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Third Hankel determinant of starlike and convex functions

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Abstract For an analytic function f of the form $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ satisfying either $\mathrm{Re}\big((f'(z))^{\alpha}(zf'(z)/f(z))^{(1-\alpha)}\big) > 0$ or $\mathrm{Re}\big((f'(z))^{\alpha}(1 + zf''(z)/f'(z))^{(1-\alpha)}\big) > 0$, the bounds for the third Hankel determinant $H_3(1) = a_3(a_2a_4 - a_3^2) - a_4(a_4 - a_2a_3) + a_5(a_3 - a_2^2)$ are obtained. Our results include some previously known results.

Keywords Starlike functions · Convex functions · Coefficient conditions · Subordination · Hankel determinant

Mathematics Subject Classification 30C45 · 30C55 · 30C80

1 Introduction

Let $\mathcal{A}$ be the class of all normalized analytic functions of the form $f(z) = z + \sum_{n=2}^{\infty} a_nz^n$ in the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of univalent functions in $\mathbb{D}$. The qth Hankel determinant (denoted by $H_q(n)$) for $q = 1, 2, \ldots$ and $n = 1, 2, 3, \ldots$ of the function f is the $q \times q$ determinant given by $H_q(n) := \det(a_{n+i+j-2})$. Here $a_{n+i+j-2}$ denotes the entry for the ith row and

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jth column of the matrix. The second Hankel determinant $H_2(2) := a_2a_4 - a_3^2$ for the class of functions whose derivative has positive real part, the classes of starlike and convex functions, close-to-starlike and close-to-convex functions with respect to symmetric points have been studied in [3, 4] respectively. One may refer to the survey given by Liu et al. [7] for the other work done in the research of Hankel determinant for univalent functions. Other than the survey, their paper also contains bounds on the second Hankel determinant for some other subclasses of analytic functions. Other interesting papers on this topic include [6, 9].

The third Hankel determinant for the class of starlike and convex functions was studied by Babalola [1]. Shanmugam et al. [12] obtained the third Hankel determinant $H_3(1)$ for the class of α-starlike functions. The third Hankel determinant for the class of close to convex functions can be referred to in [10], for a subclass of p-valent functions has been studied in [13], for a class of analytic functions associated with the lemniscate of Bernoulli in [11] and for starlike and convex functions with respect to symmetric points in [8]. One can refer to [14] for the third Hankel determinant for the inverse of a function whose derivative has a positive real part and [12] for α-starlike functions.

An analytic function f is said to be subordinate to F, written $f \prec F$ or $f(z) \prec F(z)$, $(z \in \mathbb{D})$ if there exists an analytic function $w : \mathbb{D} \to \mathbb{D}$ satisfying $w(0) = 0$ and $f(z) = F(w(z))$ in $\mathbb{D}$. If F is univalent in $\mathbb{D}$, then $f(z) \prec F(z)$ if and only if $f(0) = F(0)$ and $f(\mathbb{D}) \subseteq F(\mathbb{D})$. Let φ be a univalent function with positive real part, $\varphi(0) = 1$ and $\varphi'(0) > 0$. In this paper, we determine the bounds on the third Hankel determinant $H_3(1)$ for the functions f in the classes $\mathcal{M}_\alpha$ and $\mathcal{L}_\alpha$ defined by:

$$\mathcal{M}_\alpha := \left\{ f \in \mathcal{S} : \mathrm{Re}\left((f'(z))^\alpha \left(\frac{zf'(z)}{f(z)} \right)^{1-\alpha} \right) > 0 \right\},$$

and

$$\mathcal{L}_\alpha := \left\{ f \in \mathcal{S} : \mathrm{Re}\left((f'(z))^\alpha \left(1 + \frac{zf''(z)}{f'(z)} \right)^{1-\alpha} \right) > 0 \right\}.$$

2 Third Hankel determinant

The first theorem gives the coefficient bounds for the first five coefficients for the functions in the class $\mathcal{M}_\alpha$ which is the class of all analytic functions $f \in \mathcal{S}$ satisfying the following inequality

$$\mathrm{Re}\left((f'(z))^\alpha \left(\frac{zf'(z)}{f(z)} \right)^{1-\alpha} \right) > 0$$

Note that

$$\mathcal{S}^* = \mathcal{M}_0 = \left\{ f \in \mathcal{S} : \mathrm{Re}\left(\frac{zf'(z)}{f(z)} \right) > 0 \right\} \qquad \text{and} \qquad \mathcal{R} = \mathcal{M}_1 = \{ \mathrm{f} \in \mathcal{S} : \mathrm{Re}(\mathrm{f}'(z)) > 0 \}$$

are the classes of starlike functions and the class of functions whose derivative has positive real part respectively. The latter class is a subclass of the close-to-convex functions. Thus as α varies from 0 to 1, our class $\mathcal{M}_\alpha$ provides a continuous passage from the class of $\mathcal{S}^*$ of starlike functions to the the class $\mathcal{R}$ of functions whose derivative has positive real part.

Theorem 2.1 *If the function* $f \in \mathcal{M}_\alpha$*, then the coefficients* a_n $(n = 2, 3, 4, 5)$ *of* f *satisfy:*

$$|a_2| \le \frac{2}{(1+\alpha)},$$

$$|a_3| \le \frac{2(3+\alpha)}{(2+\alpha)(1+\alpha)^2},$$

$$|a_4| \le \frac{2\left(36 + 19\alpha + 11\alpha^2 + 5\alpha^3 + \alpha^4\right)}{3(1+\alpha)^3(2+\alpha)(3+\alpha)},$$

and

$$|a_5| \le \frac{2\left(360 + 433\alpha + 437\alpha^2 + 331\alpha^3 + 137\alpha^4 + 28\alpha^5 + 2\alpha^6\right)}{3(1+\alpha)^4(2+\alpha)^2(3+\alpha)(4+\alpha)}.$$

Proof Since $f \in \mathcal{M}_\alpha$, there is an analytic function $p(z) = 1 + c_1 z + c_2 z^2 + \cdots \in \mathcal{P}$ such that

$$(f'(z))^\alpha \left(\frac{zf'(z)}{f(z)}\right)^{1-\alpha} = p(z). \tag{2.1}$$

The Taylor series expansion of the function f gives

$$\begin{aligned}(f'(z))^\alpha \left(\frac{zf'(z)}{f(z)}\right)^{1-\alpha} &= 1 + a_2(1+\alpha)z + \frac{1}{2}((2+\alpha)(2a_3 - (1-\alpha)a_2^2)z^2 \\ &\quad + \frac{1}{6}(3+\alpha)(6a_4 - 6(1-\alpha)a_2a_3 + (1-\alpha)(2-\alpha)a_2^3))z^3 + \cdots.\end{aligned} \tag{2.2}$$

Then using Eqs. (2.1), (2.2) and the expansion of the function p, the coefficients a_2–a_5 can be expressed as a function of the coefficients c_i of $p \in \mathcal{P}$:

$$a_2 = \frac{c_1}{(1+\alpha)}, \tag{2.3}$$

$$a_3 = \frac{1}{2(2+\alpha)(1+\alpha)^2}\left(2(1+\alpha)^2 c_2 + (1-\alpha)(2+\alpha)c_1^2\right), \tag{2.4}$$

$$a_4 = \frac{1}{6(1+\alpha)^3(2+\alpha)(3+\alpha)}\big((1-\alpha)(2+\alpha)(3+\alpha)(1-2\alpha)c_1^3 \\ + 6(1+\alpha)^3(2+\alpha)c_3 + 6(1+\alpha)^2(1-\alpha)(3+\alpha)c_1c_2\big), \tag{2.5}$$

and

$$a_5 = \frac{1}{24(1+\alpha)^4(2+\alpha)^2(3+\alpha)(4+\alpha)}\big(24(1+\alpha)^3(2+\alpha)^2(1-\alpha)(4+\alpha)c_1c_3 \\ + 24(1+\alpha)^4(3+\alpha)(2+\alpha)^2c_4 + 12(1+\alpha)^4(3+\alpha)(1-\alpha)(4+\alpha)c_2^2 \\ + 12(1+\alpha)^2(1-\alpha)(2+\alpha)(3+\alpha)(4+\alpha)(1-2\alpha)c_1^2c_2 \\ + (2+\alpha)^2(1-\alpha)(3+\alpha)(4+\alpha)(1-2\alpha)(1-3\alpha)c_1^4\big). \tag{2.6}$$

Consequently, using the triangle inequality and the fact that $|c_k| \leq 2$ $(k = 1, 2, 3, \ldots)$, we arrive at the desired bounds for a_2, a_3, a_4 and a_5.

We now prove some results which will be required to estimate the third Hankel determinant $H_3(1)$ for functions in the class $\mathcal{M}_\alpha$. We make use of the following lemma in proving our next result:

Lemma 2.2 [2] *If the function* $p \in \mathcal{P}$ *and is given by*

$$p(z) = 1 + c_1z + c_2z^2 + c_3z^3 + \cdots \tag{2.7}$$

then,

$$2c_2 = c_1^2 + x(4 - c_1^2), \tag{2.8}$$

$$4c_3 = c_1^3 + 2(4 - c_1^2)c_1x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)y, \tag{2.9}$$

for some x, y with $|x| \leq 1$ *and* $|y| \leq 1$.

Theorem 2.3 *Let*

$$\alpha_0 = \frac{1}{3}\left((77 + 2\sqrt{1489})^{1/3} - \frac{3}{(77 + 2\sqrt{1489})^{1/3}} - 4\right) \approx 0.267554$$

For the function $f \in \mathcal{M}_\alpha$*, the following coefficient bounds hold:*

1. *When* $0 \leq \alpha \leq \alpha_0$*, then* $|a_2a_3 - a_4| \leq \dfrac{2(18 - \alpha - 4\alpha^2 - \alpha^3)}{3(1+\alpha)^2(2+\alpha)(3+\alpha)}$.
2. *When* $\alpha_0 \leq \alpha \leq 1$*, then* $|a_2a_3 - a_4| \leq \dfrac{2(6 + 3\alpha + \alpha^2)\sqrt{6 + 9\alpha + 4\alpha^2 + \alpha^3}}{3(1+\alpha)(2+\alpha)(3+\alpha)\sqrt{7\alpha + 4\alpha^2 + \alpha^3}}$.

Proof Using the expressions for a_2, a_3 and a_4 from Eqs. (2.3) to (2.5), we see that

$$|a_2a_3 - a_4| = \frac{1}{3(1+\alpha)^2(2+\alpha)(3+\alpha)}\big|(3(1+\alpha)^2(2+\alpha)c_3 + 3\alpha(1+\alpha)(3+\alpha)c_1c_2 - (1-\alpha)(2+\alpha)(3+\alpha)c_1^3)\big|.$$

Substituting the values for c_2 and c_3 from Lemma 2.2 in the above expression, we have

$$\begin{aligned}|a_2a_3 - a_4| &= \frac{1}{12(1+\alpha)^2(2+\alpha)(3+\alpha)}\big|(18-\alpha-4\alpha^2-\alpha^3)c_1^3 - 12(4-c^2)(1+\alpha)c_1x \\ &+ 3(4-c^2)(1+\alpha)^2(2+\alpha)c_1x^2 - 6(4-c^2)(1+\alpha)^2(2+\alpha)(1-|x|^2)y\big|. \\ &\leq \frac{1}{12(1+\alpha)^2(2+\alpha)(3+\alpha)}\big((18-\alpha-4\alpha^2-\alpha^3)c_1^3 + 12(4-c^2)(1+\alpha)c_1|x| \\ &+ 3(4-c^2)(1+\alpha)^2(2+\alpha)c_1|x|^2 + 6(4-c^2)(1+\alpha)^2(2+\alpha)(1-|x|^2)|y|\big).\end{aligned}$$

Choosing $c_1 = c \in [0,2]$, replacing $|x|$ by μ and using the fact that $|y| \leq 1$ in the above inequality, we get:

$$\begin{aligned}|a_2a_3 - a_4| &\leq \frac{1}{12(1+\alpha)^2(2+\alpha)(3+\alpha)}\big((18-\alpha-4\alpha^2-\alpha^3)c^3 + 12(4-c^2)(1+\alpha)c\mu \\ &+3(4-c^2)(1+\alpha)^2(2+\alpha)c\mu^2 + 6(4-c^2)(1+\alpha)^2(2+\alpha)(1-\mu^2)\Big). \\ &=: F(c,\mu).\end{aligned}$$

We shall now maximize the function $F(c,\mu)$ for $(c,\mu) \in [0,2] \times [0,1]$. Differentiating $F(c,\mu)$ partially with respect to μ, we get:

$$\frac{\partial F}{\partial \mu} = \frac{(4-c^2)}{2(2+\alpha)(3+\alpha)}\big(2c + (2+\alpha)\mu(c-2)\big).$$

Then $\partial F/\partial\mu = 0$ for $\mu_0 = (2c)/((2-c)(1+\alpha)(2+\alpha)) \in [0,1]$ when $c \in [0,1]$. As observed from the graph of the function $F(c,\mu)$, when $c \in [0,1]$, maximum of $F(c,\mu)$ occurs at μ_0 and for $c \in [1,2]$, maximum occurs at $\mu = 1$. Thus, we maximize the function $G(c)$ given by:

$$G(c) = \begin{cases} G_1(c), & 0 \leq c \leq 1; \\ G_2(c), & 1 \leq c \leq 2, \end{cases}$$

where

$$G_1(c) = \frac{24(1+\alpha)^2(2+\alpha)^2 + 6c^2\alpha(3+\alpha)(4+3\alpha+\alpha^2) + c^3(3+\alpha)(-16+3\alpha^2+\alpha^3)}{12(1+\alpha)^2(2+\alpha)(3+\alpha)},$$

and

$$G_2(c) = \frac{12c(4-c^2)(1+\alpha) + 3c(4-c^2)(1+\alpha)^2(2+\alpha) + c^3(18-\alpha-4\alpha^2-\alpha^3)}{12(1+\alpha)^2(2+\alpha)(3+\alpha)}.$$

Firstly, we observe that $G_1'(c) = -c\big(4\alpha(1+\alpha)(2+\alpha) + c(-16+3\alpha^2+\alpha^3)\big)/(4(1+\alpha)^2(2+\alpha)^2)$. When $\alpha \in [\alpha^*, 1]$, $G_1(c)$ is an increasing function of c as $G_1'(c) > 0$ for all values of $c \in [0,1]$ and when $\alpha \in [\alpha^*, 1]$, $G_1(c)$ is a decreasing function of c as $G_1'(c) < 0$ for all $c \in [0,1]$. Thus, for $\alpha \in [0, \alpha^*]$, maximum is attained at $c = 1$ and for $\alpha \in [\alpha^*, 1]$, maximum is at $c = 0$, and is given by:

$$\max_{0\le c\le 1} G_1(c) = \begin{cases} \dfrac{144+232\alpha+225\alpha^2+102\alpha^3+17\alpha^4}{12(1+\alpha)^2(2+\alpha)^2(3+\alpha)}, & 0 \le \alpha \le \alpha^*, \\ \dfrac{2(2+\alpha)^2}{(2+\alpha)^2(3+\alpha)}, & \alpha^* \le \alpha \le 1. \end{cases} \tag{2.10}$$

Here α^* is the root of the equation $144+232\alpha+225\alpha^2+102\alpha^3+17\alpha^4 = 24(1+\alpha)^2(2+\alpha)^2$.

We now maximize $G_2(c)$. It is seen that $G_2'(c) = \big(-\alpha(7+4\alpha+\alpha^2)c^2 + (1+\alpha)(6+3\alpha+\alpha^2)\big)/\big((1+\alpha)^2(2+\alpha)(3+\alpha)\big)$. When $\alpha \in [0, \alpha']$, $G_2'(c) > 0$ for all $c \in [1,2]$, thereby implying that $G_2(c)$ is an increasing function of c. For $\alpha \in [\alpha', 1]$, it can be seen that:

$$\max_{1\le c\le 2} G_2(c) = \max_{1\le c\le 2}\{G_2(c_0), G_2(1), G_2(2)\} = G_2(c_0),$$

where $c_0 = ((6+9\alpha+4\alpha^2+\alpha^3)/\alpha(7+4\alpha+\alpha^2))^{1/2}$ is the positive root of $G_2'(c) = 0$. Thus, it is seen that

$$\max_{1\le c\le 2} G_2(c) = \begin{cases} \dfrac{2(18-\alpha-4\alpha^2-\alpha^3)}{3(1+\alpha)^2(2+\alpha)(3+\alpha)}, & 0 \le \alpha \le \alpha', \\ \dfrac{2(6+3\alpha+\alpha^2)\sqrt{6+9\alpha+4\alpha^2+\alpha^3}}{3(1+\alpha)(2+\alpha)(3+\alpha)\sqrt{\alpha(7+4\alpha+\alpha^2)}}, & \alpha' \le \alpha \le 1. \end{cases} \tag{2.11}$$

where α' is the root of $(18-\alpha-4\alpha^2-\alpha^3)(\alpha(7+4\alpha+\alpha^2))^{1/2} = (1+\alpha)(6+3\alpha+\alpha^2)(6+9\alpha+4\alpha^2+\alpha^3)^{1/2}$. The absolute maximum value of $G(c)$ over the interval $c \in [0,2]$ is given by:

$$\begin{aligned} \max_{0\le c\le 2} G(c) &= \max_{0\le c\le 2}\{G_1(c), G_2(c)\} \\ &= \begin{cases} \dfrac{2(18-\alpha-4\alpha^2-\alpha^3)}{3(1+\alpha)^2(2+\alpha)(3+\alpha)}, & 0 \le \alpha \le \alpha_0, \\ \dfrac{2(6+3\alpha+\alpha^2)\sqrt{6+9\alpha+4\alpha^2+\alpha^3}}{3(1+\alpha)(2+\alpha)(3+\alpha)\sqrt{\alpha(7+4\alpha+\alpha^2)}}, & \alpha_0 \le \alpha \le 1. \end{cases} \end{aligned} \tag{2.12}$$

where α_0 is the root in [0, 1] of $(18-\alpha-4\alpha^2-\alpha^3)(\alpha(7+4\alpha+\alpha^2))^{1/2} = (1+\alpha)(6+3\alpha+\alpha^2)(6+9\alpha+4\alpha^2+\alpha^3)^{1/2}$ which on solving gives the expression for α_0 given in the statement of the theorem.

For the third Hankel determinant, we have the following theorem:

Corollary 2.4 *If $f \in \mathcal{M}_\alpha$, then the third Hankel determinant $H_3(1)$ satisfies*

$$|H_3(1)| \le \begin{cases} R, & 0 \le \alpha \le \alpha_0, \\ S, & \alpha_0 \le \alpha \le 1. \end{cases}$$

where

$$R = \frac{4}{9(1+\alpha)^5(2+\alpha)^3(3+\alpha)^2(4+\alpha)}\big(10368 + 22815\alpha + 27229\alpha^2 + 22644\alpha^3 + 12505\alpha^4 + 4190\alpha^5 + 739\alpha^6 + 32\alpha^7 - 9\alpha^8 - \alpha^9\big),$$

$$S = \frac{4}{9(1+\alpha)^4(2+\alpha)^3(3+\alpha)^2(4+\alpha)(\alpha(7+4\alpha+\alpha^2))^{1/2}}\{(2+\alpha)(4+\alpha) \times (6+9\alpha+4\alpha^2+\alpha^3)^{1/2}(216+222\alpha+159\alpha^2+82\alpha^3+32\alpha^4+8\alpha^5+\alpha^6) + 3(3+\alpha)(\alpha(7+4\alpha+\alpha^2))^{1/2}(576+1063\alpha+1109\alpha^2+655\alpha^3+209\alpha^4 + 34\alpha^5+2\alpha^6)\},$$

and

$$\alpha_0 = \frac{1}{3}\left((77+2\sqrt{1489})^{1/3} - \frac{3}{(77+2\sqrt{1489})^{1/3}} - 4\right) \approx 0.267554.$$

Proof Since $f \in \mathcal{A}$, $a_1 = 1$, so that we have

$$|H_3(1)| \le |a_3||a_2a_4 - a_3^2| + |a_4||a_4 - a_2a_3| + |a_5||a_3 - a_2^2|. \tag{2.13}$$

By substituting $B_i = 2$ $(i = 1, 2, 3, \ldots)$ and $\mu = 1$ in [5, Theorem 2.11], we get the following bound for the expression $|a_3 - a_2^2|$ for $f \in \mathcal{M}_\alpha$:

$$|a_3 - a_2^2| \le 2/(2+\alpha).$$

Similarly, [5, Theorem 2.9] gives the following bound for $f \in \mathcal{M}_\alpha$:

$$|a_2a_4 - a_3^2| \le 4/(2+\alpha)^2.$$

Using these two bounds, the bound for the expression $|a_4 - a_2a_3|$ from Theorem 2.3 and the bounds for $|a_k|$ $(k = 1, 2, 3, \ldots)$ from Theorem 2.1 in the equation (2.13), the desired estimates for the thrid Hankel determinant follows.

Remark 2.5 For $\alpha = 0$, Corollary 2.4 reduces to the following estimate for starlike functions given in [1]: $H_3(1) \le 16$.

Our next theorem gives bounds for the first five coefficients for functions in the class $\mathcal{L}_\alpha$ which is the class of all analytic functions $f \in \mathcal{S}$ satisfying

$$\operatorname{Re}\left((f'(z))^\alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)^{1-\alpha}\right) > 0.$$

Note that

$$\mathcal{K} = \mathcal{L}_0 = \left\{ f \in \mathcal{S} : \mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0 \right\} \qquad \text{and} \qquad \mathcal{R} = \mathcal{L}_1 = \{ \mathrm{f} \in \mathcal{S} : \mathrm{Re}(\mathrm{f}'(\mathrm{z})) > 0 \}$$

are the classes of convex functions and a subclass of close-to-convex functions respectively. Thus as α varies from 0 to 1, our class $\mathcal{L}_\alpha$ provides a continuous passage from the class $\mathcal{K}$ of convex functions to the the class $\mathcal{R}$ of functions whose derivative has positive real part.

Theorem 2.6 *If the function $f \in \mathcal{L}_\alpha$, then the coefficients a_2–a_5 satisfy*

$$\begin{aligned} |a_2| &\leq 1, \\ |a_3| &\leq \frac{2}{3(2-\alpha)}(3-2\alpha), \\ |a_4| &\leq \frac{1}{2(2-\alpha)(3-2\alpha)}\big((8-7\alpha) + 4(1-\alpha)|1-2\alpha|\big), \end{aligned}$$

and

$$\begin{aligned} |a_5| \leq \frac{1}{10(2-\alpha)^2(3-2\alpha)(4-3\alpha)} &\big(4(56 - 101\alpha + 54\alpha^2 - 8\alpha^3) \\ &+ 16(1-\alpha)|1-2\alpha|((12-7\alpha) + |4 - 13\alpha + 6\alpha^2|)\big). \end{aligned}$$

Proof Since the function $f \in \mathcal{L}_\alpha$, there is an analytic function $p(z) = 1 + c_1 z + c_2 z^2 + \cdots \in \mathcal{P}$, such that

$$(f'(z))^\alpha \left(1 + \frac{zf''(z)}{f'(z)}\right)^{1-\alpha} = p(z). \tag{2.14}$$

The Taylor series expansion of the function f gives

$$\begin{aligned} (f'(z))^\alpha \left(1 + \frac{zf''(z)}{f'(z)}\right)^{1-\alpha} &= 1 + 2a_2 z + (3(2-\alpha)a_3 - 4(1-\alpha)a_2^2)z^2 \\ &\quad + (4(3-2\alpha)a_4 - 18(1-\alpha)a_2 a_3 + 8(1-\alpha)a_2^3)z^3 + \cdots. \end{aligned} \tag{2.15}$$

Then using (2.14), (2.15) and the expansion for the function p, we express a_n in terms of the coefficients c_i of $p \in \mathcal{P}$:

$$a_2 = \frac{c_1}{2}, \tag{2.16}$$

$$a_3 = \frac{1}{3(2-\alpha)}\big(c_2 + (1-\alpha)c_1^2\big), \tag{2.17}$$

$$a_4 = \frac{1}{4(2-\alpha)(3-2\alpha)}\left((1-\alpha)(1-2\alpha)c_1^3 + (2-\alpha)c_3 + 3(1-\alpha)c_1c_2\right), \quad (2.18)$$

and

$$a_5 = \frac{1}{10(2-\alpha)^2(3-2\alpha)(4-3\alpha)}\left(8(2-\alpha)^2(1-\alpha)c_1c_3 + 2(2-\alpha)^2(3-2\alpha)c_4 \right.$$
$$+ (1-\alpha)(4+\alpha)(3-2\alpha)c_2^2 + 2(1-\alpha)(1-2\alpha)(12-7\alpha)c_1^2c_2$$
$$\left. + (1-\alpha)(1-2\alpha)(4-13\alpha+6\alpha^2)c_1^4\right). \quad (2.19)$$

Therefore, by making use of the triangle inequality and the fact that $|c_k| \leq 2$ $(k = 1, 2, 3, \ldots)$, for $p \in \mathcal{P}$, we get the desired bounds for a_2, a_3, a_4 and a_5.

Next, we prove certain results which will be required to estimate the third hankel determinant $H_3(1)$ for the class $\mathcal{L}_\alpha$. Firstly, we find an upper bound for $|a_2a_3 - a_4|$ for the function $f \in \mathcal{L}_\alpha$.

Theorem 2.7 *Let* $\alpha_0 \approx 0.852183$ *be the root in [0, 1] of the equation*

$$(24-19\alpha)^{3/2} = 9\sqrt{3}(2-\alpha)^{3/2}(3-2\alpha)^{1/2}.$$

If $f \in \mathcal{L}_\alpha$ *, then*

$$|a_2a_3 - a_4| \leq \begin{cases} \dfrac{(24-19\alpha)^{3/2}}{18\sqrt{3}(2-\alpha)(3-2\alpha)\sqrt{6-7\alpha+2\alpha^2}}, & 0 \leq \alpha < \alpha_0; \\[2ex] \dfrac{1}{2(3-2\alpha)}, & \alpha_0 \leq \alpha \leq 1. \end{cases}$$

Proof By making use of the Eqs. (2.16)– (2.18), we get

$$a_2a_3 - a_4 = \frac{1}{12(2-\alpha)(3-2\alpha)}\left(-3(2-\alpha)c_3 - (3-5\alpha)c_1c_2 + (1-\alpha)(3-2\alpha)c_1^3\right).$$

Substituting the values for c_2 and c_3 from Lemma 2.2 in the above expression, we have

$$|a_2a_3 - a_4| = \frac{1}{48(2-\alpha)(3-2\alpha)}\left|\alpha(9-8\alpha)c_1^3 - 2(4-c^2)(9-8\alpha)c_1x \right.$$
$$\left. + 3(4-c^2)(2-\alpha)c_1x^2 - 6(4-c^2)(2-\alpha)(1-|x|^2)y\right|.$$
$$\leq \frac{1}{48(2-\alpha)(3-2\alpha)}\left(\alpha(9-8\alpha)c_1^3 + 2(4-c^2)(9-8\alpha)c_1|x| \right.$$
$$\left. + 3(4-c^2)(2-\alpha)c_1|x|^2 + 6(4-c^2)(2-\alpha)(1-|x|^2)|y|\right).$$

Choosing $c_1 = c \in [0, 2]$, replacing $|x|$ by μ and using the fact that $|y| \leq 1$ in the above inequality, we get

$$|a_2a_3 - a_4| \leq \frac{1}{48(2-\alpha)(3-2\alpha)}\big(\alpha(9-8\alpha)c^3 + 2(4-c^2)(9-8\alpha)c\mu \\ + 3(4-c^2)(2-\alpha)c\mu^2 + 6(4-c^2)(2-\alpha)(1-\mu^2)\big). \\ = F(c,\mu).$$

We shall now maximize the function $F(c,\mu)$ for $(c,\mu) \in [0,2] \times [0,1]$. Differentiating $F(c,\mu)$ partially with respect to μ, we get

$$\frac{\partial F}{\partial \mu} = \frac{(4-c^2)}{48(2-\alpha)(3-2\alpha)}\big(2(9-8\alpha)c + 6\mu(2-\alpha)(c-2)\big).$$

Then $\partial F/\partial\mu = 0$ for $\mu_0 = ((9-8\alpha)c)/(3(2-c)(2-\alpha)) \in [0,1]$ when $c \in [0, 0.8]$. As observed from the graph of the function $F(c,\mu)$, when $c \in [0, 0.8]$, maximum of $F(c,\mu)$ occurs at μ_0 and for $c \in [0.8, 2]$, maximum occurs at $\mu = 1$. Thus, we have:

$$\max_{0\leq\mu\leq 1} F(c,\mu) = G(c) = \begin{cases} G_1(c), & 0 \leq c \leq 0.8; \\ G_2(c), & 0.8 \leq c \leq 2, \end{cases}$$

where

$$G_1(c) = \frac{72(2-\alpha)^2 + 2c^2(3-5\alpha)(15-11\alpha) - c^3(9-8\alpha)(-9+2\alpha+3\alpha^2)}{144(2-\alpha)^2(3-2\alpha)},$$

and

$$G_2(c) = \frac{4c((24-19\alpha) - c^2(2-\alpha)(3-2\alpha))}{48(2-\alpha)(3-2\alpha)}.$$

Note that $G_1'(c) = \big(4c(3-5\alpha)(15-11\alpha) - 3c^2(9-8\alpha)(3\alpha^2+2\alpha-9)\big)/(144(2-\alpha)^2(3-2\alpha))$. The function $G_1'(c) = 0$ implies $c = 0$ and $c_0 = (4(3-5\alpha)(15-11\alpha))/((9-8\alpha)(3\alpha^2+2\alpha-9))$. In order to find the maximum value for $G_1(c)$, we check the behaviour of $G_1(c)$ at the end points of the interval [0, 0.8] and at $c = c_0$. It can be observed that there exists some $\alpha^* \in [0, 0.8]$ such that for all values of $c \in [0, 0.8]$ and $\alpha \in [0, \alpha^*]$, $G_1'(c) > 0$, thereby implying that $G_1(c)$ is an increasing function of $c \in [0, 0.8]$ and maximum occurs at $c = 0.8$. Similarly, using a similar argument, it is observed that when $\alpha \in [\alpha^*, 0.8]$, $G_1(c)$ decreases as $c \in [0, 0.8]$ and hence, maximum occurs at $c = 0$ and is given as:

$$\max_{0\leq c\leq 0.8} G_1(c) = \begin{cases} G_1(0.8), & 0 \leq \alpha \leq \alpha^*; \\ \dfrac{1}{2(3-2\alpha)}, & \alpha^* \leq \alpha \leq 1. \end{cases} \quad (2.20)$$

Here α^* is the root of the equation $2(3-2\alpha)G_1(0.8) = 1$. We now maximize $G_2(c)$. It is seen that $G_2'(c) = \big((24-19\alpha) - 3c^2(2-\alpha)(3-2\alpha)\big)/\big(12(2-\alpha)(3-2\alpha)\big)$. On solving $G_2'(c) = 0$, the critical points as obtained are $c = \pm\sqrt{(24-19\alpha)}/\sqrt{3(2-\alpha)(3-2\alpha)}$. Since c cannot be negative, thus the only points of consideration in finding the maximum of $G_2(c)$ are the end points of the interval [0.8, 2] and $c_0 = ((24-19\alpha)/3(2-\alpha)(3-2\alpha))^{1/2} \in [0.8, 2]$ for all $\alpha \in [0, 1]$. It is observed that $G_2'(c) > 0$ for $c \in [0.8, c_0]$ and $G_2'(c) < 0$ when $c \in [c_0, 1]$, thereby implying that the function $G_2(c)$ increases first in the interval $[0.8, c_0]$ and then decreases in the interval $[c_0, 1]$. Hence the maximum occurs at $c = c_0$ and is given by:

$$\max_{0.8 \le c \le 2} G_2(c) = \frac{1}{18\sqrt{3}} \left(\frac{(24-19\alpha)}{(2-\alpha)(3-2\alpha)} \right)^{3/2}. \tag{2.21}$$

In order to find the find the absolute maximum value of $G(c)$ over the interval $c \in [0, 2]$, we compare the maximum values of $G_1(c)$ and $G_2(c)$ as obtained in (2.20) and (2.21) to get:

$$\max_{0 \le c \le 2} G(c) = \begin{cases} \frac{1}{18\sqrt{3}} \left(\frac{(24-19\alpha)}{(2-\alpha)(3-2\alpha)} \right)^{3/2}, & 0 \le \alpha \le \alpha_0; \\ \frac{1}{2(3-2\alpha)}, & \alpha_0 \le \alpha \le 1. \end{cases} \tag{2.22}$$

where α_0 is the root of $(24-19\alpha)^{3/2} = 9\sqrt{3}(2-\alpha)^{3/2}(3-2\alpha)^{1/2}$.

For the third Hankel determinant for the function $f \in \mathcal{L}_\alpha$, we have the following theorem:

Corollary 2.8 *If $f \in \mathcal{L}_\alpha$, then the third Hankel determinant $H_3(1)$ satisfies*

$$|H_3(1)| \le \frac{1}{540(2-\alpha)^3} (P + Q + R)$$

where

$$P = \frac{5\sqrt{3}(24-19\alpha)^{3/2}(2-\alpha)\{8-7\alpha+4(1-\alpha)|1-2\alpha|\}}{(6-7\alpha+2\alpha^2)^{1/2}(3-2\alpha)^2},$$

$$Q = \frac{5\{(72-78\alpha+17\alpha)^2 - 32\alpha(3-2\alpha)|18-27\alpha+8\alpha^2|\}}{48-62\alpha+17\alpha^2-\alpha|18-27\alpha+8\alpha^2|},$$

$$R = \frac{144\{56-101\alpha+54\alpha^2-8\alpha^3+4(1-\alpha)|1-2\alpha|(12-7\alpha+|4-13\alpha+6\alpha^2|)\}}{(4-3\alpha)(3-2\alpha)}.$$

Proof By substituting $B_i = 2$ $(i = 1, 2, 3, \cdots)$ and $\mu = 1$ in [5, Theorem 2.15], we get the following bound for the expression $|a_3 - a_2^2|$ for $f \in \mathcal{L}_\alpha$:

$$|a_3 - a_2^2| \le 2/(3(2-\alpha)).$$

Similarly, [5, Theorem 2.13] gives the following bound for $f \in \mathcal{L}_\alpha$:

$$|a_2a_4 - a_3^2| \leq \frac{32\alpha(3-2\alpha)|-18+27\alpha-8\alpha^2| - (72-78\alpha+17\alpha^2)^2}{72(2-\alpha)^2(3-2\alpha)\{\alpha|18-27\alpha+8\alpha^2| + (-48+62\alpha-17\alpha^2)\}}.$$

Using these two bounds, the bound for the expression $|a_4 - a_2a_3|$ from Theorem 2.7 and the bounds for $|a_k| \quad (k = 1, 2, 3, \cdots)$ from Theorem 2.6 in the equation (2.13), the desired estimates for the thrid Hankel determinant follows.

Remark 2.9 For $\alpha = 0$, Corollary 2.8 reduces to $H_3(1) \leq 1/8$ obtained in [1] for convex functions.

Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

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