi\in L^2_{\mu,\alpha,\nu}(I)$ with the parameters $a>0$, $b\geq 0$ is mathematically defined as

$$\begin{aligned}&\psi_{b,a,\theta}(t)\\&=D_a\left(\tau^\theta_b\psi\right)(t)=D_a\psi^\theta(b,t)=a^{-\frac{1}{2}-2\nu+2\alpha}e^{\frac{i\beta}{2}\left(\frac{1}{a^{2\nu}}-1\right)\left(t^{2\nu}+b^{2\nu}\right)\cot\theta}\psi^\theta\left(\frac{b}{a},\frac{t}{a}\right)\\&=a^{-\frac{1}{2}-2\nu+2\alpha}e^{\frac{i\beta}{2}\left(\frac{1}{a^{2\nu}}-1\right)\left(t^{2\nu}+b^{2\nu}\right)\cot\theta}\nu\beta C^\theta_{\mu,\alpha}\int_0^\infty\psi(z)D^\theta_{\mu,\alpha,\beta,\nu}\left(\frac{b}{a},\frac{t}{a},z\right)\\&\quad\times e^{\frac{i\beta}{2}z^{2\nu}\cot\theta}z^{\nu\mu-\alpha+2\nu-1}dz,\end{aligned}\tag{11}$$

where D_a denotes the fractional dilation operator.

A fractional Hankel-type integral wavelet is a function $\psi\in L^2_{\mu,\alpha,\nu}(I)$, which satisfies the following condition:

$$C^{\theta,\mu,\alpha,\beta,\nu}_\psi=\int_0^\infty\frac{\left|\left(H^\theta_{\mu,\alpha,\beta,\nu}e^{\frac{-i\beta}{2}z^{2\nu}\cot\theta}z^{\nu\mu+\alpha}\psi(z)\right)(w)\right|^2}{\nu^2\beta^2w^{1+2\nu\mu+2\alpha}}dw<\infty,\tag{12}$$

and it is known as the admissibility condition of the fractional Hankel-type integral wavelet.

Proposition 1 *If $\psi\in L^2_{\mu,\alpha,\nu}(I)$, then the fractional powers of the Hankel-type integral transformation of $\psi_{b,a,\theta}$ is given by*

$$\begin{aligned}\left(H^\theta_{\mu,\alpha,\beta,\nu}\psi_{b,a,\theta}\right)(w)=&\frac{1}{\sqrt{a}}(aw)^{\nu\mu-\alpha}e^{-\frac{i\beta}{2}\left((a^{2\nu}-1)w^{2\nu}+b^{2\nu}\right)\cot\theta}(bw\csc\theta)^\alpha\\&\times J_\mu\left(\beta(bw)^\nu\csc\theta\right)\left(H^\theta_{\mu,\alpha,\beta,\nu}e^{\frac{-i\beta}{2}z^{2\nu}\cot\theta}z^{\nu\mu+\alpha}\psi(z)\right)(aw).\end{aligned}$$

Proof See [7]. □

Lemma 1 *If $K^{\theta}(t,w)$ is kernel of the transformation (2) then*

$$(a)\ \overline{K^{\theta}}(w,t) = K^{-\theta}(w,t),$$
$$(b)\ \int_0^{\infty} K^{\theta_1}(t,w)K^{\theta_2}(w,z)dw = K^{\theta_1+\theta_2}(t,z),$$
$$(c)\ \int_0^{\infty} K^{\theta}(a,w)K^{-\theta}(w,t)dw = \delta(t-a).$$

Proof (a) Since for $\theta \neq n\pi$

$$\begin{aligned} K^{-\theta}(w,t) &= \nu\beta C_{\mu,\alpha}^{-\theta}(tw)^{\alpha} J_{\mu}(-\beta(wt)^{\nu}\csc(-\theta))e^{\frac{i\beta}{2}(t^{2\nu}+\omega^{2\nu})\cot(-\theta)}w^{-1-2\alpha+2\nu} \\ &= \nu\beta(-1)^{1+\mu}\frac{e^{-i(1+\mu)(\theta+\frac{\pi}{2})}}{\sin\theta}J_{\mu}(\beta(wt)^{\nu}\csc\theta)e^{-\frac{i\beta}{2}(t^{2\nu}+\omega^{2\nu})\cot\theta}w^{-1-2\alpha+2\nu} \\ &= \overline{\nu\beta\frac{e^{i(1+\mu)(\theta-\frac{\pi}{2})}}{\sin\theta}J_{\mu}(\beta(wt)^{\nu}\csc\theta)e^{\frac{i\beta}{2}(t^{2\nu}+\omega^{2\nu})\cot\theta}w^{-1-2\alpha+2\nu}} \\ &= \overline{K^{\theta}}(w,t). \end{aligned}$$

Similarly (b) and (c) can be proved easily by using the property (a). □

2 Fractional Hankel-type integral wavelet packet transformation

As per [2, 3, 5, 8], the fractional Hankel-type integral wavelet packet transformation is defined as:

$$\left(WP_{\psi}^{\theta}f\right)(u,b,a) = \int_0^{\infty} K^{\theta}(t,u)\overline{\psi}_{b,a,\theta}(t)f(t)dt, \tag{13}$$

where $\psi_{b,a,\theta}$ as fractional wavelet.

More precisely, the fractional Hankel-type integral wavelet packet transformation (FrHWPT) is given by

$$\begin{aligned} \left(WP_{\psi}^{\theta}f\right)(u,b,a) = \nu\beta C_{\mu,\alpha}^{\theta}\int_0^{\infty}(tu)^{\alpha}J_{\mu}\left(\beta(tu)^{\nu}\csc\theta\right)e^{\frac{i\beta}{2}(t^{2\nu}+u^{2\nu})\cot\theta} \\ \times t^{-1-2\alpha+2\nu}\overline{\psi}_{b,a,\theta}(t)f(t)dt. \end{aligned} \tag{14}$$

Now assume that $f_u(t) = \nu\beta C_{\mu,\alpha}^{\theta}(tu)^{\alpha}J_{\mu}\left(\beta(tu)^{\nu}\csc\theta\right)e^{\frac{i\beta}{2}(t^{2\nu}+u^{2\nu})\cot\theta}f(t)$ and using Parseval's identity in (14) we get

$$\left(WP_{\psi}^{\theta}f\right)(u,b,a) = \int_0^{\infty}\hat{f}_u^{\theta}(w)\overline{\hat{\psi}}_{b,a,\theta}(w)w^{-1-2\alpha+2\nu}dw. \tag{15}$$

Now we see that

$$\begin{aligned}\hat{f}_u^\theta(w) &= \nu\beta C_{\mu,\alpha}^\theta \int_0^\infty (tu)^\alpha J_\mu\left(\beta(tu)^\nu \csc\theta\right) e^{\frac{i\beta}{2}(t^{2\nu}+u^{2\nu})\cot\theta} f_u(t) t^{-1-2\alpha+2\nu} dt \\ &= \nu^2\beta^2 (C_{\mu,\alpha}^\theta)^2 \int_0^\infty (tw)^\alpha J_\mu\left(\beta(tu)^\nu \csc\theta\right) e^{\frac{i\beta}{2}(t^{2\nu}+u^{2\nu})\cot\theta} \\ &\quad \times \left((tu)^\alpha J_\mu\left(\beta(tu)^\nu \csc\theta\right) e^{\frac{i\beta}{2}(t^{2\nu}+u^{2\nu})\cot\theta} f(t)\right) t^{-1-2\alpha+2\nu} dt.\end{aligned}$$

Using (9), (2) and (7), we get

$$\begin{aligned}\hat{f}_u^\theta(w) &= \nu\beta C_{\mu,\alpha}^\theta (\csc\theta)^{-2\alpha} e^{i\beta(w^{2\nu}+u^{2\nu})\cot\theta} \\ &\quad \times \tau_u^\theta\left(z^{-\nu\mu-\alpha} e^{-\frac{i\beta}{2}z^{2\nu}\cot\theta} H_{\mu,\alpha,\beta,\nu}^\theta\left(t^{\nu\mu+\alpha} e^{\frac{i\beta}{2}t^{2\nu}\cot\theta} f(t)\right)(z)\right)(w).\end{aligned} \tag{16}$$

Therefore

$$\begin{aligned}&\left(WP_\psi^\theta f\right)(u,b,a) \\ &= \frac{\nu\beta C_{\mu,\alpha}^\theta e^{i\beta u^{2\nu}\cot\theta}}{(\csc\theta)^{2\alpha}\sqrt{a}} \int_0^\infty \tau_u^\theta\left(z^{-\nu\mu-\alpha} e^{-\frac{i\beta}{2}z^{2\nu}\cot\theta} H_{\mu,\alpha,\beta,\nu}^\theta\left(t^{\nu\mu+\alpha} e^{\frac{i\beta}{2}t^{2\nu}\cot\theta} f(t)\right)(z)\right)(w) \\ &\times (aw)^{-\nu\mu-\alpha} e^{\frac{i\beta}{2}a^{2\nu}w^{2\nu}} e^{\frac{i\beta}{2}(w^{2\nu}+b^{2\nu})\cot\theta} (bw)^\alpha J_\mu\left(\beta(bw)^\nu \csc\theta\right) \\ &\times \overline{\left(H_{\mu,\alpha,\beta,\nu}^\theta e^{\frac{-i\beta}{2}z^{2\nu}\cot\theta} z^{\nu\mu+\alpha}\psi(z)\right)(aw)} w^{-1-2\alpha+2\nu} dw.\end{aligned} \tag{17}$$

Remark 1 From (17) we can express the fractional Hankel- type integral wavelet packet transformation $\left(WP_\psi^\theta f\right)(u,b,a)$ of a function $f \in L_{\mu,\alpha,\nu}^2(I)$ as:

$$\begin{aligned}&\left(WP_\psi^\theta f\right)(u,b,a) \\ &= \frac{e^{i\beta u^{2\nu}\cot\theta}}{(\csc\theta)^{2\alpha}\sqrt{a}} H_{\mu,\alpha,\beta,\nu}^\theta\Big[\tau_u^\theta\left(z^{-\nu\mu-\alpha} e^{-\frac{i\beta}{2}z^{2\nu}\cot\theta} H_{\mu,\alpha,\beta,\nu}^\theta\left(t^{\nu\mu+\alpha} e^{\frac{i\beta}{2}t^{2\nu}\cot\theta} f(t)\right)(z)\right) \\ &\quad (w)\Big](aw)^{-\mu\nu-\alpha} e^{\frac{i\beta}{2}a^{2\nu}w^{2\nu}\cot\theta} \overline{\left(H_{\mu,\alpha,\beta,\nu}^\theta e^{\frac{-i\beta}{2}z^{2\nu}\cot\theta} z^{\nu\mu+\alpha}\psi(z)\right)(aw)}\Big](b).\end{aligned}$$

Theorem 1 *If* ψ_1 and ψ_2 *are two fractional Hanlel- type integral wavelets,* $f_1, f_2 \in L_{\mu,\alpha,\nu}^2(I)$ *and for scalars* α_1 and α_2 *we have*

$$(a)\left(WP_{\psi_1}^\theta(\alpha_1 f_1 + \alpha_2 f_2)\right)(u,b,a) = \alpha_1\left(WP_{\psi_1}^\theta f_1\right)(u,b,a) + \alpha_2\left(WP_{\psi_2}^\theta f_2\right)(u,b,a),$$

$$(b)\left(WP_{\alpha_1\psi_1+\alpha_2\psi_2}^\theta f_1\right)(u,b,a) = \overline{\alpha_1}\left(WP_{\psi_1}^\theta f_1\right)(u,b,a) + \overline{\alpha_2}\left(WP_{\psi_2}^\theta f_1\right)(u,b,a).$$

Proof (*a*) Using (13), we have

$$\begin{aligned}&\left(WP^{\theta}_{\psi_1}(\alpha_1 f_1+\alpha_2 f_2)\right)(u,b,a)\\&=\int_0^\infty K^\theta(t,u)\overline{\psi_1}_{b,a,\theta}(t)(\alpha_1 f_1+\alpha_2 f_2)(t)dt\\&=\alpha_1\int_0^\infty K^\theta(t,u)\overline{\psi_1}_{b,a,\theta}(t)f_1(t)dt+\alpha_2\int_0^\infty K^\theta(t,u)\overline{\psi_1}_{b,a,\theta}(t)f_2(t)dt\\&=\alpha_1\left(WP^{\theta}_{\psi_1}f_1\right)(u,b,a)+\alpha_2\left(WP^{\theta}_{\psi_2}f_2\right)(u,b,a).\end{aligned}$$

Similarly (b) can be proved as (a). □

Theorem 2 *If* ψ_1 and ψ_2 *are two fractional wavelets and* $(WP^\theta_{\psi_1}f)(u,b,a)$ *and* $(WP^\theta_{\psi_2}g)(u,b,a)$ *denote the fractional Hankel-type integral wavelet packet transformations of the functions f and g respectively, then*

$$\begin{aligned}&\int_0^\infty\int_0^\infty (WP^\theta_{\psi_1}f)(u,b,a)\overline{(WP^\theta_{\psi_2}g)(u,b,a)}b^{-1-2\alpha+2\nu}dbda\\&\qquad=(\csc\theta)^{4\alpha}C^{\theta,\mu,\alpha,\beta,\nu}_{\psi_1,\psi_2}\int_0^\infty\left((ut)^\alpha J_\mu\left(\beta(ut)^\nu\csc\theta\right)\right)^2 f(x)\overline{g(x)}x^{-1-2\alpha+2\nu}dx,\end{aligned}$$

where

$$\begin{aligned}C^{\theta,\mu,\alpha,\beta,\nu}_{\psi_1,\psi_2}&=(\nu\beta)^{-2}\int_0^\infty w^{-1-2\nu\mu-2\alpha}\left(H^\theta_{\mu,\alpha,\beta,\nu}\left(z^{\nu\mu+\alpha}e^{\frac{-i\beta}{2}z^{2\nu}\cot\theta}\psi_1(z)\right)\right)(w)\\&\quad\times\overline{\left(H^\theta_{\mu,\alpha,\beta,\nu}\left(z^{\nu\mu+\alpha}e^{\frac{-i\beta}{2}z^{2\nu}\cot\theta}\psi_2(z)\right)\right)(w)}dw.\end{aligned}$$

Proof Using Remark 1 and (5), we get

$$\begin{aligned}&\int_0^\infty\int_0^\infty (WP^\theta_{\psi_1}f)(u,b,a)\overline{(WP^\theta_{\psi_2}g)(u,b,a)}b^{-1-2\alpha+2\nu}dbda\\&=\frac{\nu^2\beta^2C^{\theta,\mu,\alpha,\beta,\nu}_{\psi_1,\psi_2}}{(\csc\theta)^{-4\alpha}}\int_0^\infty\tau^\theta_u\left(z^{-\nu\mu-\alpha}e^{-\frac{i\beta}{2}z^{2\nu}\cot\theta}H^\theta_{\mu,\alpha,\beta,\nu}\left(t^{\nu\mu+\alpha}e^{\frac{i\beta}{2}t^{2\nu}\cot\theta}f(t)\right)(z)\right)(w)\\&\times\overline{\tau^\theta_u\left(z^{-\nu\mu-\alpha}e^{-\frac{i\beta}{2}z^{2\nu}\cot\theta}H^\theta_{\mu,\alpha,\beta,\nu}\left(t^{\nu\mu+\alpha}e^{\frac{i\beta}{2}t^{2\nu}\cot\theta}g(t)\right)(z)\right)(w)}w^{-1-2\alpha+2\nu}dw\\&=\frac{\nu^2\beta^2C^{\theta,\mu,\alpha,\beta,\nu}_{\psi_1,\psi_2}}{(\csc\theta)^{-4\alpha}}\int_0^\infty H^\theta_{\mu,\alpha,\beta,\nu}\left[\tau^\theta_u\left(z^{-\nu\mu-\alpha}e^{-\frac{i\beta}{2}z^{2\nu}\cot\theta}H^\theta_{\mu,\alpha,\beta,\nu}\left(t^{\nu\mu+\alpha}e^{\frac{i\beta}{2}t^{2\nu}\cot\theta}\right.\right.\right.\\&\left.\left.\left.\times f(t)\right)(z)\right)(w)\right]\overline{H^\theta_{\mu,\alpha,\beta,\nu}\left[\tau^\theta_u\left(z^{-\nu\mu-\alpha}e^{-\frac{i\beta}{2}z^{2\nu}\cot\theta}H^\theta_{\mu,\alpha,\beta,\nu}\left(t^{\nu\mu+\alpha}e^{\frac{i\beta}{2}t^{2\nu}\cot\theta}\right.\right.\right.}\\&\overline{\left.\left.\left.\times g(t)\right)(z)\right)(w)\right]}x^{-1-2\alpha+2\nu}dx.\end{aligned}\tag{18}$$

Now we see that

$$\begin{aligned} &H^{\theta}_{\mu,\alpha,\beta,\nu}\left[\tau^{\theta}_{u}\left(z^{-\nu\mu-\alpha}e^{-\frac{i\beta}{2}z^{2\nu}\cot\theta}H^{\theta}_{\mu,\alpha,\beta,\nu}\left(t^{\nu\mu+\alpha}e^{\frac{i\beta}{2}t^{2\nu}\cot\theta}f(t)\right)(z)\right)(w)\right](x)\\ &=\nu\beta\int_0^\infty(xw)^{\alpha}J_{\mu}\left(\beta(xw)^{\nu}\csc\theta\right)e^{\frac{i\beta}{2}(x^{2\nu}+w^{2\nu})\cot\theta}w^{-1-2\alpha+2\nu}\Big(\nu\beta C^{\theta}_{\mu,\alpha}\\ &\quad\times\int_0^\infty D^{\theta}_{\mu,\alpha,\beta,\nu}(u,w,z)H^{\theta}_{\mu,\alpha,\beta,\nu}\left(t^{\nu\mu+\alpha}e^{\frac{i\beta}{2}t^{2\nu}\cot\theta}f(t)\right)(z)z^{-1-2\alpha+2\nu}dz\Big)dw\\ &=(\csc\theta)^{2\alpha}e^{2i(1+\mu)(\theta-\frac{\pi}{2})}e^{-\frac{i\beta}{2}u^{2\nu}\cot\theta}e^{\frac{3i}{2}\beta x^{2\nu}\cot\theta}(ux)^{\alpha}J_{\mu}\left(\beta(ux)^{\nu}\csc\theta\right)f(x). \end{aligned} \tag{19}$$

Similarly proceeding as the above we get

$$\begin{aligned} &\overline{H^{\theta}_{\mu,\alpha,\beta,\nu}\left[\tau^{\theta}_{u}\left(z^{-\nu\mu-\alpha}e^{-\frac{i\beta}{2}z^{2\nu}\cot\theta}H^{\theta}_{\mu,\alpha,\beta,\nu}\left(t^{\nu\mu+\alpha}e^{\frac{i\beta}{2}t^{2\nu}\cot\theta}g(t)\right)(z)\right)(w)\right](x)}\\ &=(\csc\theta)^{2\alpha}e^{-2i(1+\mu)(\theta-\frac{\pi}{2})}e^{\frac{i\beta}{2}u^{2\nu}\cot\theta}e^{-\frac{3i}{2}\beta x^{2\nu}\cot\theta}(ux)^{\alpha}J_{\mu}\left(\beta(ux)^{\nu}\csc\theta\right)\overline{g(x)}. \end{aligned} \tag{20}$$

Using (19) and (20) in (18) we get

$$\begin{aligned} &\int_0^\infty\int_0^\infty(WP^{\theta}_{\psi_1}f)(u,b,a)\overline{(WP^{\theta}_{\psi_2}g)(u,b,a)}b^{-1-2\alpha+2\nu}dbda\\ &\qquad=(\csc\theta)^{4\alpha}C^{\theta,\mu,\alpha,\beta,\nu}_{\psi_1,\psi_2}\int_0^\infty\left((ut)^{\alpha}J_{\mu}\left(\beta(ut)^{\nu}\csc\theta\right)\right)^2f(x)\overline{g(x)}x^{-1-2\alpha+2\nu}dx. \end{aligned}$$

□

Remark 2 The following are deductions of Theorem 2:

(i) If $f=g$, then

$$\begin{aligned} &\int_0^\infty\int_0^\infty(WP^{\theta}_{\psi_1}f)(u,b,a)\overline{(WP^{\theta}_{\psi_2}f)(u,b,a)}b^{-1-2\alpha+2\nu}dbda\\ &\qquad=(\csc\theta)^{4\alpha}C^{\theta,\mu,\alpha,\beta,\nu}_{\psi_1,\psi_2}\int_0^\infty\left((ut)^{\alpha}J_{\mu}\left(\beta(ut)^{\nu}\csc\theta\right)\right)^2|f(x)|^2x^{-1-2\alpha+2\nu}dx. \end{aligned}$$

(ii) If $\psi_1=\psi_2=\psi$, then

$$\begin{aligned} &\int_0^\infty\int_0^\infty(WP^{\theta}_{\psi}f)(u,b,a)\overline{(WP^{\theta}_{\psi}g)(u,b,a)}b^{-1-2\alpha+2\nu}dbda\\ &\qquad=(\csc\theta)^{4\alpha}C^{\theta,\mu,\alpha,\beta,\nu}_{\psi}\int_0^\infty\left((ut)^{\alpha}J_{\mu}\left(\beta(ut)^{\nu}\csc\theta\right)\right)^2f(x)\overline{g(x)}x^{-1-2\alpha+2\nu}dx. \end{aligned}$$

(iii) If $f=g$ and $\psi_1=\psi_2=\psi$, then

$$\begin{aligned} &\int_0^\infty\int_0^\infty\left|\left(W^{\theta}_{\psi}f\right)(u,b,a)\right|^2b^{-1-2\alpha+2\nu}dbda\\ &\qquad=(\csc\theta)^{4\alpha}C^{\theta,\mu,\alpha,\beta,\nu}_{\psi}\int_0^\infty\left((ut)^{\alpha}J_{\mu}\left(\beta(ut)^{\nu}\csc\theta\right)\right)^2|f(x)|^2x^{-1-2\alpha+2\nu}dx \end{aligned}$$

where $C^{\theta,\mu,\alpha,\beta,\nu}_{\psi}$ is given by (11).

Theorem 3 (Inversion Formula) *Let* $f \in L^2_{\mu,\alpha,\nu}(I)$. *Then f can be reconstructed involving the fractional Hankel-type transformation by the formula*

$$f(t) = \frac{\nu\beta\overline{C^{\theta}_{\mu,\alpha}}e^{-\frac{i\beta}{2}(u^{2\nu}+t^{2\nu})\cot\theta}}{(\csc\theta)^{4\alpha}C^{\theta,\mu,\alpha,\beta,\nu}_{\psi_1,\psi_2}(ut)^{\alpha}J_{\mu}\big(\beta(ut)^{\nu}\csc\theta\big)}\int_0^{\infty}\int_0^{\infty}\Big(W^{\theta}_{\psi}f\Big)(u,b,a)$$
$$\times(\psi_2)_{b,a,\theta}(t)b^{-1-2\alpha+2\nu}dbda$$

where $C^{\theta,\mu,\alpha,\beta,\nu}_{\psi_1,\psi_2}$ *is as in Theorem* 2.

Proof Form Theorem 2, we have

$$(\csc\theta)^{4\alpha}C^{\theta,\mu,\alpha,\beta,\nu}_{\psi_1,\psi_2}\int_0^{\infty}\Big((ut)^{\alpha}J_{\mu}\big(\beta(ut)^{\nu}\csc\theta\big)\Big)^2 f(t)\overline{g(t)}t^{-1-2\alpha+2\nu}dt$$
$$=\int_0^{\infty}\int_0^{\infty}(WP^{\theta}_{\psi_1}f)(u,b,a)\overline{(WP^{\theta}_{\psi_2}g)(u,b,a)}b^{-1-2\alpha+2\nu}dbda$$
$$=\int_0^{\infty}\Big(\nu\beta C^{\theta}_{\mu,\alpha}\int_0^{\infty}\int_0^{\infty}(WP^{\theta}_{\psi_1}f)(u,b,a)(tw)^{\alpha}J_{\mu}\big(\beta(tu)^{\nu}\csc\theta\big)e^{\frac{i\beta}{2}(t^{2\nu}+w^{2\nu})\cot\theta}$$
$$\times(\psi_2)_{b,a,\theta}b^{-1-2\alpha+2\nu}dbda\Big)\overline{g(t)}t^{-1-2\alpha+2\nu}dt.$$

On equating we get

$$f(t) = \frac{\nu\beta\overline{C^{\theta}_{\mu,\alpha}}e^{-\frac{i\beta}{2}(u^{2\nu}+t^{2\nu})\cot\theta}}{(\csc\theta)^{4\alpha}C^{\theta,\mu,\alpha,\beta,\nu}_{\psi_1,\psi_2}(ut)^{\alpha}J_{\mu}\big(\beta(ut)^{\nu}\csc\theta\big)}\int_0^{\infty}\int_0^{\infty}\Big(W^{\theta}_{\psi}f\Big)(u,b,a)$$
$$\times(\psi_2)_{b,a,\theta}(t)b^{-1-2\alpha+2\nu}dbda. \tag{21}$$

□

3 Some examples

Example 1 Assume that $f(t) = \frac{\delta(t-c)}{t^{2\nu-1}}, 0 < c < \infty$. Then the fractional Hankel-type integral wavelet packet transformation of *f*(*t*) is given by

$$\begin{aligned}
&\left(W_{\psi}^{\theta}f\right)(u,b,a)\\
&=\int_0^\infty K^{\theta}(t,u)\overline{\psi}_{b,a,\theta}(t)f(t)dt\\
&=v^3\beta^3C_{\mu,\alpha}^{\theta}(\csc\theta)^{2+2\alpha}a^{-\frac{1}{2}-2v}(bu)^{\alpha}e^{\frac{i\beta}{2}(u^{2v}+b^{2v})\cot\theta}\int_0^\infty\overline{\psi(z)}z^{v\mu+2v-1}dz\\
&\quad\times\int_0^\infty x^{-1+2v-v\mu}J_\mu\left(\beta\left(\frac{b}{a}x\right)^v\csc\theta\right)J_\mu\left(\beta(xz)^v\csc\theta\right)dx\\
&\quad\times\left(\int_0^\infty J_\mu\left(\beta\left(\frac{t}{a}x\right)^v\csc\theta\right)J_\mu\left(\beta(tu)^v\csc\theta\right)e^{i\beta t^{2v}\cot\theta}\delta(t-c)dt\right)\\
&=v^3\beta^3C_{\mu,\alpha}^{\theta}(\csc\theta)^{2+2\alpha}a^{-\frac{1}{2}-2v}(bu)^{\alpha}e^{\frac{i\beta}{2}(u^{2v}+b^{2v}+2c^{2v})\cot\theta}J_\mu\left(\beta(tu)^v\csc\theta\right)\\
&\quad\times\int_0^\infty\overline{\psi(z)}z^{v\mu+2v-1}dz\int_0^\infty x^{-1+2v-v\mu}J_\mu\left(\beta\left(\frac{b}{a}x\right)^v\csc\theta\right)\\
&\quad\times J_\mu\left(\beta(xz)^v\csc\theta\right)J_\mu\left(\beta\left(\frac{c}{a}x\right)^v\csc\theta\right)dx.
\end{aligned}$$

Using [10, p.41], we get

$$\begin{aligned}
&\left(W_{\psi}^{\theta}f\right)(u,b,a)\\
&=\frac{v^2\beta^{3-3\mu}C_{\mu,\alpha}^{\theta}}{\Gamma(\mu+\frac{1}{2})\Gamma\frac{1}{2}}(\csc\theta)^{2+2\alpha-3\mu}u^{\alpha}b^{\alpha-v}c^{-v}2^{\mu-1}a^{-\frac{1}{2}}e^{\frac{i\beta}{2}(u^{2v}+b^{2v}+2c^{2v})\cot\theta}\\
&\quad\times J_\mu\left(\beta(uc)^v\csc\theta\right)\int_0^\infty\overline{\psi(z)}z^{2v-1}\Delta^{2\mu-1}dz,
\end{aligned}$$

where $\triangle$ denotes the area of a triangle having sides $\left(\beta\left(\frac{b}{a}\right)\csc\theta\right)$, $\left(\beta\left(\frac{c}{a}\right)\csc\theta\right)$, $(\beta z^v\csc\theta)$ such a triangle exists.

Example 2 Assume $f(t)=e^{-it\cot\theta}$, then

$$\begin{aligned}
&\left(W_{\psi}^{\theta}f\right)(u,b,a)\\
&=v^3\beta^3C_{\mu,\alpha}^{\theta}(\csc\theta)^{2+2\alpha}a^{-\frac{1}{2}-2v}(bu)^{\alpha}e^{\frac{i\beta}{2}(u^{2v}+b^{2v})\cot\theta}\int_0^\infty\overline{\psi(z)}z^{v\mu+2v-1}dz\\
&\quad\times\int_0^\infty x^{-1+2v-v\mu}J_\mu\left(\beta\left(\frac{b}{a}x\right)^v\csc\theta\right)J_\mu\left(\beta(xz)^v\csc\theta\right)dx\\
&\quad\times\left(\int_0^\infty t^{2v-1}J_\mu\left(\beta\left(\frac{t}{a}x\right)^v\csc\theta\right)J_\mu\left(\beta(tu)^v\csc\theta\right)dt\right).
\end{aligned}\tag{22}$$

Using [1, p.6] and (4) in (22), we get

$$\begin{aligned}
&\left(W_{\psi}^{\theta}f\right)(u,b,a)\\
&= v^2\beta^3 C_{\mu,\alpha}^{\theta}(\csc\theta)^{2\alpha+2}a^{-\frac{1}{2}-2v}(bu)^{\alpha}e^{\frac{i\beta}{2}(u^{2v}+b^{2v})\cot\theta}\int_0^\infty \overline{\psi(z)}z^{v\mu+2v-1}dz\\
&\quad\times\int_0^\infty x^{-v\mu+2v-1}J_\mu\left(\beta\left(\frac{b}{a}x\right)^v\csc\theta\right)J_\mu\left(\beta(xz)^v\csc\theta\right)\frac{a^v}{\beta x^v\csc\theta}\\
&\quad\times\delta\left(\frac{\beta x^v\csc\theta}{a^v}-\beta u^v\csc\theta\right)dx\\
&= (\csc\theta)^{2\alpha}u^{-v\mu+2\alpha-2v+1}a^{-v\mu-\frac{1}{2}}K^{\theta}(u,b)\int_0^\infty\overline{\psi(z)}z^{v\mu+2v-1}J_\mu\left(\beta(auz)^v\csc\theta\right)dz.
\end{aligned}$$

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