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RESEARCH ARTICLE

Mathematical modeling as citizenry

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Abstract
Mathematical modeling involves authentic connections between mathematics and the real world as students explain phenomena and make predictions in ways that do not include obvious approaches or distinct, single answers. Although school mathematics is not often connected to democracy, I argue that mathematical modeling can serve as a curricular means for developing democratic citizens and that its superseding purpose is *mathematical modeling as citizenry*. In doing so, I explore connections between mathematical modeling and democratic habits in this conceptual paper. I illustrate the ways in which mathematical modeling can be a curricular means for democracy by providing opportunities for students to engage in the democratic habits of problem solving and inquiry, collaboration and communication, nonrepression, deliberation, and critical thinking and critique. Further, I discuss considerations – both areas of emphasis and areas of caution – for the uptake of mathematical modeling as citizenry using its connections to mathematical practices. Examples from the Common Core Standards in the United States and the Australian Curriculum are also utilized. The connections between mathematical modeling and democratic habits have the potential to deepen students' understanding of the modeling process, of mathematics, of the world around them, and of their roles as citizens.

Keywords Mathematical modeling · Democratic habits · Citizenship · Curriculum interpretations

Introduction

Democracy is something that needs to be developed and nourished. As Dewey (1937/1987) asserted, "Democracy as a form of life cannot stand still…if it is to live, [it] must go forward to meet the changes that are here and that are coming" (p. 182). Yet, within the United States, scholars have raised concerns about the minimal and superficial ways that democracy has been viewed and practiced (e.g., Hytten, 2017) and the assumptions that democracy and citizen development just happen without effort (e.g., Barber,1993). Worldwide, there are also concerns about the decline of democracy. The International Institute for Democracy and Electoral Assistance's (2023) annual report found a pattern of decline in democratic practices for the sixth year in a row and asserted, "democracy is still in trouble, stagnant at best, and declining in many places" (p. 8).

In the contexts of schools, the concept of democratic citizenship is often experienced in minimal ways. This includes the acquisition of knowledge and facts about how government works (Johnson, 2020; Stuteville & H. Johnson, 2016) and teaching about citizenship through facts and skills (Down et al., 2008), rather than through the nourishment of civic habits, practices, and ideals. Classrooms, particularly school mathematics classrooms, do not typically espouse democratic education focused on cultivating civic habits that students can exercise as they encounter issues and develop as citizens (e.g., Apple, 1995; Atweh & Goos, 2011; Graven et al., 2023; Stemhagen, 2016). That is, school mathematics classrooms often offer few opportunities for student agency and choice (e.g., Noddings, 2013), reflect exclusionary environments (e.g., Adiredja & Louie, 2020), and portray mathematics as an objective, value-free discipline (e.g., Ernest, 2018). Indeed, as Feinberg (2021) points out, "Mathematics and democracy are not considered good traveling companions, and of all the subjects taught in school, mathematics are seen to be the least amenable to democratic norms" (p. ix).

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Similarly, according to Stemhagen (2016), the development of informed and engaged democratic citizens is a "conspicuous absence from mathematics class [that] hurts both school mathematics and the broader democratic education project" (p. 100). To counter the belief that cultivating democratic citizens is separate from the mathematics classroom, the purpose of this paper is to illustrate how mathematical modeling can serve as a curricular means for developing democratic citizens. Specifically, I argue that the superseding purpose of mathematical modeling is *mathematical modeling as citizenry*. In doing so, I draw on Dishon's (2018) assertion:

> education that is meant to cultivate democratic habits is not a matter of *habituating the appropriate behaviours*, rather, it entails *facilitating productive challenges.* Democratic habits can be cultivated by presenting students with situations that do not directly introduce civic knowledge or demand specific obligations, but rather indirectly call for certain modes of behaviour such as collaboration, thoughtfulness and compromise. (p. 490, emphasis in original)

Mathematical modeling problems reflect the situations for which Dishon (2018) advocated, because they are open-ended, ill-defined, and introduce a productive challenge for students. As students work to solve these problems, they engage in actions and behaviors that foster democratic habits.

Unlike the traditional use of the term habit to describe something that is done repetitively in the same context, democratic habits are based in an individual's interactions with the environment and with others, are restructured through experience, and are adapted across time and contexts (Dewey, 1922b; Dishon, 2018). They position citizens to engage in current democratic societies, yet are also adaptable to future societies (Dishon, 2018; Stitzlein, 2014). This conceptual paper explores connections between mathematical modeling and the democratic habits of problem solving and inquiry, collaboration and communication, nonrepression, deliberation, and critical thinking and critique. I describe mathematical modeling and the ways that it can be utilized to engage students in these democratic habits and argue for the superseding purpose of mathematical modeling as citizenry. This paper also includes areas of emphasis and areas of caution to be considered when leveraging mathematical modeling as citizenry using examples drawn from curricular documents and educational standards from the United States and Australia.

Mathematical modeling

Mathematical modeling is "a process that uses mathematics to represent, analyze, make predictions or otherwise provide insight into real-world phenomena" (Garfunkel & Montgomery, 2019, p. 8). It is more than just an application of mathematical ideas by students (Stillman, 2019); rather, it involves translating messy, authentic, real-world contexts into mathematics in ways that do not include obvious approaches or distinct, single answers (Cirillo et al., 2016). Students can engage in mathematical modeling at all levels of their schooling. As examples that span a range of students, Brown and Stillman (2017) asked primary students how many brass numerals a hardware store should have in stock, Hyunyi et al. (2021) asked middle years students about the size and density of the Great Pacific Garbage Patch, and Stillman (2019) asked tertiary students about the removal of a eucalypt forest. Both school-aged and adult citizens engage in mathematical modeling to solve problems through schooling and society.

Mathematical modeling problems are complex, authentic, real-world problems. It is important to note that there can be many different legitimate solutions to a mathematical modeling problem (Cirillo et al., 2016), depending on the assumptions and decisions made by different students. Following Bliss et al. (2014), I refer to "a solution" throughout this article, rather than "the solution." I do this to acknowledge that groups of students do arrive at "a solution." That solution might be different from the solutions generated by other groups of students, and it is not the only valid solution to the mathematical modeling problem (i.e. "the solution"). To arrive at a solution to a mathematical modeling problem, students go through a nonlinear, iterative process comprised of different phases (e.g., Bliss et al., 2014; Garfunkel & Montgomery, 2019). Students may need to revisit several phases within the process before arriving at a solution and reporting results. These phases include understanding and simplifying, mathematizing, working mathematically, interpreting, and validating (Maaß, 2006) as detailed below.

When engaged in the *understanding and simplifying phase* of the process, students work to identify what they are being asked to solve and make assumptions and decisions based on real-world context and the mathematics required to work towards a solution (Bliss et al., 2014; Garfunkel & Montgomery, 2019). Students are *mathematizing* when they translate the real-world context into a mathematical model using different mathematical representations, such as equations, tables, graphs, or drawings (Maaß, 2006). *Working mathematically* involves students performing mathematical operations to arrive at mathematical results (Bliss et al., 2014; Garfunkel & Montgomery, 2019; Maaß, 2006). Students engage in *interpreting* their mathematical results in terms of the real-world context to communicate their results (Maaß, 2006). Throughout the mathematical modeling process, students are also *validating*, which involves analyzing and evaluating their models, analyzing and evaluating the process taken to arrive at a solution, and, perhaps, revisiting initial assumptions and/or extending the model used to solve the problem (Bliss et al., 2014; Garfunkel & Montgomery, 2019; Maaß, 2006).

As students engage in mathematical modeling, different purposes of modeling can be emphasized. Three potential purposes of mathematical modeling include modeling as a vehicle, modeling as content, and modeling as critic. Mathematical modeling can be used as a vehicle for developing students' mathematical understanding (Julie & Mudaly, 2007). In this case, the process of mathematical modeling serves to deepen students' mathematical knowledge. When mathematical modeling serves to develop skills and competencies related to mathematical modeling itself, then the purpose is modeling as content (Julie & Mudaly, 2007). The purpose, modeling as critic, uses mathematical modeling problems related to social issues to focus on the mathematical decisions made in society (Barbosa, 2006).

Regardless of the purpose of modeling being emphasized, the process of mathematical modeling can be used to cultivate democratic habits and foster the development of democratic citizens. As such, I argue that there is fourth purpose of mathematical modeling, *mathematical modeling as citizenry*. Modeling as citizenry supersedes the other three purposes, because regardless of which other purpose is being targeted (i.e., vehicle, content, or critic), students engage in modeling as citizenry by undergoing problem solving and inquiry, collaborating and communicating, remaining open to new ideas and opinions (nonrepression), deliberating about decisions and assumptions, and engaging in critical thinking and critique. These democratic habits can be fostered in conjunction to using mathematical modeling to deepen mathematical knowledge (modeling as vehicle), to develop a better understanding of the modeling process (modeling as content), or to deepen understanding of mathematics' role in society (modeling as critic). (See Fig. 1.) As such, the process of mathematical modeling can be a curricular means for democracy and citizen development.

Mathematical modeling as citizenry

The process of mathematical modeling incorporates practices that can aid in the development of democratic citizens. Thus, a purpose of mathematical modeling can be citizens' growth and development, or *mathematical modeling as citizenry*. Here, I describe specific elements of mathematical modeling and the mathematical modeling phases connected with habits of democratic citizenship, including problem solving and inquiry, collaboration and communication, nonrepression, deliberation, and critical thinking and critique.

Problem solving and inquiry

At its heart, mathematical modeling involves problem solving and inquiry. Students investigate a broad and complex problem connected to the real world and arrive at possible solutions which are influenced by the assumptions and choices they make (e.g., Bliss et al., 2014; Cirillo et al., 2016; Garfunkel & Montgomery, 2019). Similarly, as Stitzlein (2021) points out, "Inquiry brings citizens together to make sense of and solve problems together. Inquiry is invoked to investigate the world, hypothesize ways to solve our problems, and experiment with solutions" (p. 49). As citizens consider community issues and work together to solve community problems, they may mathematically represent a scenario in order "to gain qualitative and/or quantitative understanding of some real-world problems and to predict future behavior" (Bliss et al., 2014, p. 3). That is, citizens may engage in mathematical modeling.

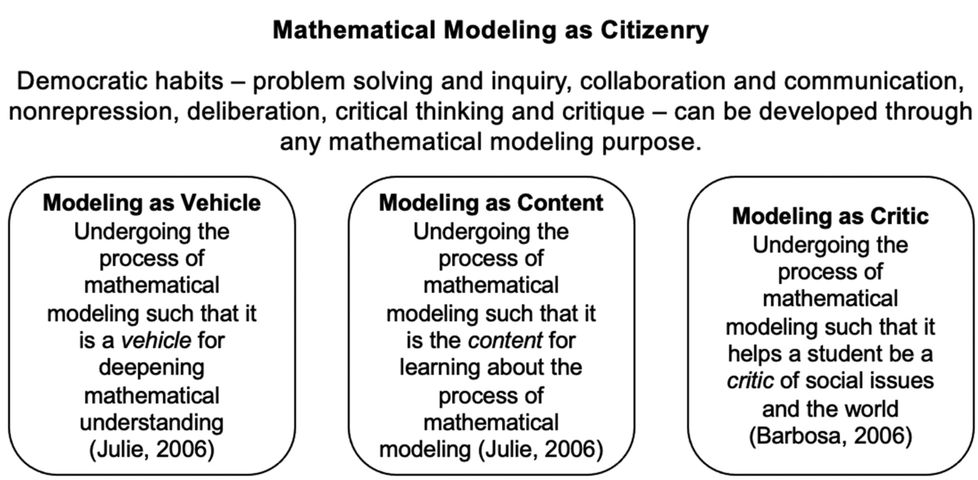

Fig. 1 Relationships across purposes of mathematical modeling

The problems that students face as they engage in mathematical modeling reflect the types of problems that citizens face. This is particularly the case with the purpose of mathematical modeling as critic, in which students use mathematical modeling to investigate and develop a better understanding of different social issues (Barbosa, 2006). Yet, regardless of the purpose being employed for mathematical modeling, when students engage in mathematical modeling, they experience "[r]eal-world problems [that] do not all come packaged the same way" (Garfunkel & Montgomery, 2019, p. 14). That is, these problems are messy and ill-defined, may not provide complete information, and have a variety of approaches and several valid answers (Bliss et al., 2014; Cirillo et al., 2016).

This is also the case for the problems that citizens face and for which they work to find solutions. Citizens work to find solutions through inquiry. Similar to the process of mathematical modeling, inquiry is "experimental in nature and invites multiple, and often conflicting, perspectives into communication with each other to imagine, create, and test potential solutions" (Stitzlein, 2021, pp. 28–29). As such, mathematical modeling fosters the democratic habits of problem solving and inquiry.

Collaboration and communication

The process of mathematical modeling is often a collaborative process that does not occur in solitude (Garfunkel & Montgomery, 2019). Students' collaboration during mathematical modeling helps them to generate different ways of thinking about problems and different solution approaches. Students need to work as a group to determine how they will combine their individual perspectives and lived experiences when making assumptions and choices, deciding how to best approach the problem, and evaluating the reasonableness of the solutions and models during the modeling process. That is, collaboration during the mathematical modeling process "does not just entail 'groupwork' but also mutual responsibility for understanding each other, responsibility for the collective progress, and collective agency in decision making" (Frejd & Vos, 2024, p. 283).

Students' collaboration during mathematical modeling described by Frejd and Vos (2024) supports democratic citizen development. As Noddings (2013) asserted:

> We want to develop citizens who can do more than use formal procedures of a democracy; we want citizens who respect their interdependence and can work cooperatively across groups with whom they share some values but have different central interests. (p. 23)

Mathematical modeling supports this objective by providing a platform for students to practice collaboration and work cooperatively as they negotiate different ideas – both theirs and the ideas of others. In this way, the collaboration that takes place as students engage in mathematical modeling reflects Dewey's (1922a) view of democracy among people with varied interests, perspectives, and concerns in which communication is essential.

Through collaboration during the mathematical modeling process, students also work on the democratic habit of communication as they work together to arrive at a solution. During the mathematical modeling process, students describe, explain, and justify as they construct and validate their models. They generate and communicate ideas, and they listen to and consider others' ideas. As such, engaging in the mathematical modeling process helps to develop students' communication skills. Communicating ideas and being open to the ideas of others are essential parts of mathematical modeling and democratic citizenship. In describing the democratic habit of communication, Stitzlein (2017) noted, "Generating and sharing ideas is also essential to a healthy democracy, where knowledge and viewpoints must be free and openly accessible" (p. 185). This open exchange of ideas is related to the next democratic habit that mathematical modeling helps to develop – nonrepression.

Nonrepression

During the mathematical modeling process, students need to be open to considering different ideas and viewpoints. As such, they need to practice nonrepression, which according to Gutmann (1990) is "the prevention of repressive practices, that is, practices that stifle rational understanding and inquiry" (p. 16). In other words, the democratic habit of nonrepression allows for the sharing of and listening to all ideas with open minds. Nonrepression in mathematical modeling means that students consider all assumptions, models, and solutions with open minds, even those that are different from their own. This can allow students to see that assumptions, models, and solutions that are different from theirs may also be valid (Garfunkel & Montgomery, 2019).

The democratic habit of nonrepression can be fostered throughout the mathematical modeling process, and it is particularly apparent in the understanding and simplifying phase. In working to understand and simplify the problem, students engage in the democratic practice of nonrepression by listening to, considering, and working through others' perspectives as they make sense of the problem. Nonrepression is also evident when students are validating their solutions and models, because they need to consider the viewpoints of others in relation to their work to determine whether their solution is valid and reasonable in relation to the real-world context and in relation to the needs of the client, who posed the problem and is the intended recipient of the group's solution (Bliss et al., 2014; Frejd & Vos, 2024).

Developing the habit of nonrepression is particularly important in mathematical modeling, because each modeler who approaches the same mathematical modeling problem brings with them different perspectives and lived experiences which influence the solution process. That is, the choices and assumptions made by each group of students leads to different, yet valid, solutions (Bliss et al., 2014; Cirillo et al., 2016). By engaging in mathematical modeling, students are afforded opportunities to practice the democratic habit of nonrepression by being open to and considering different viewpoints. This helps students to make rational decisions as well as develop other civic virtues, such as seeking the truth, tolerance towards differences, and mutual respect despite differences (Gutmann, 1993).

Deliberation

Through the mathematical modeling process, students engage in the democratic habit of deliberation. Noddings (2013) defined deliberation as "a political process of analyzing, debating, and evaluating social/political practices that give voices to a wide range of ideals without fastening on any one" (p. 16). Similarly, in mathematical modeling, deliberation involves listening to different points of view, analyzing and evaluating these different points of view, acknowledging disagreements, and working towards common understandings. The process of deliberation helps students engage at a deeper level with the situation and with the mathematics of the task as they consider and evaluate other perspectives. This is also the case for citizens involved in deliberation, as they seek out information and alternative perspectives when solving problems and considering issues.

Relatedly, Frejd and Vos (2024) emphasize consultation as a method to explore alternative perspectives and as a necessary component of mathematical modeling. This consultation exists in several settings: within the group of students, with experts to inform the group's work, and with the client who posed the problem to make sure their needs are being met. Consultation within each of these settings helps students to develop skills in collective deliberation and decision-making (Gutmann, 1993). Additionally, consultation with different groups of people exposes students to different viewpoints which they need to reconcile as they make decisions and construct of their mathematical model. Carefully considering and deliberating about different perspectives during the mathematical modeling process allows students to develop participatory democratic citizenship skills involving analysis and making decisions (Patrick, 2002). In this way, developing skills in deliberation while engaging in mathematical modeling helps to foster thoughtful democratic citizens and reflects the "primary purpose of schooling in a democratic society [which] is to produce thoughtful citizens who can deliberate and make wise choices" (Noddings, 2013, p. 25).

Critical thinking and critique

Engaging in mathematical modeling also helps to foster the democratic habits of critical thinking and critique. This is particularly evident in the validating phase of the mathematical modeling process, in which students undergo a self-assessment of their model and of their process. When engaging in the validating phase, students evaluate and assess their models by looking for alternatives, strengths, and weaknesses (Bliss et al., 2014).

These practices help develop the democratic principle of critical thinking and critique. According to Schroeder (2017), developing the principle of critical thinking and critique helps citizens to question, analyze, and express dissent, instead of automatically deferring to authority. Stitzlein (2017) asserted:

> Many young people experience frustration with social and political issues but lack the know-how or desire to engage in political dissent well. They are unprepared not only for an ideal democracy, where dissent is carefully employed to improve social life, but also for democracy as it is currently being lived outside school walls. (p. 194)

Thus, as students evaluate their mathematical models, they are building skills of citizenship related to critical thinking, questioning, analyzing, and critiquing. These are skills that need to be developed, and we want to provide students with opportunities to critique already-constructed models to nurture these skills. As Skovsmose (1990) reminds us, "it is not possible to develop a critical attitude towards the application of mathematics solely by improving the modeling capability of students…It is not sufficient to [only] become a model builder" (p. 112). We must encourage critical thinking and critique of mathematical models for students to develop as mathematical modelers and as citizens.

Extending the critical questioning and evaluation of mathematical models beyond the classroom is advocated for by proponents of mathematical modeling (e.g., Barbosa, 2006; Skovsmose, 1990). This involves creating and critiquing models in ways that reveal and illustrate the power behind mathematics to foster a critical understanding of the world (e.g., Barbosa, 2006; Blomhøj, 2009; Skovsmose, 1990). According to Blomhøj (2009), developing the competency to "critique mathematical models and the ways in which they are used in decision making is becoming imperative for the developing and maintaining of societies based on equality and democracy" (p. 11). Likewise, Barbosa (2006) argued that as students engage in critique of mathematical models, helps to "produce critical, politically engaged citizens" (p. 296).

Summarizing mathematical modeling as citizenry

As students engage in the process of mathematical modeling, they are cultivating democratic habits of problem solving and inquiry, collaboration and critique, nonrepression, deliberation, and critical thinking and critique. Mathematical modeling provides a fertile environment in which to develop these habits. The open-ended and messy nature of mathematical modeling problems parallel the open-ended and messy problems that we face as citizens. In this way, mathematical modeling can support the cultivation of democratic habits by "presenting students with a trajectory of interactions that demand responding in flexible and varied ways to complex problems that cannot be addressed by mere repetition" (Dishon, 2018, p. 490).

Further, each time students participate in mathematical modeling, they are afforded opportunities to foster these democratic habits. This is the case regardless of the whether the purpose of the mathematical modeling is to deepen mathematical understanding (modeling as vehicle, Julie & Mudaly, 2007), to develop mathematical modeling skills and competencies (modeling as content, Julie & Mudaly, 2007) or to focus on social issues (modeling as critic, Barbosa, 2006). Thus, the superseding purpose of mathematical modeling should be viewed as *modeling as citizenry* in which mathematical modeling is a curricular means for the development and nourishment of democratic citizens.

Yet, care needs to be taken to foster democratic habits through mathematical modeling in intentional ways. To do this, curricular considerations for engaging in mathematical modeling as citizenry are discussed next.

Table 1 Sources of mathematical practices

Strands of Mathematical Proficiency (NRC, 2001, *p.* 115)	Mathematical Process Standards (NCTM, 2000)
• Conceptual understanding • Procedural fluency • Strategic competence • Adaptive reasoning • Productive disposition	• Problem solving • Reasoning and proof • Communication • Connections • Representation
Aspects of Mathematical Proficiency (ACARA, 2022a)	**Standards for Mathematical Practice** (NGA Center & CCSSO, 2010, pp. 6–8)
• Understanding • Fluency • Reasoning • Problem solving	• Make sense of problems and persevere in solving them. • Reason abstractly and quantitatively. • Construct viable arguments and critique the reasoning of others. • Model with mathematics. • Use appropriate tools strategically. • Attend to precision. • Look for and make use of structure. • Look for and express regularity in repeated reasoning.

Curricular considerations for engaging in mathematical modeling as citizenry

The process of mathematical modeling provides opportunities for students to develop democratic habits, such as problem solving and inquiry, collaboration and communication, nonrepression, deliberation, and critical thinking and critique. This section provides considerations – both areas of emphasis and caution – for the deliberate uptake of mathematical modeling as citizenry using examples from the Australian Curriculum: Mathematics [Australian Curriculum, Assessment and Reporting Authority (ACARA), 2022a] and the Common Core State Standards [National Governors Association Center for Best Practices & Council of Chief State School Officers (NGA Center & CCSSO), 2010] from the United States. It is important to note that although the Common Core State Standards are not a national curriculum, these standards are the foundation of many school mathematics curricula in the United States. Additionally, the examples discussed in this section are meant only for illustrative purposes. Although the examples are not meant to serve as comparisons, there is value in noticing similarities in how mathematical modeling is approached in these documents.

Emphasis areas for engaging in mathematical modeling as citizenry

The mathematical practices that students use as they engage in mathematical modeling can be leveraged to cultivate democratic habits. Mathematical practices describe skills, habits, and dispositions for students to develop as they engage in mathematics, and they are included as aspects of mathematical proficiency in the Australian Curriculum (ACARA, 2022a) and as Standards for Mathematical Practice in the Common Core State Standards (NGA Center & CCSSO, 2010). The mathematical practices in these curricular and standards documents are based on the germinal piece, *Adding it Up* [National Research Council (NRC), 2001], and the Mathematical Process Standards [National Council of Teachers of Mathematics (NCTM), 2000]. Table 1 provides the mathematical practices from each of these sources.

Students develop and utilize mathematical practices as they engage in mathematical modeling. Explicitly pointing out how these mathematical practices reflect the work of citizens outside of the classroom is suggested to leverage mathematical modeling as citizenry. For example, mathematical modeling requires students to problem solve. As students participate in mathematical problem solving through mathematical modeling they are developing, exploring, representing, and solving problems for which a solution and the method(s) of obtaining the solution(s) to the problem are not known in advance (NCTM, 2000; NGA Center & CCSSO, 2010). Stitzlein (2021) describes civic inquiry as "experimental in nature and [it] invites multiple, and often conflicting, perspectives into communication with each other to imagine, create, and test potential solutions" (pp. 28–29). As students participate in mathematical modeling as citizenry, deliberate connections can be made among the thinking processes and investigations involved in mathematical problem solving and those involved in civic problem solving. Mathematical problem solving through mathematical modeling is a way for students to explore how mathematical thinking can help them as democratic citizens.

Similarly, the mathematical practices of reasoning and communication can help students develop their deliberation skills. It is through deliberation that citizens work together to discuss and debate shared problems while being open to various viewpoints and making collective decisions (Noddings, 2013; Schroeder, 2017; Stitzlein, 2017). Preparing for and undergoing deliberation mirror the process of mathematical reasoning, which involves students comparing "their ideas with others' ideas, which may cause them to modify, consolidate, or strengthen their arguments or reasoning" (NCTM, 2000, p. 58). Similar to civic deliberation, when reasoning mathematically, students also evaluate each other's ideas, "listen or read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve the arguments" (NGA Center & CCSSO, 2010, p. 7). As students engage in mathematical modeling, they are utilizing reasoning as they "compare and contrast related ideas and reflect upon and explain their choices" (ACARA, 2022a, p. 9).

Deliberation also goes beyond the sharing and evaluating of reasoning to collective decision making and acknowledgement. Mathematically, this is implied as part of the practice of communication (NCTM, 2000) through the development and use of shared mathematical knowledge. As part of the practice of mathematical communication, students must "learn to question and probe one another's thinking in order to clarify underdeveloped ideas…since not all methods have equal merit, students must learn to examine the methods and ideas of others in order to determine their strengths and limitations" (NCTM, 2000, p. 63). This process leads to collective acknowledgement of different strategies for solving problems. Students communicate and reason through different assumptions and decisions that need to be made during the mathematical modeling process. They also use the practices of communication and reasoning when interpreting mathematical results in relation to the real-world context and when evaluating their model and their process. Students' uses of communication and reasoning are opportunities to make explicit connections to the ways in which citizens outside of the classroom need to reason about decisions, analyze and evaluate claims and evidence, and communicate their perspectives to others.

Examples of the mathematical practices involved in mathematical modeling

Mathematical modeling is included in the Common Core Standards (NGA Center & CCSSO, 2010) and the Australian Curriculum: Mathematics (ACARA, 2022a). In fact, the Australian Curriculum connects mathematical modeling to citizenship. Mathematical modeling is described in the Australian Curriculum (ACARA, 2022a) in this way:

> Students develop an understanding of mathematical modelling when they use mathematics to gain insight into and make predictions about real-world phenomena. Mathematical models are used to inform judgements and make decisions in personal, civic and work life. When using mathematical modelling to solve problems, students make assumptions, recognise, connect and apply mathematical structures. The modelling process utilises mathematics to formulate, analyse, solve, interpret, generalise and communicate their results in response to a real-world situation. Mathematical modelling is an essential dimension of the contemporary discipline of mathematics and is key to informed and participating citizenship. (p. 10)

As mathematical modeling tasks are facilitated in classrooms, emphasizing the ways in which the mathematical practices support mathematical modeling as citizenry is encouraged. Here, I provide an illustration of how that this can be done using contexts from the Australian Curriculum and the Common Core Standards.

A context that could be used in primary classrooms is creating a budget for a school fete. This mathematical modeling situation aligns with a Year 4 elaboration for content descriptor AC9M4N08 that suggests that students should be "modelling practical problems involving money, such as a budget for a large event" (ACARA, 2022b, p. 45), Similarly, an elaboration for content descriptor AC9M6N09 recommends that Year 6 students should be "modelling and solving the problem of creating a budget for a class excursion" (p. 69). A context for a secondary classroom from the Common Core State Standards (NGA Center & CCSSO, 2010) features this scenario: "Estimating how much water and food is needed for emergency relief in a devastated city of 3 million people, and how it might be distributed" (p. 72).

In both contexts, students need to use problem-solving skills to determine how to approach each task. The strategies to use and the possible solutions are not obvious, and students will need to work to understand and simplify the problem to move forward. Connections can be made to the community within both contexts and how citizens outside of the classroom engage in similar problem-solving processes when facing with matters such as these. Additionally, deliberate connections could be made between students' work in mathematical modeling and the civic habits of problem solving and civic inquiry and critical thinking.

The problem solving required of students in these mathematical modeling tasks will lead to different assumptions and decisions, which will lead to multiple valid solutions. For example, primary students may consider the different food options, activities, and entertainment to be included at the fete, what supplies would be needed for the day, how wristbands or tickets might be priced, and/or whether donations will be sought from local businesses. The secondary students might consider how much of the city's population is made up of children and adults, how much water and food would be required daily, how many days of support would be needed, and/or possible distribution methods. In both examples, students will need to reason about the context as they generate assumptions and make decisions, and they will need to communicate with each other as they make collective decisions. These are opportunities to explicitly foster the civic habits of deliberation and collaboration and communication. The civic habit of nonrepression can also be emphasized by encouraging students to listen and consider their classmates' ideas with open minds.

The assumptions and decisions students make about either context will influence how they approach the problem and will lead to multiple, valid solutions. As students arrive at different solutions, they will need to interpret mathematical results in terms of the context with which they are working and evaluate the processes they took. Mathematical practices and democratic habits can also be emphasized here. Mathematical reasoning can be used to validate different approaches (e.g., why different pricing methods were chosen for wristbands depending on the assumptions and decisions students made about the school fete, why different amounts of food and water were used depending on the assumptions and decisions students made about the city's population). Communication is used as groups of students discuss how they approached the mathematical modeling task. It will be important for students to consider the ideas of other groups and to recognize that different solutions are valid. Here, mathematical modeling for citizenry can be leveraged to promote nonrepression, deliberation, and critical thinking and critique.

Cautions for engaging in mathematical modeling as citizenry

There are also cautions to consider related to mathematical modeling as it is described within mathematics standards and curricular documents. Within these documents, various interpretations for modeling exist, and these interpretations can lead to different ways that mathematical modeling is facilitated in a mathematics classroom. Not all interpretations of modeling reflect the view of mathematical modeling taken up in this article in which students engage in messy, authentic, real-world contexts that involve making assumptions, decisions, and interpretations about the task and about the mathematics involved. In addition to the interpretation of mathematical modeling used in this article, Felton-Koestler (2017) found two other interpretations of modeling within the Common Core State Standards (NGA Center & CCSSO, 2010): (1) equating mathematical modeling with representing (or modeling) mathematics through the use of physical manipulatives, diagrams, and/or pictures and (2) equating mathematical modeling with the use of straightforward real-world contexts or traditional mathematical word problems. These interpretations are discussed next.

Equating mathematical modeling with representing mathematics

Caution should be taken so that mathematical modeling is not conflated with using models to represent mathematics. According to mathematical modeling scholars, "*Mathematical modeling* and *modeling mathematics* are not the same" (Cirillo et al., 2016, p. 3, emphasis in original). Unlike the open, authentic problems used in mathematical modeling contexts, using models to represent mathematical ideas can take place with any mathematics task, including mathematics tasks that are not linked to real-world contexts and tasks that have only one valid answer. Although representing mathematics using different models is important, one should not equate that with mathematical modeling.

Equating modeling with representing mathematics hinders the opportunities available to students to develop democratic habits, because the task may not lend itself to authentic problem solving, critical thinking, or collaboration. For instance, as part of the Common Core State Standards, second grade students are expected to "add and subtract within 1000, using concrete *models* or drawings" (NGA Center & CCSSO, 2010, p. 19, emphasis added). When addressing this standard, students might be presented with a task that is devoid of a real-world context, such as

356 + 31 or 356 − 31. Students might have some choice how they model addition and subtraction with these tasks, but these tasks are not mathematical modeling tasks. It is important to recognize that modeling mathematical ideas is not the same as mathematical modeling.

A similar caution about interpreting the modeling of mathematical ideas should also be extended to the Australian Curriculum: Mathematics. The Year 2 Achievement Standards include the expectation that students will "use mathematical modelling to solve practical problems involving authentic situations *by representing problems with physical and virtual materials, and diagrams*, and using different calculation strategies to find solutions" (ACARA, 2022b, p. 19, emphasis added). Here, the process of solving mathematical modeling tasks is equated to representing problems with various materials and diagrams. This interpretation is illustrated in the Year 2 elaborations for the content descriptor that students will "use mathematical modelling to solve practical problems involving additive and multiplicative situations" (AC9M2N06, ACARA, 2022b, p. 22) for which the following example is provided: "modelling and solving the problem 'How many days are there left in this year?' by using a calendar" (p. 22). For this task, students may have different strategies for determining the number of days left in the year, but their mathematical ideas will be represented with a calendar. Further, there is only one valid answer to this task. This is also not a mathematical modelling task, but a task asking students to model their mathematical ideas.

Equating mathematical modeling with straightforward real-world contexts

Caution should also be taken so that mathematical modeling is not equated with the use of straightforward real-world contexts and/or traditional mathematical word problems (Felton-Koestler, 2017). Mathematical modeling tasks are not straightforward, and students will need to grapple with how to proceed. The various ways in which students proceed with the task will be based on the different assumptions and decisions they make. It would be erroneous, for example, to believe that students were engaging in mathematical modeling through this standard as it is written: "Solve *word problems involving division of whole numbers* leading to answers in the form of fractions or mixed numbers, e.g., by using visual fraction *models* or equations to represent the problem (NGA Center & CCSSO, 2010, p. 37, emphasis added). This interpretation equates mathematical modeling problems with traditional mathematical word problems. Yet, unlike word problems, mathematical modeling problems are not straightforward and lead to multiple valid answers. Students often do not have all the information they need with mathematical modeling problems, which leads to the assumptions and decisions that need to be made during the mathematical modeling process (Bliss et al., 2014).

Taking caution to not interpret mathematical modeling as working on straightforward mathematics tasks that draw on real-world context is also suggested for those using the Australian Curriculum (ACARA, 2022a). For example, the achievement standards for several primary years mention using 'mathematical modelling to solve practical problems' using the appropriate content for each year level. Students are also often asked to represent these situations in various ways. In Year 3, for example, students are expected to "use mathematical modelling to *solve practical problems* involving additive and multiplicative situations" (AC9M3N06, ACARA, 2022b, p. 32, emphasis added). The elaborations for this content description provide examples of these practical problems, which are application-based questions and situations. That is, the examples take a situation from the world outside of the classroom and ask students to apply mathematical ideas and represent the situation to arrive at an answer. The elaboration reads:

> modelling *practical multiplicative situations using materials or a diagram to represent* the problem; for example, if 4 tomato plants each have 6 tomatoes, deciding whether to use an addition or multiplication number sentence, explaining how each number in their number sentence is connected to the situation. (ACARA, 2022b, p. 32, emphasis added)

In this elaboration, the example is a straightforward context which includes all the information the students need to arrive at an answer. This is different from mathematical modeling which encourages students to make assumptions and decisions to inform an iterative process leading to multiple, valid answers (Bliss et al., 2014; Cirillo et al., 2016; Garfunkel & Montgomery, 2019).

It should be noted that although the examples used in these cautions were derived from primary standards, misinterpretations of mathematical modeling can also exist with secondary standards. Within the Common Core State Standards (NGA Center & CCSSO, 2010), for example, mathematical modeling examples for secondary students includes "Modeling saving account balance, bacterial colony growth, or investment growth" (representing mathematical ideas, p. 72), and "Planning a table tennis tournament for 7 players at a club with 4 tables, where each player plays against each other player" (straightforward real-world context, p. 72). Similarly, the Australian Curriculum (ACARA, 2022c) includes these content elaborations for Year 9 students: "modelling practical contexts using linear functions such as cooking times that include resting or cooling times, or water leakage from water tanks, using tables and graphs or digital tools and algebraically" (representing mathematical ideas and straightforward real-world contexts, p. 32). Care should be taken at all levels that mathematical modeling tasks engage students in problems that are messy and ill-defined, require students to make assumptions and decisions based on the context, and lead to multiple valid solutions.

Concluding remarks

Similar to problems that citizens face, mathematical modeling problems are open and authentically connected to the real-world, without an obvious approach for finding a solution. As such, democratic habits can be fostered as students engage in mathematical modeling. Students can develop democratic habits of problem solving and inquiry, collaboration and communication, nonrepression, deliberation, and critical thinking and critique through engagement in mathematical modeling. Mathematical modeling, then, can be considered a curricular means for the development and nourishment of democratic citizens, and the superseding purpose of mathematical modeling can be citizens' growth and development, or *mathematical modeling as citizenry*.

For democracy to be nourished through mathematical modeling, however, care must be taken to make explicit connections between the mathematical practices involved in mathematical modeling and democratic habits. Additionally, consistent interpretations that reflect authentic mathematical modeling must be used within school mathematics curricula. Care must also be taken to avoid equating mathematical modeling with representing mathematical ideas and with straightforward real-world contexts, particularly as these ideas are evident in the Common Core State Standards and the Australian Curriculum. These interpretations do not reflect authentic mathematical modeling and do not afford students opportunities to develop democratic habits by grappling with problems that are similar to those they face as citizen.

The connections between mathematical modeling and democratic habits are powerful. I believe these connections should be encouraged to foster students' development as citizens, helping them to acquire habits that will help them to nurture democracy and work towards more democratic futures. Encouraging democratic habits through authentic mathematical modeling has the potential to deepen students' understanding of the modeling process (modeling as content), of mathematics (modeling as a vehicle), of the world around them (modeling as critic). Importantly, mathematical modeling can be a curricular means for democracy and citizen development, helping students understand their roles as citizens through *mathematical modeling as citizenry*.

Declarations

Conflict of interest The author has no conflict of interest to report.

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