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ORIGINAL RESEARCH PAPER

New congruences for k-tuples t-core partitions

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Abstract Let $A_{t,k}(n)$ denote the number of partition k-tuples of n where each partition is t-core. In this paper, we prove some Ramanujan-type congruences for the partition function $A_{t,k}(n)$ when $(t,k) = (3,4)$, (3,9), (4,8), (5, 6), (8, 4), (9, 3) and (9, 6) by employing q-series identities.

Keywords t-core partition · k-tuple · Partition congruence · q-series identities

Mathematics Subject Classification 05A17 · 11P83

1 Introduction

A partition of a positive integer n is a non-increasing sequence of positive integers, called parts, whose sum equals n. Let $p(n)$ denote the number of partition of n. For convenience, we shall set $p(0) = 1$. The generating function for $p(n)$ is given by

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$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty}, \tag{1}$$

where

$$(a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n), |q|<1. \tag{2}$$

Most popular congruences of $p(n)$ that discovered by Ramanujan for $n \geq 0$,

$$p(5n+4) \equiv 0 \pmod 5 \tag{3}$$

$$p(7n+5) \equiv 0 \pmod 7 \tag{4}$$

$$p(11n+6) \equiv 0 \pmod{11} \tag{5}$$

Ramanujan's work inspired scholars to study the arithmetic properties for the other types of partition functions such as t-core partition. A partition of n is called a t-core of n if none of its hook number is a multiple of t. Let $A_t(n)$ denote the number of t-core partitions of n, the generating function of $A_t(n)$ is given by

$$\sum_{n=0}^{\infty} A_t(n)q^n = \frac{(q^t;q^t)_\infty^t}{(q;q)_\infty}. \tag{6}$$

The arithmetic properties of t-core partition function have been studied by several authors, see [2, 4, 5, 8, 10, 15]. A partition k-tuple $(\lambda_1, \lambda_2, \ldots, \lambda_k)$ of a positive integer n is a k-tuple of partitions $\lambda_1, \lambda_2, \ldots, \lambda_k$ such that the sum of all the parts equals n. A partition k-tuple $(\lambda_1, \ldots, \lambda_k)$ of n with t-cores means that each λ_i is t-core. Let $A_{t,k}(n)$ denote the number of partition k-tuples of n with t-cores. The generating function of $A_{t,k}(n)$ can be obtained as

$$\sum_{n=0}^{\infty} A_{t,k}(n)q^n = \frac{(q^t;q^t)_\infty^{kt}}{(q;q)_\infty^k}. \tag{7}$$

Wang [12] proved some arithmetic identities and congruences for partition triples with 3-cores. Recently, Chern [9] studied the function $A_{t,k}(n)$ and proved some identities by employing the method of modular form. In sequel, in this paper we study the arithmetic properties of $A_{t,k}(n)$ for $(t,k) = (3,4), (3,9), (4,8)$, (5, 6), (8, 4), (9, 3) and (9, 6) by using q-series identities and prove some Ramanujan-type congruences.

In Sect. 3, we prove some congruence and infinite family of congruences for $A_{3,4}$ for modulo 4. For example, we prove for $\alpha \geq 0$,

$$A_{3,4}\left(2^{2(\alpha+1)+1}n + \frac{11 \cdot 2^{2\alpha+1} - 4}{3}\right) \equiv 0 \pmod 4. \tag{8}$$

In Sect. 4, we prove arithmetic identities and congruences for $A_{3,9}$ modulo 3 and 9. For example, we prove, for $k \geq 0$,

$$A_{3,9}\left(9\cdot 2^{2k+2}n+30\cdot 2^{2k}-3\right)\equiv 0\,(\mathrm{mod}\,3). \tag{9}$$

In Sect. 5, we prove congruences for $A_{4,8}$ modulo 4. In Sect. 6, we prove congruences for $A_{8,4}$ modulo 2. In Sects. 7 and 8, we prove some congruences for $A_{9,3}$ and $A_{9,6}$. Section 2 is devoted to record some preliminary results.

2 Preliminaries

Lemma 2.1 *For any prime p, we have*

$$(q^p;q^p)_\infty\equiv(q;q)_\infty^p\,(\mathrm{mod}\,p).$$

Proof Follows easily from binomial theorem. □

Lemma 2.2 [1, Lemma 1.4] *For any prime p, we have*

$$(q;q)_\infty^{p^2}\equiv(q^p;q^p)_\infty^p\,(\mathrm{mod}\,p^2).$$

Proof Follows easily from binomial theorem. □

Lemma 2.3 [13, Eq. (2.11)] *We have*

$$\frac{1}{(q;q)_\infty^4}=\frac{(q^4;q^4)_\infty^{14}}{(q^2;q^2)_\infty^{14}(q^8;q^8)_\infty^4}+4q\frac{(q^4;q^4)_\infty^2(q^8;q^8)_\infty^4}{(q^2;q^2)_\infty^{10}}. \tag{10}$$

Lemma 2.4 [3, p. 648, Eq. (2.9)] *For any integer* $k\geq 1$, *we have*

$$p_8\left(2^{2k}n+\frac{2^{2k}-1}{3}\right)=(-8)^k p_8(n), \tag{11}$$

where

$$\sum_{n=0}^{\infty}p_8(n)q^n=(q;q)_\infty^8. \tag{12}$$

Lemma 2.5 [13, Eq. (3.75)] *We have*

$$\frac{(q^3;q^3)_\infty^3}{(q;q)_\infty}=\frac{(q^4;q^4)_\infty^3(q^6;q^6)_\infty^2}{(q^2;q^2)_\infty^2(q^{12};q^{12})_\infty}+q\frac{(q^{12};q^{12})_\infty^3}{(q^4;q^4)_\infty}. \tag{13}$$

Lemma 2.6 [14, Lemma 2.1, Eq. (2.3)] *We have*

$$(q;q)_\infty^8=\frac{(q^4;q^4)_\infty^{20}}{(q^2;q^2)_\infty^4(q^8;q^8)_\infty^8}+16q^2\frac{(q^2;q^2)_\infty^4(q^8;q^8)_\infty^8}{(q^4;q^4)_\infty^4}-8q(q^4;q^4)_\infty^8. \tag{14}$$

Lemma 2.7 [11] *We have*

$$\begin{aligned}\frac{1}{(q;q)_\infty}=\frac{(q^{25};q^{25})_\infty^5}{(q^5;q^5)_\infty^6}\big(&F^{-4}(q^5)+qF^{-3}(q^5)+2q^2F^{-2}(q^5)\\&+3q^3F^{-1}(q^5)+5q^4-3q^5F(q^5)\\&+2q^6F^2(q^5)-q^7F^3(q^5)+q^8F^4(q^5)\big),\end{aligned} \tag{15}$$

where $F(q):=q^{-1/5}R(q)$ *and* $R(q)$ *is Roger's Ramanujan continued fraction defined by*

$$R(q):=\frac{q^{1/5}}{1}\,{}_{+}\frac{q}{1}\,{}_{+}\frac{q^2}{1}\,{}_{+}\frac{q^3}{1}\,{}_{+\cdots},\quad |q|<1.$$

Lemma 2.8 [7, p. 345, Entry 1(iv)] *We have*

$$(q;q)_\infty^3=(q^9;q^9)_\infty^3\big(4q^3W^2(q^3)-3q+W^{-1}(q^3)\big), \tag{16}$$

where $W(q)=q^{-1/3}G(q)$ *and* $G(q)$ *is the Ramanujan's cubic continued fraction defined by*

$$G(q):=\frac{q^{1/3}}{1}\,{}_{+}\frac{q+q^2}{1}\,{}_{+}\frac{q^2+q^4}{1}\,{}_{+\cdots},\quad |q|<1.$$

Lemma 2.9 [6, Eq. (3.9)] *We have*

$$\begin{aligned}\frac{1}{(q;q)_\infty^3}=\frac{(q^9;q^9)_\infty^9}{(q^3;q^3)_\infty^{12}}\bigg(&\frac{1}{w^2(q^3)}+\frac{3q}{w(q^3)}+9q^2\\&+8q^3w(q^3)+12q^4w^2(q^3)+16q^6w^4(q^3)\bigg),\end{aligned} \tag{17}$$

where $w(q)=\frac{(q;q)_\infty(q^6;q^6)_\infty^3}{(q^2;q^2)_\infty(q^3;q^3)_\infty^3}$.

3 Congruences for $A_{3,4}(n)$ modulo 4

Theorem 3.1 *For* $n\geq 0$, *we have*

$$(i)A_{3,4}(2n+1) \equiv 0 \pmod 4,$$
$$(ii)A_{3,4}(8n+6) \equiv 0 \pmod 4,$$
$$(iii)A_{3,4}(16n+8) \equiv 0 \pmod 4.$$

Proof Setting $t = 3$ and $k = 4$ in (7), we obtain

$$\sum_{n=0}^{\infty} A_{3,4}(n)q^n = \frac{(q^3;q^3)_\infty^{12}}{(q;q)_\infty^4}. \tag{18}$$

Using Lemma 2.2 in (18), we obtain

$$\sum_{n=0}^{\infty} A_{3,4}(n)q^n \equiv \frac{(q^6;q^6)_\infty^6}{(q^2;q^2)_\infty^2} \pmod 4. \tag{19}$$

Since there are no terms containing q^{2n+1} in (19), we complete the proof (i).

Extracting terms involving q^{2n} and replacing q^2 by q from (19), we have

$$\sum_{n=0}^{\infty} A_{3,4}(2n)q^n \equiv \frac{(q^3;q^3)_\infty^6}{(q;q)_\infty^2} \pmod 4. \tag{20}$$

Using Lemma 2.5 in (20) and squaring, we obtain

$$\sum_{n=0}^{\infty} A_{3,4}(2n)q^n \equiv \left(\frac{(q^4;q^4)_\infty^6 (q^6;q^6)_\infty^4}{(q^2;q^2)_\infty^4 (q^{12};q^{12})_\infty} + 2q \frac{(q^4;q^4)_\infty^3 (q^6;q^6)_\infty^2 (q^{12};q^{12})_\infty^2}{(q^2;q^2)_\infty^2 (q^{12};q^{12})_\infty (q^2;q^2)_\infty^2} + q^2 \frac{(q^{12};q^{12})_\infty^6}{(q^4;q^4)_\infty^2} \right) \pmod 4. \tag{21}$$

Extracting terms involving q^{2n+1} in (21), dividing by q and replacing q^2 by q, we obtain

$$\sum_{n=0}^{\infty} A_{3,4}(4n+2)q^n \equiv 2(q;q)_\infty^2 (q^3;q^3)_\infty^6 \pmod 4. \tag{22}$$

Equation (22) can be written as

$$\sum_{n=0}^{\infty} A_{3,4}(4n+2)q^n \equiv 2(q;q)_\infty^4 \frac{(q^3;q^3)_\infty^6}{(q;q)_\infty^2} \pmod 4. \tag{23}$$

Again using Lemma 2.5 in (23), we have

$$\sum_{n=0}^{\infty} A_{3,4}(4n+2)q^n \equiv 2\frac{(q^4;q^4)_\infty^6}{(q^2;q^2)_\infty^2} + 2q^2 \frac{(q^2;q^2)_\infty^2 (q^{12};q^{12})_\infty^6}{(q^4;q^4)_\infty^2} \pmod 4. \tag{24}$$

Extracting terms containing q^{2n+1} in (24), we arrive at (ii).

Extracting terms involving q^{2n} in (21) and replacing q^2 by q, we obtain

$$\sum_{n=0}^{\infty} A_{3,4}(4n)q^n \equiv (q^2;q^2)_\infty^4 + q\frac{(q^6;q^6)_\infty^6}{(q^2;q^2)_\infty^2} \pmod 4. \tag{25}$$

Extracting terms involving q^{2n} in (25) and replacing q^2 by q , we obtain

$$\sum_{n=0}^{\infty} A_{3,4}(8n)q^n \equiv (q;q)_\infty^4 \equiv (q^2;q^2)_\infty^2 \pmod 4. \tag{26}$$

Again, extracting terms containing q^{2n+1} in (26), we arrive at (iii). □

Theorem 3.2 *For any positive integer* n, *and* $\alpha \geq 0$, *we have*

$$A_{3,4}(2n) \equiv A_{3,4}\left(2^{2\alpha+1}n + 4\cdot\frac{2^{2\alpha}-1}{3}\right) \pmod 4. \tag{27}$$

Proof Extracting terms involving q^{2n+1} in (25), dividing by q, and replacing q^2 by q, we have

$$\sum_{n=0}^{\infty} A_{3,4}(8n+4)q^n \equiv \frac{(q^3;q^3)_\infty^6}{(q;q)_\infty^2} \pmod 4. \tag{28}$$

From (20) and (28), we can deduce that

$$A_{3,4}(2n) \equiv A_{3,4}(8n+4) \pmod 4. \tag{29}$$

Replacing n by $4n+2$ in (29) and iterating, we arrive at the desired result. □

Theorem 3.3 *For any positive integer* n, *and* $\alpha \geq 0$, *we have*

$$A_{3,4}\left(2^{2(\alpha+1)+1}n + \frac{11\cdot 2^{2\alpha+1}-4}{3}\right) \equiv 0 \pmod 4. \tag{30}$$

Proof Replacing n by $4n+3$ in (27) and employing Theorem 3.1(ii), we complete the proof. □

4 Congruences for $A_{3,9}(n)$ modulo 3 and 9

Theorem 4.1 *For any positive integer* n *and* $k \geq 0$, *we have*

$$\sum_{n=0}^{\infty} A_{3,9}(9\cdot 2^{2k}n + 3(2^{2k}-1))q^n \equiv (q;q)_\infty^8 \pmod 3. \tag{31}$$

Proof Setting $t=3$ and $k=9$ in (7), we obtain

$$\sum_{n=0}^{\infty} A_{3,9}(n)q^n = \frac{(q^3;q^3)_\infty^{27}}{(q;q)_\infty^9}. \tag{32}$$

Using Lemma 2.1 in (32), we obtain

$$\sum_{n=0}^{\infty} A_{3,9}(n)q^n \equiv \frac{(q^9;q^9)_\infty^9}{(q^9;q^9)_\infty} (\text{mod } 3). \tag{33}$$

Extracting terms involving q^{9n} in (33) and replacing q^9 by q, we obtain

$$\sum_{n=0}^{\infty} A_{3,9}(9n)q^n \equiv (q;q)_\infty^8 (\text{mod } 3). \tag{34}$$

Using (12) in (34), we obtain

$$\sum_{n=0}^{\infty} A_{3,9}(9n)q^n \equiv p_8(n)(\text{mod } 3). \tag{35}$$

Employing Lemma 2.4 in (35), we arrive at the desired result. □

Theorem 4.2 *For $n \geq 0$, we have*

$$A_{3,9}(9n+j) \equiv 0\,(\text{mod } 3),$$

where $j = 1,2,3,4,5,6,7,8$

Proof Extracting terms containing q^{9n+j} for $1 \leq j \leq 8$, from both sides of (33), we arrive at the desired result. □

Theorem 4.3 *For any positive integer n and $k \geq 0$, we have*

$$A_{3,9}(9 \cdot 2^{2k+2}n + 30 \cdot 2^{2k} - 3) \equiv 0\,(\text{mod } 3). \tag{36}$$

Proof Using Lemma 2.6 in (31) and then extracting terms involving q^{4n+3}, dividing by q^3 and replacing q^4 by q ,we complete the proof. □

Theorem 4.4 *For $n \geq 0$, we have*

$$A_{3,9}(3n) \equiv \tau(n+1)\,(\text{mod } 9)$$

where τ is the Ramanujan's tau function defined by

$$q(q;q)_\infty^{24} = \sum_{n=1}^{\infty} \tau(n)q^n \tag{37}$$

Proof Setting $t = 3$ and $k = 9$ in (7), we obtain

$$\sum_{n=0}^{\infty} A_{3,9}(n)q^n = \frac{(q^3;q^3)_\infty^{27}}{(q;q)_\infty^9}. \tag{38}$$

Using Lemma 2.2 in (38), we obtain

$$\sum_{n=0}^{\infty} A_{3,9}(n)q^n \equiv \frac{(q^3;q^3)_\infty^{27}}{(q^3;q^3)_\infty^3} \pmod 9. \tag{39}$$

Extracting terms involving q^{3n} in (39) and replacing q^3 by q, we obtain

$$\sum_{n=0}^{\infty} A_{3,9}(3n)q^n \equiv (q;q)_\infty^{24} \pmod 9. \tag{40}$$

From (40) and (37), we deduce that

$$\sum_{n=0}^{\infty} A_{3,9}(3n)q^{n+1} \equiv \sum_{n=0}^{\infty} \tau(n+1) \pmod 9. \tag{41}$$

From (41), we easily arrive at the desired result. □

5 Congruences for $A_{4,8}(n)$ modulo 4

Theorem 5.1 *For $n \geq 0$, we have*

$$A_{4,8}(4n+j) \equiv 0 \pmod 4; \quad j = 1,2,3.$$

Proof Setting $t = 4$ and $k = 8$ in (7), we obtain

$$\sum_{n=0}^{\infty} A_{4,8}(n)q^n = \frac{(q^4;q^4)_\infty^{32}}{(q;q)_\infty^8}. \tag{42}$$

Applying Lemma 2.2 in (42), we obtain

$$\sum_{n=0}^{\infty} A_{4,8}(n)q^n \equiv (q^4;q^4)_\infty^{30} \pmod 4. \tag{43}$$

Extracting terms involving q^{4n+j} for $j = 1,2$, and 3 in (43), we complete the proof. □

6 Congruences for $A_{5,6}(n)$ modulo 3 and 5

Theorem 6.1 *For $n \geq 0$, we have*

$$(i)\ A_{5,6}(3n+1) \equiv 0 \pmod 3$$
$$(ii)\ A_{5,6}(3n+2) \equiv 0 \pmod 3$$
$$(iii)\ A_{5,6}(5n+4) \equiv 0 \pmod 5$$

Proof Setting $t = 5$ and $k = 6$ in (7), we obtain

$$\sum_{n=0}^{\infty} A_{5,6}(n)q^n = \frac{(q^5;q^5)_\infty^{30}}{(q;q)_\infty^6}. \tag{44}$$

Applying Lemma 2.1 in (44) we obtain

$$\sum_{n=0}^{\infty} A_{5,6}(n)q^n \equiv \frac{(q^{15};q^{15})_\infty^{10}}{(q^3;q^3)_\infty^3}(q;q)_\infty^3 \pmod 3. \tag{45}$$

Using Lemma 2.8 in (45), we obtain

$$\sum_{n=0}^{\infty} A_{5,6}(n)q^n \equiv \frac{(q^{15};q^{15})_\infty^{10}}{(q^3;q^3)_\infty^3}\left(4q^3w^2(q^3) - 3q + w^{-1}(q^3)\right) \pmod 3. \tag{46}$$

Extracting terms containing q^{3n+1} and q^{3n+2} in (46), we complete the proof of (i) and (ii), respectively.

Applying Lemma 2.1 in (44), we obtain

$$\sum_{n=0}^{\infty} A_{5,6}(n)q^n \equiv \frac{(q^5;q^5)_\infty^{30}}{(q^5;q^5)_\infty(q;q)_\infty} \pmod 5. \tag{47}$$

Using Lemma 2.7 in (47) and extracting terms involving q^{5n+4}, dividing by q^4 and replacing q^5 by q, we can easily obtain (iii). □

7 Congruences for $A_{8,4}(n)$ modulo 2

Theorem 7.1 *For* $n \geq 0$, *we have*

$$(i)\ A_{8,4}(4n+1) \equiv 0 \pmod 2$$
$$(ii)\ A_{8,4}(4n+2) \equiv 0 \pmod 2$$
$$(iii)\ A_{8,4}(4n+3) \equiv 0 \pmod 2$$

Proof Setting $t = 8$ and $k = 4$ in (7), we obtain

$$\sum_{n=0}^{\infty} A_{8,4}(n)q^n = \frac{(q^8;q^8)_\infty^{32}}{(q;q)_\infty^4}. \tag{48}$$

Applying Lemma 2.3 in (48) and extracting the terms involving q^{4n+1}, q^{4n+2} and q^{4n+3}, we complete the proof of (i), (ii), and (iii), respectively. □

8 Congruences for $A_{9,3}(n)$ modulo 3

Theorem 8.1 *For $n \geq 0$, we have*

$$(i)\ A_{9,3}(3n+1) \equiv 0 \pmod 3$$
$$(ii)\ A_{9,3}(3n+2) \equiv 0 \pmod 3$$

Proof Setting $t = 9$ and $k = 3$ in (7), we obtain

$$\sum_{n=0}^{\infty} A_{9,3}(n)q^n = \frac{(q^9;q^9)_\infty^{27}}{(q;q)_\infty^3}. \tag{49}$$

Using Lemma 2.9 in (49) and extracting the terms involving q^{3n+1} and q^{3n+2}, we complete the proof of (i) and (ii), respectively. □

9 Congruences for $A_{9,6}(n)$ modulo 3

Theorem 9.1 *For $n \geq 0$, we have*

$$(i)\ A_{9,6}(3n+1) \equiv 0 \pmod 3$$
$$(ii)\ A_{9,6}(3n+2) \equiv 0 \pmod 3$$

Proof Setting $t = 9$ and $k = 6$ in (7), we obtain

$$\sum_{n=0}^{\infty} A_{9,6}(n)q^n = \frac{(q^9;q^9)_\infty^{54}}{(q;q)_\infty^6} = \frac{(q^9;q^9)_\infty^{54}(q;q)_\infty^3}{(q;q)_\infty^9}. \tag{50}$$

Using Lemma 2.8 in (50), we obtain

$$\sum_{n=0}^{\infty} A_{9,6}(n)q^n \equiv \frac{(q^9;q^9)_\infty^{54}}{(q^3;q^3)_\infty^3}\left(4q^3w^2(q^3) - 3q + w^{-1}(q^3)\right) \pmod 3. \tag{51}$$

Extracting terms involving q^{3n+1} and q^{3n+2} in (51), we complete the proof of (i) and (ii), respectively. □

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