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ORIGINAL RESEARCH PAPER

# Spectral theorem for quaternionic compact normal operators

**G. Ramesh[1] · P. Santhosh Kumar[1]**

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**Abstract** In this article, we prove two versions of the spectral theorem for quaternionic compact normal operators, namely the series representation and the resolution of identity form. Though the series representation form already appeared in [5], we prove this by using simultaneous diagonalization. Whereas the resolution of identity is new in the literature for the quaternion case, we prove this by associating a complex linear operator to the given right linear operator and applying the classical result. In this process we prove some spectral properties of compact operators parallel to the classical theory. We also establish the singular value decomposition of a compact operator.

**Keywords** Standard eigenvalue · Slice complex plane · Minimum modulus · Generalized standard eigenvalue

**Mathematics Subject Classification** 47S10 · 47B15 · 35P05

## 1 Introduction and preliminaries

The spectral theorem for quaternionic compact normal operators on right quaternionic Hilbert space was recently proved by Ghiloni et al. [6], in which the authors established the left multiplication to prove the series representation of such operators [6, Theorem 1.4].

✉ P. Santhosh Kumar
ma12p1004@iith.ac.in

G. Ramesh
rameshg@iith.ac.in

1 Department of Mathematics, IIT Hyderabad, Kandi, Sangareddy, Medak, Telangana 502285, India

We prove two versions of the spectral theorem. The first version is the series representation of quaternionic compact normal operator and the second version is the resolution of identity form. First we observe that spherical point spectrum of self-adjoint operator coincides with classical definition of point spectrum. This helps us to prove the series representation of quaternionic compact self-adjoint operators. Using the cartesian decomposition and simultaneous diagonalization, we prove the series representation for quaternionic compact normal operators. This approach is similar to the classical one. Moreover, we prove several important spectral properties of quaternionic compact operators. We, then prove the singular value decomposition theorem similar to the classical result. In proving these, we establish several results related to compact operators which are similar to the classical case.

Next, we establish the resolution of identity for the quaternionic compact normal operators. This is a new result in this paper. In this case, we associate a complex linear operator to the given operator and use the classical result to get the representation for the complex linear operator and lift this result to the given operator.

Throughout, we consider right eigenvalues for the operators. The concept of right eigenvalues of quaternion matrices is discussed in [1] with topological approach. Brenner and Lee proved that every $n$- dimensional quaternion matrix have exactly $n$-complex right eigenvalues with nonnegative imaginary parts (See [2, 8] for details). Such eigenvalues are known as standard eigenvalues. The spectral theorem for quaternion matrices is proved in [3].

We prove that a quaternionic compact normal operator has series representation. We observe that the standard eigenvalues are enough to describe the spectral properties of the operator. This generalizes the result of Brenner and Lee.

We organize this article into four sections. In the first section we recall some of the basic properties of quaternions, definitions, properties of compact operators on quaternionic Hilbert spaces.

In the second section we discuss the spectral theorem for quaternionic compact self-adjoint operators, singular value decomposition and the simultaneous diagonalization.

In the third section, the spectral theorem for quaternionic compact normal operator is proved by using the cartesian decomposition.

In the final section, the resolution of identity on quaternionic Hilbert space is given.

### 1.1 Quaternions

Let $i, j, k$ be three vectors that satisfy $i^2 = j^2 = k^2 = -1 = i \cdot j \cdot k$. Let $\mathbb{H} = \{q = q_0 + q_1 i + q_2 j + q_3 k : q_n \in \mathbb{R},\ n = 0, 1, 2, 3\}$ denote the division ring (skew field) of all real quaternions. The conjugate of $q$ is $\overline{q} = q_0 - q_1 i - q_2 j - q_3 k$. The real part of $\mathbb{H}$ is denoted by $\mathrm{Re}(\mathbb{H}) = \{q \in \mathbb{H} : q = \overline{q}\}$ and the imaginary part of $\mathbb{H}$ is denoted by $\mathrm{Im}(\mathbb{H}) = \{q \in \mathbb{H} : q = -\overline{q}\}$. The set $\mathbb{S} := \{q \in \mathrm{Im}(\mathbb{H}) : |q| = 1\}$ is the unit sphere in $\mathrm{Im}(\mathbb{H})$. Here we list out some of the properties of quaternions, which we need later.

1. For $p, q \in \mathbb{H}$, $\overline{p.q} = \overline{q}.\overline{p}$ and $|q| := \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}$.
2. $|p.q| = |p|.|q|$ and $|\overline{p}| = |p|$.
3. For $p, q \in \mathbb{H}$ define $p \sim q$ if and only if $p = s^{-1}qs$, for some $s \neq 0 \in \mathbb{H}$. This is an equivalence relation and the equivalence class of $p$ is $[p] := \{s^{-1}ps : 0 \neq s \in \mathbb{H}\}$.
4. For each $m \in \mathbb{S}$, define $\mathbb{C}_m := \{\alpha + m\beta : \alpha, \beta \in \mathbb{R}\}$, a real sub algebra of $\mathbb{H}$ (It is also called as slice complex plane).
5. Let $m, n \in \mathbb{S}$. If $m \neq \pm n$, then $\mathbb{C}_m \cap \mathbb{C}_n = \mathbb{R}$. In fact, $\mathbb{H} = \bigcup_{m \in \mathbb{S}} \mathbb{C}_m$.

Let $H$ be a right $\mathbb{H}$-module with the map $\langle \cdot | \cdot \rangle : H \times H \to \mathbb{H}$ satisfying the following three properties:

1. If $x \in H$, then $\langle x|x \rangle \geq 0$ and $\langle x|x \rangle = 0$ if and only if $x = 0$.
2. $\langle x|yp + zq \rangle = \langle x|y \rangle p + \langle x|z \rangle q$, if $x, y, z \in H$ and $p, q \in \mathbb{H}$.
3. $\langle x|y \rangle = \overline{\langle y|x \rangle}$, for all $x, y \in H$.

Define $\|x\| = \sqrt{\langle x|x \rangle}$, for every $x \in H$. If the normed space $(H, \|\cdot\|)$ is complete, then we call $H$, a right quaternionic Hilbert space.

Let $x_1, x_2, x_3 \ldots x_n$ be vectors in $H$. Then the $\mathbb{H}$-linear span is denoted by $\text{span}_{\mathbb{H}}\{x_1, x_2, x_3 \ldots x_n\}$ and it is defined as

$$\text{span}_{\mathbb{H}}\{x_1, x_2, x_3 \ldots x_n\} = \left\{ \sum_{l=1}^{n} x_l q_l : q_l \in \mathbb{H}, l = 1, 2, \ldots, n \right\}.$$

Let $S$ be a subset of $H$. Then the orthogonal complement of $S$ is denoted by $S^{\perp}$ and is defined as

$$S^{\perp} := \{x \in H \mid \langle x|y \rangle = 0 \text{ for all } y \in S\}.$$

Throughout this article $\mathbb{H}$ denotes the division ring of quaternions and $H$ refers to be the right quaternionic Hilbert space.

**Proposition 1.1** *Let $\{\phi_n : n \in \mathbb{N}\}$ be an orthonormal basis for $H$. Then the following are equivalent*:

1. *For $x, y \in H$. The series*

$$\langle x|y \rangle = \sum_{n \in \mathbb{N}} \langle x|\phi_n \rangle \cdot \langle \phi_n|y \rangle$$

*converges absolutely in $\mathbb{H}$.*

2. *For every $x \in H$, we have*

$$\|x\|^2 = \sum_{n \in \mathbb{N}} |\langle x|\phi_n \rangle|^2$$

3. $(\text{span}_{\mathbb{H}}\{\phi_n : n \in \mathbb{N}\})^{\perp} = \{0\}$.

**Definition 1.2** [5, Definition 2.9] A map $T : H \to H$ is said to be a right $\mathbb{H}$-linear operator or quaternionic linear if $T(x.q + y) = Tx.q + Ty$, for every $x, y \in H$ and $q \in \mathbb{H}$. We say that $T$ is bounded (or continuous), if there exists $k > 0$ such that $\|Tx\| \leq k\|x\|$, for all $x \in H$. If $T$ is bounded, then

$$\|T\| := \sup\{\|Tx\| : x \in H, \|x\| = 1\},$$

is finite and is called the norm of $T$.

We denote the set of all bounded right linear operators between $H_1$ and $H_2$ by $\mathcal{B}(H_1, H_2)$ and $\mathcal{B}(H, H) = \mathcal{B}(H)$. If $T \in \mathcal{B}(H_1, H_2)$, the null space and the range space are denoted by $N(T)$ and $R(T)$ respectively.

**Definition 1.3** Let $T \in \mathcal{B}(H)$. The minimum modulus of $T$ is defined by

$$m(T) = \inf\{\|Tx\| : x \in H, \|x\| = 1\}.$$

**Definition 1.4** [5, Definition 2.12] Let $T \in \mathcal{B}(H)$. Then there exists a unique operator $T^* \in \mathcal{B}(H)$ such that $\langle x|Ty\rangle = \langle T^*x|y\rangle$ for all $x, y \in H$. This operator $T^*$ is called the adjoint of $T$.

**Definition 1.5** Let $T \in \mathcal{B}(H)$. Then $T$ is said to be

1. self-adjoint if $T = T^*$
2. positive if $\langle x|Tx\rangle \geq 0$, for all $x \in H$
3. anti self-adjiont if $T^* = -T$
4. normal if $TT^* = T^*T$
5. unitary if $TT^* = T^*T = I$.

**Definition 1.6** Let $T \in \mathcal{B}(H)$. A closed subspace $M$ of $H$ is said to be invariant under $T$, if $T(M) := \{Tx : x \in M\} \subseteq M$. Moreover, if $M^{\perp}$ is also invariant under $T$, then we say $M$ to be a reducing subspace for $T$.

**Theorem 1.7** [5, Theorem 2.18]

1. *Let $T \in \mathcal{B}(H)$ be positive. Then there exists a unique positive operator $S \in \mathcal{B}(H)$ such that $S^2 = T$. The operator $S$ is called the square root of $T$ and it is denoted by $S = T^{\frac{1}{2}}$.*
2. *If $T \in \mathcal{B}(H_1, H_2)$, then $|T| = (T^*T)^{\frac{1}{2}}$ is called the modulus of $T$ and is denoted by $|T|$.*

**Theorem 1.8** [5, Theorem 2.20] *Let $T \in \mathcal{B}(H)$. Then there exists a unique operator $W \in \mathcal{B}(H)$ such that*

1. $T = W|T|$
2. $N(W) = N(T)$
3. $\|W(u)\| = \|u\|$, *for all* $u \in N(T)^{\perp}$.

**Theorem 1.9** [5, Theorem 5.9] *Let $T \in \mathcal{B}(H)$ be normal. Then there exists three mutually commuting bounded operators $A$, $B$ and $J$ such that*

$$T = A + JB,$$

*where $A = \frac{T+T^*}{2}, B = \frac{|T-T^*|}{2}$ and $J$ is an anti self-adjoint unitary operator.*

Through out this article, $J$ denotes an anti self-adjoint unitary operator.

**Definition 1.10** Let $T \in \mathcal{B}(H)$. Then $T$ is said to be compact if $\overline{T(S)}$ is compact for every bounded subset $S$ of $H$. Equivalently $(T(x_n))$ has a convergent subsequence for every bounded sequence $(x_n)$ of $H$.

We denote the class of compact operators between $H_1$ and $H_2$ by $\mathcal{K}(H_1, H_2)$ and $\mathcal{K}(H, H) = \mathcal{K}(H)$.

*Examples 1.11* We give some examples of compact operators:

1. Every right linear bounded operator with finite rank is compact.
2. Let $H = \ell^2(\mathbb{N}, \mathbb{H})$. Define $D : H \to H$ by

$$D(q_1, q_2, q_3, \ldots) = \left(q_1, \frac{q_2}{2}, \frac{q_3}{3}, \ldots\right), \quad \text{for all } (q_j)_{j\in\mathbb{N}} \in H.$$

   Define $D_n : H \to H$ by

$$D_n(q_1, q_2, \ldots q_n, \ldots) = \left(q_1, \frac{q_2}{2}, \ldots, \frac{q_n}{n}, 0, 0, \ldots\right), \quad \text{for all } (q_j)_{j\in\mathbb{N}} \in H.$$

   Then $\{D_n\}_{n\in\mathbb{N}}$ converges to $D$ in the operator norm. Since each $D_n$ is compact, by [4, Theorem 2], $D$ is compact.

Let $T \in \mathcal{B}(H)$ and $q \in \mathbb{H}$. Define $\Delta_q(T) := T^2 - T(q + \overline{q}) + I.|q|^2$. This operator is used to define the spherical spectrum of $T$.

**Definition 1.12** [5, Definition 4.1] If $T \in \mathcal{B}(H)$, then the spherical spectrum and the spherical point spectrum are defined as follows:

1. the spherical spectrum:

$$\sigma_S(T) := \{q \in \mathbb{H} : \Delta_q(T) \ \text{ is not invertible in } \mathcal{B}(H)\}.$$

2. the spherical point spectrum:

$$\sigma_{p^S}(T) := \{q \in \mathbb{H} : N(\Delta_q(T)) \neq \{0\}\}.$$

**Theorem 1.13** [10, Theorem 5.4] *If $T$ is an $n \times n$ quaternion matrix, then $T$ has exactly n-right eigenvalues which are complex numbers with nonnegative imaginary parts.*

*These eigenvalues are said to be standard eigenvalues.*

## 2 Representation of compact self-adjoint operators

In this section we obtain a spectral representation of quaternionic compact self-adjoint operator. Though [4, Conjecture 1] is proved for quaternionic normal operators in [6], we reprove it for quaternionic compact self-adjoint operators inspired by the classical proof.

**Proposition 2.1** *If $T \in \mathcal{K}(H)$ is self-adjoint, then $\pm\|T\| \in \sigma_{p^S}(T)$.*

*Proof* Since $T$ is self-adjoint, there exists a sequence $(x_n)$ in $H$ such that $\|x_n\| = 1$, for every $n \in \mathbb{N}$ and $|\langle x_n|Tx_n\rangle| \to \|T\|$ as $n \to \infty$. That is there exists $r \in \mathbb{R}$ with $|r| = \|T\|$ and $\langle x_n|Tx_n\rangle \to r$, as $n \to \infty$. We see that

$$\|Tx_n - r \cdot x_n\|^2 = \|Tx_n\|^2 + r^2 - 2r\langle x_n|Tx_n\rangle \leq 2r^2 - 2r\langle x_n|Tx_n\rangle \to 0,$$

as $n \to \infty$. Since $T$ is compact $(Tx_n)$ has a convergent subsequence, say $(Tx_{n_k})$ converges to $y \in H$. Then

$$r \cdot x_{n_k} - y = (Tx_{n_k} - y) - (Tx_{n_k} - r \cdot x_{n_k}) \to 0,$$

as $n \to \infty$. By using the continuity of $T$, we have $T(x_{n_k})$ converges to $\frac{1}{r}Ty$. This implies that $Ty = ry$. Moreover,

$$\Delta_r(T)y = T^2y - 2rTy + r^2y = 2r^2y - 2r^2y = 0.$$

Hence $N(\Delta_r(T)) \neq \{0\}$. Equivalently, $r = \pm\|T\| \in \sigma_{p^S}(T)$. □

**Lemma 2.2** *If $T = T^* \in \mathcal{B}(H)$ and $r \in \mathbb{R}$, then $N(\Delta_r(T)) = N(T - r \cdot I)$. Moreover,*

$$\sigma_{p^S}(T) = \{r \in \mathbb{R} : Tx_r = r \cdot x_r, \text{ for some } 0 \neq x_r \in H\}.$$

*Proof* Let $x \in H$. Then $x \in N(\Delta_r(T))$ if and only if $(T^2 - 2rT + r^2I)x = 0$ if and only if $(T - r \cdot I)^2x = 0$. Since $T$ is self-adjoint, it is equivalent to write $x \in N(T - r \cdot I)$. By [5, Theorem 4.8(b)], $\sigma_S(T) \subseteq \mathbb{R}$. Therefore

$$\begin{aligned}\sigma_{p^S}(T) &= \{r \in \mathbb{R} : N(\Delta_r(T)) \neq \{0\}\}\\ &= \{r \in \mathbb{R} : N(T - r \cdot I) \neq \{0\}\}\\ &= \{r \in \mathbb{R} : Tx_r = r \cdot x_r, \text{ for some } 0 \neq x_r \in H\}.\end{aligned}$$ □

**Theorem 2.3** *Let $T \in \mathcal{K}(H)$ be self-adjoint. Then there exists an orthonormal system $\phi_1, \phi_2, \phi_3, \ldots$ of eigenvectors of $T$ corresponding to the eigenvalues $\lambda_1, \lambda_2, \lambda_3, \ldots$, such that $|\lambda_1| \geq |\lambda_2| \geq |\lambda_3| \geq \ldots$, and*

$$Tx = \sum_{n\in\mathbb{N}} \phi_n \lambda_n \langle \phi_n|x\rangle, \quad \text{for all } x \in H.$$

*Moreover, if $(\lambda_n)$ is infinite, then $\lambda_n \longrightarrow 0$ as $n \to \infty$.*

*Proof* By Proposition 2.1 and Lemma 2.2, the proof follows along the similar lines of classical spectral theorem for compact self-adjoint complex operators (see [7, Theorem 5.1] for classical proof). □

**Theorem 2.4** *Let $T \in \mathcal{K}(H)$. Then, there exists a sequence $(\phi_n), (\psi_n)$ of orthonormal vectors and a sequence of positive reals $(\alpha_n)$ such that*

$$Tx = \sum_{n \in \mathbb{N}} \psi_n \alpha_n \langle \phi_n | x \rangle, \quad \text{for all } x \in H. \tag{1}$$

*If $(\alpha_n)$ is infinite, then $\alpha_n \longrightarrow 0$ as $n \to \infty$. Moreover, the series in* Eq. (1) *converges in the operator norm.*

*Proof* Since $T$ is compact, by [4, Theorem 2], we have $|T| := (T^*T)^{\frac{1}{2}}$ is compact. By Theorem 2.3, there exists an orthonormal system $(\phi_n)$ of eigenvectors of $|T|$ and corresponding eigenvalues $(\alpha_n)$ such that

$$|T|x = \sum_{n \in \mathbb{N}} \phi_n \alpha_n \langle \phi_n | x \rangle, \quad \text{for all } x \in H. \tag{2}$$

If $(\alpha_n)$ is infinite, then $\alpha_n \to 0$, as $n \to \infty$. By Theorem 1.8, there exists a unique operator $W \in \mathcal{B}(H)$ such that $T = W|T|$, where $W|_{N(|T|)^{\perp}}$ is an isometry and $N(W) = N(|T|)$. Let us take $\psi_n = W\phi_n$. We show that $(\psi_n)$ forms an orthonormal system. Consider

$$\langle \psi_n | \psi_m \rangle = \langle W\phi_n | W\phi_m \rangle = \langle W^*W\phi_n | \phi_m \rangle = \langle \phi_n | \phi_m \rangle = \delta_{nm}.$$

For $x \in H$,

$$Tx = W|T|x = \sum_{n \in \mathbb{N}} W(\phi_n)\alpha_n \langle \phi_n | x \rangle = \sum_{n \in \mathbb{N}} \psi_n \alpha_n \langle \phi_n | x \rangle.$$

Since the expression of $|T|$ in Eq. (2) converges in the operator norm, it follows that the series in Eq. (1) converges in the operator norm. □

*Example 2.5* Let $H = \ell^2(\mathbb{N}, \mathbb{H})$. Define $R : H \to H$ by

$$R(x) = (0, x_1, x_2, x_3, \ldots,), \quad \text{for all } \ x = (x_1, x_2, x_3, \ldots,) \in H,$$

and let $D$ be as in Example 1.11(2). Also we have

$$\sigma_{p^S}(D) = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\} \quad \text{and} \quad \sigma_S(D) = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\} \cup \{0\}.$$

Let $T = RD$. Then $|T|^2 = T^*T = D^*R^*RD = D^*D = D^2$. Hence $|T| = D$. By Theorem 2.4, the representation of $|T|$ is,

$$|T|(x) = \left( x_1, \frac{x_2}{2}, \frac{x_3}{3}, \ldots \right) = \sum_{n \in \mathbb{N}} e_n \cdot \frac{1}{n} \langle e_n | x \rangle, \quad \forall\, x \in H.$$

Thus for all $x \in H$,

$$T(x) = R|T|(x) = R\left(\sum_{n\in\mathbb{N}} e_n \frac{1}{n}\langle e_n|x\rangle\right) = \sum_{n\in\mathbb{N}} R(e_n)\frac{1}{n}\langle e_n|x\rangle$$
$$= \sum_{n\in\mathbb{N}} e_{n+1}\frac{1}{n}\langle e_n|x\rangle.$$

Now we prove the converse of Theorem 2.3.

**Theorem 2.6** *Suppose there exists an orthonormal system of vectors* $(\phi_n)$ *and a sequence* $(\alpha_n)$ *of real numbers which is either finite or converges to* 0*. If the operator* $T$ *is defined by*

$$Tx = \sum_{n\in\mathbb{N}} \phi_n \alpha_n \langle \phi_n|x\rangle, \quad \text{for all } x \in H,$$

*then* $T$ *is a bounded quaternionic compact self-adjoint.*

*Proof* Clearly, $T$ is right $\mathbb{H}$-linear. We show that $T$ is self-adjoint. Let $x, y \in H$. Then

$$\langle Tx|y\rangle = \left\langle \sum_{n\in\mathbb{N}} \phi_n\alpha_n\langle\phi_n \mid x\rangle|y \right\rangle = \sum_{n\in\mathbb{N}} \alpha_n \overline{\langle\phi_n|x\rangle}\langle\phi_n|y\rangle$$
$$= \sum_{n\in\mathbb{N}} \alpha_n\langle x|\phi_n\rangle\langle\phi_n|y\rangle$$
$$= \left\langle x \mid \sum_{n\in\mathbb{N}} \alpha_n\phi_n\langle\phi_n|y\rangle \right\rangle$$
$$= \langle x|Ty\rangle.$$

Therefore $T = T^*$. The rest of the proof is to show $T$ is compact. This follows in the similar lines as in [7, Theorem 6.2]. Define

$$T_n x = \sum_{k=1}^{n} \phi_k\alpha_k\langle\phi_k|x\rangle, \text{ for all } x \in H.$$

Here each $T_n$ is a finite rank operator, hence compact. We see that

$$\|T - T_n\|^2 = \sup_{\|x\|=1} \|(T - T_n)x\|^2 = \sup_{\|x\|=1} \left\| \sum_{k=n+1}^{\infty} \phi_k\alpha_k\langle\phi_n|x\rangle \right\|^2$$
$$\leq \sup_{k\geq n+1} |\alpha_k|^2 \to 0,$$

as $n \to \infty$. Since $T_n$ converges to $T$ in the operator norm and $\mathcal{K}(H)$ is closed in $\mathcal{B}(H)$ by [4, Theorem 2], we conclude that $T$ is compact. □

**Theorem 2.7** (Simultaneous diagonalization) *Let $T, S \in \mathcal{K}(H)$ be self-adjoint. If $TS = ST$, then there exist an orthonormal system $\{\phi_n\}$ of eigenvectors of both $T$ and $S$ with corresponding eigenvalues $\{\lambda_n\}$* and *$\{\mu_n\}$ respectively, such that*

$$Tx = \sum_{n\in\mathbb{N}} \phi_n \lambda_n \langle \phi_n | x\rangle \quad \text{and} \quad Sx = \sum_{n\in\mathbb{N}} \phi_n \mu_n \langle \phi_n | x\rangle, \quad \text{for all } x \in H.$$

*Proof* By Theorem 2.3, there exist an orthonormal system $\{\phi_n\}$ of eigenvectors of $T$ and corresponding eigenvalues $\{\lambda_n\}$ such that

$$Tx = \sum_{n=1}^{\infty} \phi_n \lambda_n \langle \phi_n | x\rangle, \quad \text{for all } x \in H.$$

We claim that $N(\Delta_{\lambda_n}(T))$ is invariant under $S$. Let $x \in N(\Delta_{\lambda_n}(T)) = N(T - \lambda_n I)$. That is $Tx = x\lambda_n$. Then $T(Sx) = STx = S(x\lambda_n) = (Sx)\lambda_n$. This shows that $S(N(\Delta_{\lambda_n}(T))) \subseteq N(\Delta_{\lambda_n}(T))$.

Define $S_n := S|_{N(\Delta_{\lambda_n}(T))} : N(\Delta_{\lambda_n}(T)) \longrightarrow N(\Delta_{\lambda_n}(T))$, which is quaternionic compact self-adjoint operator, for each $n \in \mathbb{N}$. Then by Theorem 2.3, we can choose $\phi_n \in N(\Delta_{\lambda_n}(T))$ such that $S\phi_n = \phi_n \mu_n$. Since $H = N(T) \oplus N(T)^{\perp}$, if $x \in H$, then there exists $x_1 \in N(T)$ and $x_2 \in N(T)^{\perp}$ such that $x = x_1 + x_2$. Since $\{\phi_n\}$ forms an orthonormal basis for $N(T)^{\perp}$, we have

$$x_2 = \sum_{n\in\mathbb{N}} \phi_n \langle \phi_n | x\rangle.$$

If $S(N(T)) = \{0\}$, then

$$Sx = Sx_1 + Sx_2 = \sum_{n\in\mathbb{N}} S(\phi_n)\langle \phi_n | x\rangle = \sum_{n\in\mathbb{N}} \phi_n \mu_n \langle \phi_n | x\rangle. \tag{3}$$

If $S(N(T)) \neq \{0\}$, then as $N(T)$ is invariant under $S$, the operator $S|_{N(T)}$ is compact self-adjoint. By Theorem 2.3, there exist a system $\{\psi_k\} \subset N(T)$ of eigenvectors of $S$ and corresponding eigenvalues $\{\xi_k\}$ such that

$$Sx = \sum_{k} \psi_k \xi_k \langle \psi_k | x\rangle, \quad \text{for all } \ x \in N(T). \tag{4}$$

If $x \in H$, then $x = x_1 + x_2,\ x_1 \in N(T), x_2 \in N(T)^{\perp}$. The system $\{\phi_n\} \cup \{\psi_k\}$ give the spectral decomposition for both $S$ and $T$. By Eqs. (3) and (4), we have

$$Sx = Sx_1 + Sx_2 = \sum_{n=1}^{\infty} \phi_n \mu_n \langle \phi_n | x_1\rangle + \sum_{k} \psi_k \xi_k \langle \psi_k | x_2\rangle. \qquad \square$$

## 3 Representation of compact normal operators

In this section, we give a spectral representation for quaternionic compact normal operators by using Theorems 2.3, 2.4 and the Cartesian decomposition of a quaternionic normal operator. First, we prove few results that are needed for our purpose.

**Proposition 3.1** *Let $T \in \mathcal{B}(H)$ be normal and $q_1, q_2 \in \sigma_{p^S}(T)$ such that $[q_1] \cap [q_2] = \emptyset$. Then $N(\Delta_{q_1}(T)) \perp N(\Delta_{q_2}(T))$.*

*Proof* Since $T$ is normal, we have $\sigma_{p^S}(T) = \sigma_{p^S}(T^*)$. Let $x \in N(\Delta_{q_1}(T))$ and $y \in N(\Delta_{q_2}(T))$. Then $Tx = x.s^{-1}q_1 s$, for some $0 \neq s \in \mathbb{H}$ and since $N(\Delta_{q_2}(T^*)) = N(\Delta_{q_2}(T))$, $T^*y = y.l^{-1}q_2 l$, for some $0 \neq l \in \mathbb{H}$. Then

$$\begin{aligned} \overline{s}\ \overline{q_1}\ \overline{s}^{-1}\langle x|y\rangle = \langle x.s^{-1}q_1 s|y\rangle = \langle Tx|y\rangle &= \langle x|T^*y\rangle \\ &= \langle x|y.l^{-1}q_2 l\rangle \\ &= \langle x|y\rangle .l^{-1}q_2 l. \end{aligned}$$

We show that $\langle x|y\rangle = 0$. Suppose $\langle x|y\rangle \neq 0$, then multiplying with $\langle x|y\rangle^{-1}$ from left side of the above equation, we get

$$\langle x|y\rangle^{-1}\overline{s}\ \overline{q_1}\ \overline{s}^{-1}\langle x|y\rangle = l^{-1}q_2 l,$$

which is contradiction to $[q_1] \cap [q_2] = \emptyset$. Hence the result. □

We prove a Lemma which plays an important role in proving the spectral representation for quaternionic compact normal operator.

**Lemma 3.2** *Let $J \in \mathcal{B}(H)$ be anti self-adjoint and unitary. Let $B \in \mathcal{B}(H)$ be positive such that $JB = BJ$. Then*

$$\sigma_{p^S}(JB) = \{rq : r \in \sigma_{p^S}(B), q \in \sigma_{p^S}(J)\}.$$

*Proof* Since $JB$ is anti self-adjoint, if $q \in \sigma_{p^S}(JB)$, then $q \subseteq Im(\mathbb{H})$ and there exists $0 \neq x \in H$ such that $x \in N(\Delta_q(JB))$. This implies

$$\begin{aligned} 0 = \Delta_q(JB)(x) &= ((JB)^2 - JB(q + \overline{q}) + |q|^2 I)(x) \\ &= (-B^2 + |q|^2 I)(x) \\ &= (B + |q|I)(B - |q|I)(x). \end{aligned}$$

Since $B \geq 0$, $(B + |q|I)$ is invertible. So we conclude that $Bx = x|q|$. Therefore $|q| \in \sigma_{p^S}(B)$.

Clearly, $\frac{q}{|q|} \in \sigma_{p^S}(J)$.

Conversely, suppose that $r \in \sigma_{p^S}(B)$ and $q \in \sigma_{p^S}(J)$. We claim that $rq \in \sigma_{p^S}(JB)$. It is clear that there exists $0 \neq x \in H$ such that $Bx = xr$ and $|q| = 1$. Consider

$$\Delta_{rq}(JB)(x) = ((JB)^2 - JB(rq + \overline{rq}) + |rq|^2 I)(x) = (-B^2 + r^2|q|^2 I)(x)$$
$$= -B^2(x) + xr^2$$
$$= 0.$$

Therefore $rq \in \sigma_{p^S}(JB)$. □

**Note 3.3** Let $q \in \mathbb{H}$. Then $q \in \sigma_{p^S}(JB)\setminus\{0\} \Leftrightarrow |q| \in \sigma_{p^S}(B)\setminus\{0\}$.

We generalize Lemma 3.2 to the whole spherical spectrum.

**Lemma 3.4** *Let J and B be as in* Lemma 3.2. *Then*

$$\sigma_S(JB) = \{rq : r \in \sigma_S(B), q \in \sigma_S(J)\}.$$

*Proof* Let $q \in \sigma_S(JB)$. Then $\overline{q} = -q$ and $\Delta_q(JB) = (B + |q|I)(B - |q|I)$ is not invertible. Since $B \geq 0$, $(B + |q|I)$ is invertible. This implies $(B - |q|I)$ is not invertible. By [5, Theorem 4.8(e)], $\frac{q}{|q|} \in \sigma_S(J)$.

Conversely, suppose that $r \in \sigma_S(B)$ and $q \in \sigma_S(J)$. Consider

$$\Delta_{r.q}(JB) = -B^2 + |r.p|^2.I = -B^2 + r^2.I = (B + r.I)(B - r.I).$$

Since $(B - r.I)$ is not invertible, $rq \in \sigma_S(JB)$. □

We give a spectral representation of quaternionic compact normal operators and show that the spherical spectrum is precisely the equivalence class of standard eigenvalues. Necessarily, in order to have eigenspace to be right linear, the eigenvalues should be given in terms of equivalence class. This is a generalization of [10, Theorem 5.4].

**Theorem 3.5** *Let $T \in \mathcal{K}(H)$ be normal. Then there exists an othonormal system $\{\phi_n\}$ of eigenvectors of T and corresponding quaternion eigenvalues $\{q_n\}$ such that*

1. $Tx = \sum_{n\in\mathbb{N}} \phi_n q_n \langle \phi_n | x \rangle$, *for all $x \in H$. Moreover, if $(q_n)$ is infinite, then $q_n \to 0$, as $n \to \infty$. Hence, the series above converges in the operator norm of $\mathcal{B}(H)$.*
2. $\sigma_{p^S}(T) = \{[q_n] : n \in \mathbb{N}\} = \{[\mathrm{Re}(q_n) + |\mathrm{Im}(q_n)| \cdot i] : n \in \mathbb{N}\}$

*Furthermore, the following properties holds true:*

(a) *The system of eigenvectors $\{\phi_n\}_{n\in\mathbb{N}}$ forms an orthonormal basis for $N(T)^\perp = \overline{R(T)}$ and thus R(T) is separable.*
(b) *The matrix of $T|_{N(T)^\perp}$ with respect to $\{\phi_n\}_{n\in\mathbb{N}}$ is* $\mathrm{diag}(q_1, q_2, q_3, \ldots,)$, *the diagonal matrix with the diagonal entries $q_1, q_2, q_3, \ldots$.*
(c) $H = N(T) \bigoplus_{n=1}^{\infty} N(\Delta_{q_n}(T))$.

*Proof* Proof of (1): Since $T$ is normal, we have

$$T = A + JB,$$

where $A$, $B$ and $J$ as in Theorem 1.9. By Lemma 3.2, we have

$$\sigma_S(JB) = \{r \cdot q : r \in \sigma_{p^S}(B) \text{ and } q \in \sigma_{p^S}(J)\}.$$

Since $T$ is compact, the operators $A$ and $B$ are also compact. By [4, Theorem 2], $JB$ is compact. In fact $A$ is a quaternionic compact self-adjoint and $B$ is a quaternionic compact positive operator with $AB = BA$. If $\{\frac{\mu_n}{|\mu_n|}\} \subset \mathbb{S}$ is a sequence of eigenvalues of $JB$, then $\{|\mu_n|\}$ is a sequence of eigenvalues of $B$. Thus by Theorem 2.7, there exists an orthonormal system $\{\phi_n\}$ of eigenvectors of both $A$ and $B$ with corresponding eigenvalues $\{\lambda_n\}, \{|\mu_n|\}$ of $A$ and $B$ respectively, such that

$$Ax = \sum_{n\in\mathbb{N}} \phi_n \lambda_n \langle \phi_n | x\rangle \quad \text{and} \quad Bx = \sum_{n\in\mathbb{N}} \phi_n |\mu_n| \langle \phi_n | x\rangle, \quad \text{for all } x \in H.$$

By Lemma 3.2, we have $J(\phi_n) = \phi_n \cdot \frac{\mu_n}{|\mu_n|}$. Let $x \in H$. Then by Theorem 1.9, we have

$$\begin{aligned} Tx = Ax + JBx &= \sum_{n\in\mathbb{N}} \phi_n \lambda_n \langle \phi_n | x\rangle + J\left(\sum_{n\in\mathbb{N}} \phi_n |\mu_n| \langle \phi_n | x\rangle\right) \\ &= \sum_{n\in\mathbb{N}} \phi_n \lambda_n \langle \phi_n | x\rangle + \sum_{n\in\mathbb{N}} J(\phi_n) |\mu_n| \langle \phi_n | x\rangle \\ &= \sum_{n\in\mathbb{N}} \lambda_n \phi_n \langle \phi_n | x\rangle + \sum_{n\in\mathbb{N}} \phi_n \frac{\mu_n}{|\mu_n|} |\mu_n| \langle \phi_n | x\rangle \\ &= \sum_{n\in\mathbb{N}} \phi_n (\lambda_n + \mu_n) \langle \phi_n | x\rangle. \end{aligned}$$

Let $q_n = \lambda_n + \mu_n$. Then

$$Tx = \sum_{n\in\mathbb{N}} \phi_n q_n \langle \phi_n | x\rangle.$$

If $\{q_n\}$ is infinite, then either $\{\lambda_n\}$ or $\{\mu_n\}$ is infinite. So $q_n \to 0$, as $n \to \infty$.

*Proof of* (2) Let $0 \neq p \in H$. If $p \in [q_k]$ for some $k$, then $p = s^{-1} q_k s$, for some $0 \neq s \in \mathbb{H}$ and

$$T(\phi_k \cdot s) = \sum_{n\in\mathbb{N}} \phi_n q_n \langle \phi_n | \phi_k \rangle s = \phi_k q_k s = (\phi_k \cdot s)(s^{-1} q_k s) = (\phi_k \cdot s) \cdot p.$$

This implies that $p \in \sigma_{p^S}(T)$. Suppose $0 \neq q$ is an eigenvalue of $T$ such that $q \notin [q_k]$ for all $k \in \mathbb{N}$. Then there exists $0 \neq x \in H$ such that $Tx = x \cdot q$ that is $x \in N(\Delta_q(T))$. By the representation of $T$, we have

$$\sum_{n\in\mathbb{N}} \phi_n q_n \langle \phi_n | x\rangle = x \cdot q.$$

Since $[q] \cap [q_k] = \emptyset$, for all $k \in \mathbb{N}$, by Proposition 3.1, we have $N(\Delta_q(T)) \perp N(\Delta_{q_k}(T))$ for all $k \in \mathbb{N}$. Since $\phi_k \in N(\Delta_{q_k}(T))$, for all $k \in \mathbb{N}$, we conclude that

$$x \cdot q = 0$$

a contradiction. Therefore $\sigma_{p^S}(T) = \{[q_n] : n \in \mathbb{N}\}$.

Now we prove the properties by using the representation of quaternionic compact normal operator.

*Proof of* (a) It is clear that $\{\phi_n\}$ is an orthonormal set. We prove that $N(T) = \text{span}_{\mathbb{H}}\{\phi_1, \phi_2, \phi_3, \ldots,\}^{\perp}$. Let $x \in N(T)$. Then, $\langle \phi_n | x \rangle = 0$ for each $n \in \mathbb{N}$. Equivalently, $x \in \text{span}_{\mathbb{H}}\{\phi_1, \phi_2, \phi_3, \ldots,\}^{\perp}$. Conversely, suppose that $x \in \text{span}_{\mathbb{H}}\{\phi_1, \phi_2, \phi_3, \ldots,\}^{\perp}$. Then, $Tx = 0$.

Therefore $N(T)^{\perp} = \text{span}_{\mathbb{H}}\{\phi_1, \phi_2, \phi_3, \ldots,\}^{\perp\perp} = \overline{\text{span}}_{\mathbb{H}}\{\phi_1, \phi_2, \phi_3, \ldots,\}$. Since $T$ is normal, $\overline{R(T)} = N(T^*)^{\perp} = N(T)^{\perp} = \overline{\text{span}}_{\mathbb{H}}\{\phi_1, \phi_2, \phi_3, \ldots,\}$. Thus $R(T)$ is separable.

*Proof of* (b) It is clear from (1), that $\{\phi_n\}_{n \in \mathbb{N}}$ is an orthonormal basis for $N(T)^{\perp}$. The matrix representation of $T|_{N(T)^{\perp}}$ with respect to $\{\phi_n\}_{n \in \mathbb{N}}$ is a diagonal matrix with the diagonal entries $\langle \phi_j | T\phi_i \rangle = \delta_{ij} q_i, \ i, j = 1, 2, 3, \ldots$ .

*Proof of* (c) By the projection theorem, $H = N(T) \oplus N(T)^{\perp}$. It is clear that, $\text{span}_{\mathbb{H}}\{\phi_{n_j}\} = N(\Delta_{q_n}(T))$, for each $n \in \mathbb{N}$. From (1), we can write

$$H = N(T) \bigoplus_{n=1}^{\infty} N(\Delta_{q_n}(T)).$$

□

*Remark 3.6* By using spectral representation in Theorem 3.5, we can prove the following:

1. The dimension of $N(\Delta_{q_k}(T))$ is finite for $q_k \neq 0$ and $k \in \mathbb{N}$.
2. $\sigma_S(T) \setminus \{0\} = \sigma_{p^S}(T) \setminus \{0\}$.

As a consequence of Theorem 3.5, we prove the result by Brenner [2] and Lee [6], that an $n \times n$ normal matrix with quaternion entries have exactly $n$- standard eigenvalues.

**Corollary 3.7** *Let $A \in M_n(\mathbb{H})$ be normal. Then $A$ has exactly n-standard eigenvalues.*

*Proof* By Theorem 3.5, there exists an orthonormal system $\{\phi_1, \phi_2, \ldots \phi_n\}$ of eigenvectors of $A$ with corresponding eigenvalues $\{q_1, q_2, \ldots q_n\}$ such that

$$Ax = \sum_{j=1}^{n} \phi_j q_j \langle q_j | x \rangle, \quad \text{for all } x \in H,$$

where $q_j = \lambda_j + \mu_j$, for $j \in \{1, 2, 3, \ldots, n\}$. Here $\lambda_j$ is an eigenvalue of $\frac{A+A^*}{2}$, $|\mu_j|$ is an eigenvalue of $\frac{|A-A^*|}{2}$ and $\frac{\mu_j}{|\mu_j|}$ is an eigenvalue of $J$, as in Theorem 3.5. Here each $[q_j]$ is an eigensphere corresponding to an eigenvector $\phi_j$. By Remark 3.6, we have

$$\sigma_S(A) = \sigma_{p^S}(A) = \{[q_j] : j = 1, 2, \ldots, n\}.$$

Each class $[q_j]$ is represented by a complex number $(q_j) + i \cdot |\text{Im } (q_j)|$. So $A$ has $n$-standard eigenvalues. □

**Note 3.8** Let $A \in M_n(\mathbb{H})$. If $A = A^*$, then $A$ has $n$- real eigenvalues. In fact, these are standard eigenvalues of $A$ with the imaginary part zero. In particular, if $A = I$, the identity matrix then $\{1\}$ is the only standard eigenvalue of $A$.

## 4 Resolution of identity

We present a second version of the spectral theorem, namely the resolution of identity for a quaternionic compact normal operator. We restrict the given operator to the slice complex Hilbert space, use classical theorem given in [9, Theorem 6.11.1], later by using [5, Proposition 3.11] extend this result to the quaternionic operator.

**Definition 4.1**

1. If $K$ is Hilbert sapce over the slice $\mathbb{C}_m$, for some $m \in \mathbb{S}$, then $K$ is said to be $\mathbb{C}_m$-Hilbert space.
2. Let $K$ be $\mathbb{C}_m$-Hilbert space. A map $T : K \to K$ is said to be $\mathbb{C}_m$-linear if

$$T(x + \lambda y) = Tx + \lambda Ty, \quad \text{for every } x, y \in K \quad \text{and} \quad \lambda \in \mathbb{C}_m.$$

We need the following facts to establish the resolution of identity.

**Lemma 4.2** [5, Lemma 3.9] *Let $m \in \mathbb{S}$ and $J \in \mathcal{B}(H)$. Define $\mathbb{C}_m$-linear spaces $H_{\pm}^{Jm} = \{x \in H : J(x) = \pm x \cdot m\}$. Then $H_{\pm}^{Jm} \neq \{0\}$, the restriction of Hermitian scalar product $\langle \cdot | \cdot \rangle$ to $H_{\pm}^{Jm}$ is $\mathbb{C}_m$-valued and therefore $H_{\pm}^{Jm}$ is $\mathbb{C}_m$-Hilbert space.*

**Lemma 4.3** [5, Lemma 3.10] *As a $\mathbb{C}_m$-Hilbert space, $H$ admits the following direct sum decomposition:*

$$H = H_{+}^{Jm} \oplus H_{-}^{Jm}.$$

*Remark 4.4* It is not necessary to consider $H$ to be $\mathbb{C}_m$-Hilbert space in Lemma 4.3. We show that, $H_{+}^{Jm} \oplus H_{-}^{Jm}$ is quaternionic Hilbert space. Let $n \in \mathbb{S}$ be such that $mn = -nm$. If $q \in \mathbb{H}$, then $q = \alpha + \beta \cdot n$ where $\alpha, \beta \in \mathbb{C}_m$. Let $x \in H$. Then $x = a + b$, where $a \in H_{+}^{Jm}$ and $b \in H_{-}^{Jm}$. Moreover,

$$\begin{aligned} x \cdot q &= (a+b) \cdot (\alpha + \beta \cdot n) \\ &= a\alpha + a \cdot \beta \cdot n + b\alpha + b \cdot \beta \cdot n \\ &= (a\alpha + b \cdot \beta \cdot n) + (a \cdot \beta \cdot n + b\alpha). \end{aligned}$$

It is enough to show $(a\alpha + b \cdot \beta \cdot n) \in H_+^{Jm}$ and $(a \cdot \beta \cdot n + b\alpha) \in H_-^{Jm}$. But it is clear from the definition of $H_\pm^{Jm}$ that

$$J(a\alpha + b \cdot \beta \cdot n) = J(a)\alpha + J(b)\beta \cdot n = a \cdot m\alpha - b \cdot m \cdot \beta \cdot n = (a\alpha + b \cdot \beta \cdot n) \cdot m.$$

and

$$J(a \cdot \beta \cdot n + b\alpha) = J(a)\beta \cdot n + J(b)\alpha = a \cdot m \cdot \beta \cdot n - b \cdot m\alpha = -(a \cdot \beta \cdot n + b\alpha) \cdot m.$$

Hence $x \cdot q \in H$.

**Proposition 4.5** [5, Proposition 3.11] *If $T : H_+^{Jm} \to H_+^{Jm}$ is a bounded $\mathbb{C}_m$-linear operator, then there exists unique bounded, right $\mathbb{H}$-linear operator $\widetilde{T} : H \to H$ such that $\widetilde{T}(x) = T(x)$, for every $x \in H_+^{Jm}$. Furthermore*

1. $\|\widetilde{T}\| = \|T\|$
2. $J\widetilde{T} = \widetilde{T}J$
3. *Let $V : H \to H$ be bounded right linear quaternionic operator. Then $V = \widetilde{U}$, for some bounded $\mathbb{C}_m$-linear operator $U : H_+^{Jm} \to H_+^{Jm}$ if and only if $JV = VJ$*
4. $(\widetilde{T})^* = \widetilde{T^*}$
5. *If $S : H_+^{Jm} \to H_+^{Jm}$ is bounded $\mathbb{C}_m$-linear operator, then $\widetilde{ST} = \widetilde{S}\widetilde{T}$*

(6) *If $S$ is the inverse of $T$, then $\widetilde{S}$ is the inverse of $\widetilde{T}$.*

*Remark 4.6* If $T_+$ is a $\mathbb{C}_m$-linear operator on $H_+^{Jm}$ such that $T = \widetilde{T_+}$, then for $a \in H_+^{Jm}, b \in H_-^{Jm}$, we have

$$T(a+b) = T_+(a) - T_+(b \cdot n) \cdot n.$$

Note that if $T \in \mathcal{B}(H)$ is normal but not self-adjoint, then by Theorem 1.9, there exists an anti self-adjoint unitary operator $J \in \mathcal{B}(H)$ such that $TJ = JT$. Also, if $T$ is self-adjoint operator then the existence of an anti self-adjoint unitary operator $J$ commuting with $T$ is guaranteed by [5, Theorem 5.7(b)]. So Proposition 4.5 holds true for quaternionic normal operator.

**Theorem 4.7** *Let $T \in \mathcal{K}(H)$ be normal and $m \in \mathbb{S}$.* Then *there exists a system of non-zero eigenvalues $\{\lambda_t\} \subset \mathbb{C}_m$ of $T$ such that*

$$T = \sum_{t=1}^{\infty} \lambda_t \widetilde{P}_t, \tag{5}$$

*where $\widetilde{P}_t$ is an orthogonal projection onto $N(\Delta_{\lambda_t}(T))$. If $\{\lambda_t\}$ is infinite, then $\lambda_t \to 0$ as $t \to \infty$. The series in* Eq. (5) *converges in the operator norm.*

*Proof* Since $T$ is normal, there exists $J \in \mathcal{B}(H)$ such that $JT = TJ$ and $JT^* = T^*J$. By Proposition 4.5, $T_+$ is compact normal with $\widetilde{T_+} = T$. By [9, Theorem 6.11.1], there exists a system of eigenvalues $\{\lambda_t\} \subset \mathbb{C}_m$ of $T_+$ and let $P_t$ be an orthogonal projection onto $N(\lambda_t I - T_+)$ such that

$$T_+ = \sum_{t=1}^{\infty} \lambda_t P_t \quad \text{and} \quad \sum_{t=1}^{\infty} P_t = I.$$

Here $\lambda_t \to 0$ if $\{\lambda_t\}$ is infinite and the series converges in the operator norm. Let $x = a + b \in H$, where $a \in H_+^{Jm}$ and $b \in H_-^{Jm}$. Then

$$\begin{aligned} T(x) = T_+(a) - T_+(b \cdot n) \cdot n &= \sum_{t=1}^{\infty} \lambda_t P_t(a) - \sum_{t=1}^{\infty} \lambda_t P_t(b \cdot n) \cdot n \\ &= \sum_{t=1}^{\infty} \lambda_t [P_t(a) - P_t(b \cdot n) \cdot n] \\ &= \sum_{t=1}^{\infty} \lambda_t \widetilde{P}_t(x). \end{aligned}$$

From (4) and (5) of Proposition 4.5, $\widetilde{P}_t$ is a quaternionic orthogonal projection. We claim that $R(\widetilde{P}_t) = N(\Delta_{\lambda_t}(T))$. To see this, let $x_1 + x_2 \cdot n \in R(\widetilde{P}_t)$. Then $x_1, x_2 \in R(P_{\lambda_t}) = N(\lambda_t I - T_+)$ and

$$\Delta_{\lambda_t}(T)(x_1 + x_2 \cdot n) = \Delta_{\lambda_t}(T)(x_1) + \Delta_{\lambda_t}(T)(x_2) \cdot n = 0.$$

It is enough to show $N(\Delta_{\lambda_t}(T)) \subseteq R(\widetilde{P}_t)$. By Theorem 3.5(c), there exist a linearly independent set $\{\phi_i : 1 \le i \le k_t\} \subset H_+^{Jm}$ such that

$$T(\phi_i) = T_+(\phi_i) = \phi_i \cdot \lambda_t, \text{ for } 1 \le i \le k_t,$$

and $\text{span}_{\mathbb{H}}\{\phi_i : 1 \le i \le k_t\} = N(\Delta_{\lambda_t}(T))$. Since $\{\phi_i : 1 \le i \le k_t\} \subset N(\lambda_t I - T_+) = R(P_t)$ and $R(\widetilde{P}_t)$ is right $\mathbb{H}$-linear space of $H$, we conclude that $\text{span}_{\mathbb{H}}\{\phi_i : 1 \le i \le k_t\} \subseteq R(\widetilde{P}_t)$. Thus $R(\widetilde{P}_t) = N(\Delta_{\lambda_t}(T))$. It is clear from Proposition 4.5(1), that

$$\lim_{n\to\infty} \left\| T - \sum_{t=1}^{n} \lambda_t \widetilde{P}_t \right\| = \lim_{n\to\infty} \left\| T_+ - \sum_{t=1}^{n} \lambda_t P_t \right\| = 0.$$

The series in Eq. (5) converges in the operator norm.

It remains to show that $\sum_{t=1}^{\infty} \widetilde{P}_t = I$, where $I$ denote the identity operator on $H$. For this, let $x = a + b$, where $a \in H_+^{Jm}$ and $b \in H_-^{Jm}$. Then

$$\begin{aligned}\sum_{t=1}^{\infty} \widetilde{P}_t(x) = \lim_{k\to\infty} \sum_{t=1}^{k} \widetilde{P}_t(x) &= \lim_{k\to\infty} \sum_{t=1}^{k} P_t(a) - P_t(b \cdot n) \cdot n \\ &= \sum_{t=1}^{\infty} P_t(a) - \sum_{t=1}^{\infty} P_t(b \cdot n) \cdot n \\ &= a - (b \cdot n) \cdot n \\ &= x.\end{aligned}$$

Therefore

$$\sum_{t=1}^{\infty} \widetilde{P}_t = I.$$

Note that the above series converges in the strong operator topology of $\mathcal{B}(H)$. □

*Remark 4.8* In Theorem 4.7, the meaning of $\lambda_t \widetilde{P}_t$ is the extension of $\mathbb{C}_m$- linear operator $\lambda_t P_t$ to $H$. By the definition

$$\lambda_t \widetilde{P}_t(a + b) = \lambda_t P_t(a) - \lambda_t P_t(b \cdot n) \cdot n, \quad \text{for all } a \in H_+^{Jm}, \quad b \in H_+^{Jm}.$$

Clearly, $\lambda_t \widetilde{P}_t$ is a right $\mathbb{H}$-linear operator.

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