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ORIGINAL PAPER

Use of weights in mixed randomized response model

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Abstract In this paper, we have suggested a weighted unbiased estimator based on mixed randomized response model. Some unbiased estimators are generated from the proposed weighted estimator. The variance of the proposed weighted estimator is obtained and relevant condition is obtained in which the proposed weighted estimator is superior to Singh and Tarray (Sociol Methods Res 44(4):706–722, 2014) estimator. It is interesting to mention that we have investigated an estimator $\hat{\pi}_{\mathrm{HS}(1)}$ which is the member of the suggested weighted estimator $\hat{\pi}_{\mathrm{HS}}$ provide better efficiency than the Singh and Tarray's (2014) estimator $\hat{\pi}_{\mathrm{h}}$ and close to the optimum estimator $\hat{\pi}_{\mathrm{HS}}^{(\mathrm{o})}$. Thus, the estimator $\hat{\pi}_{\mathrm{HS}(1)}$ is an alternative to optimum estimator $\hat{\pi}_{\mathrm{HS}}^{(\mathrm{o})}$. The study is further extended in case of stratified random sampling.

Keywords Weighted estimator · Mixed randomized response model · Efficiency comparison

Mathematics Subject Classification 62D05

1 Introduction

Respondents sometimes come across sensitive questions, such as gambling, alcoholism, sexual and physical abuse, drug addiction, abortion, tax evasion, illegal income, mobbing, political view, doping usage, homosexual activities and many others. When respondents are asked directly with such questions, they may refuse to answer

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the question or give untruthful answers, which would significantly affect the quantity and quality of the survey. Warner (1965) introduced the randomized response technique (RRT) to address this problem.

Several variations of the original RRT models, both binary response and quantitative response models, have been discussed by researchers, including Mangat and Singh (1990) Fox and Tracy (1986), Chaudhuri and Mukerjee (1987, 1988), Hedayat and Sinha (1991), Tracy and Mangat (1996), Mangat and Singh (1990),Mangat (1994), Mahmood et al. (1998), Singh et al. (2000), Chang and Huang (2001), Christofides (2003), Huang (2004), Chang et al. (2004a, b), and Singh and Tarray (2012).

To implement the privacy problem with the Moors (1997) model, Mangat et al. (1997) and Singh et al. (2000) have given several strategies as alternatives to Moors (1997) model, but their models may lose a large portion of data information and require a high cost to obtain confidentiality of the respondents. These drawbacks with the previous alternative models for the Moors model motivated Kim and Warde (2005) to envisage a mixed RR model using simple random sampling with replacement that modifies the privacy problem. The work of this paper based on mixed randomized response model due to Singh and Tarray's (2014). So the description of their model is given below.

1.1 Singh and Tarray's (2014) mixed randomized response model

In the model given by Singh and Tarray (2014), a single sample with size n is selected by simple random sampling with replacement (SRSWR) from the population. Each respondent from the sample is instructed to answer the direct question, "I am a member of the innocuous trait group". If a respondent answers "Yes" to direct question, then he or she is instructed to go to randomization device R_1 consisting of the statements (i) "I am a member of the sensitive trait group" and (ii) "I am a member of the innocuous trait group" with probabilities of selection P_1 and $\left(1-P_1\right)$, respectively. If a respondent answers "No" to the direct question, then the respondent is instructed to use a randomization procedure due to Mangat (1994). In the Mangat's (1994) RR procedure, each respondent is instructed to say "Yes" if he or she is a member of the sensitive trait group. If he or she is not a member of the sensitive trait group, then the respondent is required to use the Warner's (1965) randomization device R_2 consisting of statements: (a) "I belong to the sensitive trait group" and (b) "I do not belong to the sensitive trait group" represented with probabilities P and $(1-P)$, respectively. Then he or she is to report "Yes" or "No" according to the outcome of the randomization device R_2 and the actual status that he or she has with respect to the sensitive trait group. The survey procedures are performed under the assumption that both the sensitive and the innocuous questions are unrelated and independent in a randomization device R_1. To protect the respondent's privacy, the respondents should not disclose to the interviewer the question they answered from either R_1 or R_2.

Let n be the sample size confronted with a direct question, and n_1 and $n_2\left(=n-n_1\right)$ denote the number of "Yes" and "No" answers from the sample. Since all the respondents using a randomization device R_1 already responded "Yes" from the initial direct innocuous question, the proportion "Y" of getting "Yes" answers from the respondents using randomization device R_1 is expressed as

$$Y = P_1\pi_s + \left(1 - P_1\right)\pi_1 = P_1\pi_s + \left(1 - P_1\right), \tag{1}$$

where π_s is the proportion of "Yes" answers from the sensitive trait and π_1 is the proportion of "Yes" answer from the innocuous question.

An unbiased estimator of π_s, in terms of the sample proportion of "Yes" responses $\hat{Y}$, is given by

$$\hat{\pi}_1 = \frac{\hat{Y} - \left(1 - P_1\right)}{P_1}, \tag{2}$$

with variance

$$V\left(\hat{\pi}_1\right) = \frac{Y(1-Y)}{n_1 P_1^2} = \frac{1}{n_1}\left[\left(1 - \pi_s\right)\pi_s + \frac{\left(1 - \pi_s\right)\left(1 - P_1\right)}{P_1}\right]. \tag{3}$$

The proportion of "Yes" answers from the respondents using Mangat's (1994) randomization device R_2

$$X = \pi_s + \left(1 - \pi_s\right)(1 - P). \tag{4}$$

An unbiased estimator of π_s, in terms of the sample proportion of "Yes" responses $\hat{X}$ is given by

$$\hat{\pi}_2 = \frac{\hat{X} - (1 - P)}{P}. \tag{5}$$

The variance of $\hat{\pi}_2$ is given by

$$V\left(\hat{\pi}_2\right) = \frac{X(1-X)}{n_2 P^2} = \frac{1}{n_2}\left[\pi_s\left(1 - \pi_s\right) + \frac{(1 - P)\left(1 - \pi_s\right)}{P}\right]. \tag{6}$$

Giving weight $\lambda = n_1/n$ to the estimator $\hat{\pi}_1$ and $(1-\lambda) = \left(n - n_1\right)/n$ to the estimator $\hat{\pi}_2$, Singh and Tarray (2014) suggested an unbiased estimator for π_s as

$$\hat{\pi}_h = \lambda\hat{\pi}_1 + (1-\lambda)\hat{\pi}_2, \quad \text{for} \quad 0 < \lambda < 1. \tag{7}$$

with the variance

$$\begin{aligned} V\left(\hat{\pi}_h\right) =& \frac{\lambda}{n}\left[\pi_s\left(1 - \pi_s\right) + \frac{\left(1 - \pi_s\right)\left(1 - P_1\right)}{P_1}\right] \\ &+ \frac{(1-\lambda)}{n}\left[\pi_s\left(1 - \pi_s\right) + \frac{\left(1 - \pi_s\right)(1 - P)}{P}\right]. \end{aligned} \tag{8}$$

For $P = (2 - P_1)^{-1}$, Singh and Tarray (2014) obtained the variance of $\hat{\pi}_{\mathrm{h}}$ as

$$V(\hat{\pi}_{\mathrm{h}}) = \frac{1}{n}\left[\lambda V_1 + (1-\lambda)V_2\right], \tag{9}$$

where

$$V_1 = \left[\pi_{\mathrm{s}}(1-\pi_{\mathrm{s}}) + \frac{(1-\pi_{\mathrm{s}})(1-P_1)}{P_1}\right],$$

$$V_2 = \left[\pi_{\mathrm{s}}(1-\pi_{\mathrm{s}}) + (1-\pi_{\mathrm{s}})(1-P_1)\right].$$

In Sect. 2, we have suggested a weighted unbiased estimator for π_{s} and studied its properties.

2 Proposed class of unbiased estimators

We define a weighted unbiased estimator for π_{s} as

$$\hat{\pi}_{\mathrm{HS}} = \eta_1\hat{\pi}_1 + \eta_2\hat{\pi}_2, \tag{10}$$

where η_1 and η_2 are suitably chosen weights such that $\eta_1 + \eta_2 = 1$.

For suitable values of (η_1, η_2), a set of estimators can be identified, for instance, see Table 1

It is known that the two randomization devices are independent, therefore, the variance of $\hat{\pi}_{\mathrm{HS}}$ is given by

$$V(\hat{\pi}_{\mathrm{HS}}) = \eta_1^2 V(\hat{\pi}_1) + \eta_2^2 V(\hat{\pi}_2),$$

$$= \frac{1}{n}\left\{\frac{\eta_1^2}{\lambda}\left[\pi_{\mathrm{s}}(1-\pi_{\mathrm{s}}) + \frac{(1-\pi_{\mathrm{s}})(1-P_1)}{P_1}\right] + \frac{\eta_2^2}{(1-\lambda)}\left[\pi_{\mathrm{s}}(1-\pi_{\mathrm{s}}) + \frac{(1-\pi_{\mathrm{s}})(1-P)}{P}\right]\right\}. \tag{11}$$

Table 1 Different weights of (η_1, η_2) and the resulting estimators of $\hat{\pi}_{\mathrm{s}}$

S. no.	Values of weights		Estimators
	η_1	η_2	
1	λ	$(1-\lambda)$	$\hat{\pi}_{\mathrm{h}} = \lambda\hat{\pi}_1 + (1-\lambda)\hat{\pi}_2$ Singh and Tarray (2014) estimator
2	$(1-\lambda)$	λ	$\hat{\pi}_{\mathrm{HS1}} = (1-\lambda)\hat{\pi}_1 + \lambda\hat{\pi}_2$
3	$\frac{1}{(1+\lambda)}$	$\frac{\lambda}{(1+\lambda)}$	$\hat{\pi}_{\mathrm{HS2}} = \frac{1}{(1+\lambda)}\hat{\pi}_1 + \frac{\lambda}{(1+\lambda)}\hat{\pi}_2$
4	$-\lambda$	$(1+\lambda)$	$\hat{\pi}_{\mathrm{HS3}} = (1+\lambda)\hat{\pi}_2 - \lambda\hat{\pi}_1$
5	$\frac{\lambda}{(1+\lambda)}$	$\frac{1}{(1+\lambda)}$	$\hat{\pi}_{HS4} = \frac{\lambda}{(1+\lambda)}\hat{\pi}_1 + \frac{1}{(1+\lambda)}\hat{\pi}_2$
6	$(1+\lambda)$	$-\lambda$	$\hat{\pi}_{\mathrm{HS5}} = (1+\lambda)\hat{\pi}_1 - \lambda\hat{\pi}_2$

Inserting $P = \left(2 - P_1\right)^{-1}$ in (11) we get

$$\begin{aligned}V\left(\hat{\pi}_{\mathrm{HS}}\right) &= \frac{1}{n}\left\{\frac{\eta_1^2}{\lambda}\left[\pi_{\mathrm{s}}\left(1-\pi_{\mathrm{s}}\right) + \frac{\left(1-\pi_{\mathrm{s}}\right)\left(1-P_1\right)}{P_1}\right] + \frac{\eta_2^2}{(1-\lambda)}\left[\pi_{\mathrm{s}}\left(1-\pi_{\mathrm{s}}\right) + \left(1-\pi_{\mathrm{s}}\right)\left(1-P_1\right)\right]\right\} \\ &= \frac{1}{n}\left[\frac{\eta_1^2}{\lambda}V_1 + \frac{\eta_2^2}{(1-\lambda)}V_2\right] = \frac{1}{n}\left[\frac{\eta_1^2}{\lambda}V_1 + \frac{\left(1+\eta_1^2-2\eta_1\right)}{(1-\lambda)}V_2\right] \\ &= \frac{1}{n}\left[\eta_1^2\left\{\frac{V_1}{\lambda} + \frac{V_2}{(1-\lambda)}\right\} - \frac{2\eta_1 V_2}{(1-\lambda)} + \frac{V_2}{(1-\lambda)}\right] \\ &= \frac{1}{n\lambda(1-\lambda)}\left[\eta_1^2\left\{(1-\lambda)V_1 + \lambda V_2\right\} - 2\eta_1\lambda V_2 + \lambda V_2\right].\end{aligned} \tag{12}$$

The variance of $\hat{\pi}_{HS}$ at (12) is minimised for

$$\left.\begin{aligned}\eta_1 &= \frac{\lambda V_2}{\left[(1-\lambda)V_1 + \lambda V_2\right]} = \eta_{10}(\text{say}) \\ \eta_2 &= \frac{(1-\lambda)V_1}{\left[(1-\lambda)V_1 + \lambda V_2\right]} = \eta_{20}(\text{say})\end{aligned}\right\}, \tag{13}$$

Inserting (13) in (10) we get the optimum estimator (OE) for π_{s} as

$$\hat{\pi}_{\mathrm{HS}}^{(\mathrm{o})} = \left(w_{10}\hat{\pi}_1 + w_{20}\hat{\pi}_2\right). \tag{14}$$

Thus, the resulting minimum variance of $\hat{\pi}_{\mathrm{HS}}$ (or the variance of the OE $\left(\hat{\pi}_{\mathrm{HS}}^{\mathrm{o}}\right)$ is given by

$$\begin{aligned}\min.\ V\left(\hat{\pi}_{\mathrm{HS}}\right) &= \frac{V_1 V_2}{n\left[(1-\lambda)V_1 + \lambda V_2\right]} \\ &= V\left(\hat{\pi}_{\mathrm{HS}}^{(\mathrm{o})}\right).\end{aligned} \tag{15}$$

Thus, we state the following Theorem.

Theorem 2.1 The variance of the weighted estimator $\hat{\pi}_{HS}$,

$$V\left(\hat{\pi}_{HS}\right) \geq \frac{V_1 V_2}{n\left[(1-\lambda)V_1 + \lambda V_2\right]}$$

with equality holding if

$$\eta_1 = \eta_{10} \text{ and } \eta_2 = \eta_{20}$$

Putting $\eta_1 = \lambda$ *and* $\eta_2 = (1-\lambda)$ *in* (12) *one can easily get the variance of* Singh and Tarray (2014) *estimator as given in* (9).

2.1 Special case

For $\eta_1 = \frac{\lambda P_1}{1-\lambda}$, the proposed estimator $\hat{\pi}_{\mathrm{HS}}$ defined by (10) reduces to an unbiased estimator

$$\hat{\pi}_{\mathrm{HS(1)}} = \frac{\lambda P_1}{(1-\lambda)}\hat{\pi}_1 + \frac{\left\{1-\lambda\left(1+P_1\right)\right\}}{(1-\lambda)}\hat{\pi}_2. \tag{16}$$

Here, we note that $\left(\lambda, P_1\right)$ are known.

Putting $\eta_1 = \frac{\lambda P_1}{1-\lambda}$ in (12) we get the variance of the unbiased estimator $\hat{\pi}_{\mathrm{HS(1)}}$ as

$$V\left(\hat{\pi}_{\mathrm{HS(1)}}\right) = \frac{1}{n(1-\lambda)}\left[\frac{\lambda P_1^2\left\{(1-\lambda)V_1 + \lambda V_2\right\}}{(1-\lambda)^2} - \frac{2\lambda P_1 V_2}{(1-\lambda)} + V_2\right]. \tag{17}$$

Putting $\eta_1 = \lambda \Rightarrow \eta_2 = (1-\lambda)$ in (12) we get the variance of the Singh and Tarray's (2014) estimator $\hat{\pi}_{\mathrm{h}}$ as

$$V\left(\hat{\pi}_{\mathrm{h}}\right) = \frac{1}{n}\left[\lambda V_1 + (1-\lambda)V_2\right]. \tag{18}$$

From (17) and (18) we have

$$V\left(\hat{\pi}_{\mathrm{h}}\right) - V\left(\hat{\pi}_{\mathrm{HS(1)}}\right) = \frac{\lambda}{n(1-\lambda)^2}\left(1 - \frac{P_1}{1-\lambda}\right)\left[\left\{(1-\lambda)V_1 + \lambda V_2\right\}\left(1-\lambda+P_1\right) - 2(1-\lambda)V_2\right],$$

which is positive if

$$\left.\begin{array}{ll}\text{either } \left[\lambda^2\left(V_1 - V_2\right) + \lambda\left\{V_2 - \left(2+P_1\right)\left(V_1 - V_2\right)\right\} + \left\{V_1\left(1+P_1\right) - 2V_2\right\}\right] > 0, & P_1 < (1-\lambda) \\ \text{or } \left[\lambda^2\left(V_1 - V_2\right) + \lambda\left\{V_2 - \left(2+P_1\right)\left(V_1 - V_2\right)\right\} + \left\{V_1\left(1+P_1\right) - 2V_2\right\}\right] < 0, & P_1 > (1-\lambda)\end{array}\right\}. \tag{19}$$

To see the performance of the suggested estimator $\hat{\pi}_{\mathrm{HS(1)}}$ at (16) relative to Singh and Tarray (2014) estimator $\hat{\pi}_{\mathrm{h}}$ given by (7) we have computed the percent relative efficiency (PRE) of $\hat{\pi}_{\mathrm{HS(1)}}$ with respect to $\hat{\pi}_{\mathrm{h}}$ using the formula given in Sect. 3 for different values of $\left(\lambda, P_1, \pi_{\mathrm{s}}\right)$.

3 Efficiency comparison

In this section, we have made the comparison of the proposed weighted mixed randomized response model, under completely truthful reporting case, with Singh and Tarray's (2014) model.

We have from (9) and (16) that

$$V(\hat{\pi}_{\mathrm{h}}) - \min.\, V(\hat{\pi}_{\mathrm{HS}}) = \left[= V\left(\pi_{\mathrm{HS}}^{(\mathrm{o})}\right)\right] = \frac{\lambda(1-\lambda)(V_1 - V_2)^2}{n[(1-\lambda)V_1 + \lambda V_2]}$$
$$= \frac{\lambda(1-\lambda)(1-\pi_{\mathrm{s}})^2(1-P_1)^2\left[\frac{1}{P_1} - 1\right]^2}{n[(1-\lambda)V_1 + \lambda V_2]}, \tag{20}$$

which is always positive.

It follows that the proposed class of estimators $\hat{\pi}_{\mathrm{HS}}$ is more efficient than Singh and Tarray (2014) estimator $\hat{\pi}_{\mathrm{h}}$ at optimum condition. Thus, we infer that to get estimator better than Singh and Tarray (2014) estimator $\hat{\pi}_{\mathrm{h}}$ one has to choose the values of (η_1, η_2) in the vicinity of the exact optimum values (η_{10}, η_{20}) of (η_1, η_2).

The percent relative efficiency of the OE $\hat{\pi}_{\mathrm{HS}}^{(\mathrm{o})}$ with respect to Singh and Tarray (2014) estimator $\hat{\pi}_{\mathrm{h}}$ is given by

$$\mathrm{PRE}\left(\hat{\pi}_{\mathrm{HS}}^{(\mathrm{o})}, \hat{\pi}_{\mathrm{h}}\right) = \frac{V(\hat{\pi}_{\mathrm{h}})}{V\left(\hat{\pi}_{\mathrm{HS}}^{(\mathrm{o})}\right)} \times 100$$
$$= \left[1 + \frac{\lambda(1-\lambda)(V_1 - V_2)^2}{V_1 V_2}\right] \times 100, \tag{21}$$

Further, the PRE of $\hat{\pi}_{\mathrm{HS(1)}}$ with respect to $\hat{\pi}_{\mathrm{h}}$ is given by

$$\mathrm{PRE}\left(\hat{\pi}_{\mathrm{HS(1)}}, \hat{\pi}_{\mathrm{h}}\right) = \frac{V(\hat{\pi}_{\mathrm{h}})}{V\left(\hat{\pi}_{\mathrm{HS(1)}}\right)} \times 100$$
$$= \frac{[\lambda V_1 + (1-\lambda)V_2](1-\lambda)}{\left[V_2 - \frac{2\lambda P_1 V_2}{(1-\lambda)} + \frac{\lambda P_1^2\{(1-\lambda)V_1 + \lambda V_2\}}{(1-\lambda)^2}\right]} \times 100, \tag{22}$$

with the help of the formula given in (13), we have computed the optimum values of η_{10} and η_{20} for different values of $(\lambda, \pi_{\mathrm{s}}, P_1)$ and findings are shown in Table 2.

Table 2 Optimum values of weights η_{10} and η_{20} of η_1 and η_2

π_s	λ	$n = 1000$		$P_1 = 0.1$		$P_1 = 0.14$		$P_1 = 0.18$		$P_1 = 0.22$		$P_1 = 0.26$		$P_1 = 0.30$		$P_1 = 0.34$		$P_1 = 0.38$		$P_1 = 0.42$	
		n_1	n_2	η_{10}	η_{20}	η_{10}	η_{20}	η_{10}	η_{20}	η_{10}	η_{20}	η_{10}	η_{20}	η_{10}	η_{20}	η_{10}	η_{20}	η_{10}	η_{20}	η_{10}	η_{20}
0.1	0.1	100	900	0.01	0.99	0.02	0.98	0.02	0.98	0.03	0.97	0.03	0.97	0.04	0.96	0.04	0.96	0.04	0.96	0.05	0.95
0.1	0.3	300	700	0.04	0.96	0.06	0.94	0.08	0.92	0.09	0.91	0.11	0.89	0.12	0.88	0.14	0.86	0.15	0.85	0.16	0.84
0.1	0.4	400	600	0.07	0.93	0.09	0.91	0.12	0.88	0.14	0.86	0.16	0.84	0.18	0.82	0.20	0.80	0.22	0.78	0.23	0.77
0.1	0.5	500	500	0.10	0.90	0.13	0.87	0.17	0.83	0.19	0.81	0.22	0.78	0.25	0.75	0.27	0.73	0.29	0.71	0.31	0.69
0.2	0.1	100	900	0.01	0.99	0.02	0.98	0.02	0.98	0.03	0.97	0.03	0.97	0.04	0.96	0.04	0.96	0.05	0.95	0.05	0.95
0.2	0.3	300	700	0.05	0.95	0.07	0.93	0.08	0.92	0.10	0.90	0.12	0.88	0.13	0.87	0.15	0.85	0.16	0.84	0.17	0.83
0.2	0.4	400	600	0.07	0.93	0.10	0.90	0.13	0.87	0.15	0.85	0.17	0.83	0.19	0.81	0.21	0.79	0.23	0.77	0.25	0.75
0.2	0.5	500	500	0.11	0.89	0.14	0.86	0.18	0.82	0.21	0.79	0.24	0.76	0.26	0.74	0.29	0.71	0.31	0.69	0.33	0.67
0.3	0.1	100	900	0.01	0.99	0.02	0.98	0.02	0.98	0.03	0.97	0.04	0.96	0.04	0.96	0.05	0.95	0.05	0.95	0.05	0.95
0.3	0.3	300	700	0.05	0.95	0.07	0.93	0.09	0.91	0.11	0.89	0.12	0.88	0.14	0.86	0.16	0.84	0.17	0.83	0.18	0.82
0.3	0.4	400	600	0.08	0.92	0.11	0.89	0.13	0.87	0.16	0.84	0.18	0.82	0.20	0.80	0.22	0.78	0.24	0.76	0.26	0.74
0.3	0.5	500	500	0.11	0.89	0.15	0.85	0.19	0.81	0.22	0.78	0.25	0.75	0.28	0.72	0.30	0.70	0.32	0.68	0.34	0.66
0.4	0.1	100	900	0.02	0.98	0.02	0.98	0.03	0.97	0.03	0.97	0.04	0.96	0.04	0.96	0.05	0.95	0.05	0.95	0.06	0.94
0.4	0.3	300	700	0.06	0.94	0.08	0.92	0.10	0.90	0.11	0.89	0.13	0.87	0.15	0.85	0.16	0.84	0.18	0.82	0.19	0.81
0.4	0.4	400	600	0.08	0.92	0.11	0.89	0.14	0.86	0.17	0.83	0.19	0.81	0.21	0.79	0.23	0.77	0.25	0.75	0.27	0.73
0.4	0.5	500	500	0.12	0.88	0.16	0.84	0.20	0.80	0.23	0.77	0.26	0.74	0.29	0.71	0.31	0.69	0.33	0.67	0.35	0.65

Table 3 Percent relative efficiency of the suggested optimum estimator $\left(\hat{\pi}_{\mathrm{HS}}^{(\mathrm{o})}\right)$ with respect to Singh and Tarray (2014) estimator $\left(\hat{\pi}_{\mathrm{h}}\right)$

π_s	λ	$n = 1000$		$\mathrm{PRE}\left(\hat{\pi}_{\mathrm{HS}}^{(\mathrm{o})}, \hat{\pi}_{\mathrm{h}}\right)$								
		n_1	n_2	$P_1 = 0.1$	$P_1 = 0.14$	$P_1 = 0.18$	$P_1 = 0.22$	$P_1 = 0.26$	$P_1 = 0.30$	$P_1 = 0.34$	$P_1 = 0.38$	$P_1 = 0.42$
0.1	0.1	100	900	164.89	141.91	129.32	121.46	116.13	112.33	109.52	107.39	105.73
0.1	0.3	300	700	251.41	197.79	168.42	150.06	137.64	128.78	122.22	117.24	113.38
0.1	0.4	400	600	273.04	211.76	178.19	157.22	143.02	132.89	125.39	119.70	115.29
0.1	0.5	500	500	280.25	216.42	181.45	159.60	144.81	134.26	126.45	120.52	115.93
0.2	0.1	100	900	158.35	137.36	125.89	118.75	113.94	110.53	108.02	106.13	104.68
0.2	0.3	300	700	236.15	187.17	160.41	143.75	132.53	124.57	118.72	114.31	110.92
0.2	0.4	400	600	255.60	199.62	169.04	150.01	137.18	128.08	121.39	116.35	112.49
0.2	0.5	500	500	262.08	203.77	171.92	152.09	138.73	129.25	122.28	117.03	113.01
0.3	0.1	100	900	152.91	133.61	123.09	116.57	112.20	109.12	106.87	105.18	103.90
0.3	0.3	300	700	223.46	178.42	153.89	138.67	128.47	121.27	116.02	112.09	109.11
0.3	0.4	400	600	241.10	189.62	161.58	144.19	132.54	124.31	118.31	113.82	110.41
0.3	0.5	500	500	246.98	193.36	164.15	146.04	133.89	125.33	119.07	114.40	110.84
0.4	0.1	100	900	148.32	130.47	120.77	114.78	110.79	107.99	105.95	104.44	103.31
0.4	0.3	300	700	212.75	171.09	148.47	134.50	125.17	118.63	113.89	110.37	107.72
0.4	0.4	400	600	228.86	181.25	155.39	139.42	128.77	121.29	115.87	111.85	108.82
0.4	0.5	500	500	234.23	184.63	157.70	141.07	129.97	122.18	116.54	112.35	109.19

Table 4 Percent relative efficiency of the suggested estimator $\left(\hat{\pi}_{HS(1)}\right)$ with respect to Singh and Tarray (2014) estimator $\left(\hat{\pi}_{h}\right)$

π_s	λ	$n = 1000$		$PRE\left(\hat{\pi}_{HS}^{(o)}, \hat{\pi}_{h}\right)$								
		n_1	n_2	$P_1 = 0.1$	$P_1 = 0.14$	$P_1 = 0.18$	$P_1 = 0.22$	$P_1 = 0.26$	$P_1 = 0.30$	$P_1 = 0.34$	$P_1 = 0.38$	$P_1 = 0.42$
0.1	0.1	100	900	164.88	141.90	129.31	121.44	116.12	112.32	109.51	107.38	105.73
0.1	0.3	300	700	251.38	197.78	168.42	150.06	137.63	128.75	122.15	117.11	113.18
0.1	0.4	400	600	273.03	211.76	178.17	157.13	142.82	132.52	124.79	118.78	113.97
0.1	0.5	500	500	280.24	216.33	181.15	158.94	143.60	132.30	123.51	116.34	110.25
0.2	0.1	100	900	158.30	137.30	125.83	118.69	113.88	110.47	107.96	106.07	104.62
0.2	0.3	300	700	235.97	187.03	160.31	143.69	132.50	124.56	118.72	114.31	110.90
0.2	0.4	400	600	255.41	199.52	169.00	150.00	137.17	128.02	121.22	115.99	111.85
0.2	0.5	500	500	261.95	203.76	171.91	151.94	138.28	128.30	120.60	114.35	109.05
0.3	0.1	100	900	152.81	133.49	122.97	116.44	112.06	108.97	106.72	105.04	103.77
0.3	0.3	300	700	223.05	178.06	153.58	138.42	128.28	121.14	115.94	112.06	109.10
0.3	0.4	400	600	240.58	189.24	161.34	144.06	132.49	124.31	118.30	113.73	110.15
0.3	0.5	500	500	246.48	193.12	164.09	146.04	133.80	124.94	118.17	112.70	108.05
0.4	0.1	100	900	148.16	130.28	120.57	114.56	110.56	107.75	105.72	104.21	103.08
0.4	0.3	300	700	212.06	170.45	147.90	134.00	124.76	118.31	113.65	110.22	107.64
0.4	0.4	400	600	227.93	180.50	154.83	139.04	128.54	121.20	115.86	111.85	108.75
0.4	0.5	500	500	233.22	184.01	157.40	140.98	129.97	122.08	116.10	111.30	107.21

Using the formulae given by (21) and (22) we have computed the values of $\text{PRE}\left(\hat{\pi}_{\text{HS}}^{(\text{o})}, \hat{\pi}_{\text{h}}\right)$ and $\text{PRE}\left(\hat{\pi}_{\text{HS}(1)}, \hat{\pi}_{\text{h}}\right)$ for different values of $\left(\lambda,\ \pi_{\text{s}},\ P_1\right)$ and findings are tabulated in Tables 3 and 4, respectively.

Table 2 depicts the optimum values $\left(\eta_{10}, \eta_{20}\right)$ of weights $\left(\eta_1, \eta_2\right)$ in the proposed estimator $\hat{\pi}_{\text{HS}}$ for the various values of π_{s}, λ, P_1, and $n = 1000$. Table 2 reveals that for fixed values of $\left(\pi_{\text{s}}, \lambda\right)$, the value of η_{10} increases as P_1 increases while η_{20} decreases as P_1 increases. On the other hand, it is looked upon that for fixed values of $\left(\lambda, P_1\right)$ the value of η_{10} increases as π_s increases and η_{20} decreases as π_{s} increases. It follows from Table 3 that $\text{PRE}\left(\hat{\pi}_{\text{HS}}^{(\text{o})}, \hat{\pi}_{\text{h}}\right)$ decreases as P_1 increases and it decreases as π_{s} increases. For fixed values of $\left(\pi_{\text{s}}, P_1\right)$ the $\text{PRE}\left(\hat{\pi}_{\text{HS}}^{(\text{o})}, \hat{\pi}_{\text{h}}\right)$ increases as λ decreases.

There is considerable gain in efficiency using the proposed OE $\hat{\pi}_{\text{HS}}^{(\text{o})}$ over Singh and Tarray (2014) estimator $\hat{\pi}_{\text{h}}$ as long as $P_1 < \frac{1}{2}$. However, in general, the PRE of the proposed OE is larger than 100%.

Further, from Table 4 it is observed that

1. there is substantial gain in efficiency using the envisaged estimator $\hat{\pi}_{\text{HS}(1)}$ over Singh and Tarray's (2014) estimator $\hat{\pi}_{\text{h}}$ where $P_1 \leq 0.42$.
2. the proposed estimator $\hat{\pi}_{\text{HS}(1)}$ is always better than Singh and Tarray's (2014) estimator $\hat{\pi}_{\text{h}}$ as long as $0 < P_1 \leq 0.42$ and $\lambda \in (0.1,\ 0.5)$.
3. the $\text{PRE}\left(\hat{\pi}_{\text{HS}(1)}, \hat{\pi}_{\text{h}}\right)$ decreases as P_1 increases.

Thus, the proposed estimator $\hat{\pi}_{\text{HS}(1)}$ is to be preferred over Singh and Tarray's (2014) estimator $\hat{\pi}_{\text{h}}$ under the parametric restrictions (i) and (ii).

Further comparing results of Tables 3 and 4 we observed that the values of the Table 3 is very close to the values of Table 4. Thus, we infer that the proposed estimator $\hat{\pi}_{\text{HS}(1)}$ would be used as an alternative to the optimum estimator $\hat{\pi}_{\text{HS}}^{(\text{o})}$. There is practical difficulty in using the proposed optimum estimator $\hat{\pi}_{\text{HS}}^{(\text{o})}$ as it depends on the unknown parameter π_{s} under investigation while the proposed estimator $\hat{\pi}_{\text{HS}(1)}$ does not face any such difficulty. So the estimator $\hat{\pi}_{\text{HS}(1)}$ would be preferred over the optimum estimator $\hat{\pi}_{\text{HS}}^{(\text{o})}$ and Singh and Tarray (2014) estimator $\hat{\pi}_{\text{h}}$.

3.1 Analytical comparison between the estimator $\hat{\pi}_{\text{h}}$ and $\hat{\pi}_{\text{HS}}$

From (9) and (12) we have

$$n\lambda(1-\lambda)\left[V\left(\hat{\pi}_{\text{h}}\right) - V\left(\hat{\pi}_{\text{HS}}\right)\right] = \left[(1-\lambda)\left(\lambda^2 - \eta_1^2\right)V_1 + \lambda\left\{(1-\lambda)^2 - \eta_2^2\right\}V_2\right],$$

which is positive if

$$\left[(1-\lambda)\left(\lambda^2 - \eta_1^2\right)V_1 + \left\{\lambda(1-\lambda)^2 - \lambda\eta_2^2\right\}V_2\right] > 0$$

Table 5 Range of η_1 for different value of π_s, λ, P_1 and $n = 1000$

π_s	$n = 1000$		Upper limit of η_1 (i.e. $<\lambda$)	Lower limit of η_1								
	n_1	n_2	λ	$P_1 = 0.1$	$P_1 = 0.14$	$P_1 = 0.18$	$P_1 = 0.22$	$P_1 = 0.26$	$P_1 = 0.30$	$P_1 = 0.34$	$P_1 = 0.38$	$P_1 = 0.42$
0.1	100	900	0.1	−0.08	−0.07	−0.06	−0.05	−0.04	−0.03	−0.02	−0.01	0.00
0.1	300	700	0.3	−0.21	−0.18	−0.14	−0.11	−0.08	−0.05	−0.02	0.00	0.03
0.1	400	600	0.4	−0.26	−0.21	−0.17	−0.12	−0.08	−0.04	0.00	0.03	0.07
0.1	500	500	0.5	−0.30	−0.23	−0.17	−0.11	−0.06	−0.01	0.04	0.09	0.13
0.2	100	900	0.1	−0.20	−0.17	−0.13	−0.10	−0.07	−0.04	−0.01	0.02	0.05
0.2	300	700	0.3	−0.20	−0.17	−0.13	−0.10	−0.07	−0.04	−0.01	0.02	0.05
0.2	400	600	0.4	−0.25	−0.20	−0.15	−0.10	−0.06	−0.02	0.02	0.06	0.10
0.2	500	500	0.5	−0.29	−0.21	−0.15	−0.09	−0.03	0.02	0.07	0.12	0.16
0.3	100	900	0.1	−0.20	−0.16	−0.12	−0.09	−0.05	−0.02	0.01	0.04	0.07
0.3	300	700	0.3	−0.20	−0.16	−0.12	−0.09	−0.05	−0.02	0.01	0.04	0.07
0.3	400	600	0.4	−0.24	−0.19	−0.13	−0.08	−0.04	0.00	0.04	0.08	0.12
0.3	500	500	0.5	−0.27	−0.19	−0.13	−0.06	0.00	0.05	0.10	0.15	0.19
0.4	100	900	0.1	−0.07	−0.06	−0.05	−0.04	−0.02	−0.01	0.00	0.01	0.02
0.4	300	700	0.3	−0.19	−0.15	−0.11	−0.07	−0.04	−0.01	0.03	0.05	0.08
0.4	400	600	0.4	−0.23	−0.17	−0.12	−0.07	−0.02	0.02	0.06	0.10	0.14
0.4	500	500	0.5	−0.26	−0.18	−0.10	−0.04	0.02	0.07	0.12	0.17	0.21

i.e. if $\left\{-(1-\lambda)V_1-\lambda V_2\right\}\eta_1^2+2\eta_1\lambda V_2+\left\{\lambda^2(1-\lambda)V_1+\lambda(1-\lambda)^2V_2-\lambda V_2>0\right\}$
i.e. if $-\eta_1^2D+2\eta_1\lambda V_2+\lambda\left\{(1-\lambda)\lambda V_1+(1-\lambda)^2V_2-V_2\right\}>0$
i.e. if $-\eta_1^2D+2\eta_1\lambda V_2+\lambda\left\{\lambda\left[(1-\lambda)V_1+\lambda V_2-\lambda V_2\right]+(1-\lambda)^2V_2-V_2\right\}>0$
i.e. if $-\eta_1^2D+2\eta_1\lambda V_2+\lambda\left\{\lambda D-\lambda^2V_2+(1-\lambda)^2V_2-V_2\right\}>0$
i.e. if $-\eta_1^2D+2\eta_1\lambda V_2+\lambda\left\{\lambda D-2\lambda V_2\right\}>0$
i.e. if $-\eta_1^2D+2\eta_1\eta_{10}D+\lambda\left\{\lambda D-2\eta_{10}D\right\}>0$
i.e. if $-\eta_1^2+2\eta_1\eta_{10}+\lambda\left\{\lambda-2\eta_{10}\right\}>0$
i.e. if $\eta_1^2-2\eta_1\eta_{10}-\lambda\left\{\lambda-2\eta_{10}\right\}<0$
i.e. if $\left(\eta_1-\eta_{10}\right)^2-\left(\lambda-\eta_{10}\right)^2<0$
i.e. if $\left(\eta_1-\eta_{10}\right)^2<\left(\lambda-\eta_{10}\right)^2$
i.e. if $\left|\eta_1-\eta_{10}\right|<\left|\lambda-\eta_{10}\right|$ (23)
where $D=\left[(1-\lambda)V_1+\lambda V_2\right]$.

It is observed that the OE $\hat{\pi}_{\mathrm{HS}}^{(\mathrm{o})}$ is hard to apply in practice as the optimum weights involve the unknown parameter π_{s}. However, one can generate estimators from $\hat{\pi}_{\mathrm{HS}}$ better than Singh and Tarray (2014) estimator $\hat{\pi}_{\mathrm{h}}$ with help of (23) even when exact optimum value of η_1 is unknown.

We have computed the range of η_1 using (23) for different values of $\pi_{\mathrm{s}}, \lambda, P_1$ and n=1000 and findings are shown in Table 5. It is observed from Table 5 that the value of lower limit of η_1 increases as P_1 increases for fixed values of $\left(\pi_{\mathrm{s}}, \lambda\right)$ resulting in the shorter range of η_1. We note from Tables 2 and 5 that one can obtain efficient estimator of π_{s} from the proposed class of estimators $\hat{\pi}_{\mathrm{HS}}$ even if the value of η_1 deviates from its exact optimum value η_{10}. Thus, the proposed class of estimators $\hat{\pi}_{\mathrm{HS}}$ can be used in practice even if the investigator is less experienced or has less association with the population under investigation.

The range of λ can be obtained from (23) in which the estimators shown in Table 1 are better than the Singh and Tarray estimator $\hat{\pi}_{\mathrm{h}}$. For example, if we set $w_1=(1-\lambda)$, we find that the estimator $\hat{\pi}_{\mathrm{HS}1}$ is more efficient than the Singh and Tarray's (2014) estimator $\hat{\pi}_{\mathrm{h}}$ if

$$\lambda=\frac{1}{2}. \tag{24}$$

4 Estimation that utilizes approximate optimum value

In this section, we study the "robustness" of the OE $\hat{\pi}_{\mathrm{HS}}^{(\mathrm{o})}$ in (14) against departure from the true optimum values $\left(\eta_{10}, \eta_{20}\right)$ of $\left(\eta_1, \eta_2\right)$.

It is to be mentioned that the OE $\hat{\pi}_{\mathrm{HS}}^{(\mathrm{o})}$ in (14) is of little practical utility as it depends on optimum values $\left(\eta_{10}, \eta_{20}\right)$ in (13) which are functions of the unknown parameter π_{s} (under study) and the known probability P_1. However, in many practical situations, investigator has prior information regarding the parameter π_{s} and hence of $\left(\eta_{10}, \eta_{20}\right)$ due to either long association with the experimental material or through past data. One can also obtain the values of $\left(\eta_{10}, \eta_{20}\right)$ from the sample data at hand. Thus, the assumption that the investigator has prior information or guessed or approximate values of $\left(\eta_{10}, \eta_{20}\right)$ is quite reasonable. The estimator $\hat{\pi}_{\mathrm{HS}}^{(\mathrm{o})}$ which substitute the approximate value $\tilde{\eta}_{10} = \alpha\eta_{10} \Rightarrow \tilde{\eta}_{20} = \left(1 - \alpha\eta_{20}\right)$, where $\alpha(> 0)$ is the departure from the true optimum value in the estimator $\hat{\pi}_{HS}^{(o)}$ at (14) is defined by

$$\hat{\pi}_{\mathrm{HS}}^{(\mathrm{o})*} = \left(\tilde{\eta}_{10}\hat{\pi}_1 + \tilde{\eta}_{20}\hat{\pi}_2\right). \tag{25}$$

The variance of $\hat{\pi}_{\mathrm{HS}}^{(\mathrm{o})*}$ is given by

$$V\left(\hat{\pi}_{\mathrm{HS}}^{(\mathrm{o})*}\right) = \frac{1}{n}\left[\frac{\tilde{\eta}_{10}}{\lambda}V_1 + \frac{\tilde{\eta}_{20}}{\lambda}V_2\right]. \tag{26}$$

From (9) and (26) we have

$$n\left[V\left(\hat{\pi}_{\mathrm{h}}\right) - V\left(\hat{\pi}_{\mathrm{HS}}^{(\mathrm{o})*}\right)\right] = \left[\left(\lambda - \frac{\tilde{\eta}_{10}^2}{\lambda}\right)V_1 + \left((1-\lambda) - \frac{\tilde{\eta}_{20}^2}{(1-\lambda}\right)V_2\right], \tag{27}$$

which is always positive if

$$\alpha^2 - 2\alpha + \frac{\lambda}{\eta_{10}}\left(2 - \frac{\lambda}{\eta_{10}}\right) < 0$$

i.e. if $(\alpha - 1)^2 - \left(1 - \frac{\lambda}{\eta_{10}}\right)^2 < 0$

i.e. if $(\alpha - 1)^2 < \left(1 - \frac{\lambda}{\eta_{10}}\right)^2$

i.e. if $|\alpha - 1| < \left|1 - \frac{\lambda}{\eta_{10}}\right|$

$$\text{i.e. if } \left(2 - \frac{\lambda}{\eta_{10}}\right) < \alpha < \frac{\lambda}{\eta_{10}} \tag{28}$$

From (9) and (26) the percent relative efficiency of the proposed estimator $\hat{\pi}_{\mathrm{HS}}^{(\mathrm{o})*}$ for the approximate values $\left(\tilde{\eta}_{10}, \tilde{\eta}_{20}\right)$, with respect to Singh and Tarray (2014) estimator is given as

$$\mathrm{PRE}\left(\hat{\pi}_{\mathrm{HS}}^{(\mathrm{o})*}, \hat{\pi}_{\mathrm{h}}\right) = \frac{V\left(\hat{\pi}_{\mathrm{h}}\right)}{V\left(\hat{\pi}_{\mathrm{HS}}^{(\mathrm{o})*}\right)} \times 100$$

$$= \frac{\lambda(1-\lambda)\left[\lambda V_1 + (1-\lambda)V_2\right]}{\left[(1-\lambda)\alpha^2\eta_{10}^2 V_1 + \lambda\left(1 - \alpha\eta_{10}\right)^2 V_2\right]} \times 100. \tag{29}$$

Table 6 Range of α for different values of π_s, λ, P_1 and $n = 1000$

π_s	λ	$n = 1000$		$P_1 = 0.1$		$P_1 = 0.14$		$P_1 = 0.18$		$P_1 = 0.22$		$P_1 = 0.26$		$P_1 = 0.30$		$P_1 = 0.34$		$P_1 = 0.38$		$P_1 = 0.42$	
		n_1	n_2	U	L	U	L	U	L	U	L	U	L	U	L	U	L	U	L	U	L
0.1	0.1	100	900	8.29	−6.29	5.95	−3.95	4.65	−2.65	3.83	−1.83	3.26	−1.26	2.84	−0.84	2.52	−0.52	2.26	−0.26	2.06	−0.06
0.1	0.3	300	700	6.67	−4.67	4.85	−2.85	3.84	−1.84	3.20	−1.20	2.76	−0.76	2.43	−0.43	2.18	−0.18	1.98	0.02	1.82	0.18
0.1	0.4	400	600	5.86	−3.86	4.30	−2.30	3.44	−1.44	2.89	−0.89	2.50	−0.50	2.23	−0.23	2.01	−0.01	1.84	0.16	1.71	0.29
0.1	0.5	500	500	5.05	−3.05	3.75	−1.75	3.03	−1.03	2.57	−0.57	2.25	−0.25	2.02	−0.02	1.84	0.16	1.70	0.30	1.59	0.41
0.2	0.1	100	900	7.63	−5.63	5.49	−3.49	4.30	−2.30	3.54	−1.54	3.02	−1.02	2.63	−0.63	2.34	−0.34	2.11	−0.11	1.92	0.08
0.2	0.3	300	700	6.15	−4.15	4.49	−2.49	3.56	−1.56	2.98	−0.98	2.57	−0.57	2.27	−0.27	2.04	−0.04	1.86	0.14	1.72	0.28
0.2	0.4	400	600	5.42	−3.42	3.99	−1.99	3.20	−1.20	2.69	−0.69	2.34	−0.34	2.09	−0.09	1.89	0.11	1.74	0.26	1.62	0.38
0.2	0.5	500	500	4.68	−2.68	3.49	−1.49	2.83	−0.83	2.41	−0.41	2.12	−0.12	1.91	0.09	1.74	0.26	1.62	0.38	1.51	0.49
0.3	0.1	100	900	7.08	−5.08	5.10	−3.10	4.00	−2.00	3.30	−1.30	2.82	−0.82	2.47	−0.47	2.20	−0.20	1.99	0.01	1.82	0.18
0.3	0.3	300	700	5.73	−3.73	4.19	−2.19	3.33	−1.33	2.79	−0.79	2.42	−0.42	2.14	−0.14	1.93	0.07	1.77	0.23	1.64	0.36
0.3	0.4	400	600	5.05	−3.05	3.73	−1.73	3.00	−1.00	2.54	−0.54	2.22	−0.22	1.98	0.02	1.80	0.20	1.66	0.34	1.55	0.45
0.3	0.5	500	500	4.38	−2.38	3.28	−1.28	2.67	−0.67	2.28	−0.28	2.01	−0.01	1.82	0.18	1.67	0.33	1.55	0.45	1.46	0.54
0.4	0.1	100	900	6.61	−4.61	4.77	−2.77	3.76	−1.76	3.11	−1.11	2.66	−0.66	2.34	−0.34	2.09	−0.09	1.89	0.11	1.74	0.26
0.4	0.3	300	700	5.36	−3.36	3.93	−1.93	3.14	−1.14	2.64	−0.64	2.29	−0.29	2.04	−0.04	1.85	0.15	1.69	0.31	1.57	0.43
0.4	0.4	400	600	4.74	−2.74	3.52	−1.52	2.84	−0.84	2.41	−0.41	2.11	−0.11	1.89	0.11	1.73	0.27	1.60	0.40	1.49	0.51
0.4	0.5	500	500	4.12	−2.12	3.10	−1.10	2.53	−0.53	2.17	−0.17	1.92	0.08	1.74	0.26	1.60	0.40	1.50	0.50	1.41	0.59

Table 7 Percent relative efficiency of the suggested optimum estimator $\left(\hat{\pi}_{HS}^{(o)}\right)$ with respect to Singh and Tarray (2014) estimator $\left(\hat{\pi}_{h}\right)$ when $\alpha = 0.5$

π_s	λ	$n = 1000$		$PRE\left(\hat{\pi}_{HS}^{(o)}, \hat{\pi}_h\right)$								
		n_1	n_2	$P_1 = 0.1$	$P_1 = 0.14$	$P_1 = 0.18$	$P_1 = 0.22$	$P_1 = 0.26$	$P_1 = 0.30$	$P_1 = 0.34$	$P_1 = 0.38$	$P_1 = 0.42$
0.1	0.1	100	900	164.39	141.31	128.62	120.65	115.22	111.32	108.40	106.16	104.40
0.1	0.3	300	700	248.48	194.59	164.93	146.28	133.56	124.40	117.53	112.24	108.06
0.1	0.4	400	600	268.13	206.47	172.51	151.13	136.53	125.99	118.07	111.94	107.09
0.1	0.5	500	500	272.75	208.41	172.91	150.52	135.18	124.06	115.68	109.17	103.99
0.2	0.1	100	900	157.82	136.72	125.15	117.90	112.97	109.45	106.83	104.83	103.27
0.2	0.3	300	700	233.16	183.88	156.81	139.83	128.29	120.00	113.82	109.08	105.36
0.2	0.4	400	600	250.60	194.21	163.21	143.74	130.47	120.92	113.78	108.27	103.94
0.2	0.5	500	500	254.47	195.60	163.17	142.75	128.79	118.71	111.13	105.25	100.60
0.3	0.1	100	900	152.37	132.94	122.31	115.67	111.18	107.98	105.61	103.81	102.41
0.3	0.3	300	700	220.41	175.04	150.17	134.62	124.08	116.53	110.93	106.65	103.31
0.3	0.4	400	600	236.02	184.10	155.60	137.75	125.62	116.91	110.43	105.45	101.55
0.3	0.5	500	500	239.26	185.03	155.20	136.46	123.67	114.46	107.55	102.22	98.01
0.4	0.1	100	900	147.75	129.77	119.95	113.84	109.72	106.79	104.64	103.01	101.75
0.4	0.3	300	700	209.64	167.63	144.65	130.32	120.63	113.73	108.62	104.74	101.72
0.4	0.4	400	600	223.70	175.61	149.27	132.80	121.65	113.67	107.74	103.21	99.68
0.4	0.5	500	500	226.40	176.15	148.56	131.25	119.48	111.01	104.69	99.82	95.98

Table 8 Percent relative efficiency of the suggested optimum estimator $\left(\hat{\pi}_{\mathrm{HS}}^{(\mathrm{o})}\right)$ with respect to Singh and Tarray (2014) estimator $\left(\hat{\pi}_{\mathrm{h}}\right)$ when $\alpha = 0.7$

π_s	λ	$n = 1000$		$\mathrm{PRE}\left(\hat{\pi}_{\mathrm{HS}}^{(\mathrm{o})}, \hat{\pi}_{\mathrm{h}}\right)$								
		n_1	n_2	$P_1 = 0.1$	$P_1 = 0.14$	$P_1 = 0.18$	$P_1 = 0.22$	$P_1 = 0.26$	$P_1 = 0.30$	$P_1 = 0.34$	$P_1 = 0.38$	$P_1 = 0.42$
0.1	0.1	100	900	164.71	141.69	129.07	121.16	115.80	111.97	109.12	106.94	105.25
0.1	0.3	300	700	250.35	196.63	167.14	148.68	136.14	127.17	120.49	115.39	111.40
0.1	0.4	400	600	271.25	209.83	176.10	154.97	140.61	130.32	122.65	116.79	112.20
0.1	0.5	500	500	277.50	213.46	178.28	156.21	141.19	130.40	122.35	116.17	111.33
0.2	0.1	100	900	158.16	137.13	125.62	118.44	113.59	110.14	107.59	105.66	104.17
0.2	0.3	300	700	235.06	185.97	159.10	142.32	130.97	122.89	116.91	112.37	108.85
0.2	0.4	400	600	253.78	197.64	166.90	147.69	134.69	125.41	118.54	113.31	109.25
0.2	0.5	500	500	259.29	200.75	168.66	148.59	134.98	125.25	118.02	112.50	108.20
0.3	0.1	100	900	152.71	133.37	122.81	116.25	111.83	108.70	106.41	104.68	103.36
0.3	0.3	300	700	222.35	177.19	152.53	137.18	126.85	119.52	114.14	110.07	106.95
0.3	0.4	400	600	239.24	187.60	159.38	141.81	129.96	121.54	115.35	110.66	107.05
0.3	0.5	500	500	244.14	190.27	160.81	142.44	130.02	121.19	114.65	109.69	105.85
0.4	0.1	100	900	148.12	130.22	120.48	114.44	110.40	107.55	105.48	103.92	102.74
0.4	0.3	300	700	211.62	169.83	147.07	132.96	123.50	116.82	111.94	108.27	105.48
0.4	0.4	400	600	226.97	179.18	153.13	136.97	126.11	118.44	112.81	108.58	105.34
0.4	0.5	500	500	231.35	181.49	154.28	137.37	125.99	117.91	111.97	107.49	104.04

Table 9 Percent relative efficiency of the suggested optimum estimator $\left(\hat{\pi}_{\mathrm{HS}}^{(\mathrm{o})}\right)$ with respect to Singh and Tarray (2014) estimator $\left(\hat{\pi}_h\right)$ when $\alpha = 0.8$

π_s	λ	$n = 1000$		$\mathrm{PRE}\left(\hat{\pi}_{\mathrm{HS}}^{(\mathrm{o})}, \hat{\pi}_{\mathrm{h}}\right)$								
		n_1	n_2	$P_1 = 0.1$	$P_1 = 0.14$	$P_1 = 0.18$	$P_1 = 0.22$	$P_1 = 0.26$	$P_1 = 0.30$	$P_1 = 0.34$	$P_1 = 0.38$	$P_1 = 0.42$
0.1	0.1	100	900	164.81	141.81	129.21	121.33	115.99	112.17	109.34	107.19	105.52
0.1	0.3	300	700	250.93	197.27	167.85	149.44	136.97	128.06	121.44	116.41	112.49
0.1	0.4	400	600	272.24	210.90	177.26	156.21	141.94	131.74	124.16	118.39	113.89
0.1	0.5	500	500	279.02	215.10	180.03	158.07	143.18	132.52	124.60	118.55	113.84
0.2	0.1	100	900	158.26	137.26	125.77	118.61	113.79	110.36	107.83	105.92	104.45
0.2	0.3	300	700	235.66	186.64	159.83	143.11	131.84	123.82	117.91	113.44	109.99
0.2	0.4	400	600	254.78	198.74	168.08	148.97	136.06	126.88	120.11	114.98	111.02
0.2	0.5	500	500	260.83	202.42	170.46	150.51	137.04	127.44	120.35	114.97	110.82
0.3	0.1	100	900	152.82	133.50	122.97	116.43	112.04	108.93	106.66	104.96	103.66
0.3	0.3	300	700	222.97	177.87	153.28	138.01	127.75	120.49	115.18	111.18	108.14
0.3	0.4	400	600	240.27	188.72	160.60	143.12	131.38	123.07	116.97	112.39	108.89
0.3	0.5	500	500	245.71	191.97	162.65	144.41	132.15	123.45	117.07	112.26	108.57
0.4	0.1	100	900	148.23	130.36	120.64	114.63	110.62	107.79	105.74	104.21	103.06
0.4	0.3	300	700	212.25	170.53	147.85	133.81	124.42	117.82	113.01	109.43	106.71
0.4	0.4	400	600	228.02	180.32	154.38	138.32	127.57	120.01	114.49	110.37	107.25
0.4	0.5	500	500	232.94	183.22	156.17	139.40	128.17	120.25	114.46	110.13	106.84

Table 10 Percent relative efficiency of the suggested optimum estimator $\left(\hat{\pi}_{HS}^{(o)}\right)$ with respect to Singh and Tarray (2014) estimator $\left(\hat{\pi}_{h}\right)$ when $\alpha = 0.9$

π_s	λ	$n = 1000$		$\mathrm{PRE}\left(\hat{\pi}_{HS}^{(o)}, \hat{\pi}_h\right)$								
		n_1	n_2	$P_1 = 0.1$	$P_1 = 0.14$	$P_1 = 0.18$	$P_1 = 0.22$	$P_1 = 0.26$	$P_1 = 0.30$	$P_1 = 0.34$	$P_1 = 0.38$	$P_1 = 0.42$
0.1	0.1	100	900	164.87	141.89	129.29	121.42	116.10	112.29	109.48	107.34	105.68
0.1	0.3	300	700	251.29	197.66	168.28	149.91	137.47	128.60	122.03	117.03	113.16
0.1	0.4	400	600	272.84	211.55	177.96	156.96	142.75	132.60	125.08	119.37	114.94
0.1	0.5	500	500	279.94	216.09	181.09	159.21	144.40	133.82	125.98	120.02	115.40
0.2	0.1	100	900	158.33	137.33	125.86	118.72	113.90	110.49	107.97	106.08	104.62
0.2	0.3	300	700	236.03	187.04	160.27	143.59	132.36	124.38	118.52	114.09	110.69
0.2	0.4	400	600	255.39	199.40	168.80	149.74	136.90	127.78	121.07	116.01	112.12
0.2	0.5	500	500	261.77	203.43	171.55	151.69	138.30	128.79	121.80	116.51	112.45
0.3	0.1	100	900	152.89	133.58	123.06	116.54	112.16	109.07	106.82	105.13	103.84
0.3	0.3	300	700	223.34	178.28	153.73	138.50	128.29	121.08	115.81	111.86	108.86
0.3	0.4	400	600	240.89	189.39	161.34	143.93	132.25	124.00	117.97	113.46	110.02
0.3	0.5	500	500	246.66	193.01	163.77	145.63	133.45	124.85	118.56	113.85	110.26
0.4	0.1	100	900	148.30	130.44	120.74	114.75	110.74	107.94	105.90	104.39	103.24
0.4	0.3	300	700	212.62	170.95	148.31	134.32	124.98	118.43	113.67	110.13	107.47
0.4	0.4	400	600	228.65	181.02	155.14	139.15	128.47	120.97	115.53	111.48	108.42
0.4	0.5	500	500	233.90	184.28	157.32	140.65	129.51	121.69	116.01	111.78	108.59

Table 11 Percent relative efficiency of the suggested optimum estimator $\left(\hat{\pi}_{\mathrm{HS}}^{(\mathrm{o})}\right)$ with respect to Singh and Tarry (2014) estimator $\left(\hat{\pi}_{\mathrm{h}}\right)$ when $\alpha = 1.3$

π_s	λ	$n = 1000$		$\mathrm{PRE}\left(\hat{\pi}_{\mathrm{HS}}^{(\mathrm{o})}, \hat{\pi}_{\mathrm{h}}\right)$								
		n_1	n_2	$P_1 = 0.1$	$P_1 = 0.14$	$P_1 = 0.18$	$P_1 = 0.22$	$P_1 = 0.26$	$P_1 = 0.30$	$P_1 = 0.34$	$P_1 = 0.38$	$P_1 = 0.42$
0.1	0.1	100	900	164.71	141.69	129.07	121.16	115.80	111.97	109.12	106.94	105.25
0.1	0.3	300	700	250.35	196.63	167.14	148.68	136.14	127.17	120.49	115.39	111.40
0.1	0.4	400	600	271.25	209.83	176.10	154.97	140.61	130.32	122.65	116.79	112.20
0.1	0.5	500	500	277.50	213.46	178.28	156.21	141.19	130.40	122.35	116.17	111.33
0.2	0.1	100	900	158.16	137.13	125.62	118.44	113.59	110.14	107.59	105.66	104.17
0.2	0.3	300	700	235.06	185.97	159.10	142.32	130.97	122.89	116.91	112.37	108.85
0.2	0.4	400	600	253.78	197.64	166.90	147.69	134.69	125.41	118.54	113.31	109.25
0.2	0.5	500	500	259.29	200.75	168.66	148.59	134.98	125.25	118.02	112.50	108.20
0.3	0.1	100	900	152.71	133.37	122.81	116.25	111.83	108.70	106.41	104.68	103.36
0.3	0.3	300	700	222.35	177.19	152.53	137.18	126.85	119.52	114.14	110.07	106.95
0.3	0.4	400	600	239.24	187.60	159.38	141.81	129.96	121.54	115.35	110.66	107.05
0.3	0.5	500	500	244.14	190.27	160.81	142.44	130.02	121.19	114.65	109.69	105.85
0.4	0.1	100	900	148.12	130.22	120.48	114.44	110.40	107.55	105.48	103.92	102.74
0.4	0.3	300	700	211.62	169.83	147.07	132.96	123.50	116.82	111.94	108.27	105.48
0.4	0.4	400	600	226.97	179.18	153.13	136.97	126.11	118.44	112.81	108.58	105.34
0.4	0.5	500	500	231.35	181.49	154.28	137.37	125.99	117.91	111.97	107.49	104.04

Table 12 Percent relative efficiency of the suggested optimum estimator $\left(\hat{\pi}_{\mathrm{HS}}^{(\mathrm{o})}\right)$ with respect to Singh and Tarray (2014) estimator $\left(\hat{\pi}_{\mathrm{h}}\right)$ when $\alpha = 1.5$

π_s	λ	$n = 1000$		$\mathrm{PRE}\left(\hat{\pi}_{\mathrm{HS}}^{(\mathrm{o})}, \hat{\pi}_{\mathrm{h}}\right)$								
		n_1	n_2	$P_1 = 0.1$	$P_1 = 0.14$	$P_1 = 0.18$	$P_1 = 0.22$	$P_1 = 0.26$	$P_1 = 0.30$	$P_1 = 0.34$	$P_1 = 0.38$	$P_1 = 0.42$
0.1	0.1	100	900	164.39	141.31	128.62	120.65	115.22	111.32	108.40	106.16	104.40
0.1	0.3	300	700	248.48	194.59	164.93	146.28	133.56	124.40	117.53	112.24	108.06
0.1	0.4	400	600	268.13	206.47	172.51	151.13	136.53	125.99	118.07	111.94	107.09
0.1	0.5	500	500	272.75	208.41	172.91	150.52	135.18	124.06	115.68	109.17	103.99
0.2	0.1	100	900	157.82	136.72	125.15	117.90	112.97	109.45	106.83	104.83	103.27
0.2	0.3	300	700	233.16	183.88	156.81	139.83	128.29	120.00	113.82	109.08	105.36
0.2	0.4	400	600	250.60	194.21	163.21	143.74	130.47	120.92	113.78	108.27	103.94
0.2	0.5	500	500	254.47	195.60	163.17	142.75	128.79	118.71	111.13	105.25	100.60
0.3	0.1	100	900	152.37	132.94	122.31	115.67	111.18	107.98	105.61	103.81	102.41
0.3	0.3	300	700	220.41	175.04	150.17	134.62	124.08	116.53	110.93	106.65	103.31
0.3	0.4	400	600	236.02	184.10	155.60	137.75	125.62	116.91	110.43	105.45	101.55
0.3	0.5	500	500	239.26	185.03	155.20	136.46	123.67	114.46	107.55	102.22	98.01
0.4	0.1	100	900	147.75	129.77	119.95	113.84	109.72	106.79	104.64	103.01	101.75
0.4	0.3	300	700	209.64	167.63	144.65	130.32	120.63	113.73	108.62	104.74	101.72
0.4	0.4	400	600	223.70	175.61	149.27	132.80	121.65	113.67	107.74	103.21	99.68
0.4	0.5	500	500	226.40	176.15	148.56	131.25	119.48	111.01	104.69	99.82	95.98

We have computed the range of $\alpha(\%)$ for different values of $\left(\pi_s, P_1, \lambda\right)$ in Table 6. It is observed from Table 6 that the upper limit of α decreases while lower limit of α increases as P_1 increases for the fixed values of $\left(\pi_s, \lambda\right)$. Table 6 also exhibits that for fixed values of $\left(\lambda, P_1\right)$, the value of upper limit of α decreases while the lower limit of increases as increases.

Further to appreciate the idea of robustness, we have computed the relative efficiency (%) of the proposed estimator $\hat{\pi}_{HS}^{(o)*}$ with respect to Singh and Tarray (2014) estimator $\hat{\pi}_h$ for various values of $\left(\pi_s, P_1, \lambda\right)$ and α demonstrated in Tables 7, 8, 9, 10, 11, 12. It is observed that the values of $\mathrm{PRE}\left(\hat{\pi}_{HS}^{(o)*}, \hat{\pi}_h\right)$ are more than 100. Further from Tables 7, 8, 9, 10, 11, 12 we note that the value of percent relative efficiency $\mathrm{PRE}\left(\hat{\pi}_{HS}^{(o)*}, \hat{\pi}_h\right)$ decreases as the value of P_1 increases and it increases with increasing value of π_s. Thus, we conclude that the proposed estimator $\hat{\pi}_{HS}^{(o)}$ has practical utility in practice even if α departs from 'unity'.

5 Stratified mixed randomized response model

Stratified random sampling is usually applied by decomposing the population into distinct homogeneous groups called strata. It gives reasonably representative sample of the population. Many researchers have suggested RR techniques using stratified random sampling, for instance, Hong et al. (1994), Kim and Elam (2005) and Singh and Tarray (2014). We now present estimator under stratified estimation method proposed by Singh and Tarray (2014) to be used later for comparison purposes.

5.1 Singh and Tarray (2014) stratified mixed randomized response model

Singh and Tarray (2014) assumed that the population is partitioned into "*r*" nonoverlapping strata, and a sample is selected by simple random sampling with replacement from each stratum. To get the full benefit from stratification, they assumed that the number of units in each stratum is known. In this model, an individual respondent in a sample from each stratum is instructed to answer a direct question "I am a member of the innocuous trait group." Respondents answer the direct question by "Yes" or "No." If a respondent answers "Yes," then he or she is instructed to go to the randomization device R_{k1} consisting of statements: (i) "I am the member of the sensitive trait group" and (ii) "I am a member of the innocuous trait group" with preassigned probabilities Q_k and $\left(1 - Q_k\right)$, respectively. If a respondent answers "No," then the respondent is instructed to use a randomization procedure due to Mangat (1994). In the Mangat's (1994) RR procedure, each respondent is instructed to say "Yes" if he or she is a member of the sensitive trait group. If he or she is not a member of the sensitive trait group, then the respondent is required to use the Warner's (1965) randomization device R_{k2} consisting of the statement: (i) "I belong to the sensitive trait group" and (b) "I do not belong to the sensitive trait group" with preassigned probabilities P_k and $\left(1 - P_k\right)$, respectively. Then he or she is to report "Yes" or "No" according to the outcome of the randomization device R_{k2} and the actual status that he or she has with respect to the sensitive trait group. The survey procedures are performed under the assumption that both the sensitive and the innocuous questions are unrelated and independent in a randomization device R_{k1}. To protect the respondent's privacy, the respondents should not disclose to the interviewer the question they answered from either R_{k1} or R_{k2}. Suppose we denote m_k as the number of units in the sample from stratum k and n as the total number of units in samples from all strata. Let m_{k1} be the number of people responding "Yes" when respondents in a sample m_k were asked the direct question and m_{k2} be the number of people responding "No" when respondents in a sample m_k were asked the direct question so that $\sum_{k=1}^{r} m_k = \sum_{k=1}^{r} \left(m_{k1} + m_{k2}\right)$. Under the assumption that these "Yes" or "No" reports are made truthfully, and Q_k and $P_k(\neq 0.5)$ are set by the researcher, then the proportion of "Yes" answer from the respondents using the randomization device R_{k1} will be

$$\begin{aligned} Y_k &= Q_k \pi_{S_k} + \left(1 - Q_k\right)\pi_{1k} \quad \text{for} \quad k = 1, 2, \ldots, r, \\ &= Q_k \pi_{S_k} + \left(1 - Q_k\right) \text{ (i.e. } \pi_{1k} = 1). \end{aligned} \tag{30}$$

The estimator of π_{S_k} in terms of the sample proportion of "Yes" response $\hat{Y}_k$ is given as

$$\hat{\pi}_{h1_k} = \frac{\hat{Y}_k - \left(1 - Q_k\right)}{Q_k} \quad \text{for} \quad k = 1, 2, \ldots, r, \tag{31}$$

with the variance

$$V\left(\hat{\pi}_{h1_k}\right) = \frac{\left(1 - \pi_{S_k}\right)\left[Q_k \pi_{S_k} + \left(1 - Q_k\right)\right]}{m_{k1} Q_k} = \frac{V_{1k}}{m_{k1}}, \tag{32}$$

where $V_{1k} = \frac{\left(1-\pi_{S_k}\right)\left[Q_k \pi_{S_k} + \left(1-Q_k\right)\right]}{Q_k}$.

The proportion of "Yes" answers from the respondents using Mangat (1994) randomization device R_{2k}:

$$X_k = \pi_{S_k} + \left(1 - \pi_{S_k}\right)\left(1 - P_k\right). \tag{33}$$

The estimator of π_{S_k} in terms of the sample proportion of "Yes" response $\hat{X}_k$ is given by

$$\hat{\pi}_{h2_k} = \frac{\hat{X}_k - \left(1 - P_k\right)}{P_k}, \tag{34}$$

with the variance

$$V\left(\hat{\pi}_{h2_k}\right) = \frac{1}{\left(m_k - m_{k1}\right)}\left[\pi_{S_k}\left(1 - \pi_{S_k}\right) + \left(1 - Q_k\right)\left(1 - \pi_{S_k}\right)\right] = \frac{V_{2k}}{m_{k2}}, \tag{35}$$

where $V_{2k} = \left[\pi_{S_k}\left(1 - \pi_{S_k}\right) + \left(1 - Q_k\right)\left(1 - \pi_{S_k}\right)\right]$

The pooled unbiased estimator of π_{S_k} in terms of the sample proportion of "Yes" response $\hat{Y}_k$ and $\hat{X}_k$ is given as

$$\hat{\pi}_{mS_k} = \frac{m_{k1}}{m_k}\hat{\pi}_{h1_k} + \frac{m_k - m_{k1}}{m_k}\hat{\pi}_{h2_k}, \quad \text{for} \quad 0 < \frac{m_{k1}}{m_k} < 1. \tag{36}$$

with the variance

$$V\left(\hat{\pi}_{mS_k}\right) = \frac{\pi_{S_k}\left(1 - \pi_{S_k}\right)}{m_k} + \frac{1}{m_k}\left[\left(1 - \pi_{S_k}\right)\left(1 - Q_k\right)\left\{\frac{\lambda_k}{Q_k} + \left(1 - \lambda_k\right)\right\}\right], \tag{37}$$

where $m_k = m_{k1} + m_{k2}$ and $\lambda_k = m_{k1}/m_k$.

Thus, the unbiased estimator of $\pi_S = \sum_{k=1}^{r} w_k \pi_{S_k}$ is given as

$$\hat{\pi}_{mS} = \sum_{k=1}^{r} w_k \hat{\pi}_{mS_k} = \sum_{k=1}^{r} w_k \left[\frac{m_{k1}}{m_k}\hat{\pi}_{h1_k} + \frac{m_k - m_{k1}}{m_k}\hat{\pi}_{h2_k}\right]. \tag{38}$$

The variance of π_{mS} is given as

$$V\left(\hat{\pi}_{mS}\right) = \sum_{k=1}^{r} \frac{w_k^2}{m_k}\left[\frac{\pi_{S_k}\left(1 - \pi_{S_k}\right)}{m_k} + \frac{1}{m_k}\left(\left(1 - \pi_{S_k}\right)\left(1 - Q_k\right)\left\{\frac{\lambda_k}{Q_k} + \left(1 - \lambda_k\right)\right\}\right)\right]. \tag{39}$$

In the next section, we have proposed a weighted unbiased estimator for Singh and Tarray (2014) stratified estimator π_S and studied its properties.

6 Proposed Stratified Mixed Randomized Response Model Using Weights

Moving along the direction for stratified mixed RR model traced by Singh and Tarray (2014), we introduce a weighted unbiased estimator for π_s as

$$\hat{\pi}_{mh_k} = \eta_{1k}\hat{\pi}_{h1k} + \eta_{2k}\hat{\pi}_{h2k}, \tag{40}$$

where η_{1k} and η_{2k} are suitably chosen constant such that $\eta_{1k} + \eta_{2k} = 1$.

For $\eta_{1k} = \frac{\lambda_k Q_k}{(1-\lambda_k)}$ and $\eta_{2k} = (1 - \eta_{1k}) = \frac{\{1-\lambda_k(1+Q_k)\}}{(1-\lambda_k)}$, in (40) we get an unbiased estimator $\hat{\pi}_{\mathrm{mh}_k}$ for π_{s} as

$$\hat{\pi}_{\mathrm{mh}_k} = \frac{\lambda_k^2 Q_k^2}{(1-\lambda_k)}\hat{\pi}_{\mathrm{h}1k} + \frac{\{1-\lambda_k(1+Q_k)\}}{(1-\lambda_k)}\hat{\pi}_{\mathrm{h}2k}. \tag{41}$$

We mention that in (41) $\lambda_k' s$ and $Q_k' s$ are known.

The variance of the estimator $\hat{\pi}_{\mathrm{mh}_k}$ is given as

$$\begin{aligned} V(\hat{\pi}_{\mathrm{mh}_k}) &= \frac{\lambda_k^2 Q_k^2}{(1-\lambda_k)} V(\hat{\pi}_{\mathrm{h}1k}) + \frac{\{1-\lambda_k(1+Q_k)\}}{(1-\lambda_k)} V(\hat{\pi}_{\mathrm{h}2k}) \\ &= \frac{\lambda_k Q_k^2}{(1-\lambda_k)^2}\frac{V_{1k}}{m_k} + \frac{\{1-\lambda_k(1+Q_k)\}^2}{(1-\lambda_k)^2}\frac{V_{2k}}{m_k(1-\lambda_k)} \\ &= \frac{1}{m_k}\left[\frac{\lambda_k Q_k^2 V_{1k}}{(1-\lambda_k)^2} + \frac{\{1-\lambda_k(1+Q_k)\}^2 V_{2k}}{(1-\lambda_k)^3}\right], \end{aligned} \tag{42}$$

$$= \frac{D_k}{m_k}, \tag{43}$$

where $D_k = \frac{1}{(1-\lambda_k)^2}\left[\lambda_k Q_k^2 V_{1k} + \frac{\{1-\lambda_k(1+Q_k)\}^2 V_{2k}}{(1-\lambda_k)}\right]$ and V_{1k} and V_{2k} are same as defined earlier

The unbiased estimator of $\pi_S = \sum_{k=1}^{r} w_k \pi_{S_k}$ is given by

$$\hat{\pi}_{\mathrm{mh}} = \sum_{k=1}^{r} w_k \hat{\pi}_{\mathrm{mh}_k}, \tag{44}$$

with the variance

$$\begin{aligned} V(\hat{\pi}_{\mathrm{mh}}) &= \sum_{k=1}^{r} w_k^2 V(\hat{\pi}_{\mathrm{mh}_k}) \\ &= \sum_{k=1}^{r} \frac{w_k^2}{m_k} D_k. \end{aligned} \tag{45}$$

6.1 Variance of $\hat{\pi}_{\mathrm{mh}}$ under Neyman allocation

Information on π_{S_k} is usually unavailable. But if prior information about them is available from past experience then we may derive the Neyman allocation formula.

The Neyman allocation of n to $m_1, m_2, \ldots, m_{r-1}$ and m_r, to derive the minimum variance of $\hat{\pi}_{\mathrm{mh}}$ subject to $n = \sum_{k=1}^{r} m_k$ is approximately given by

$$m_k \propto w_k \sqrt{D_k},$$

$$\Rightarrow m_k = \frac{n w_k \sqrt{D_k}}{\sum\limits_{k=1}^{r} w_k \sqrt{D_k}}. \tag{46}$$

Using (45) and (46) the minimal variance of $\hat{\pi}_{\mathrm{mh}}$ is given by

$$V\left(\hat{\pi}_{\mathrm{mh}}\right)_{\mathrm{opt}} = \sum_{k=1}^{r} \frac{w_k^2}{n w_k \sqrt{D_k}} \left(\sum_{k=1}^{r} w_k \sqrt{D_k} \right) D_k,$$

$$= \frac{1}{n} \left(\sum_{k=1}^{r} w_k \sqrt{D_k} \right)^2 \tag{47}$$

6.2 Efficiency comparison with Singh and Tarray (2014) model

In this section, we have made the comparison of the proposed mixed randomized response model using Singh and Tarray's (2014) model by way of variance comparison.

To compare the proposed estimator $\hat{\pi}_{\mathrm{mh}}$ with that of Singh and Tarray's (2014) estimator $\hat{\pi}_{\mathrm{mS}}$, we write the variance of Singh and Tarray's (2014) estimator $\hat{\pi}_{\mathrm{mS}}$ under Neyman allocation as

$$V\left(\hat{\pi}_{\mathrm{mS}}\right)_{\mathrm{opt}} = \frac{1}{n} \left(\sum_{k=1}^{r} w_k \sqrt{S_k^*} \right)^2, \tag{48}$$

where $S_k^* = \left[\pi_{\mathrm{S}_k}\left(1 - \pi_{\mathrm{S}_k}\right) + \frac{\left(1-\pi_{\mathrm{S}_k}\right)\left(1-Q_k\right)\lambda_k}{Q_k} + \left(1 - \pi_{\mathrm{S}_k}\right)\left(1 - Q_k\right)\left(1 - \lambda_k\right) \right]$.

From Eq. (47) and (48), we have

$$n\left[V\left(\hat{\pi}_{\mathrm{mS}}\right)_{\mathrm{opt}} - V\left(\hat{\pi}_{\mathrm{mh}}\right)_{\mathrm{opt}}\right] = \left(\sum_{k=1}^{r} w_k \sqrt{S_k^*}\right)^2 - \left(\sum_{k=1}^{r} w_k \sqrt{D_k}\right)^2 > 0$$

i.e. if $\left(\sum_{k=1}^{r} w_k \sqrt{S_k^*}\right)^2 > \left(\sum_{k=1}^{r} w_k \sqrt{D_k}\right)^2$

i.e. if $\sum_{k=1}^{r} w_k \sqrt{D_k} < \sum_{k=1}^{r} w_k \sqrt{S_k^*}$

i.e. if $\sum_{k=1}^{r} w_k \left(\sqrt{D_k} - \sqrt{S_k^*}\right) < 0$

i.e. if $\left(\sqrt{D_k} - \sqrt{S_k^*}\right) < 0 \quad \forall \quad k = 1, 2, \ldots, r$

i.e. if $D_k < S_k^* \quad \forall \quad k = 1, 2, \ldots, r$ (49)

Thus, we state the following theorem.

Theorem 6.1 The proposed mixed randomized response model based on stratified random sampling is more efficient than the Singh and Tarray's (2014) stratified mixed randomized response model as long as the condition $D_k < S_k^* \; \forall \; k = 1, 2, \ldots, r$; is satisfied.

To have an idea about the efficiency gain of the proposed stratified estimator $\hat{\pi}_{\mathrm{mh}}$, we perform a numerical study. Their performance is evaluated through percent relative efficiency with respect to Singh and Tarray (2014) stratified estimator $\hat{\pi}_{\mathrm{mS}}$ in case of two strata (i.e., $r=2$) using formula:

$$\mathrm{PRE}\left(\hat{\pi}_{\mathrm{mh}}, \hat{\pi}_{\mathrm{mS}}\right) = \frac{V\left(\hat{\pi}_{\mathrm{mS}}\right)}{V\left(\hat{\pi}_{\mathrm{mh}}\right)} \times 100 = \frac{\left(\sum_{k=1}^{r} w_k \sqrt{S_k^*}\right)^2}{\left(\sum_{k=1}^{r} w_k \sqrt{D_k}\right)^2} \times 100, \tag{50}$$

where

$\pi_{\mathrm{S}} = w_1 \pi_{\mathrm{S}_1} + w_2 \pi_{\mathrm{S}_2}$,

$\sqrt{S_k^*} = \left[\pi_{\mathrm{S}_k}\left(1 - \pi_{\mathrm{S}_k}\right) + \frac{\left(1-\pi_{\mathrm{S}_k}\right)\left(1-Q_k\right)\lambda_k}{Q_k} + \left(1 - \pi_{\mathrm{S}_k}\right)\left(1 - Q_k\right)\left(1 - \lambda_k\right)\right]^{1/2}$, and

$\sqrt{D_k} = \frac{1}{\left(1-\lambda_k\right)}\left[\lambda_k Q_{1k}^2 V_{1k} + \frac{\left\{1-\lambda_k\left(1+Q_k\right)\right\}^2 V_{2k}}{\left(1-\lambda_k\right)}\right]^{1/2}$

Table 13 Percent relative efficiency of the proposed stratified estimator $\hat{\pi}_{mh}$ with respect to Singh and Tarray (2014) estimator $\hat{\pi}_{mS}$

π_{S1}	π_{S2}	π_S	w_1	w_2	$\lambda = \lambda_1 = \lambda_2$	Q_1							
						0.1		0.2		0.3		0.4	
						Q_2							
						0.2	0.3	0.3	0.4	0.5	0.6	0.7	0.8
0.08	0.13	0.10	0.6	0.4	0.1	149.27	144.83	120.20	118.27	109.19	108.81	104.69	104.74
0.08	0.13	0.10	0.6	0.4	0.2	199.61	204.65	145.17	152.69	135.68	149.38	144.37	168.21
0.08	0.13	0.10	0.6	0.4	0.3	228.28	232.24	156.75	163.46	140.04	152.63	142.61	162.37
0.08	0.13	0.10	0.6	0.4	0.4	245.44	248.67	163.20	168.75	140.35	150.37	134.89	147.99
0.18	0.23	0.20	0.6	0.4	0.1	153.97	159.38	125.94	133.01	125.03	136.56	135.86	155.03
0.18	0.23	0.20	0.6	0.4	0.2	190.07	193.95	140.33	146.66	131.27	142.53	137.42	155.18
0.18	0.23	0.20	0.6	0.4	0.3	215.77	218.59	150.48	156.11	135.08	145.50	136.29	151.23
0.18	0.23	0.20	0.6	0.4	0.4	231.24	233.49	156.29	161.01	135.65	144.03	130.19	140.06
0.28	0.33	0.30	0.6	0.4	0.1	149.84	154.47	123.97	130.18	122.82	132.63	131.47	146.57
0.28	0.33	0.30	0.6	0.4	0.2	182.48	185.56	136.63	142.12	128.09	137.68	132.76	146.84
0.28	0.33	0.30	0.6	0.4	0.3	205.71	207.78	145.59	150.46	131.42	140.34	131.92	143.84
0.28	0.33	0.30	0.6	0.4	0.4	219.71	221.35	150.81	154.96	132.11	139.31	126.85	134.57
0.38	0.43	0.40	0.6	0.4	0.1	146.73	150.80	122.67	128.24	121.41	129.99	128.62	141.08
0.38	0.43	0.40	0.6	0.4	0.2	176.53	179.06	133.90	138.78	125.91	134.28	129.63	141.30
0.38	0.43	0.40	0.6	0.4	0.3	197.69	199.27	141.85	146.19	128.80	136.62	128.92	138.81
0.38	0.43	0.40	0.6	0.4	0.4	210.43	211.69	146.53	150.30	129.51	135.84	124.50	130.72
0.48	0.53	0.50	0.6	0.4	0.1	144.66	148.31	122.08	127.15	120.78	128.45	126.99	137.65
0.48	0.53	0.50	0.6	0.4	0.2	172.10	174.27	132.12	136.55	124.62	132.12	127.74	137.73
0.48	0.53	0.50	0.6	0.4	0.3	191.54	192.83	139.20	143.17	127.12	134.14	127.04	135.49
0.48	0.53	0.50	0.6	0.4	0.4	203.19	204.26	143.39	146.90	127.78	133.46	123.00	128.12

Findings for the percent relative efficiency are given in Table 13 for the different cases of π_S, λ_k and Q_k respectively.

It is observed from Table 13 that, the values of $\text{PRE}\left(\hat{\pi}_{\text{mh}}, \hat{\pi}_{\text{mS}}\right)$ are more than 100. Thus, the proposed estimator $\hat{\pi}_{\text{mh}}$ is more efficient than the one earlier considered by Singh and Tarray's (2014) estimator $\hat{\pi}_{\text{mS}}$ for given parametric values. Further, we note that the $\text{PRE}\left(\hat{\pi}_{\text{mh}}, \hat{\pi}_{\text{mS}}\right)$ decreases as Q_1 increases. For the fixed values of $\left(\pi_S, Q_1\right)$ the $\text{PRE}\left(\hat{\pi}_{\text{mh}}, \hat{\pi}_{\text{mS}}\right)$ increases as λ increases. Larger gain in efficiency is observed as long as Q_1 lies between 0.1 and 0.3 (i.e. $0.1 \le Q_1 \le 0.3$) and Q_2 lies between 0.2 and 0.5 (i.e. $0.2 \le Q_2 \le 0.5$).

6.3 Estimation of population proportion using mixed randomized response model when weights η_{1k} and η_{2k} are scalars

Using the estimator $\hat{\pi}_{\text{mh}k}$ defined at (40) we define a weighted unbiased estimator of population proportion $\pi_S = \sum_{k=1}^{r} w_k \pi_{S_k}$ as

$$\left.\begin{aligned} \hat{\pi}_{m\eta} &= \sum_{k=1}^{r} w_k \hat{\pi}_{\text{mh}k} \\ &= \sum_{k=1}^{r} w_k \left\{\eta_{1k}\hat{\pi}_{\text{h}1k} + \left(1-\eta_{1k}\right)\hat{\pi}_{\text{h}2k}\right\} \end{aligned}\right. . \tag{51}$$

The variance of $\hat{\pi}_{\text{m}\eta}$ is given by

$$V\left(\hat{\pi}_{\text{m}\eta}\right) = \sum_{k=1}^{r} \frac{w_k^2}{m_k\left(1-\lambda_k\right)\lambda_k}\left[\lambda_k V_{2k} + \eta_{1k}^2\left\{\left(1-\lambda_k\right)V_{1k} + \lambda_k V_{2k}\right\} - 2\eta_{1k}\lambda_k V_{2k}\right]. \tag{52}$$

The variance $V\left(\hat{\pi}_{\text{m}\eta}\right)$ at (51) is minimized for

$$\left.\begin{aligned} \eta_{1k} &= \frac{\lambda_k V_{2k}}{\left[\left(1-\lambda_k\right)V_{1k} + \lambda_k V_{2k}\right]} = \eta_{1ko} \\ \eta_{2k} &= \frac{\left(1-\lambda_k\right)V_{1k}}{\left[\left(1-\lambda_k\right)V_{1k} + \lambda_k V_{2k}\right]} = \eta_{2ko} \end{aligned}\right\}. \tag{53}$$

Thus, the resulting minimum variance of $\hat{\pi}_{\text{m}\eta}$ is given by

$$V_{\text{min}}\left(\hat{\pi}_{\text{m}\eta}\right) = \sum_{k=1}^{r} \frac{w_k^2}{m_k} V_{ko}, \tag{54}$$

where

$$V_{\mathrm{ko}} = \frac{V_{1k}V_{2k}}{\left[\left(1-\lambda_k\right)V_{1k} + \lambda_k V_{2k}\right]}. \tag{55}$$

Thus, the resulting optimum estimator for π_{S} is given by

$$\hat{\pi}_{\mathrm{m}\eta_{\mathrm{ko}}} = \eta_{1k0}\hat{\pi}_{h1k} + \left(1-\eta_{1k0}\right)\hat{\pi}_{\mathrm{h}2k}. \tag{56}$$

whose variance is

$$V\left(\hat{\pi}_{\mathrm{m}\eta_{ko}}\right) = V_{\min}\left(\hat{\pi}_{\mathrm{m}\eta}\right) = \sum_{k=1}^{r} \frac{w_k^2}{m_k} V_{\mathrm{ko}}. \tag{57}$$

The variance of the optimum estimator (OE) under Neyman allocation $m_k \propto w_k\sqrt{S_k}$, $k=1,2,\ldots r$.

$$\text{i.e. } \frac{m_k}{n} = \frac{w_k\sqrt{V_{\mathrm{ko}}}}{\sum\limits_{k=1}^{r} w_k\sqrt{V_{\mathrm{ko}}}}, \quad k=1,2,\ldots,r, \tag{58}$$

is given by

$$V\left(\hat{\pi}_{\mathrm{m}\eta_{\mathrm{ko}}}\right)_{\mathrm{opt}} = \frac{1}{n}\left(\sum_{k=1}^{r} w_k\sqrt{V_{ko}}\right)^2. \tag{59}$$

From (39) and (50) we have

$$V\left(\hat{\pi}_{\mathrm{mS}}\right) - V\left(\hat{\pi}_{\mathrm{m}\eta_{\mathrm{ko}}}\right) = \frac{1}{n}\left\{\left(\sum_{k=1}^{r} w_k\sqrt{S_k^*}\right)^2 - \left(\sum_{k=1}^{r} w_k\sqrt{V_{ko}}\right)^2\right\},$$

which is positive if

$$\left(\sum_{k=1}^{r} w_k\sqrt{S_k^*}\right)^2 > \left(\sum_{k=1}^{r} w_k\sqrt{V_{ko}}\right)^2,$$

i.e. if $\sum\limits_{k=1}^{r} w_k\sqrt{S_k^*} - \sum\limits_{k=1}^{r} w_k\sqrt{V_{\mathrm{ko}}} > 0$

i.e. if $\sum\limits_{k=1}^{r} w_k\left(\sqrt{S_k^*} - \sqrt{V_{\mathrm{ko}}}\right) > 0$

i.e. if $\sqrt{S_k^*} - \sqrt{V_{ko}} > 0 \quad \forall \quad k = 1, 2, \ldots, r$
i.e. if $\sqrt{S_k^*} > \sqrt{V_{ko}} \quad \forall \quad k = 1, 2, \ldots, r$
i.e. if $S_k^* > V_{ko} \quad \forall \ldots k = 1, 2, \ldots, r$

i.e. if $\left[\pi_{S_k}\left(1 - \pi_{S_k}\right) + \left(1 - \pi_{S_k}\right)\left(1 - Q_k\right) + \left(1 - \pi_{S_k}\right)\left(1 - Q_k\right)\lambda_k\left(\frac{1}{Q_k - 1}\right)\right] > \frac{V_{1k}V_{2k}}{\left[\left(1 - \lambda_k\right)V_{1k} + \lambda_k V_{2k}\right]} \quad \forall \quad k = 1, 2, \ldots, r$

i.e. if $V_{2k} + \left(1 - \pi_{S_k}\right)\left(1 - Q_k\right)\lambda_k\left(\frac{1}{Q_k} - 1\right) - \frac{V_{1k}V_{2k}}{\left\{\left(1-\lambda_k\right)V_{1k} + \lambda_k V_{2k}\right\}} > 0 \quad \forall \quad k = 1, 2, \ldots, r$

i.e. if $\frac{\lambda_k V_{2k}\left(1-\pi_{S_k}\right)\left(1-Q_k\right)^2}{Q_k\left[\left(1-\lambda_k\right)V_{1k} + \lambda_k V_{2k}\right]} + \frac{\left(1-\pi_{S_k}\right)\left(1-Q_k\right)^2\lambda_k}{Q_k} > 0 \quad \forall \quad k = 1, 2, \ldots, r$

$$\text{i.e. if } \frac{V_{2k}}{\left\{\left(1 - \lambda_k\right)V_{1k} + \lambda_k V_{2k}\right\}} + 1 > 0 \quad \forall \quad k = 1, 2, \ldots, r, \tag{60}$$

which is always true.

Thus, the proposed optimum estimator (OE) $\hat{\pi}_{m\eta_{ko}}$ is more efficient than the Singh and Tarray's (2014) estimator $\hat{\pi}_{mS}$ under Neyman allocation.

From (47) and (59) we have

$$n\left[V\left(\hat{\pi}_{mh}\right) - V\left(\hat{\pi}_{m\eta_{ko}}\right)\right] = \left(\sum_{k=1}^{r} w_k\sqrt{D_k}\right)^2 - \left(\sum_{k=1}^{r} w_k\sqrt{V_{ko}}\right)^2$$

$$= \left\{\sum_{k=1}^{r} w_k\sqrt{D_k} + \sum_{k=1}^{r} w_k\sqrt{V_{ko}}\right\}\left\{\sum_{k=1}^{r} w_k\sqrt{D_k} - \sum_{k=1}^{r} w_k\sqrt{V_{ko}}\right\}$$

$$= \left\{\sum_{k=1}^{r} w_k\left(\sqrt{D_k} + \sqrt{V_{ko}}\right)\right\}\left\{\sum_{k=1}^{r} w_k\left(\sqrt{D_k} - \sqrt{V_{ko}}\right)\right\},$$

which is positive if

$$\left(\sqrt{D_k} + \sqrt{V_{ko}}\right) > 0, \quad \forall \quad k = 1, 2, \ldots, r$$

i.e. if $\sqrt{D_k} > \sqrt{V_{ko}}, \quad \forall \quad k = 1, 2, \ldots, r;$
i.e. if $D_k > V_{ko}, \quad \forall \quad k = 1, 2, \ldots, r;$

$$\text{i.e. if } \left(D_k - V_{ko}\right) > 0, \quad \forall \quad k = 1, 2, \ldots, r. \tag{61}$$

Now we have

$$\begin{aligned}\left(D_k-V_{ko}\right)&=\frac{1}{\left(1-\lambda_k\right)^3}\left[\left(1-\lambda_k\right)\lambda_k Q_k^2 V_{1k}+\left\{1-\lambda_k\left(1+Q_k\right)\right\}^2 V_{2k}-\frac{\left(1-\lambda_k\right)^3 V_{1k}V_{2k}}{\left\{\left(1-\lambda_k\right)V_{1k}+\lambda_k V_{2k}\right\}}\right]\\&=\frac{1}{\left(1-\lambda_k\right)^3}\left[\left(1-\lambda_k\right)\lambda_k Q_k^2 V_{1k}+\left\{\left(1-\lambda_k\right)^2+\lambda_k^2-2\lambda_k\left(1-\lambda_k\right)Q_k\right\}V_{2k}-\frac{\left(1-\lambda_k\right)^3 V_{1k}V_{2k}}{\left\{\left(1-\lambda_k\right)V_{1k}+\lambda_k V_{2k}\right\}}\right]\\&=\frac{1}{\left(1-\lambda_k\right)^3\left\{\left(1-\lambda_k\right)V_{1k}+\lambda_k V_{2k}\right\}}\left[\lambda_k\left(1-\lambda_k\right)^2 Q_k^2 V_{1k}^2+\lambda_k\left(1-\lambda_k-\lambda_k Q_k\right)^2 V_{2k}^2\right.\\&\quad\left.-2\lambda_k\left(1-\lambda_k\right)\left(1-\lambda_k-\lambda_k Q_k\right)Q_k V_{1k}V_{2k}\right]\\&=\frac{\lambda_k}{\left(1-\lambda_k\right)^3\left\{\left(1-\lambda_k\right)V_{1k}+\lambda_k V_{2k}\right\}}\left[\left(1-\lambda_k\right)^2 Q_k^2 V_{1k}^2-2\left(1-\lambda_k\right)\left(1-\lambda_k-\lambda_k Q_k\right)Q_k V_{1k}V_{2k}\right.\\&\quad\left.+\left(1-\lambda_k-\lambda_k Q_k\right)^2 V_{2k}^2\right]\\&=\frac{\lambda_k\left[\left(1-\lambda_k\right)Q_k V_{1k}-\left(1-\lambda_k-\lambda_k Q_k\right)V_{2k}\right]^2}{\left(1-\lambda_k\right)^3\left\{\left(1-\lambda_k\right)V_{1k}+\lambda_k V_{2k}\right\}}>0.\end{aligned}\tag{62}$$

From (61) and (62) we have

$$D_k-V_{ko}>0,\quad\forall\quad k=1,2,\ldots,r\tag{63}$$

which is always true.

Thus,

$$\left[V\left(\hat{\pi}_{mh}\right)-V\left(\hat{\pi}_{m\eta_{ko}}\right)\right]>0.\tag{64}$$

It follows from (64) that the proposed estimator $\hat{\pi}_{m\eta_{ko}}$ is more efficient than the proposed estimator $\hat{\pi}_{mh}$.

Now we established the following theorem.

Theorem 6.2 The proposed optimum estimator $\hat{\pi}_{m\eta_{ko}}$ is more efficient than the Singh and Tarray's (2014) estimator $\hat{\pi}_{ms}$ and the proposed estimator $\hat{\pi}_{mh}$.

To see the performance of the proposed optimum estimator (OE) $\hat{\pi}_{m\eta_{ko}}$ relative to Singh and Tarray's (2014) estimator $\hat{\pi}_{mS}$ under Neyman allocation, we have computed the percent relative efficiency (PRE) of the estimator $\hat{\pi}_{m\eta_{ko}}$ with respect to the estimator $\hat{\pi}_{mS}$ using the formula:

$$\begin{aligned}\text{PRE}\left(\hat{\pi}_{m\eta_{ko}},\hat{\pi}_{mS}\right)&=\frac{V\left(\hat{\pi}_{mS}\right)}{V\left(\hat{\pi}_{m\eta_{ko}}\right)}\times 100,\\&=\frac{\left(\sum_{k=1}^{r}w_k\sqrt{S_k^*}\right)^2}{\left(\sum_{k=1}^{r}w_k\sqrt{V_{ko}}\right)^2}\times 100,\end{aligned}\tag{65}$$

Table 14 Percent relative efficiency of the proposed estimator $\hat{\pi}_{m\eta_{ko}}$ with respect to Singh and Tarray (2014) estimator $\hat{\pi}_{mS}$

π_{S1}	π_{S2}	π_S	w_1	w_2	$\lambda = \lambda_1 = \lambda_2$	Q_1 0.1		0.2		0.3		0.4	
						Q_2 0.2	0.3	0.3	0.4	0.5	0.6	0.7	0.8
0.08	0.13	0.10	0.6	0.4	0.1	149.29	144.85	120.21	118.28	109.20	108.82	104.69	104.74
0.08	0.13	0.10	0.6	0.4	0.2	199.64	204.75	145.21	152.91	136.11	150.69	146.88	174.88
0.08	0.13	0.10	0.6	0.4	0.3	228.32	232.45	156.88	164.03	141.27	155.81	148.52	176.21
0.08	0.13	0.10	0.6	0.4	0.4	245.55	249.29	163.70	170.41	143.86	158.06	148.32	175.09
0.18	0.23	0.20	0.6	0.4	0.1	154.02	159.44	125.98	133.07	125.10	136.80	136.30	156.30
0.18	0.23	0.20	0.6	0.4	0.2	190.15	194.03	140.38	146.75	131.44	143.16	138.71	158.64
0.18	0.23	0.20	0.6	0.4	0.3	215.85	218.66	150.52	156.30	135.63	147.19	139.72	159.23
0.18	0.23	0.20	0.6	0.4	0.4	231.27	233.63	156.41	161.73	137.66	148.84	139.28	157.96
0.28	0.33	0.30	0.6	0.4	0.1	149.93	154.56	124.06	130.26	122.89	132.77	131.70	147.23
0.28	0.33	0.30	0.6	0.4	0.2	182.67	185.72	136.76	142.22	128.18	137.98	133.42	148.76
0.28	0.33	0.30	0.6	0.4	0.3	205.95	207.90	145.68	150.53	131.64	141.21	134.01	148.89
0.28	0.33	0.30	0.6	0.4	0.4	219.89	221.37	150.83	155.21	133.24	142.43	133.40	147.53
0.38	0.43	0.40	0.6	0.4	0.1	146.88	150.95	122.83	128.37	121.52	130.10	128.75	141.44
0.38	0.43	0.40	0.6	0.4	0.2	176.85	179.34	134.16	138.95	126.00	134.43	129.98	142.41
0.38	0.43	0.40	0.6	0.4	0.3	198.14	199.55	142.08	146.27	128.87	137.05	130.24	142.22
0.38	0.43	0.40	0.6	0.4	0.4	210.86	211.80	146.61	150.35	130.12	137.92	129.49	140.80
0.48	0.53	0.50	0.6	0.4	0.1	144.88	148.53	122.30	127.35	120.94	128.57	127.08	137.85
0.48	0.53	0.50	0.6	0.4	0.2	172.58	174.69	132.51	136.82	124.77	132.22	127.92	138.41
0.48	0.53	0.50	0.6	0.4	0.3	192.23	193.30	139.60	143.34	127.16	134.34	127.91	137.96
0.48	0.53	0.50	0.6	0.4	0.4	203.94	204.55	143.60	146.90	128.10	134.92	127.03	136.45

for two strata (i.e. $r=2$), $\lambda_1 = \lambda_2 = \lambda$ and different values of $(\pi_{S1}, \pi_{S2}, \pi_S)$ and Q_1 and Q_2.

Findings are shown in Table 14

Table 14 shows that the proposed optimum estimator $\hat{\pi}_{m\eta_{ko}}$ is more efficient than Singh and Tarray (2014) estimator $\hat{\pi}_{mS}$. We further note that the values of $PRE(\hat{\pi}_{mh}, \hat{\pi}_{mS})$ is very close to the value of $PRE(\hat{\pi}_{m\eta_{ko}}, \hat{\pi}_{mS})$. There is practical difficulty in using the proposed optimum estimator $\hat{\pi}_{m\eta_{ko}}$ as it depends on the unknown parameter π_s under investigation while the proposed estimator $\hat{\pi}_{mh}$ does not face any such difficulty, so the estimator $\hat{\pi}_{mh}$ would be preferred over the optimum estimator $\hat{\pi}_{m\eta_{ko}}$ and Singh and Tarray (2014) estimator $\hat{\pi}_h$. Thus, we infer that the proposed estimator $\hat{\pi}_{mh}$ would be used as an alternative to the optimum estimator $\hat{\pi}_{m\eta_{ko}}$.

7 Conclusion

In this article, we have suggested a weighted unbiased estimator based on mixed randomized response model and its Stratified RR model which are more efficient than the Singh and Tarray (2014) model. We have also discussed a particular case by giving a suitable weight in the proposed weighted estimator and found that the relative efficiency of the estimator for the different parametric choices is close to the proposed optimum estimator. Thus, our mixed RR model and Stratified mixed RR model are good alternative to the Singh and Tarray's (2014) model.

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