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ORIGINAL RESEARCH PAPER

Fixed point theorems for generalized multivalued contraction

N. Chandra[1] · Mahesh C. Joshi[1] · Narendra K. Singh[1]

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Abstract In this paper, a fixed point theorem for multi-valued mappings on a complete metric space is established taking a general contractive condition which generalizes several contractive conditions. Many generalizations of some well known results are also obtained as corollaries. Further, we give an application to the existence and uniqueness of solutions for certain classes of functional equations arising in dynamic programming.

Keywords Multi-valued mapping · Fixed points and generalized contraction

Mathematics Subject Classification 47H10 · 54H25

1 Introduction and preliminaries

Let (X, d) be a metric space and $CB(X)$ the collection of all nonempty closed and bounded subsets of X. The Hausdorff metric H on $CB(X)$ induced by the metric d is given by

$$H(A,B) = \max\left\{\sup_{a\in A} D(a,B), \sup_{b\in B} D(b,A)\right\}$$

for $A, B \in CB(X)$, where $D(x,A) = \inf_{y\in A} d(x,y)$.

✉ N. Chandra
cnaveen329@gmail.com

Mahesh C. Joshi
mcjoshi69@gmail.com

Narendra K. Singh
nsijwali@gmail.com

[1] Department of Mathematics, D. S. B. Campus, Kumaun University, Nainital 263002, India

Following the Banach contraction principle, Nadler [9] introduced the concept of multivalued contraction and established a fixed point theorem. In fact, Nadler [9] proved the following result.

Theorem 1.1 (Nadler [9]) *Let* (X, d) *be a complete metric space and* $T : X \to CB(X)$ *be a mapping. Assume there exists* $r \in [0, 1)$ *such that* $H(Tx, Ty) \leq r\, d(x, y)$ for all $x, y \in X$. *Then, there exists* $z \in X$ *such that* $z \in Tz$.

Subsequently a number of fixed point theorems have been obtained by the researchers for multivalued mappings in different settings of spaces (see [2, 4, 7, 10, 11] and references therein). Kikkawa and Suzuki [7] proved a new version of Nadler's result taking Suzuki type contractive condition for multivalued mappings. Đorić and Lazović [5] obtained a generalization of Kikkawa–Suzuki theorem for Ćirić type generalized multivalued mappings [2]. This result also extends some well known theorems on the existence of fixed points for multivalued mappings.

Theorem 1.2 (Đorić and Lazović [5]) *Let* (X, d) *be a complete metric space and* $T : X \to CB(X)$. *Define a non-increasing function* $\phi : [0, 1) \to (\frac{1}{2}, 1]$ *by*

$$\phi(r) = \begin{cases} 1 & \text{if } 0 \leq r < \dfrac{1}{2} \\ 1 - r & \text{if } \dfrac{1}{2} \leq r < 1. \end{cases} \tag{1}$$

Assume there exists $r \in [0, 1)$ *such that* $\phi(r)D(x, Tx) \leq d(x, y)$ *implies*

$$H(Tx, Ty) \leq r \max\left\{ d(x, y), D(x, Tx), D(y, Ty), \frac{D(x, Ty) + D(y, Tx)}{2} \right\} \tag{2}$$

for all $x, y \in X$. *Then, there exists* $z \in X$ *such that* $z \in Tz$.

In this paper, we extend and generalize the result due to Đorić et al. [5] for multivalued mappings satisfying generalized contractive type condition. For rest of the paper, we use following notation.

$$\begin{aligned} M(x, y) = {} & a(x, y)\, d(x, y) + b(x, y) \max\{D(x, Tx), D(y, Ty)\} \\ & + \; c(x, y)\, [D(x, Ty) + D(y, Tx)]. \end{aligned}$$

2 Main results

Theorem 2.1 *Let* (X, d) *be a complete metric space and* $T : X \to CB(X)$. *If there exists* $r \in [0, 1)$ *such that*

$$\phi(r)D(x, Tx) \leq d(x, y) \quad \text{implies} \quad H(Tx, Ty) \leq M(x, y) \tag{3}$$

for all $x, y \in X$, *where* $a(x, y), b(x, y), c(x, y) \geq 0$ *with* $\sup_{x,y \in X}[a(x, y) + b(x, y) + 2c(x, y)] = r < 1$ *and the function* $\phi : [0, 1) \to (\frac{1}{2}, 1]$ *is defined as in Theorem 1.2, then* T *has a fixed point, i.e., there exists* $z \in X$ *such that* $z \in Tz$.

Proof Without loss of generality, choose r_1 such that $0 \le r < r_1 < 1$. Suppose $y_1 \in X$ and $y_2 \in Ty_1$ are arbitrary, then $D(y_2, Ty_2) \le H(Ty_1, Ty_2)$. Since $\phi(r)D(y_1, Ty_1) \le d(y_1, Ty_1) \le d(y_1, y_2)$, hence by (3),

$$\begin{aligned} H(Ty_1, Ty_2) &\le M(y_1, y_2) \\ &\le a\, d(y_1, y_2) + b\, max\{d(y_1, y_2), D(y_2, Ty_2)\} + c\, D(y_1, Ty_2) \\ &\le a\, d(y_1, y_2) + b\, max\{d(y_1, y_2), D(y_2, Ty_2)\} \\ &+ \; c\,[d(y_1, y_2) + D(y_2, Ty_2)] \\ &\le (a + b + 2c) max\{d(y_1, y_2), D(y_2, Ty_2)\}, \\ &\text{where } a, b \text{ and } c \text{ are evaluated at the point } (y_1, y_2). \end{aligned}$$

Hence,

$$\begin{aligned} D(y_2, Ty_2) &\le H(Ty_1, Ty_2) \le r\, max\{d(y_1, y_2), D(y_2, Ty_2)\} \\ &\Rightarrow D(y_2, Ty_2) \le r\, d(y_1, y_2) < r_1 d(y_1, y_2). \end{aligned}$$

Which implies that there exists $y_3 \in Ty_2$ such that $d(y_2, y_3) \le r_1\, d(y_1, y_2)$. Continuing this process, we can construct a sequence $\{y_n\}$ in X such that

$$y_{n+1} \in Ty_n \text{ and } \; d(y_{n+1}, y_{n+2}) \le r_1\, d(y_n, y_{n+1}).$$

Hence,

$$\sum_{n=1}^{\infty} d(y_n, y_{n+1}) \le \sum_{n=1}^{\infty} r_1^{n-1} d(y_1, y_2) < \infty$$

which shows that $\{y_n\}$ is a Cauchy sequence in X. Since X is complete, there exists a point $z \in X$ such that $\lim_{n\to\infty} y_n = z$.

Now, we will show that

$$D(z, Tx) \le r \;\max\{d(z, x), D(x, Tx)\} \text{ for all } x \in X\backslash\{z\}. \tag{4}$$

As $\lim_{n\to\infty} y_n = z$, then there exists $m \in \mathbb{N}$ such that

$$d(z, y_n) \le \frac{1}{3} d(x, z) \quad \forall n \ge m.$$

Now,

$$\begin{aligned} \phi(r)D(y_n, Ty_n) &\le D(y_n, Ty_n) \\ &\le d(y_n, y_{n+1}) \\ &\le d(y_n, z) + d(y_{n+1}, z) \\ \Rightarrow \phi(r)D(y_n, Ty_n) &\le \frac{2}{3} d(x, z) \\ &\le d(x, z) - \frac{1}{3} d(x, z) \\ &\le d(x, z) - d(z, y_n) \\ &\le d(y_n, x). \end{aligned}$$

Using (3),

$$\begin{aligned} & H(Ty_n, Tx) \le M(y_n, x) \quad \forall\, n \in \mathbb{N}, \quad n \ge m \\ \Rightarrow\ & H(Ty_n, Tx) \le a\, d(y_n, x) + b\ \max\{D(y_n, Ty_n), D(x, Tx)\} \\ & \qquad c\,[D(y_n, Tx) + D(x, Ty_n)], \\ & \qquad \text{where } a, b \text{ and } c \text{ are evaluated at the point } (y_n, x). \end{aligned}$$

Since $y_{n+1} \in Ty_n$, $D(y_{n+1}, Tx) \le H(Ty_n, Tx)$. Hence,

$$\begin{aligned} D(y_{n+1}, Tx) &\le a\, d(y_n, x) + b\ \max\{d(y_n, y_{n+1}), D(x, Tx)\} \\ &\quad + c\,[D(y_n, Tx) + d(x, y_{n+1})] \\ &\le (a + b + 2c)\ \max\{d(y_n, x), d(y_n, y_{n+1}), \\ &\quad D(x, Tx), D(y_n, Tx), d(x, y_{n+1})\} \\ &\le r\ \max\{d(y_n, x), d(y_n, y_{n+1}), \\ &\quad D(x, Tx), D(y_n, Tx), d(x, y_{n+1})\}. \end{aligned}$$

Taking $n \to \infty$,

$$\begin{aligned} & D(z, Tx) \le r\ \max\{d(z, x), D(x, Tx), D(z, Tx)\} \\ & \Rightarrow D(z, Tx) \le r\ \max\{d(z, x), D(x, Tx)\}. \end{aligned}$$

Now, we prove that z is a fixed point of T.

(i) Consider $0 \le r < \frac{1}{2}$ and assume that $z \notin Tz$. Let $a \in Tz$ be such that $2r\, d(a, z) < D(z, Tz)$. Since $a \in Tz \Rightarrow a \ne z$, hence by (4),

$$D(z, Ta) \le r \max\{d(z, a), D(a, Ta)\}. \tag{5}$$

For $0 \le r < \frac{1}{2}$, $\phi(r) D(z, Tz) = D(z, Tz) \le d(z, a)$. Therefore, by (3)

$$\begin{aligned} H(Tz, Ta) &\le M(z, a) \\ &\le a\, d(z, a) + b\ \max\{D(z, Tz), D(a, Ta)\} + c\,[d(z, a) + D(a, Ta)] \\ &\le (a + b + 2c)\ \max\{d(z, a), D(z, Tz), D(a, Ta)\}, \\ &\quad \text{where } a, b \text{ and } c \text{ are evaluated at the point } (z, a). \end{aligned}$$

Hence,

$$\begin{aligned} & D(a, Ta) \le H(Tz, Ta) \le r\ \max\{d(z, a), D(a, Ta)\} \quad (\text{as } a \in Tz) \\ & \Rightarrow D(a, Ta) \le r\, d(z, a) < d(z, a). \end{aligned}$$

From (5), $D(z, Ta) \le r\, d(z, a)$. Thus,

$$\begin{aligned} D(z, Tz) &\le D(z, Ta) + H(Ta, Tz) \\ &\le D(z, Ta) + r\ \max\{d(z, a), D(a, Ta)\} \\ &\le 2r\, d(z, a) \\ &< D(z, Tz), \text{ a contraction.} \end{aligned}$$

Hence $z \in Tz$.

(ii) Now, consider $\frac{1}{2} \leq r < 1$. Firstly, we prove that

$$H(Tx, Tz) \leq r \max\left\{d(x, z), D(x, Tx), D(z, Tz), \frac{D(x, Tz) + D(z, Tx)}{2}\right\} \quad (6)$$

for all $x \in X$. For $x = z$, (6) is obvious. Taking $x \neq z$, there exists $z_n \in Tx$ such that

$$d(z, z_n) \leq D(z, Tx) + \frac{1}{n} d(x, z) \quad \forall n \in \mathbb{N}.$$

Using (5), we get

$$\begin{aligned} D(x, Tx) &\leq d(x, z_n) \\ &\leq d(x, z) + d(z, z_n) \\ &\leq d(x, z) + D(z, Tx) + \frac{1}{n} d(x, z) \\ &\leq d(x, z) + r \max\{d(x, z), D(x, Tx)\} + \frac{1}{n} d(x, z). \end{aligned}$$

Case (i) If $d(x, z) \geq D(x, Tx)$, then

$$D(x, Tx) \leq d(x, z) + r\, d(x, z) + \frac{1}{n} d(x, z) = \left(1 + r + \frac{1}{n}\right) d(x, z).$$

Making $n \to \infty$, we have $D(x, Tx) \leq (1 + r) d(x, z)$. Thus,

$$\phi(r) D(x, Tx) = (1 - r) D(x, Tx) \leq \frac{1}{1 + r} D(x, Tx) \leq D(x, Tx) \leq d(x, z)$$

Again, using (3),

$$\begin{aligned} H(Tx, Tz) &\leq M(x, z) \\ &\leq (a + b + 2c) \max\left\{d(x, z), D(x, Tx), D(z, Tz), \frac{D(x, Tz) + D(z, Tx)}{2}\right\}, \end{aligned}$$

where a, b and c are evaluated at the point (x, z).

Hence,

$$H(Tx, Tz) \leq r \max\left\{d(x, z), D(x, Tx), D(z, Tz), \frac{D(x, Tz) + D(z, Tx)}{2}\right\}.$$

Case (ii) If $d(x, z) < d(x, Tx)$, then

$$d(x,Tx) \le d(x,z) + r\,d(x,Tx) + \frac{1}{n}d(x,z)$$
$$\Rightarrow (1-r)d(x,Tx) \le (1+\frac{1}{n})d(x,z).$$

Taking $n \to \infty$, we have $(1-r)d(x,Tx) \le d(x,z)$, i.e., $\phi(r)d(x,Tx) \le d(x,z)$. Therefore, condition (3) implies

$$\begin{aligned} H(Tx,Tz) &\le M(x,z) \\ &\le (a+b+2c)\max\Big\{d(x,z), D(x,Tx), D(z,Tz), \\ &\frac{D(x,Tz)+D(z,Tx)}{2}\Big\}, \end{aligned}$$

where a, b and c are evaluated at the point (x,z).

Hence, we obtain (6) for all $x \in X$.

Now, from (6), we get

$$\begin{aligned} D(z,Tz) &= \lim_{n\to\infty} D(y_{n+1},Tz) \le \lim_{n\to\infty} H(Ty_n,Tz) \\ &\le \lim_{n\to\infty} r\max\left\{d(y_n,z), D(y_n,Ty_n), D(z,Tz), \frac{D(y_n,Tz)+D(z,Ty_n)}{2}\right\} \\ &\le \lim_{n\to\infty} r\max\left\{d(y_n,z), d(y_n,y_{n+1}), D(z,Tz), \frac{D(y_n,Tz)+d(z,y_{n+1})}{2}\right\} \\ &= r\max\left\{D(z,Tz), \frac{1}{2}D(z,Tz)\right\} = r\,D(z,Tz), \\ &\Rightarrow d(z,Tz) = 0. \end{aligned}$$

Since Tz is closed, hence $z \in Tz$. □

Remark 2.2 If we take $a(x, y)$, $b(x, y)$ and $c(x, y)$ as constants with $a+b+2c = r$ in Theorem 2.1, we get Theorem 1.2 due to Đorić et al. [5].

Taking $b(x,y) = c(x,y) = 0$ in Theorem 2.1, we get the following corollary which is generalization of Suzuki type contraction theorem [12, Theorem 2] for multivalued mappings.

Corollary 2.3 *Let* (X, d) *be a complete metric space and* $T: X \to CB(X)$*. Define a non-increasing function* $\phi: [0,1) \to (\frac{1}{2}, 1]$ *as in Theorem 1.2. Assume there exists* $a(x,y) \in [0,1)$ *such that*

$$\phi(r)D(x,Tx) \le d(x,y) \text{ implies } H(Tx,Ty) \le a(x,y)\,d(x,y)$$

for all $x, y \in X$*. Then* T *has a fixed point.*

Taking $a(x,y) = c(x,y) = 0$ in Theorem 2.1, we obtain the generalization of Kikkawa and Suzuki [6, Theorem 2.2] for multivalued mappings.

Corollary 2.4 *Let (X, d) be a complete metric space and $T : X \to CB(X)$. Define a non-increasing function $\phi : [0, 1) \to (\frac{1}{2}, 1]$ as in Theorem 1.2. Assume there exists $b(x, y) \in [0, 1)$ such that*

$$\phi(r)D(x, Tx) \leq d(x, y) \text{ implies } H(Tx, Ty) \leq b(x, y)\, max\{D(x, Tx), D(y, Ty)\}$$

for all $x, y \in X$. Then T has a fixed point.

Taking $a(x, y) = b(x, y) = 0$ in Theorem 2.1, then we get following result which is generalization of [3] for multivalued mappings.

Corollary 2.5 *Let (X, d) be a complete metric space and $T : X \to CB(X)$. Define a non-increasing function $\phi : [0, 1) \to (\frac{1}{2}, 1]$ as in Theorem 1.2. Suppose that there exists $c(x, y) \in [0, \frac{1}{2})$ such that*

$$\phi(r)D(x, Tx) \leq d(x, y) \text{ implies } H(Tx, Ty) \leq c(x, y)[D(x, Ty) + D(y, Tx)]$$

for all $x, y \in X$. Then T has a fixed point.

Taking T, a single valued mapping, we obtain following corollary.

Corollary 2.6 Let (X, d) be a complete metric space and $T : X \to X$. Define a non-increasing function $\phi : [0, 1) \to (\frac{1}{2}, 1]$ as in Theorem 1.2. Assume there exists $r \in [0, 1)$ such that $\phi(r)d(x, Tx) \leq d(x, y)$ implies $d(Tx, Ty) \leq M(x, y)$ for all $x, y \in X$, where $\sup_{x,y \in X}[a(x, y) + b(x, y) + 2c(x, y)] = r < 1$. Then T has a unique fixed point.

Here we give an example in the support of Theorem 2.1. Here it is to be noted that $a(x, y)$, $b(x, y)$ and $c(x, y)$ play an important role as variables justifying the Remark 2.2.

Example 2.7 Let $X = \{-1, 0, 1\}$. Taking d as usual metric, consider a mapping $T : X \to CB(X)$ defined by

$$Tx = \begin{cases} \{0, -1\} & \text{if } x \neq -1 \\ \{0, 1\} & \text{if } x = -1. \end{cases}$$

Taking $r = \sup_{x,y \in X}[a(x, y) + b(x, y) + 2c(x, y)] = \frac{2}{3} < 1$, we have following cases.

(i) For $x = 0$, $y = -1$, we have $H(Tx, Ty) = 1$ and $M(x, y) = 1$ with $a(x, y) = c(x, y) = 0, b(x, y) = \frac{1}{2}$.

(ii) For $x = 0$, $y = 1$, we have $H(Tx, Ty) = 0$ and $M(x, y) = \frac{1}{2}$ with $a(x, y) = c(x, y) = 0, b(x, y) = \frac{1}{2}$.

(iii) For $x = 1$, $y = 0$, we have $H(Tx, Ty) = 0$ and $M(x, y) = \frac{1}{2}$ with $a(x, y) = \frac{1}{2}, b(x, y) = c(x, y) = 0$.

(iv) For $x = 1$, $y = -1$, we have $H(Tx, Ty) = 1$ and $M(x, y) = 1$ with $a(x, y) = \frac{1}{2}, b(x, y) = c(x, y) = 0$.

(v) For $x = -1$, $y = 0$, we have $H(Tx, Ty) = 1$ and $M(x, y) = 1$ with $a(x, y) = b(x, y) = \frac{1}{3}, c(x, y) = 0$.

(vi) For $x = -1$, $y = 1$, we have $H(Tx, Ty) = 1$ and $M(x, y) = 1$ with $a(x, y) = b(x, y) = \frac{1}{3}, c(x, y) = 0$.

Hence for all $x, y \in X$, we have $\phi(r)D(x, Tx) \leq d(x, y)$ implies $H(Tx, Ty) \leq M(x, y)$, i.e., T satisfies the condition of Theorem 2.1 and has 0 as a fixed point.

Example 2.8 Let $X = \{-1, 0, 2\}$. Taking d as usual metric, consider a mapping $T : X \to CB(X)$ defined by

$$Tx = \begin{cases} \{0, 2\} & \text{if } x \neq 2 \\ \{-1, 2\} & \text{if } x = 2. \end{cases}$$

Define a non-increasing function $\phi : [0, 1) \to (\frac{1}{2}, 1]$ as in Theorem 1.2. In this example, if we assume $a(x, y)$, $b(x, y)$ and $c(x, y)$ as constants defined by $a(x, y) = \frac{1}{8}, b(x, y) = \frac{1}{8}$ and $c(x, y) = \frac{1}{8}$ for all $x, y \in X$, then T satisfies the condition of Theorem 1.2 with $r = a(x, y) + b(x, y) + 2c(x, y) = \frac{1}{2}$ but it does not satisfy condition of Theorem 2.1. For if, $(x, y) = (2, 0)$ then $\phi(r)D(x, Tx) \leq d(x, y)$, $H(Tx, Ty) = 1$ and $M(x, y) = \frac{1}{8} \cdot 2 + \frac{1}{8} \cdot 0 + \frac{1}{8} \cdot 1 = \frac{3}{8}$ which implies $H(Tx, Ty) > M(x, y)$. However, if we consider $a(x, y)$, $b(x, y)$ and $c(x, y)$ as variables, then T satisfies the condition of Theorem 2.1. To see this, we have the following calculations:

(i) For $x = 2$, $y = 0$, we have $\phi(r)D(x, Tx) \leq d(x, y)$, $H(Tx, Ty) = 1$ and $M(x, y) = 1$ with $a(x, y) = \frac{1}{2}, b(x, y) = 0, c(x, y) = 0$.

(ii) For $x = 2$, $y = -1$, we have $\phi(r)D(x, Tx) \leq d(x, y)$, $H(Tx, Ty) = 1$ and $M(x, y) = 1$ with $a(x, y) = \frac{1}{3}, b(x, y) = 0, c(x, y) = 0$.

(iii) For $x = 0$, $y = -1$, we have $\phi(r)D(x, Tx) \leq d(x, y)$, $H(Tx, Ty) = 0$ and $M(x, y) = 0$ with $a(x, y) = 0, b(x, y) = 0, c(x, y) = 0$.

(iv) For $x = 0$, $y = 2$, we have $\phi(r)D(x, Tx) \leq d(x, y)$, $H(Tx, Ty) = 1$ and $M(x, y) = 1$ with $a(x, y) = \frac{3}{2}, b(x, y) = 0, c(x, y) = 0$.

(v) For $x = -1$, $y = 0$, we have $\phi(r)D(x, Tx) \leq d(x, y)$, $H(Tx, Ty) = 0$ and $M(x, y) = 0$ with $a(x, y) = 0, b(x, y) = 0, c(x, y) = 0$.

(vi) For $x = -1$, $y = 2$, we have $\phi(r)D(x, Tx) \leq d(x, y)$, $H(Tx, Ty) = 1$ and $M(x, y) = 1$ with $a(x, y) = \frac{1}{3}, b(x, y) = 0, c(x, y) = 0$.

Hence for all $x, y \in X$, we have $\phi(r)D(x, Tx) \leq d(x, y)$ implies $H(Tx, Ty) \leq M(x, y)$ with $r = \sup_{x,y \in X}[a(x, y) + b(x, y) + 2c(x, y)] = \frac{1}{2} < 1$. Here, 0 and 2 are fixed points of T.

Remark 2.9 Let $X = \{-1, 0, 1\}$ be a usual metric space with metric d. Consider a mapping T as single valued mapping on X defined as

$$Tx = \begin{cases} 0 & \text{if } x \neq 1 \\ -1 & \text{if } x = 1. \end{cases}$$

Then by simple calculation, we get $\phi(r)d(x, Tx) \leq d(x, y)$ implies $d(Tx, Ty) \leq M(x, y)$ with $r = \sup_{x,y \in X}[a(x, y) + b(x, y) + 2c(x, y)] = \frac{3}{4} < 1$ and 0 is only fixed point of T.

3 An application to dynamic programming

There exists many applications of various fixed point theorems in dynamic programming for the existence and uniqueness of solutions of functional equations/ system of functional equations (see, [8, 11] and the refenences therein). Here, we apply above theorem to prove a result which gives a solution for a class of functional equations.

Let U and V be Banach spaces and $W \subset U, D \subset V$ over the field $\mathbb{R}$. Let $B(W)$ denote the set of all bounded real valued functions on W. It is well known that $B(W)$ endowed with the metric

$$d_B(h,k) = \sup_{x \in W} |h(x) - k(x)|, \quad h,k \in B(W) \tag{7}$$

is a complete metric space. Bellman and Lee [1] gave the following basic form of the functional equation of dynamic programming:

$$p(x) = \sup_y H(x,y,p(\tau(x,y))), \tag{8}$$

where x and y respresent the state and decision vectors respectively. $\tau : W \times D \to W$ represents the transformation of the process and $p(x)$ represents the optimal return function with initial state x.

Now, we will study the existence and uniqueness of the solution of the following functional equation:

$$p(x) = \sup_y [g(x,y) + G(x,y,p(\tau(x,y)))], \quad x \in W \tag{9}$$

where $g : W \times D \to \mathbb{R}$ and $G : W \times D \times \mathbb{R} \to \mathbb{R}$ are bounded functions. Following [5], let a function ϕ be defined as in Theorem 1.2 and the mapping $T : B(W) \to B(W)$ be defined by

$$T(h(x)) = \sup_y \{g(x,y) + G(x,y,h(\tau(x,y)))\} \quad h \in B(W), \quad x \in W. \tag{10}$$

Theorem 3.1 *If there exists* $\sup_{t \in W}[a(h(t),k(t)) + b(h(t),k(t)) + 2c(h(t),k(t))] = r \in [0,1)$ *such that*

$$\begin{aligned} &\phi(r) d_B(T(h),h) \le d_B(h,k) \text{ implies} \\ &\quad |G(x,y,h(t)) - G(x,y,k(t))| \le M(h(t),k(t)) \end{aligned} \tag{11}$$

for every $(x,y) \in W \times D$, $h,k \in B(W)$ *and* $t \in W$, *where*

$$\begin{aligned} M(h(t),k(t)) = {} & a(h(t),k(t))|h(t) - k(t)| \\ & + \; b(h(t),k(t)) \max\{|h(t) - T(h(t))|, |k(t) - T(k(t))|\} \\ & + \; c(h(t),k(t)) \, [|h(t) - T(k(t))| + |k(t) - T(h(t))|], \end{aligned}$$

then the functional equation (9) *has a unique bounded solution in* $B(W)$.

Proof Let ϵ be an arbitrary positve real number and $h_1, h_2 \in B(W)$. Then for $x \in W$, we can choose $y_1, y_2 \in D$ so that

$$T(h_1(x)) < g(x, y_1) + G(x, y, h_1(\tau(x, y_2))) + \epsilon \tag{12}$$

$$T(h_2(x)) < g(x, y_1) + G(x, y, h_2(\tau(x, y_1))) + \epsilon \tag{13}$$

Also from (10),

$$T(h_1(x)) \geq g(x, y_2) + G(x, y_2, h_1(\tau(x, y_2))) \tag{14}$$

$$T(h_2(x)) \geq g(x, y_1) + G(x, y_1, h_2(\tau(x, y_1))) \tag{15}$$

If the inequality (11) holds, then from (12) and (15),

$$\begin{aligned} T(h_1(x)) - T(h_2(x)) &< G(x, y_1, h_1(\tau(x, y_1))) - G(x, y_1, h_2(\tau(x, y_1))) + \epsilon \\ &\leq |G(x, y_1, h_1(\tau(x, y_1))) - G(x, y_1, h_2(\tau(x, y_1)))| + \epsilon \\ &\leq M(h_1(x), h_2(x)) + \epsilon. \end{aligned} \tag{16}$$

Similarly from (13) and (14), we obatin

$$T(h_2(x)) - T(h_1(x)) \leq M(h_1(x), h_2(x)) + \epsilon \tag{17}$$

From (16) and (17), we establish

$$|T(h_1(x)) - T(h_2(x))| \leq M(h_1(x), h_2(x)) + \epsilon \tag{18}$$

which is true for each $x \in W$ and arbitrary $\epsilon > 0$.

Hence

$$\phi(r) d_B(T(h_1), h_1) \leq d_B(h_1, h_2) \Rightarrow d_B(T(h_1), T(h_2)) \leq M(h_1, h_2),$$

where a, b and c are evaluated at the point $(h_1(x), h_2(x))$. Hence, the conditions of Corollary 2.6 are satisfied for the mapping T and so, the functional equation (9) has a unique bounded solution. □

Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

Ethical approval This chapter does not contain any studies with human participants or animals performed by any of the authors.

Informed consent Informed consent was obtained from all individual participants included in the study.

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