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RESEARCH ARTICLE

An Interesting Class of Non-Kac Random Polynomials

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Abstract

As evident from classical results on random polynomials, it is difficult to derive the probability distribution of the number of real roots $N_n(\mathbb{R})$ of a random polynomial of degree n, and even if derived, the distribution is not of any standard form. In this article, we construct a class of random polynomials of degree $2(n+1)$ such that the distribution of $N_{2(n+1)}(\mathbb{R})$ belongs to the scale family of binomial distributions. For the constructed class of random polynomials, we further notice that as $n \to \infty$, the expected proportion of real roots $E\left(\frac{N_{2(n+1)}(\mathbb{R})}{2(n+1)}\right)$ need not converge to 0, in contrast to most of the existing literature on random polynomials which show $E(N_n(\mathbb{R})) = o(n)$ as $n \to \infty$ that, in turn, implies that asymptotically the majority of the roots of the random polynomial are non-real. The second result of this article shows that in fact for any given $p \in [0,1]$, the construction can be engineered in such a way that the random polynomial has light-tailed coefficients and $E(N_{2(n+1)}(\mathbb{R})) \sim 2(n+1)p$ as $n \to \infty$. Hence, for the class of random polynomials, that we have constructed in this article, asymptotically the number of real roots can be arbitrarily large. Compared to Kac polynomials, which consist of light-tailed random coefficients, the amount of research done for random polynomials whose coefficients are non-identical/dependent/heavy-tailed, is relatively scarce. In the final part of the present article, we give the third and final result that concerns random polynomials with heavy-tailed coefficients. We extend the second result to show that for any given $p \in (0,1]$, we can construct non-Kac, random polynomials with heavy-tailed, stochastically dependent coefficients for which $E(N_{2(n+1)}(\mathbb{R})) \sim 2(n+1)p$ as $n \to \infty$. All these results are based on the assumption that all the coefficients of the constructed class of random polynomials are continuous random variables. We conclude the article with a discussion of how they would change if instead, we assume that the coefficients are general random variables and how far the results derived in this article can be extended to some higher degree random polynomials of the same structure.

Keywords Random polynomials · Kac–Rice formula · Binomial distribution · Heavy-tailed random coefficients

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1 Introduction

Polynomials are one of the most simple mathematical functions. For a polynomial $p_n(x) = \sum_{i=0}^{i=n} a_i x^i$ of degree n defined over $\mathbb{R}$, α is called a root of $p_n(x)$ if $p_n(\alpha) = 0$. Extensive research has been done to study the properties of different characteristics of polynomials. Examples of such characteristics include the roots themselves, the number of real roots, irreducibility, etc. When the coefficients A_is are random variables, all the characteristics of a polynomial (e.g. the number of real roots) become random. Such polynomials denoted as $F_n(x) = A_0 + A_1 x + \cdots + A_n x^n$ are called random polynomials. They have applications in the theory of polynomials. Properties of random polynomials lead to the formulation of new hypotheses about polynomials, which are otherwise difficult to discover. This approach to understanding statistical properties to compute a deterministic real picture is a typical example of a probabilistic method. Apart from that, random polynomials have applications in mathematical physics. Wigner (1955, 1958) modeled heavy atom energies with eigenvalues of random matrix and a vast literature studying the behaviour of those eigenvalues emerged subsequently. Eigenvalues of random matrices are roots of random characteristic polynomials and hence, the theory of random polynomials finds applications in their study too. Random polynomials are also used in the study of quantum systems. Multidimensional quantum systems are approximated by mathematical equations and in these approximations, one often needs to locate the roots of polynomials of high degree whose coefficients are rapidly-varying erratic functions of the energy (Bogomolny et al. 1996). As a result, these coefficients may be considered as random variables, even in a small energy interval; therefore making the underlying polynomial a random one. Sometimes, random polynomials arise as solutions of stochastic differential equations. Bharucha-Reid and Sambandham (2014) have discussed random Legendre polynomial which arises as a solution to a stochastic version of Legendre equation. Finally, random polynomials are also useful in complexity theory, where they are utilized to calculate the average case complexity of numerical algorithms. Emiris et al. (2010) used properties of random polynomials, in order to calculate the average case complexity of the bisection method.

To contrast with trigonometric random polynomials, orthogonal random polynomials, and other classes of random functions, often random polynomials are also referred to as algebraic random polynomials. The simplest form an algebraic random polynomial can take is referred to as Kac polynomial.

Definition 1 (*Kac Polynomial*) Let n be a positive integer, $c_0, \dots, c_n$ be deterministic numbers, and A be a random variable (which we call the atom distribution) of mean zero and finite nonzero variance. Consider the random polynomial $F_n(x) = c_0 A_0 + c_1 A_1 x + \cdots + c_n A_n x^n$, where $A_0, \dots, A_n$ are jointly independent copies of A. It is referred to as Kac Polynomial if $c_0 = c_1 = \cdots = c_n = 1$.

In practice, one usually normalizes the atom distribution A to have unit variance. However, the normalization does not affect the zeroes of F_n. A detailed exposition of Kac Polynomials is available in the books (Bharucha-Reid and Sambandham 2014) and (Farahmand 1998). Other choices of values for $c_0, \dots, c_n$ lead to non-Kac polynomials. Among them, Weyl polynomials and Elliptic Polynomials demand separate attention. These random polynomials are investigated along several different lines. The distribution of the roots on the complex plane is of significant interest to Mathematicians. Another line of work studies the no of real roots of the random polynomials of degree n, which is denoted by $N_n(\mathbb{R})$. As the polynomial is random, the number of real roots $N_n(\mathbb{R})$ is also random. A significant amount of literature exists on random polynomials which studied the asymptotic behaviour of the expected number of real roots $E(N_n(\mathbb{R}))$ (Bloch and Pólya 1932; Littlewood and Offord 1939, 1943, 1938). Subsequently, Kac (1943) was able to derive an exact expression of $E(N_n(\mathbb{R}))$ for finite n albeit when all the coefficients of the random polynomial are Gaussians with mean zero. In a different line of work, Wang (1983) and Yamrom (1972) gave a more accurate asymptotic representation of $E(N_n(\mathbb{R}))$ that ultimately culminated with the work (Wilkins 1988) where the authors obtained an asymptotic series for $E(N_n(\mathbb{R}))$.

In stark contrast, there exist very few works that studied the distribution of the random variable $N_n(\mathbb{R})$. Maslova (1975) proved that if coefficients satisfy the conditions $P(A_i = 0) = 0, E(A_i) = 0$, and $E(|A_i|^{2+\epsilon}) < \infty$ for some $\epsilon > 0$ then $N_n(\mathbb{R})$ asymptotically follows some Gaussian distribution. To the best of our knowledge, there does not exist any article studying the exact distribution of $N_n(\mathbb{R})$ for finite n. In this article, we construct a class of random polynomials such that the distribution of $N_n(\mathbb{R})$ belongs to the scale family of binomial distributions. We observe that for the constructed class of random polynomials as $n \to \infty$, the expected proportion of real roots $E(\frac{N_n(\mathbb{R})}{n})$ need not converge to 0, in contrast to most of the existing literature on random polynomials which show $E(N_n(\mathbb{R})) = o(n)$ as $n \to \infty$ that, in turn, implies that asymptotically the majority of the roots of the random polynomial are non-real. Curious by the observation then we investigate whether for any given $p \in [0, 1]$ we can construct random polynomials for which $E(N_n(\mathbb{R})) \sim np$ as $n \to \infty$. The second result of this article shows that indeed such a construction is possible albeit using light-tailed random coefficients.

Compared to Kac polynomials, the amount of research done for random polynomials whose coefficients are non-identical/dependent/heavy-tailed is relatively scarce. Recently, Matayoshi (2012) considered the case where the coefficients of a random polynomial are dependent but form a stationary sequence of $N(0, 1)$ distributions and obtained that $E(N_n(\mathbb{R})) \sim \frac{2}{\pi} \log n$ as $n \to \infty$. Nezakati and Farahmand (2010) considered the case where the sequence of coefficients is distributed according to a Gaussian process with stationary covariance function $Cov(A_i, A_j) = 1 - \frac{|i-j|}{n}$ and obtained that $E(N_n(\mathbb{R})) = O(\sqrt{\log n})$ for $n \to \infty$. Rezakhah and Shemehsavar (2005, 2008) studied random polynomials and $N_n(\mathbb{R})$ when the coefficients are generated by Brownian motion process and hence non-stationary. Their work was further extended by Mukeru (2019), who considered coefficients generated by successive increments of the fractional Brownian motion process.

A different line of work explored random polynomials with heavy-tailed coefficients. While most of the existing works considered coefficients that are either Gaussians or follow some distributions (may not be the same though for every coefficient) that have all order moments finite, Ibragimov and Maslova (1971a, 1971b) relaxed this condition considerably to establish asymptotic behaviour of $E(N_n(\mathbb{R}))$ for iid coefficients which only have finite variance. Recently, Do et al. (2018) derived the asymptotic behaviour of $E(N_n(\mathbb{R}))$ for A_i such that $E|A_i|^{2+\epsilon}$ are uniformly bounded but possibly with non-identical distributions. The scope of the result (Do et al. 2018) is much wider since it is derived for generalized random polynomials which are more general functions than random polynomials and can accommodate fractional degrees, unlike random polynomials whose degree can only be a positive integer. In the final part of the present article, we give the third and final result that concerns random polynomials with heavy-tailed coefficients. We extend the second result to show that for any given $p \in (0, 1]$, we can construct non-Kac, random polynomials with heavy-tailed, stochastically dependent coefficients for which $E(N_n(\mathbb{R})) \sim np$ as $n \to \infty$.

All the results derived in this article are based on the assumption that all coefficients of the constructed class of random polynomials are continuous random variables. We conclude the article with a discussion of how they would change if instead, we assume that the coefficients are general random variables and how far the results derived in this article can be extended to some higher degree random polynomials of the same structure.

2 Expected Number of Real Roots, Kac–Rice Formula and Beyond

Any discussion on random polynomials is incomplete without the Kac–Rice formula. It is, in particular used to count the expected number of real roots of a random polynomial whose all coefficients are Gaussians. It is built on the following result from real analysis which counts the number of real roots of a continuously differentiable function $F(x)$. Let $F(x)$ be continuous for $a \leqslant x \leqslant b$, continuously differentiable for $a < x < b$, and have a finite number of turning points (that is, only a finite number of points at which $F'(x)$ vanishes in (a, b)). Then the number of real roots of $F(x)$ in the interval (a, b) is denoted by $N(a, b)$, and is given by the formula

$$N(a,b) = (2\pi)^{-1} \int_{-\infty}^{\infty} d\xi \int_{a}^{b} \cos\left[\xi F(x)\right] \left|F'(x)\right| dx.$$

In this formula, multiple roots are counted once, and if either a or b is a zero it is counted as $\frac{1}{2}$. This formula is then applied to calculate the number of real roots $N_n(a,b)$ in the interval (a, b) and $E(N_n(a,b))$ of a random polynomial of degree n whose all coefficients are Gaussians.

Definition 2 (*Kac–Rice Formula*) Kac–Rice formula gives an integral representation of $E(N_n(a,b))$ as follows

$$E(N_n(a,b)) = \int_{\mathbb{R}^{n+1}} N_n(a,b) d\mu(\mathbf{g}) = \int_{a}^{b} \int_{-\infty}^{\infty} |t| f(0,t;x) dt$$

where $\mathbf{g} = (a_0, a_1, \dots, a_n)$ is a point in $\mathbb{R}^{n+1}$,$f(s,t;x)$ denotes the joint probability density of $h(x,\omega)$ and $h'(x,\omega)$ for $x \in \mathbb{R}$ at $h(x,\omega) = s$, $h'(x,\omega) = t$ where $h(x,\omega), x \in \mathbb{R}$ is a real-valued function. Further simplification of Kac–Rice formula when all the coefficients are iid Gaussian leads to the following results (see Bharucha-Reid and Sambandham 2014).

Case I Suppose that all the coefficients of the random polynomial $F_n(x)$ are identically but not necessarily independent Gaussians with mean $m(\neq 0)$, variance 1; and let the joint density function of the coefficients at the point $(a_0, a_1, \dots, a_n)$ be

$$|M|^{1/2}(2\pi)^{-(n+1)/2} \exp\left[-\frac{1}{2}(\mathbf{a}-\mathbf{m})'M(\mathbf{a}-\mathbf{m})\right]$$

where M^{-1} is the moment matrix with $\rho_{ij} = \rho, 0 < \rho < 1, i \neq j$. Then using the Kac–Rice formula one gets

$$N_n(\alpha,\beta) = \pi^{-1}\int_\alpha^\beta e^{-\tau_1}\left\{\frac{\left(A_nC_n - \mathrm{B}_n^2\right)^{1/2}}{A_n}e^{-\gamma_1^2} + \left(\frac{\pi}{2A_n}\right)^{1/2}\beta_1 \mathrm{erf}(\gamma_1)\right\}dx,$$

where

$$A_n(x) \equiv A_n = (1-\rho)\sum_{k=0}^n x^{2k} + \rho\left(\sum_{k=0}^n x^k\right)^2,$$

$$B_n(x) \equiv B_n = (1-\rho)\sum_{k=0}^n kx^{2k-1} + \rho\left(\sum_{k=0}^n x^k\right)\left(\sum_{k=0}^n kx^{k-1}\right)$$

$$C_n(x) \equiv C_n = (1-\rho)\sum_{k=0}^n k^2x^{2k-2} + \rho\left(\sum_{k=0}^n kx^{k-1}\right)^2,$$

$$T_1 = \frac{m}{A_n}\left(\sum_{k=0}^n x^k\right)^2, \quad \gamma_1 = \frac{\beta_1}{2\sqrt{\alpha_1}}, \quad \alpha_1 = \frac{A_nC_n - B_n}{2A_n},$$

$$\beta_1 = \frac{m}{A_n}\left(\left(\sum_{k=0}^n x^k\right)B_n - \left(\sum_{k=0}^n kx^{k-1}\right)A_n\right), \quad \mathrm{erf}(\gamma_1) = \frac{2}{\sqrt{\pi}}\int_0^{\gamma_1} e^{-t^2}dt.$$

Then one obtains the following:

$$E(N_n(\mathbb{R})) \sim (1/\pi)\log n, n \to \infty$$

Case II When all the coefficients of $F_n(x)$ are identically but not necessarily independent Gaussians with mean $m = 0$, variance 1; one gets

$$|M|^{1/2}(2\pi)^{-(n+1)/2} \exp\left[-\frac{1}{2}\mathbf{a}'M\mathbf{a}\right]$$

where M^{-1} is the moment matrix with $\rho_{ij} = \rho, 0 < \rho < 1, i \neq j$. Then using the Kac–Rice formula, Bharucha-Reid and Sambandham (2014) proved that

$$N_n(\alpha, \beta) = \frac{1}{\pi} \int_{\alpha}^{\beta} \frac{\left(A_n C_n - B_n^2\right)^{1/2}}{A_n} dx$$

where A_n, B_n, C_n are similar in values as stated in *Case I*. For large n

$$E(N_n(\mathbb{R})) \sim (1/\pi)\log n, \quad n \to \infty.$$

Subcase I If all the random coefficients are iid normal random variables with mean $m(\neq 0)$ and variance one; that is the density function of each A_k is

$$(1/\sqrt{2\pi})e^{-(t-m)^2/2}$$

then

$$A_n = \sum_{k=0}^{n} x^{2k}, B = \sum_{k=0}^{n} kx^{2k-1}, C = \sum_{k=0}^{n} k^2 x^{2k-2}$$

Then one can obtain

$$E(N_n(\mathbb{R})) \sim (1/\pi)\log n, n \to \infty$$

Subcase II If all the random coefficients are iid standard normal random variables.

Then one gets

$$E(N_n(\mathbb{R})) \sim (2/\pi)\log n, n \to \infty$$

While all these results concern Gaussian random polynomials, some classical papers studied $F_n(x)$ with coefficients that are iid uniformly distributed on $(-1, 1)$ or iid discrete random variables that take values $+1$ or -1 with probability $\frac{1}{2}$ (except the leading coefficient A_n which is 1, a.s.). Under those conditions one can show that for each $n \geqslant 0$, for some $n_0 > 0$,

$$P\left(N_n(\mathbb{R}) > 25(\log n)^2\right) \leqslant \frac{12\log n}{n}$$

and

$$P\left(N_n(\mathbb{R}) < \frac{\alpha \log n}{(\log\log n)^2}\right) < \frac{K}{\log n},$$

where α and K are absolute constants.

3 Main Results

In this section, we present the main results of this article. We construct a random polynomial as follows

$$F_{2(n+1)}(x) = \left(A_1 + A_2x + A_3x^2\right)\left(A_4 + A_5x + A_6x^2\right)\cdots\left(A_{3n+1} + A_{3n+2}x + A_{3n+3}x^2\right)$$

where $\left\{A_{3n-2}\right\}_{n\geqslant 1} = \left\{A_1, A_4, A_7, \ldots\right\}$ is a sequence of iid continuous random variables following common CDF F_1, $\left\{A_{3n-1}\right\}_{n\geqslant 1} = \left\{A_2, A_5, A_8, \ldots\right\}$ is a sequence of iid continuous random variables following common CDF F_2, and $\left\{A_{3n}\right\}_{n\geqslant 1} = \left\{A_3, A_6, A_9, \ldots\right\}$ is a sequence of iid continuous random variables following common CDF F_3. The CDFs F_1, F_2, and F_3 need not be the same. We further assume that the three sequences of random coefficients are jointly independent. This particular form is considered mainly out of mathematical curiosity. However, note that, they are the characteristic polynomials associated with block diagonal and block triangular random matrices with 2×2 blocks and the associated real roots are the real eigenvalues of these block random matrices. Hence, studying these roots shed light on the behaviour of eigenvalues of certain random matrices. Owing to this special form of the coefficients we can bypass the Kac–Rice formula and a direct calculation yields a closed-form expression for $E(N_{2(n+1)}(\mathbb{R}))$. In fact, direct calculation leads to a nice expression for the distribution of the random variable $N_{2(n+1)}(\mathbb{R})$.

3.1 Exact Distribution of $N_{2(n+1)}(\mathbb{R})$

Theorem 1 *Let $N_{2(n+1)}(\mathbb{R})$ be the no. of real roots of $F_{2(n+1)}(x)$. Then*

$$\frac{N_{2(n+1)}(\mathbb{R})}{2} \sim \text{Bin}\left(n+1, P\left(A_2^2 - 4A_1A_3 > 0\right)\right).$$

A consequence of the above result is that the expected number of real roots of $F_{2(n+1)}(x)$ is

$$E(N_{2(n+1)}(\mathbb{R})) = 2(n+1)P\left(A_2^2 - 4A_1A_3 > 0\right),$$

and if $p \in (0, 1)$ then as $n \to \infty$ we get

$$\frac{N_{2(n+1)}(\mathbb{R}) - 2(n+1)p}{\sqrt{n+1}} \xrightarrow{d} N(0, 4pq)$$

where

$$p = P\left(A_2^2 - 4A_1A_3 > 0\right) \quad \text{and} \quad q = 1 - p.$$

Note that one may also count $N_{2(n+1)}(\mathbb{R})$ excluding multiplicity of zeroes. In either case, the above-mentioned results remain unchanged.

Proof of Theorem 1 Let us consider the random quadratic function $f(x) = A_1 + A_2x + A_3x^2$ where A_1, A_2 and A_3 are continuous random variables with CDFs F_1, F_2 and F_3, respectively. Suppose $N(\mathbb{R})$ counts the number of real roots excluding multiplicity.

$$\text{Hence, } N(\mathbb{R}) = \begin{cases} 0, & \Delta < 0 \\ 1, & \Delta = 0 \\ 2, & \Delta > 0 \end{cases} \text{ where } \Delta \text{ is the discriminant}$$

or,

$$N(\mathbb{R}) = \begin{cases} 0, \ A_2^2 - 4A_1A_3 < 0 \\ 1, \ A_2^2 - 4A_1A_3 = 0 \\ 2, \ A_2^2 - 4A_1A_3 > 0 \end{cases}$$

Now, since A_1, A_2, A_3 are jointly independent continuous random variables, ${A_2}^2 - 4A_1A_3$ is also a continuous random variable and hence, $P\left(A_2^2 - 4A_1A_3 = 0\right) = 0$. If we denote $P\left(A_2^2 - 4A_1A_3 > 0\right)$ by p and set $q = 1 - p$

$$\text{Then, } N(\mathbb{R}) = \begin{cases} 0, & \text{w.p. } q \\ 2, & \text{w.p. } p \end{cases}$$

or,

$$\frac{N(\mathbb{R})}{2} = \begin{cases} 0, \ \text{w.p. } q \\ 1, \ \text{w.p. } p \end{cases}$$

Now consider

$$f_k(x) = A_{3k+1} + A_{3r+2}x + A_{3k+3}x^2; \quad k = 0, 1, 3, \ldots n,$$

and note that

$$F_{2(n+1)}(x) = \prod_{k=0}^{n} f_k(x).$$

Let $m_k(\mathbb{R})$ be the number of real roots of $f_k(n)$. Since, $N_{2(n+1)}(\mathbb{R})$ denotes the number of real roots of $F_{2(n+1)}(x)$

$$\therefore N_{2(n+1)}(\mathbb{R}) = \sum_{k=0}^{n} m_k(\mathbb{R})$$

where,

$$\frac{m_0(\mathbb{R})}{2}, \frac{m_1(\mathbb{R})}{2}, \frac{m_2(\mathbb{R})}{2}, \ldots, \frac{m_n(\mathbb{R})}{2} \overset{iid}{\sim} \text{Ber}(p).$$

With that we end up showing that

$$\frac{N_{2(n+1)}(\mathbb{R})}{2} \sim \text{Bin}\Big(n+1, P\big(A_2^2 - 4A_1A_3 > 0\big)\Big).$$

□

Then by property of Binomial distribution, it follows that

$$E\left(\frac{N_{2(n+1)}(\mathbb{R})}{2}\right) = (n+1)P\big(A_2^2 - 4A_1A_3 > 0\big)$$

or, equivalently

$$E(N_{2(n+1)}(\mathbb{R})) = 2(n+1)P\big(A_2^2 - 4A_1A_3 > 0\big).$$

Simple application of De-Moivre–Laplace CLT implies that if $p \in (0,1)$ then

$$\frac{N_{2(n+1)}(\mathbb{R}) - 2(n+1)P\big(A_2^2 - 4A_1A_3 > 0\big)}{\sqrt{4(n+1)P\big(A_2^2 - 4A_1A_3 > 0\big)\big(1 - P\big(A_2^2 - 4A_1A_3 > 0\big)\big)}} \xrightarrow{d} N(0,1).$$

In simplified notation

$$\frac{N_{2(n+1)}(\mathbb{R}) - 2(n+1)p}{2\sqrt{(n+1)pq}} \xrightarrow{d} N(0,1)$$

or, equivalently

$$\frac{N_{2(n+1)}(\mathbb{R}) - 2(n+1)p}{\sqrt{(n+1)}} \xrightarrow{d} N(0,4pq) \text{ provided } p \in (0,1).$$

Note that, we have counted $N(\mathbb{R})$ excluding the multiplicity. However, even if we count $N(\mathbb{R})$ including the multiplicity we get

$$N(\mathbb{R}) = \begin{cases} 0, & A_2^2 - 4A_1A_3 < 0 \\ 2, & A_2^2 - 4A_1A_3 \geq 0 \end{cases}.$$

Now, since A_1, A_2 and A_3 are jointly independent continuous random variables, ${A_2}^2 - 4A_1A_3$ is also a continuous random variable. Hence, $P\big(A_2^2 - 4A_1A_3 = 0\big) = 0$ and so $P\big(A_2^2 - 4A_1A_3 \geq 0\big) = P\big(A_2^2 - 4A_1A_3 > 0\big) = p$ and $q = 1 - p$.

$$\text{Hence, even including multiplicity } N(\mathbb{R}) = \begin{cases} 0, & \text{w.p. } q \\ 2, & \text{w.p. } p \end{cases}$$

Consequently, all the results of Theorem 1 remains unchanged.

3.2 Proportion of Real Roots

For the constructed class of random polynomials as $n \to \infty$, the expected proportion of real roots $E(\frac{N_{2(n+1)}(\mathbb{R})}{2(n+1)})$ need not converge to 0, in contrast to most of the existing literature on random polynomials which show $E(N_n(\mathbb{R})) = o(n)$ as $n \to \infty$ that, in turn, implies that asymptotically the majority of the roots of the random polynomial are non-real. Curious by the observation then we investigate whether for any given $p \in [0, 1]$ we can get CDFs F_1, F_2 and F_3 such that for the constructed $F_{2(n+1)}(x)$, $E(N_{2(n+1)}(\mathbb{R})) \sim 2(n+1)p$ as $n \to \infty$. The second result of this article searches for an answer to this question.

Theorem 2 *Given any $p \in [0, 1]$ it is possible to construct CDFs F_1, F_2, and F_3 of continuous light-tailed random variables such that for the constructed $F_{2(n+1)}(x)$, $E(N_{2(n+1)}(\mathbb{R})) \sim 2(n+1)p$ as $n \to \infty$.*

Proof of Theorem 2 Let us fix $p \in [0, 1]$.

Case I $p = 0$. F_1, F_2 and F_3 are CDFs associated with $U[2, 3]$, $U[0, 1]$ and $U[2, 3]$, respectively. Then see that $P\left(A_2^2 > 4A_1A_3\right) = 0$ since $4A_1A_3 \geq 16$ w.p. 1 but $A_2^2 \leq 1$ w.p. 1.

Case II $p = 1$. F_1, F_2 and F_3 are CDFs associated with $U[0, 1]$, $U[3, 4]$ and $U[0, 1]$, respectively. Then see that $P\left(A_2^2 > 4A_1A_3\right) = 1$ since $4A_1A_3 \leq 4$ w.p. 1 but $A_2^2 \geq 9$ w.p. 1.

Case III $p \in (0, 1)$. This is the most interesting case and we choose F_1 and F_3 to be CDFs associated with $U[0, 1]$ and $U[0, 1]$, respectively. Then, we choose F_2 judiciously. Let us define F_2 as follows

$$F_2(x) = \begin{cases} 0; & x \leq 0 \\ \left(\frac{x}{2}\right)^{\left(\frac{2}{\sqrt{1-p}}-2\right)}; & 0 < x < 2 \\ 1; & x \geq 2 \end{cases} \cdot$$

Firstly, it is easy to show that $F_2(x)$ is a CDF associated with a continuous random variable A_2 with density as follows

$$f_2(x) = \begin{cases} \left(\frac{1}{\sqrt{1-p}} - 1\right)\left(\frac{x}{2}\right)^{\left(\frac{2}{\sqrt{1-p}}-3\right)}; & 0 < x < 2 \\ 0; & \text{otherwise} \end{cases}.$$

Then,

$$
\begin{aligned}
P\left(A_2^2 - 4A_1A_3 > 0\right) &= 1 - P\left(A_2^2 - 4A_1A_3 \le 0\right) \\
&= 1 - P\left(A_2^2 \le 4A_1A_3\right) \\
&= 1 - P\left(A_2 \le 2\sqrt{A_1A_3}\right) \\
&= 1 - \int_0^1 \int_0^1 P(A_2 \le 2\sqrt{A_1A_3} | A_1 = a_1, A_3 = a_3) da_1 da_3 \\
&= 1 - \int_0^1 \int_0^1 P(A_2 \le 2\sqrt{a_1a_3} | A_1 = a_1, A_3 = a_3) da_1 da_3 \\
&= 1 - \int_0^1 \int_0^1 P(A_2 \le 2\sqrt{a_1a_3}) da_1 da_3 \\
&= 1 - \int_0^1 \int_0^1 F_2(2\sqrt{a_1a_3}) da_1 da_3 \\
&= 1 - \int_0^1 \int_0^1 \left(\frac{2\sqrt{a_1a_3}}{2}\right)^{\left(\frac{2}{\sqrt{1-p}} - 2\right)} da_1 da_3 \\
&= 1 - \int_0^1 \int_0^1 (a_1a_3)^{\left(\frac{1}{\sqrt{1-p}} - 1\right)} da_1 da_3 \\
&= 1 - \int_0^1 a_1^{\left(\frac{1}{\sqrt{1-p}} - 1\right)} da_1 \int_0^1 a_3^{\left(\frac{1}{\sqrt{1-p}} - 1\right)} da_3 \\
&= 1 - \left(\int_0^1 a_1^{\left(\frac{1}{\sqrt{1-p}} - 1\right)} da_1\right)^2 \\
&= 1 - \left(\left(\sqrt{1-p}\right) a_1^{\left(\frac{1}{\sqrt{1-p}}\right)} |_0^1\right)^2 \\
&= 1 - \left(\sqrt{1-p}\right)^2 \\
&= p.
\end{aligned}
$$

Therefore, for any given $p \in [0, 1]$, we can choose CDFs F_1, F_2 and F_3 associated with continuous light-tailed random variables such that for the constructed $F_{2(n+1)}(x)$, $E(N_{2(n+1)}(\mathbb{R})) = 2(n+1)p$ and hence, $E(N_{2(n+1)}(\mathbb{R})) \sim 2(n+1)p$ as $n \to \infty$. □

3.3 Heavy-Tailed Random Polynomial

In this final part of the article, we investigate the distribution of $N_n(\mathbb{R})$ and the asymptotic behaviour of $E(N_n(\mathbb{R}))$ when A_i s are heavy-tailed random variables. First, we show that when A_i s are heavy-tailed random variables then the coefficients of $F_n(x)$ are also heavy-tailed. However, note that by heavy-tailed random variables here we refer to those random variables for which the MGF does not exist in any neighbourhood of zero. There are alternative stronger definitions of heavy-tailed random variables.

Proposition 3 *If* $x_1, x_2 \geq 2$ *then* $x_1x_2 \geq x_1 + x_2$.

Proof of Proposition 3.

$$\begin{aligned}
x_1 \geqslant 2 &\Longrightarrow x_1 - 1 \geqslant 1 \Rightarrow \frac{1}{x_1 - 1} \leqslant 1 \\
&\Longrightarrow x_2 \geqslant 2 = 1 + 1 \geqslant 1 + \frac{1}{x_1 - 1} = \frac{x_1}{x_1 - 1} \\
&\Longrightarrow x_2 \geqslant \frac{x_1}{x_1 - 1} \\
&\Longrightarrow x_1x_2 - x_2 \geqslant x_1 \Longrightarrow x_1x_2 \geqslant x_1 + x_2.
\end{aligned}$$

Proposition 4 *If* $\int_0^\infty e^{tx}f(x)dx$ *diverges to infinity, when* $f(\cdot)$ *is a p.d.f, then* $\int_k^\infty e^{tx}f(x)dx$ *also diverges infinity, where* $K \in \mathbb{R}^+$.

Proof of Proposition 4.

$$\int_0^\infty e^{tx}f(x)dx = \int_0^k e^{tx}f(x)dx + \int_k^\infty e^{tx}f(x)dx$$

Now,

$$\begin{aligned}
0 \leq \int_0^k e^{tx}f(x)dx &\leq \int_0^k e^{tk}f(x)dx \\
&= e^{tk}\int_0^k f(x)dx \\
&\leq e^{tk}\int_0^\infty f(x)dx \\
&\leq e^{tk}
\end{aligned}$$

$\therefore \int_0^k e^{tx}f(x)dx$ is bounded. Since, $\int_0^k e^{tx}f(x) + \int_k^\infty e^{tx}f(x)dx$ diverges to infinity, we conclude that $\int_k^\infty e^{tx}f(x)dx$ also diverges to infinity.

Proposition 5 *If at least one of* $Y_1, Y_2, \ldots, Y_n$ *is a heavy-tailed random variable, then* $\prod_{i=1}^n Y_i$ *and* $\sum_{i=1}^n Y_i$ *are also heavy-tailed.*

Proof of Proposition 5. First, we show that Y_1Y_2 is heavy-tailed random variable. Note that, $E\left(e^{tY_1Y_2}\right) = \int_0^\infty \int_0^\infty e^{ty_1y_2}f_{Y_1}\left(y_1\right)f_{Y_2}\left(y_2\right)dy_1dy_2 \geqslant \int_2^\infty \int_2^\infty e^{ty_1y_2}f_{Y_1}\left(y_1\right)f_{Y_2}\left(y_2\right)dy_1dy_2$

$\geqslant \int_2^\infty \int_2^\infty e^{t(y_1+y_2)} f_{Y_1}(y_1) f_{Y_2}(y_2) dy_1 dy_2$
$= \{\int_2^\infty e^{ty} \cdot f_{Y_1}(y_1) dy_1\}\{\int_2^\infty e^{ty_2} f_{Y_2}(y_2) dy_2\}$

By putting $k = 2$ in Proposition 4, we have: $\int_2^\infty e^{ty_i} f_{Y_i}(y_i) dy_i$ diverges to infinity for at least one of $i = 1, 2$.

$\Longrightarrow \{\int_2^\infty e^{ty} \cdot f_{Y_1}(y_1) dy_1\}\{\int_2^\infty e^{ty_2} f_{Y_2}(y_2) dy_2\}$ diverges to infinity $\Rightarrow E(e^{tY_1 Y_2})$ is not finite, where $t > 0$.

$\Longrightarrow Y_1 Y_2$ is also a heavy-tailed random variable. By repeated application of the previous result then we can show that if at least one of $Y_1, Y_2, \ldots, Y_n$ is a heavy-tailed random variable, then $\prod_{i=1}^n Y_i$ is also heavy-tailed.

Now, we show that if at least one of $Y_1, Y_2, \ldots, Y_n$ is a heavy-tailed random variable then $\sum_{i=1}^n Y_i$ is also heavy-tailed. Note that, $E(e^{tY_i})$ diverges to infinity, $\forall t > 0$, for at least one of $i = 1(1)\text{n}$. Now, $P(e^{tY_i} \geqslant 0) = 1$ $\Longrightarrow P(e^{t(Y_1+Y_2+\ldots+Y_n)} \geqslant e^{tY_i}) = 1, \forall i = 1(1)n$ $\Longrightarrow E\left(e^{t(Y_1+Y_2+\cdots+Y_n)}\right) \geqslant E(e^{tY_i}) \forall i = 1(1)n$. Hence, $E\left(e^{t(y_1+\cdots+y_n)}\right)$ also diverges to infinity, which in turn, implies $Y_1 + Y_2 + \cdots + Y_n$ is also a heavy-tailed random variable.

Now, since each coefficient in the expanded polynomial is formed by the adding products of A_i, we conclude that they are heavy-tailed if all the A_is are heavy-tailed. Note that, Theorem 1 does not specify whether F_1, F_2, and F_3 are CDFs associated with continuous light-tailed or heavy-tailed random variables, and hence the result is directly applicable to the heavy-tailed coefficients, too. So, if we take the A_i to be heavy-tailed continuous random variables then also $\frac{N_{2(n+1)}(\mathbb{R})}{2}$ will follow a binomial distribution. However, now for any given $p \in [0, 1]$, finding F_1, F_2, and F_3 which are CDFs associated with continuous heavy-tailed random variables such that the $E(N_{2(n+1)}(\mathbb{R})) \sim 2(n+1)p$ as $n \to \infty$ may be tricky. In what follows, we show that for any given $p \in (0, 1]$ we can choose continuous heavy-tailed distributions F_1, F_2, and F_3 such that $E(N_{2(n+1)}(\mathbb{R})) \sim 2(n+1)p$ as $n \to \infty$. We further show that for no choice of F_1, F_2, and F_3 we can extend this result to $p = 0$ case.

Theorem 6 *For any given* $p \in (0, 1]$, *it is possible to design continuous heavy-tailed distributions* F_1, F_2, *and* F_3 *such that* $E(N_{2(n+1)}(\mathbb{R})) \sim 2(n+1)p$ *as* $n \to \infty$. *Furthermore, it is not possible to choose continuous heavy-tailed distributions* F_1, F_2 *and* F_3 *such that* $E(N_{2(n+1)}(\mathbb{R})) = o(n)$ *as* $n \to \infty$.

Proof of Theorem 6 Let us fix $p \in (0, 1]$.

Case I $p = 1$. F_1, F_2, and F_3 are CDFs associated with *Pareto*$(1, 1)$, *Pareto*$(1, 1)$, and $-$*Pareto*$(1, 1)$, respectively. Then see that $P(A_2^2 > 4A_1A_3) = 1$ since $4A_1A_3 < 0$ w.p. 1 but $A_2^2 \geq 0$ w.p. 1.

Case II $p \in (0, 1)$. In this more interesting case we choose both of F_1 and F_3 to be CDF of *Pareto*(α_1, β) and F_2 to be CDF of *Pareto*$(2\alpha, \beta)$ where $\alpha, \alpha_1 > 0$ and $\beta > 0$ are to be selected later. Let us denote the pdf associated with F_1 and F_3 by f_1 and f_3. Then,

$$
\begin{aligned}
&P\left(A_2^2-4A_1A_3>0\right)\\
&=1-P\left(A_2^2-4A_1A_3\leq 0\right)\\
&=1-P\left(A_2^2\leq 4A_1A_3\right)\\
&=1-P\left(A_2\leq 2\sqrt{A_1A_3}\right)\\
&=1-\int_\beta^\infty\int_\beta^\infty P(A_2\leq 2\sqrt{A_1A_3}|A_1=a_1,A_3=a_3)f_1(a_1)f_3(a_3)da_1da_3\\
&=1-\int_\beta^\infty\int_\beta^\infty P(A_2\leq 2\sqrt{a_1a_3}|A_1=a_1,A_3=a_3)f_1(a_1)f_3(a_3)da_1da_3\\
&=1-\int_\beta^\infty\int_\beta^\infty P(A_2\leq 2\sqrt{a_1a_3})f_1(a_1)f_3(a_3)da_1da_3\\
&=1-\int_\beta^\infty\int_\beta^\infty F_2(2\sqrt{a_1a_3})f_1(a_1)f_3(a_3)da_1da_3\\
&=1-\int_\beta^\infty\int_\beta^\infty\left(1-\left(\frac{\beta}{2\sqrt{a_1a_3}}\right)^{2\alpha}\right)f_1(a_1)f_3(a_3)da_1da_3\\
&=1-\int_\beta^\infty\int_\beta^\infty f_1(a_1)f_3(a_3)da_1da_3\\
&\quad+\int_\beta^\infty\int_\beta^\infty\left(\frac{\beta}{2\sqrt{a_1a_3}}\right)^{2\alpha}f_1(a_1)f_3(a_3)da_1da_3\\
&=1-\left(\int_\beta^\infty f_1(a_1)da_1\right)\left(\int_\beta^\infty f_3(a_3)da_3\right)\\
&\quad+\int_\beta^\infty\int_\beta^\infty\left(\frac{\beta^2}{4a_1a_3}\right)^{\alpha}f_1(a_1)f_3(a_3)da_1da_3\\
&=1-1+\int_\beta^\infty\int_\beta^\infty\left(\frac{\beta^2}{4a_1a_3}\right)^{\alpha}f_1(a_1)f_3(a_3)da_1da_3\\
&=\frac{\beta^{2\alpha}}{4^\alpha}\left(\int_\beta^\infty\left(\frac{1}{a_1}\right)^\alpha f_1(a_1)da_1\right)\left(\int_\beta^\infty\left(\frac{1}{a_3}\right)^\alpha f_3(a_3)da_3\right)\\
&=\frac{\beta^{2\alpha}}{4^\alpha}E\left(\frac{1}{A_1^\alpha}\right)E\left(\frac{1}{A_3^\alpha}\right).
\end{aligned}
$$

Now, we calculate $E(\frac{1}{A_1^\alpha})$. See that

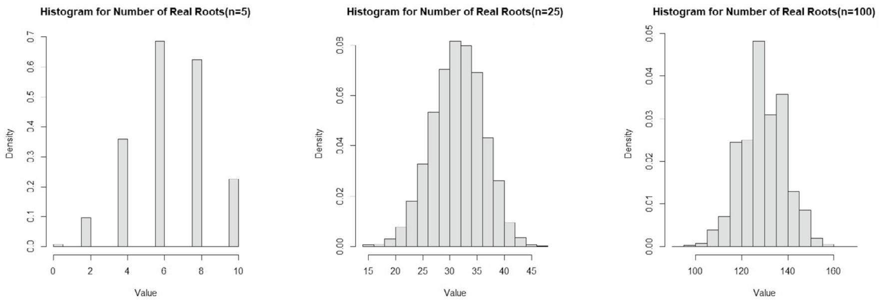
$$
\begin{aligned}
E\left(\frac{1}{A_1^{\alpha}}\right) &= \int_{\beta}^{\infty}\left(\frac{1}{a_1}\right)^{\alpha}\frac{\alpha_1\beta^{\alpha_1}}{a_1^{\alpha_1+1}}da_1\\
&= \alpha_1\beta^{\alpha_1}\int_{\beta}^{\infty}\frac{1}{a_1^{\alpha+\alpha_1+1}}da_1\\
&= \alpha_1\beta^{\alpha_1}\left[\frac{a_1^{-\alpha-\alpha_1}}{-\alpha-\alpha_1}\right]_{\beta}^{\infty}\\
&= \alpha_1\beta^{\alpha_1}\left[0-\frac{\beta^{-\alpha-\alpha_1}}{-\alpha-\alpha_1}\right]\\
&= \frac{\alpha_1}{(\alpha+\alpha_1)\beta^{\alpha}}.
\end{aligned}
$$

Hence, $P\big(A_2^2 - 4A_1A_3 > 0\big) = \frac{\beta^{2\alpha}}{4^{\alpha}}E(\frac{1}{A_1^{\alpha}})E(\frac{1}{A_3^{\alpha}}) = \frac{\alpha_1^2}{(\alpha+\alpha_1)^2 4^{\alpha}}$. We set $\alpha_1 = 1$ to get $P\big(A_2^2 - 4A_1A_3 > 0\big) = \frac{1}{(\alpha+1)^2 4^{\alpha}}$. Since the function $g(x) = \frac{1}{(x+1)^2 4^{x}}$ is continuous and strictly monotonically decreasing on $\mathbb{R}^+$ and as $\lim_{x\to 0^+} g(x) = 1$ and $\lim_{x\to\infty} g(x) = 0$ so for any $p \in (0,1)$ there exists a unique x_0 such that $g(x_0) = p$, and then we set $\alpha = x_0$. That leads to $P\big(A_2^2 - 4A_1A_3 > 0\big) = p$.

Therefore, for any given $p \in (0,1]$, we can choose CDFs F_1, F_2, and F_3 associated with continuous heavy-tailed random variables such that for the constructed $F_{2(n+1)}(x)$, $E(N_{2(n+1)}(\mathbb{R})) = 2(n+1)p$ and hence $E(N_{2(n+1)}(\mathbb{R})) \sim 2(n+1)p$ as $n \to \infty$. Now we show that it is not possible to choose continuous heavy-tailed distributions F_1, F_2, and F_3 such that $E(N_{2(n+1)}(\mathbb{R})) = o(n)$ as $n \to \infty$. We would prove it by contradiction. Suppose there are such continuous heavy-tailed distributions F_1, F_2, and F_3. Recall that $A_1 \sim F_1$ and $A_2 \sim F_2$ are independent random variables. One can get $a < b$ and $b > 0$ such that $P(a \le 4A_1A_3 \le b) > 0$. So,

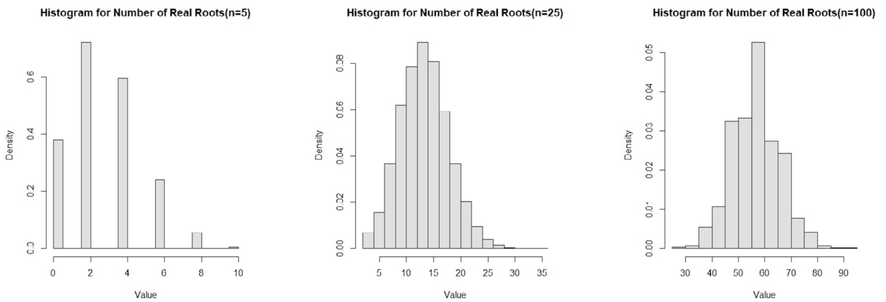
$$
\begin{aligned}
0 = p &= P\big(A_2^2 > 4A_1A_3\big)\\
&= P\big(A_2^2 > 4A_1A_3 \mid a \le 4A_1A_3 \le b\big)P(a \le 4A_1A_3 \le b)\\
&\quad + P\big(A_2^2 > 4A_1A_3 \mid (4A_1A_3 < a) \cup (4A_1A_3 > b)\big)P((4A_1A_3 < a) \cup (4A_1A_3 > b))\\
&\Longrightarrow P\big(A_2^2 > 4A_1A_3 \mid a \le 4A_1A_3 \le b\big) = 0\\
&\Longrightarrow P\big(A_2^2 > b \mid a \le 4A_1A_3 \le b\big) \le P\big(A_2^2 > 4A_1A_3 \mid a \le 4A_1A_3 \le b\big) = 0\\
&\Longrightarrow P\big(A_2^2 > b \mid a \le 4A_1A_3 \le b\big) = 0\\
&\Longrightarrow P\big(A_2^2 > b\big) = 0 \ \text{ Since } A_1, A_2, A_3 \text{ are independent.}\\
&\Longrightarrow P\big(A_2^2 \le b\big) = 1
\end{aligned}
$$

which implies that F_2 has bounded support thereby contradicting the assumption F_2 is a heavy-tailed distribution. □



(a) Distribution of $N_{10}(\mathbb{R})$ (b) Distribution of $N_{50}(\mathbb{R})$ (c) Distribution of $N_{200}(\mathbb{R})$

Fig. 1 Distribution of $N_{10}(\mathbb{R}), N_{50}(\mathbb{R})$ and $N_{200}(\mathbb{R})$, when F_1, F_2, and F_3 are $N(0, 1)$



(a) Distribution of $N_{10}(\mathbb{R})$ (b) Distribution of $N_{50}(\mathbb{R})$ (c) Distribution of $N_{200}(\mathbb{R})$

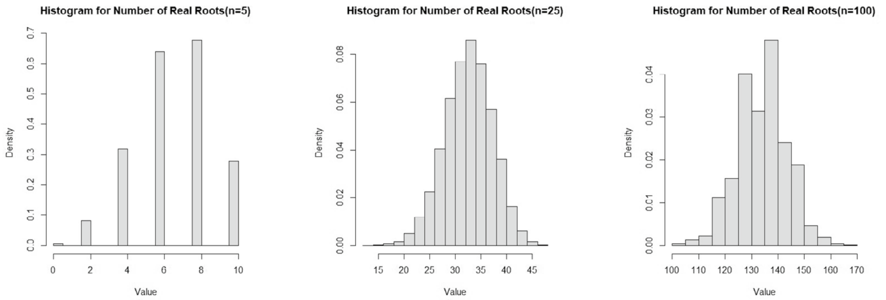
Fig. 2 Distribution of $N_{10}(\mathbb{R}), N_{50}(\mathbb{R})$ and $N_{200}(\mathbb{R})$, when F_1, F_2, and F_3 are $LN(0, 1)$

3.4 Simulation Studies

The theoretical findings obtained in the previous subsections are supplemented via simulation studies. We study the distribution of $N_{2(n+1)}(\mathbb{R})$ for three different sample sizes. We plot the relative frequency histogram of $N_{10}(\mathbb{R}), N_{50}(\mathbb{R})$ and $N_{200}(\mathbb{R})$. These quantities denote the number of real roots of random polynomials of degree 10, 50 and 200, respectively. We consider different probability distributions for the coefficients $A_0, \ldots, A_n$, to carry out the simulation studies. First, we consider the case when F_1, F_2, and F_3 are $N(0, 1)$ (Fig. 1).

Then we consider the two cases when F_1, F_2, and F_3 are $LN(0, 1)$, and F_1, F_2, and F_3 are $C(0, 1)$. Note that, $LN(0, 1)$ does not admit finite MGF, and hence, as per our definition of a heavy-tailed random variable, $LN(0, 1)$ is a heavy-tailed distribution. On the other hand, standard Cauchy distribution, denoted by $C(0, 1)$, does not admit finite mean as well as finite MGF and hence, qualifies as a heavy-tailed distribution in a much stronger sense (Figs. 2 and 3).

Although, the theorems of the previous subsections are based on the assumption that F_1, F_2, and F_3 are continuous distributions; it is curious to see how the

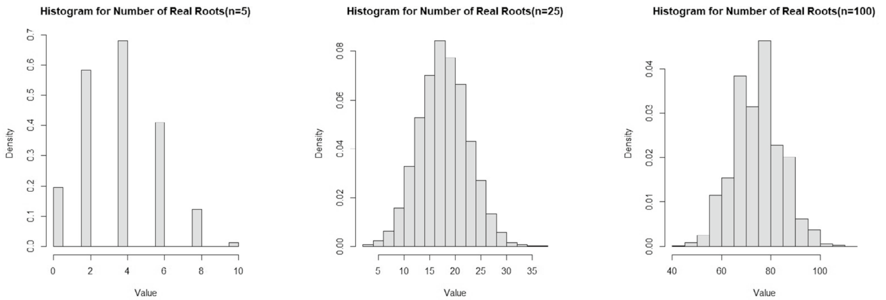


(a) Distribution of $N_{10}(\mathbb{R})$ (b) Distribution of $N_{50}(\mathbb{R})$ (c) Distribution of $N_{200}(\mathbb{R})$

Fig. 3 Distribution of $N_{10}(\mathbb{R}), N_{50}(\mathbb{R})$ and $N_{200}(\mathbb{R})$, when F_1, F_2, and F_3 are $C(0, 1)$

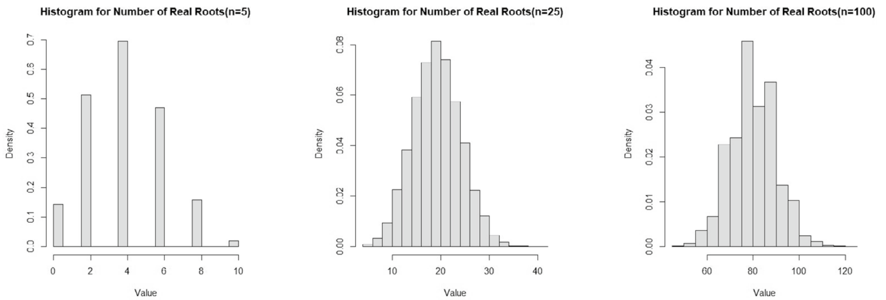
distribution of $N_{2(n+1)}(\mathbb{R})$ is modified, if instead, we assume that the coefficients are discrete random variables (Figs. 4 and 5).

As n becomes larger, the distribution $N_{2(n+1)}(\mathbb{R})$ can be approximated by normal distribution more accurately. Hence, unlike the distribution $N_{10}(\mathbb{R})$ and $N_{50}(\mathbb{R})$, which are skewed, the distribution of $N_{200}(\mathbb{R})$ is symmetric and bell-shaped. For



(a) Distribution of $N_{10}(\mathbb{R})$ (b) Distribution of $N_{50}(\mathbb{R})$ (c) Distribution of $N_{200}(\mathbb{R})$

Fig. 4 Distribution of $N_{10}(\mathbb{R}), N_{50}(\mathbb{R})$ and $N_{200}(\mathbb{R})$, when F_1, F_2, and F_3 are $Ber(\frac{1}{2})$



(a) Distribution of $N_{10}(\mathbb{R})$ (b) Distribution of $N_{50}(\mathbb{R})$ (c) Distribution of $N_{200}(\mathbb{R})$

Fig. 5 Distribution of $N_{10}(\mathbb{R}), N_{50}(\mathbb{R})$ and $N_{200}(\mathbb{R})$, when F_1, F_2, and F_3 are $Poi(1)$

each simulation study, 10,000 samples are generated to compute the respective histogram.

3.5 Discussion and Concluding Remarks

The results derived in this article are based on the assumption that the coefficients are continuous random variables. It is curious to see how are they modified if instead, we assume that the coefficients are general random variables. The exact distribution of $\frac{N_{2(n+1)}(\mathbb{R})}{2}$ remains unchanged if roots are counted including the multiplicity. However, if roots are counted excluding multiplicity, in that case, $N_{2(n+1)}(\mathbb{R})$ follows a distribution that is the $(n+1)$-fold convolution of a 3-point discrete distribution supported on $\{0, 1, 2\}$ of which Bernoulli is a special case.

Another question that arises naturally is whether the results derived in this article can be extended to some higher degree random polynomials of the same structure. A theorem similar to Theorem 1 can be stated for $F_{3(n+1)}(x) = \left(A_1 + A_2x + A_3x^2 + A_4x^3\right) \cdots \left(A_{4n+1} + A_{4n+2}x + A_{4n+3}x^2 + A_{4n+4}x^3\right)$ and one can show that $\frac{N_{3(n+1)}(\mathbb{R})-(n+1)}{2} \sim \text{Bin}\left(n+1, P\left(4(A_3^2 - 3A_2A_4)^3 - (2A_3^3 - 9A_2A_3A_4 + 27A_1A_4^2)^2 > 0\right)\right)$. However, in that case, the binomial success probability is of nontrivial form, rendering an extension of Theorems 2 and 6 difficult. Beyond the third-degree polynomial, the scaled and centered value of the number of real roots is no longer binomially distributed. When k is even, the distribution of $\frac{N_{k(n+1)}(\mathbb{R})}{2}$ is $(n+1)$-fold convolution of a $\frac{k}{2}$-point discrete distribution supported on $\{0, 1, 2, \ldots, \frac{k}{2}\}$, and when k is odd, the distribution of $\frac{N_{k(n+1)}(\mathbb{R})-(n+1)}{2}$ is $(n+1)$-fold convolution of a $\frac{k-1}{2}$-point discrete distribution supported on $\{0, 1, 2, \ldots, \frac{k-1}{2}\}$. Needless to mention for the above-mentined cases an extension of Theorems 2 and 6 seems infeasible.

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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