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RESEARCH ARTICLE

Binominal Mixture Lindley Distribution: Properties and Applications

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Abstract
In this paper, we introduce a generalized mixture distribution, so-called the binomial mixture Lindley distribution (*BMLD*). The density function of this distribution is obtained by mixing binomial probabilities with gamma distribution. *BMLD* have various distributions as its special cases and posses various shapes for its hazard rate function including increasing, decreasing, bathtub shape and upside down bathtub shape depending on its parameters. Several mathematical, structural and statistical properties of the new distribution is presented such as moments, moment generating function, hazard rate function, vitality function, mean residual life function, inequality measures, entropy and extropy etc. The parameters of the model are estimated using the method of maximum likelihood and finally real life data sets are considered to illustrate the relevance of the new model by comparing it with some other lifetime models.

Keywords Lindley distribution · Maximum likelihood estimator · Hazard rate function · Inequality measures · Entropy · Extropy · Vitality function

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1 Introduction

Numerous probability distributions are introduced in the literature by mixing, extending and modifying well known distributions and hence provide more flexible hazard rate function for modelling lifetime data. These distributions will then be more suitable for fitting appropriate real data than the base models. Knowledge of the appropriate distribution plays an important role in improving the efficiency of any statistical inference related to data sets. Hence the researchers are more keen to develop new distributions by extending classical distributions to increase model flexibility and adaptability in various aspects of modelling data.

Lindley (1958) introduced in the literature one of the most discussed lifetime distribution, the Lindley distribution, in the context of the Bayesian statistics as a counter example of the fiducial statistics. Lindley distribution (*LD*) have the probability density function (pdf),

$$f_1(x) = \frac{\theta^2}{1+\theta}(1+x)e^{-\theta x};\ x > 0,\ \theta > 0, \tag{1.1}$$

which is a mixture of exponential (θ) and gamma $(2, \theta)$ distributions. The corresponding cumulative distribution function (cdf) has been obtained as,

$$F_1(x) = 1 - \frac{\theta + 1 + \theta x}{1+\theta}e^{-\theta x};\ x > 0,\ \theta > 0, \tag{1.2}$$

where θ is the scale parameter.

Mixture models provide a mathematical based, flexible and meaningful approach for the wide variety of classification requirements. There are numerous fields in which mixture models have practical applicability. Lindley itself being a mixture model, it has gained momentum in the theoretical perspective as well as in terms of its applications. Ghitany et al. (2008) have studied various properties of this distribution and showed that (1.1) provides a better model for some applications than the exponential distribution. Mazucheli and Achcar (2011) applied the Lindley distribution to competing risk life time data. A discrete version of this distribution has been suggested by Deniz and Ojeda (2011) having its applications in count data related to insurance. Al-Mutairi et al. (2013) developed the inferential procedure of the stress-strength parameter, when both stress and strength variables follow Lindley distribution. The applicability of Lindley distribution in solving lifetime modelling problems and modelling stress strength model made researchers to develop many generalizations, modifications and extensions of this distribution. Shanker et al. (2013) introduced a two parameter Lindley distribution (LD_2) for modelling waiting and survival times data with pdf,

$$f_2(x;\alpha,\theta) = \frac{\theta^2}{\theta+\alpha}(1+\alpha x)e^{-\theta x};\ x > 0,\ \theta > 0,\ \alpha > -\theta, \tag{1.3}$$

where $f_2(x;\alpha,\theta)$ is a mixture of exponential (θ) and gamma $(2, \theta)$ with mixing probabilities $\frac{\theta}{\theta+\alpha}$ and $\frac{\alpha}{\theta+\alpha}$ respectively. Even though one parameter and two parameter Lindley distributions are mixture of $E(\theta)$ and $G(2, \theta)$, most of the further generalizations are based on two gamma models with suitable mixtures. The generalizations that we aware of are:

Zakerzadeh and Dolati (2009) introduced a generalized Lindley distribution (*GLD*) with pdf,

$$f_3(x;\alpha,\theta,\gamma) = \frac{\theta^2(\theta x)^{\alpha-1}(\alpha+\gamma x)}{(\gamma+\theta)\Gamma(\alpha+1)}e^{-\theta x};\ x>0,\ \alpha,\theta,\gamma>0, \tag{1.4}$$

$f_3(x;\alpha,\theta,\gamma)$ is a mixture of gamma (α,θ) and gamma $(\alpha+1,\theta)$ with mixing probabilities $\frac{\theta}{\gamma+\theta}$ and $\frac{\gamma}{\gamma+\theta}$ respectively.

Ghitany et al. (2011) introduced a weighted Lindley distribution (*WLD*) with pdf,

$$f_4(x;\theta,\alpha) = \frac{\theta^{\alpha+1}}{(\theta+\alpha)\Gamma(\alpha)}x^{\alpha-1}(1+x)e^{-\theta x}, \alpha,\theta,x>0, \tag{1.5}$$

$f_4(x;\theta,\alpha)$ can also be expressed as a two component mixture such that

$$f_4(x;\theta,\alpha) = pg_1(x) + (1-p)g_2(x),$$

where $p=\frac{\theta}{\theta+\alpha}$ and $g_i(x) = \frac{\theta^{\alpha+j-1}}{\Gamma(\alpha+j-1)}x^{\alpha+j-2}e^{-\theta x}, \alpha,\theta,x>0, j=1,2$, is the pdf of the gamma distribution with the shape parameter $\alpha+j-1$ and scale parameter $\theta, j=1,2$.

Elbatal et al. (2013) proposed a new generalized Lindley distribution (*NGLD*) with pdf,

$$f_5(x;\theta,\alpha,\beta) = \frac{1}{1+\theta}\left[\frac{\theta^{\alpha+1}x^{\alpha-1}}{\Gamma(\alpha)} + \frac{\theta^{\beta}x^{\beta-1}}{\Gamma(\beta)}\right]e^{-\theta x}; x>0, \alpha,\theta>0, \tag{1.6}$$

where $f_5(x;\alpha,\theta)$ is a mixture of gamma (α,θ) and gamma (β,θ) with mixing probabilities $\frac{\theta}{\theta+1}$ and $\frac{1}{\theta+1}$ respectively.

Abouammoh et al. (2015) defined another new generalized Lindley distribution ($NGLD_1$) with pdf,

$$f_6(x;\alpha,\theta) = \frac{\theta^{\alpha}x^{\alpha-2}}{(\theta+1)\Gamma(\alpha)}(x+\alpha-1)e^{-\theta x}; x>0, \theta\geq 0, \alpha\geq 1, \tag{1.7}$$

where $f_6(x;\alpha,\theta)$ is a mixture of gamma (α,θ) and gamma $(\alpha-1,\theta)$ with mixing probabilities $\frac{1}{\theta+1}$ and $\frac{\theta}{\theta+1}$ respectively.

All these generalizations play various roles in the literature both in theoretical and applied perspectives. It can be perceived that most of the further developments are based on these six models, which immensely motivates to propose a generalized family, which generalizes the afore mentioned Lindley models. Hence in this work we introduce a wider class of Lindley distribution by mixing binomial probabilities with gamma distribution and name the distribution as binomial mixture Lindley distribution (*BMLD*).

One of the main peculiarity of the *LD* is its shape of hazard rate (increasing hazard rate) function compared to the well known exponential distribution. By scrutinizing the flexibility of various variants of Lindley model in terms of the hazard rate function, Lindley-Exponential distribution (Bhati et al. 2015) possess decreasing hazard rate function, *GLD* possess bathtub shape hazard rate function and inverse Lindley distribution (Sharma et al. 2015) possess upside down bathtub shape hazard rate function. Hence another motivation of this work is to propose a flexible extension of Lindley model which possess all the available shapes of hazard rate function. During the initial stage of this work, we came across several recent articles based on Lindley models. Several authors claim that their model possess bathtub shaped hazard rate but not even a single author attempted to fit a bathtub shaped data. Hence one motivation of this work is to propose a model and successfully apply a well known bathtub shaped data of Aarset (1987). In addition to Aarset data, to prove the superiority of *BMLD* we also took two other data sets, viz., strength of glass fiber data (see, Smith and Naylor 1987) and survival times of 72 guinea pigs data (see, Bjerkedal 1960) both having increasing hazard rate function.

The rest of the paper is outlined as follows. In Sect. 2 binomial mixture Lindley distribution is defined along with its moments, model identifiability, mean, variance, a recursive relationship for moments and moment generating function. Some of the reliability properties of the model such as hazard rate function, vitality function, mean residual life function, inequality measures and some uncertainty measures are presented in Sect. 3. In Sect. 4, the parameters of the distribution are estimated using method of maximum likelihood and thus obtained observed Fisher information matrix and asymptotic confidence intervals. A simulation study is presented in Sect. 5. Finally in Sect. 6, experimental results of the proposed distribution based on real data sets are illustrated.

2 Binomial Mixture Lindley Distribution

In this section, we present definition and some important properties of the binomial mixture Lindley distribution . Here after we use the short form *BMLD* for binomial mixture Lindley distribution.

Definition 2.1 A continuous random variable X is said to follow *BMLD* if its pdf $f(x)$ has the following form,

$$f(x) = \sum_{i=0}^{g} p_i \; h_i(x), \tag{2.1}$$

where

$$h_i(x) = \frac{\theta^{\alpha_i}}{\Gamma(\alpha_i)} x^{\alpha_i - 1} e^{-\theta x},$$

for $\theta > 0$, $\alpha_i > 0$ for $i = 0, 1, \cdots, g$. We define the mixing weights p_i such that

$$p_i = \binom{g}{i}\left(\frac{\theta}{\theta+\beta}\right)^i \left(\frac{\beta}{\theta+\beta}\right)^{g-i}, \tag{2.2}$$

for $i = 0, 1, \cdots, g$ and $\sum_{i=0}^{g} p_i = 1,\ \beta > 0,\ \theta > 0$.

Special cases

(1) If g=1, α_0=1 and α_1=1, then *BMLD* becomes the exponential distribution (*ED*).
(2) If g=1, β=1 and α_0=α_1=α, then *BMLD* becomes the gamma distribution (*GD*).
(3) If g=1, β=1, α_0=2 and α_1=1, then *BMLD* becomes the Lindley distribution (*LD*).
(4) If g=1, α_0=2 and α_1=1, then *BMLD* becomes the two parameter Lindley distribution [LD_2 (Shanker et al. 2013)].
(5) If g=1, α_0=$\alpha + 1$ and α_1=α, then *BMLD* becomes the generalized Lindley distribution [*GLD* (Zakerzadeh and Dolati 2009)].
(6) If g=1 $\alpha_0 = \alpha + 1$, $\alpha_1 = \alpha$ and $\beta = \alpha$, then *BMLD* becomes the weighted Lindley distribution [*WLD* (Ghitany et al. 2011)].
(7) If g=1 and β=1, then *BMLD* becomes the new generalized Lindley distribution [*NGLD* (Elbatal et al. 2013)].
(8) If g=1, β=1, α_0=α and α_2=$\alpha - 1$, then *BMLD* becomes new generalized Lindley distribution [$NGLD_1$ (Abouammoh et al. 2015)].

The pdf of the distribution, for different values of parameters, is plotted in Fig. 1.

2.1 Identifiability

A set of parameters for a particular model is said to be identifiable if not any two sets of the parameters gives same distribution for the given x.

Result 2.1 *The identifiability condition for BMLD with pdf as given in* (2.1) *is* $\alpha_i \neq \alpha_j$ *for each* $i, j \in 0, 1, 2, ..., g$ *such that* $i \neq j$.

Proof For mathematical simplicity, first we consider the case of $g = 2$ and let

$$b_0 B_0(x) + b_1 B_1(x) + b_2 B_2(x) = 0, \tag{2.3}$$

where b_0, b_1 and b_2 are real numbers, $B_0(x) = \int_{u=0}^{x} f(u)du$, $B_1(x) = \int_{u=0}^{x} g(u)du$ and $B_2(x) = \int_{u=0}^{x} h(u)du$ with $x > 0$. Also $g(u)$ and $h(u)$ can be obtained from $f(u)$ by replacing α_i by ρ_i and α_i by μ_i respectively. Assume that for each $i = 0, 1, 2$, $\alpha_i \neq \rho_i \neq \mu_i$,

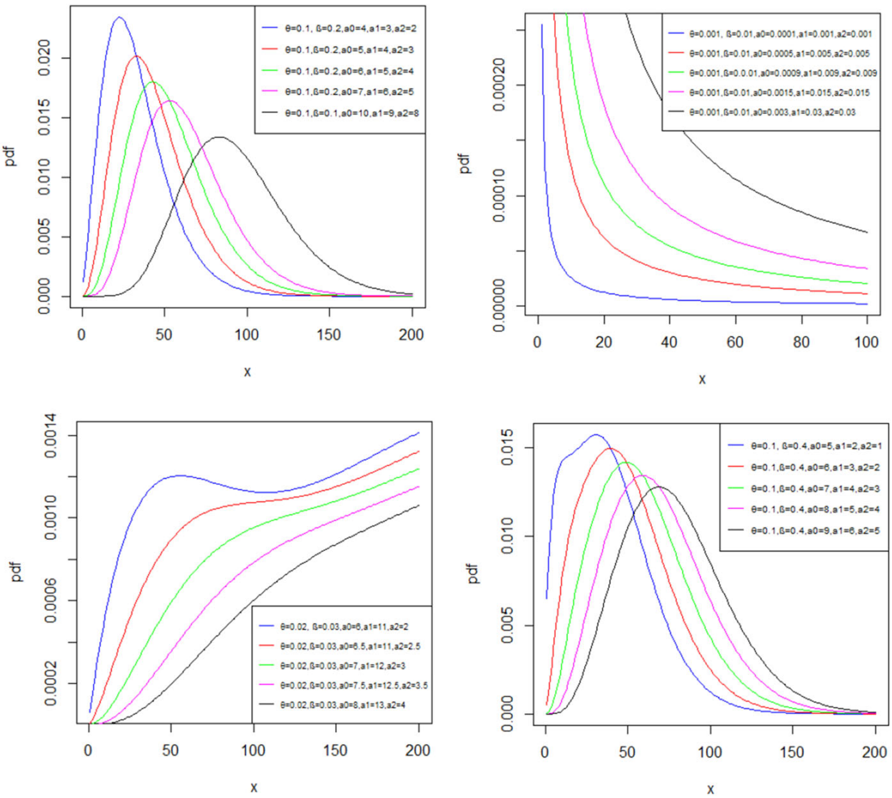

Fig. 1 The pdf of *BMLD* for g=2 and different values of θ, β, α_0, α_1, α_2

$$B_0(x) = \int_0^x \left[\left(\frac{\beta}{\theta+\beta}\right)^2 \frac{\theta^{\alpha_0}}{\Gamma(\alpha_0)} t^{\alpha_0-1} e^{-\theta t} + \frac{2\beta\theta}{(\theta+\beta)^2} \frac{\theta^{\alpha_1}}{\Gamma(\alpha_1)} t^{\alpha_1-1} e^{-\theta t} + \left(\frac{\theta}{\theta+\beta}\right)^2 \frac{\theta^{\alpha_2}}{\Gamma(\alpha_2)} t^{\alpha_2-1} e^{-\theta t} \right] dt, \tag{2.4}$$

$$B_1(x) = \int_0^x \left[\left(\frac{\beta}{\theta+\beta}\right)^2 \frac{\theta^{\rho_0}}{\Gamma(\rho_0)} t^{\rho_0-1} e^{-\theta t} + \frac{2\beta\theta}{(\theta+\beta)^2} \frac{\theta^{\rho_1}}{\Gamma(\rho_1)} t^{\rho_1-1} e^{-\theta t} + \left(\frac{\theta}{\theta+\beta}\right)^2 \frac{\theta^{\rho_2}}{\Gamma(\rho_2)} t^{\rho_2-1} e^{-\theta t} \right] dt \tag{2.5}$$

and

$$B_2(x) = \int_0^x \left[\left(\frac{\beta}{\theta+\beta} \right)^2 \frac{\theta^{\mu_0}}{\Gamma(\mu_0)} t^{\mu_0 - 1} e^{-\theta t} + \frac{2\beta\theta}{(\theta+\beta)^2} \frac{\theta^{\mu_1}}{\Gamma(\mu_1)} t^{\mu_1 - 1} e^{-\theta t} + \left(\frac{\theta}{\theta+\beta} \right)^2 \frac{\theta^{\mu_2}}{\Gamma(\mu_2)} t^{\mu_2 - 1} e^{-\theta t} \right] dt. \quad (2.6)$$

Putting the values of $B_0(x)$, $B_1(x)$ and $B_2(x)$ in (2.3) , we obtain the following,

$$\int_0^x \left[b_0 \frac{\theta^{\alpha_0}}{\Gamma(\alpha_0)} t^{\alpha_0 - 1} e^{-\theta t} + b_1 \frac{\theta^{\rho_0}}{\Gamma(\rho_0)} t^{\rho_0 - 1} e^{-\theta t} + b_2 \frac{\theta^{\mu_0}}{\Gamma(\mu_0)} t^{\mu_0 - 1} e^{-\theta t} \right] dt = 0, \quad (2.7)$$

$$\int_0^x \left[b_0 \frac{\theta^{\alpha_1}}{\Gamma(\alpha_1)} t^{\alpha_1 - 1} e^{-\theta t} + b_1 \frac{\theta^{\rho_1}}{\Gamma(\rho_1)} t^{\rho_1 - 1} e^{-\theta t} + b_2 \frac{\theta^{\mu_1}}{\Gamma(\mu_1)} t^{\mu_1 - 1} e^{-\theta t} \right] dt = 0 \quad (2.8)$$

and

$$\int_0^x \left[b_0 \frac{\theta^{\alpha_2}}{\Gamma(\alpha_2)} t^{\alpha_2 - 1} e^{-\theta t} + b_2 \frac{\theta^{\rho_2}}{\Gamma(\rho_2)} t^{\rho_2 - 1} e^{-\theta t} + b_2 \frac{\theta^{\mu_2}}{\Gamma(\mu_2)} t^{\mu_2 - 1} e^{-\theta t} \right] dt = 0. \quad (2.9)$$

On combining Eqs. (2.7), (2.8) and (2.9), we get

$$Fb = 0, \quad (2.10)$$

in which, F= $\begin{bmatrix} f_{\alpha_0} & f_{\rho_0} & f_{\mu_0} \\ f_{\alpha_1} & f_{\rho_1} & f_{\mu_1} \\ f_{\alpha_2} & f_{\rho_2} & f_{\mu_2} \end{bmatrix}$, b= $\begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix}$ and 0= $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ and we define $f_{\alpha_i} = \frac{\theta^{\alpha_i}}{\Gamma \alpha_i} \int_0^x t^{\alpha_i - 1} e^{-\theta t} dt, f_{\rho_i} = \frac{\theta^{\rho_i}}{\Gamma \rho_i} \int_0^x t^{\rho_i - 1} e^{-\theta t} dt$ and $f_{\mu_i} = \frac{\theta^{\mu_i}}{\Gamma \mu_i} \int_0^x t^{\mu_i - 1} e^{-\theta t} dt$ for $i = 0, 1, 2.$ Obviously det $F \neq 0$ shows that $b = 0$ and thereby we conclude that the distribution functions B_0, B_1 and B_2 are linearly independent over the set of real numbers (see, Titterington et al. 1985). In a similar way, the argument can be extended to the case of any positive integer $g(\geq 3)$ and thus the result follows. □

Result 2.2 *The cumulative distribution function (cdf) of the BMLD given in (2.1) has the following form,*

$$F(x) = \sum_{i=0}^{g} \binom{g}{i} \left(\frac{\theta}{\theta+\beta} \right)^i \left(\frac{\beta}{\theta+\beta} \right)^{g-i} \gamma_{\alpha_i}(\theta x). \quad (2.11)$$

Proof We have

$$
\begin{aligned}
F(x) &= \int_0^x f(t)\ dt \\
&= \sum_{i=0}^{g} \frac{p_i}{\Gamma(\alpha_i)} \int_0^x \theta(t\theta)^{\alpha_i - 1} e^{-\theta t} dt \\
&= \sum_{i=0}^{g} \binom{g}{i} \left(\frac{\theta}{\theta+\beta}\right)^i \left(\frac{\beta}{\theta+\beta}\right)^{g-i} \gamma_{\alpha_i}(\theta x),
\end{aligned}
$$

where $\gamma(s,t) = \int_0^t x^{s-1} e^{-x} dx$ is the lower incomplete gamma function and $\gamma_s(t) = \frac{\gamma(s,t)}{\Gamma(s)}$. □

Remark 2.1 The survival function of the *BMLD* is obtained as

$$
\overline{F}(x) = 1 - \sum_{i=0}^{g} \binom{g}{i} \left(\frac{\theta}{\theta+\beta}\right)^i \left(\frac{\beta}{\theta+\beta}\right)^{g-i} \gamma_{\alpha_i}(\theta x). \tag{2.12}
$$

Result 2.3 *The r^{th} raw moment about origin of the BMLD has been obtained as*

$$
\mu_r' = \sum_{i=0}^{g} \binom{g}{i} \left(\frac{\theta}{\theta+\beta}\right)^i \left(\frac{\beta}{\theta+\beta}\right)^{g-i} \frac{\Gamma(\alpha_i + r)}{\theta^r \Gamma(\alpha_i)};\ r = 1, 2, \cdots \tag{2.13}
$$

Proof By definition, we have

$$
\begin{aligned}
\mu_r' &= \int_0^\infty \sum_{i=0}^{g} p_i \frac{\theta^{\alpha_i}}{\Gamma(\alpha_i)} x^{\alpha_i + r - 1} e^{-\theta x} dx \\
&= \sum_{i=0}^{g} \frac{p_i \theta^{\alpha_i}}{\Gamma(\alpha_i)} \frac{\Gamma(\alpha_i + r)}{\theta^{\alpha_i + r}} \\
&= \sum_{i=0}^{g} \binom{g}{i} \left(\frac{\theta}{\theta+\beta}\right)^i \left(\frac{\beta}{\theta+\beta}\right)^{g-i} \frac{\Gamma(\alpha_i + r)}{\theta^r \Gamma(\alpha_i)}.
\end{aligned}
$$

□

Remark 2.2 Mean and variance of *BMLD* is given by

$$
E(X) = \sum_{i=0}^{g} \binom{g}{i} \left(\frac{\theta}{\theta+\beta}\right)^i \left(\frac{\beta}{\theta+\beta}\right)^{g-i} \frac{\alpha_i}{\theta} \tag{2.14}
$$

and

$$Var(X) = \frac{1}{\theta^2}\sum_{i=0}^{g}\binom{g}{i}\left(\frac{\theta}{\theta+\beta}\right)^{i}\left(\frac{\beta}{\theta+\beta}\right)^{g-i}\alpha_i\left\{1+\alpha_i - \sum_{i=0}^{g}\binom{g}{i}\left(\frac{\theta}{\theta+\beta}\right)^{i}\left(\frac{\beta}{\theta+\beta}\right)^{g-i}\alpha_i\right\}. \tag{2.15}$$

Result 2.4 *The moments of the BMLD can be calculated recursively through the relationship*

$$\mu'_{r+1} = \mu'_r \frac{\sum_{i=0}^{g}\frac{\binom{g}{i}}{\Gamma(\alpha_i)}\left(\frac{\theta}{\theta+\beta}\right)^{i}\left(\frac{\beta}{\theta+\beta}\right)^{g-i}\Gamma(\alpha_i+r+1)}{\theta\sum_{i=0}^{g}\frac{\binom{g}{i}}{\Gamma(\alpha_i)}\left(\frac{\theta}{\theta+\beta}\right)^{i}\left(\frac{\beta}{\theta+\beta}\right)^{g-i}\Gamma(\alpha_i+r)}. \tag{2.16}$$

Proof From (2.13), we have

$$\theta^r\mu'_r = \sum_{i=0}^{g}\frac{\binom{g}{i}}{\Gamma(\alpha_i)}\left(\frac{\theta}{\theta+\beta}\right)^{i}\left(\frac{\beta}{\theta+\beta}\right)^{g-i}\Gamma(\alpha_i+r)$$

and

$$\theta^{r+1}\mu'_{r+1} = \sum_{i=0}^{g}\frac{\binom{g}{i}}{\Gamma(\alpha_i)}\left(\frac{\theta}{\theta+\beta}\right)^{i}\left(\frac{\beta}{\theta+\beta}\right)^{g-i}\Gamma(\alpha_i+r+1).$$

$$\theta\mu'_{r+1}\sum_{i=0}^{g}\frac{\binom{g}{i}}{\Gamma(\alpha_i)}\left(\frac{\theta}{\theta+\beta}\right)^{i}\left(\frac{\beta}{\theta+\beta}\right)^{g-i}\Gamma(\alpha_i+r) = \mu'_r\sum_{i=0}^{g}\frac{\binom{g}{i}}{\Gamma(\alpha_i)}\left(\frac{\theta}{\theta+\beta}\right)^{i}\left(\frac{\beta}{\theta+\beta}\right)^{g-i}\Gamma(\alpha_i+r+1).$$

By rearranging the above equation, we get (2.16). □

Result 2.5 *If X has BMLD, then the moment generating function $M_X(t)$ has the following form,*

$$M_X(t) = \sum_{i=0}^{g} \binom{g}{i} \left(\frac{\theta}{\theta+\beta}\right)^i \left(\frac{\beta}{\theta+\beta}\right)^{g-i} \left(\frac{\theta}{\theta-t}\right)^{\alpha_i}.$$

Proof We have

$$\begin{aligned} M_X(t) &= E(e^{tX}) \\ &= \int_0^\infty e^{tx} f(x)dx \\ &= \sum_{i=0}^{g} \binom{g}{i} \left(\frac{\theta}{\theta+\beta}\right)^i \left(\frac{\beta}{\theta+\beta}\right)^{g-i} \frac{\theta^{\alpha_i}}{\Gamma(\alpha_i)} \int_0^\infty e^{-(\theta-t)x} x^{\alpha_i-1} dx \\ &= \sum_{i=0}^{g} \binom{g}{i} \left(\frac{\theta}{\theta+\beta}\right)^i \left(\frac{\beta}{\theta+\beta}\right)^{g-i} \left(\frac{\theta}{\theta-t}\right)^{\alpha_i}. \end{aligned}$$

□

Remark 2.3 The characteristic function of the *BMLD* is $\Phi_X(t) = M_X(it)$, where $i = \sqrt{-1}$ is the unit imaginary number.

3 Certain Measures of Reliability, Inequality, Entropy and Extropy

In this section we derived expressions for some reliability measures such as hazard rate function, reversed hazard rate function, cumulative hazard rate function, vitality function and mean residual life function associated with *BMLD*. Certain inequality measures, entropy and extropy measures are also obtained.

3.1 Reliability Properties

3.1.1 Hazard Rate Function

Let X denote a lifetime variable with cdf $F(x) = Pr(X \leq x)$ and pdf $f(x)$. Then the hazard rate function(hrf) is given by,

$$h(x) = \frac{f(x)}{\overline{F}(x)}, \tag{3.1}$$

where $\overline{F}(x) = 1 - F(x)$ is the survival function of X. That is, $h(x)dx$ represents the instantaneous chance that an individual will die in the interval $(x, x+dx)$ given that this individual is alive at age x.

3.1.2 Reversed Hazard Rate Function

Let X be a non-negative random variable representing lifetimes of individuals having absolutely continuous distribution function $F(x)$ and pdf $f(x)$. Then the reversed hazard rate function is given by

$$r(x) = \frac{f(x)}{F(x)}. \tag{3.2}$$

3.1.3 Cumulative Hazard Rate Function

Cumulative hazard rate function is the total number of failure or deaths over an interval of time, and it is defined as

$$R(x) = -\log \overline{F}(x), \text{ where } \overline{F}(x) \text{ is the survival function.} \tag{3.3}$$

Clearly $R(x)$ is a non-decreasing function of x satisfying; (a) $R(0) = 0$ and (b) $\lim_{x\to\infty} R(x) = \infty$.

Result 3.1 *If X has the BMLD with density function, cumulative distribution function and survival function given in Eqs.* (2.1), (2.11) *and* (2.12) *respectively, then*

(a) *Hazard rate function,*

$$h(x) = \frac{\sum_{i=0}^{g} \binom{g}{i} \left(\frac{\theta}{\theta+\beta}\right)^{i} \left(\frac{\beta}{\theta+\beta}\right)^{g-i} \frac{\theta^{\alpha_i}}{\Gamma(\alpha_i)} x^{\alpha_i - 1} e^{-\theta x}}{1 - \sum_{i=0}^{g} \binom{g}{i} \left(\frac{\theta}{\theta+\beta}\right)^{i} \left(\frac{\beta}{\theta+\beta}\right)^{g-i} \gamma_{\alpha_i}(\theta x)}. \tag{3.4}$$

(b) *Cumulative hazard rate function,*

$$R(x) = -\log\left[1 - \sum_{i=0}^{g} \binom{g}{i} \left(\frac{\theta}{\theta+\beta}\right)^{i} \left(\frac{\beta}{\theta+\beta}\right)^{g-i} \gamma_{\alpha_i}(\theta x)\right]. \tag{3.5}$$

(c) *Reversed hazard rate function,*

$$r(x) = \frac{\sum_{i=0}^{g} \binom{g}{i} \left(\frac{\theta}{\theta+\beta}\right)^{i} \left(\frac{\beta}{\theta+\beta}\right)^{g-i} \frac{\theta^{\alpha_i}}{\Gamma(\alpha_i)} x^{\alpha_i - 1} e^{-\theta x}}{\sum_{i=0}^{g} \binom{g}{i} \left(\frac{\theta}{\theta+\beta}\right)^{i} \left(\frac{\beta}{\theta+\beta}\right)^{g-i} \gamma_{\alpha_i}(\theta x)}. \tag{3.6}$$

Proof By using (2.1), (2.11) and (2.12) in the equations, $h(x) = \frac{f(x)}{\overline{F}(x)}$, $r(x) = \frac{f(x)}{F(x)}$ and $R(x) = -\log \overline{F}(x)$, the hazard rate function, reversed hazard rate function and cumulative hazard rate function are easily obtained.

The hazard rate function for *BMLD* is plotted for different values of parameters is given in Fig. 2. □

The graphs of the hazard function for various combination of parameters show various shapes including increasing, decreasing, bathtub shape (decreasing -stable-increasing) and upside down bathtub shape. This attractive flexibility of the *BMLD* hazard rate function highly suitable for non-monotone empirical hazard behaviours which are more likely to be encountered in real life situations.

3.1.4 Vitality Function

If X is a non-negative random variable having an absolutely continuous distribution function $F(x)$ with pdf $f(x)$. The vitality function associated with the random variable X is defined as,

$$v(x) = E[X|X > x]. \tag{3.7}$$

In the reliability context (3.7) can be interpreted as the average life span of components whose age exceeds x. It may be noted that the hazard rate reflects the risk of sudden death within a life span, where as the vitality function provides a more direct

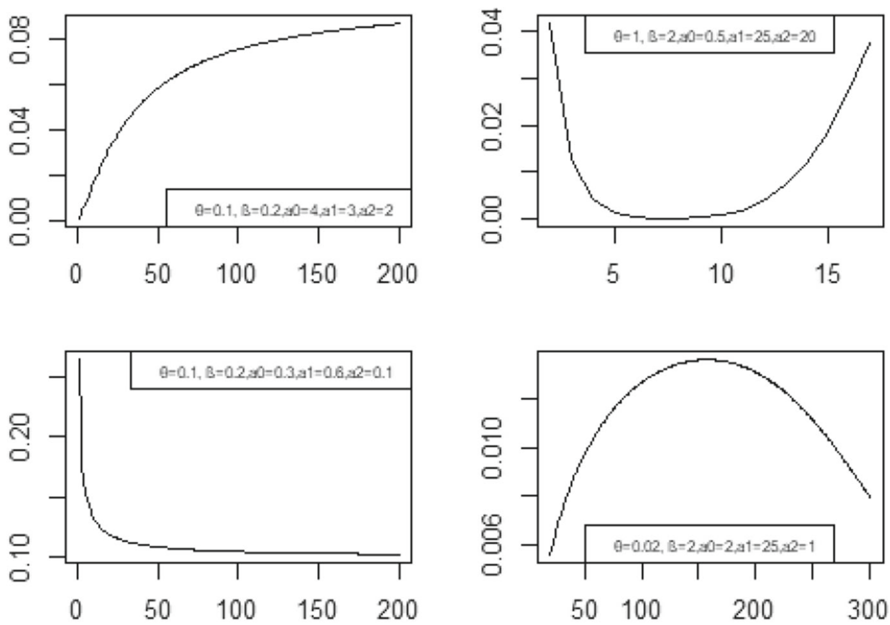

Fig. 2 The hrf of *BMLD* for $g = 2$ and different values of θ, β, α_0, α_1, α_2

measure to describe the failure pattern in the sense that it is expressed in terms of increased average life span.

Result 3.2 *The vitality function of BMLD has the following form,*

$$v(x) = \frac{\frac{1}{\theta}\sum_{i=0}^{g}\binom{g}{i}\left(\frac{\theta}{\theta+\beta}\right)^{i}\left(\frac{\beta}{\theta+\beta}\right)^{g-i}\alpha_i\Gamma_{\alpha_i+1}(\theta x)}{1-\sum_{i=0}^{g}\binom{g}{i}\left(\frac{\theta}{\theta+\beta}\right)^{i}\left(\frac{\beta}{\theta+\beta}\right)^{g-i}\gamma_{\alpha_i}(\theta x)}. \tag{3.8}$$

Proof The Eq. (3.7) can also be written as,

$$v(x) = \frac{1}{\overline{F}(x)}\int_x^{\infty} tf(t)dt. \tag{3.9}$$

Now

$$\begin{aligned}\int_x^{\infty} tf(t)dt &= \int_x^{\infty} t\sum_{i=0}^{g} p_i\frac{\theta^{\alpha_i}}{\Gamma(\alpha_i)}t^{\alpha_i-1}e^{-\theta t}dt \\ &= \frac{1}{\theta}\sum_{i=0}^{g}\frac{p_i}{\Gamma(\alpha_i)}\Gamma(\alpha_i+1,\theta x) \\ &= \frac{1}{\theta}\sum_{i=0}^{g}\binom{g}{i}\left(\frac{\theta}{\theta+\beta}\right)^{i}\left(\frac{\beta}{\theta+\beta}\right)^{g-i}\alpha_i\Gamma_{\alpha_i+1}(\theta x),\end{aligned} \tag{3.10}$$

where $\Gamma(s,t) = \int_x^{\infty} x^{s-1}e^{-x}dx$ is the upper incomplete gamma function and $\Gamma_s(t) = \frac{\Gamma(s,t)}{\Gamma(s)}$. Substituting (3.10) and (2.12) in (3.9), we get the required result. □

3.1.5 Mean Residual Life Function

Mean residual life function or remaining life expectancy function at age x is defined to be the expected remaining life given survival to age x. For a continuous random variable X, with $E(X) < \infty$, then the mean residual life function (MRLF) is defined as the Borel measurable function,

$$\begin{aligned}m(x) &= E[X-x|X>x] \\ &= \frac{1}{\overline{F}(x)}\int_x^{\infty}\overline{F}(t)dt.\end{aligned} \tag{3.11}$$

MRLF is sometimes considered as a superior measure to describe the failure pattern as compared to hazard rate function since the former focuses attention on the average lifetime over a period of time while the latter on instantaneous failure at a point of time. Also MRLF can be expressed in terms of vitality function. That is, Eq. (3.9) can also be written as

$$\begin{aligned} v(x) &= \frac{1}{\overline{F}(x)} \int_x^{\infty} \overline{F}(t)dt + x \\ &= m(x) + x. \end{aligned} \tag{3.12}$$

Result 3.3 *The mean residual life function of BMLD has the following form,*

$$m(x) = \frac{\frac{1}{\theta}\sum_{i=0}^{g}\binom{g}{i}\left(\frac{\theta}{\theta+\beta}\right)^{i}\left(\frac{\beta}{\theta+\beta}\right)^{g-i}\alpha_i\Gamma_{\alpha_i+1}(\theta x)}{1-\sum_{i=0}^{g}\binom{g}{i}\left(\frac{\theta}{\theta+\beta}\right)^{i}\left(\frac{\beta}{\theta+\beta}\right)^{g-i}\gamma_{\alpha_i}(\theta x)} - x. \tag{3.13}$$

Proof Substituting (3.8) in (3.12), we get (3.13). □

3.2 Inequality Measures

Lorenz and Bonferroni curves are income inequality measures that are widely useful and applicable to some other areas including reliability, demography, medicine and insurance (see, Bonferroni 1930). Also Zenga curve introduced by Zenga (2007) is another widely used inequality measure. In this section, we will derive Lorenz, Bonferroni and Zenga curves for the *BMLD*. The Lorenz, Bonferroni and Zenga curves are respectively given as

$L_F(x) = \frac{\int_0^x tf(t)dt}{E(X)}$, $B_F(x) = \frac{\int_0^x tf(t)dt}{F(X)E(X)}$ and $A_F(x) = 1 - \frac{\mu^-(x)}{\mu^+(x)}$, where $\mu^-(x) = \frac{\int_0^x tf(t)dt}{F(X)}$ and $\mu^+(x) = \frac{\int_x^{\infty} tf(t)dt}{\overline{F}(X)}$.

Result 3.4 *If X has the BMLD with density function, cumulative distribution function and survival function given in Eqs.* (2.1), (2.11) *and* (2.12) *respectively, then*

(a) *Lorenz curve,*

$$L_F(x) = \frac{\sum_{i=0}^{g}\binom{g}{i}\left(\frac{\theta}{\theta+\beta}\right)^{i}\left(\frac{\beta}{\theta+\beta}\right)^{g-i}\alpha_i\gamma_{\alpha_i+1}(\theta x)}{\sum_{i=0}^{g}\binom{g}{i}\left(\frac{\theta}{\theta+\beta}\right)^{i}\left(\frac{\beta}{\theta+\beta}\right)^{g-i}\alpha_i}. \tag{3.14}$$

(b) *Bonferroni curve,*

$$B_F(x) = \frac{\sum_{i=0}^{g}\binom{g}{i}\left(\frac{\theta}{\theta+\beta}\right)^{i}\left(\frac{\beta}{\theta+\beta}\right)^{g-i}\alpha_i\gamma_{\alpha_i+1}(\theta x)}{\left\{\sum_{i=0}^{g}\binom{g}{i}\left(\frac{\theta}{\theta+\beta}\right)^{i}\left(\frac{\beta}{\theta+\beta}\right)^{g-i}\alpha_i\right\}\left\{\sum_{i=0}^{g}\binom{g}{i}\left(\frac{\theta}{\theta+\beta}\right)^{i}\left(\frac{\beta}{\theta+\beta}\right)^{g-i}\gamma_{\alpha_i}(\theta x)\right\}}. \quad (3.15)$$

(c) *Zenga curve,*

$$A_F(x) = 1 - \left\{\frac{\sum_{i=0}^{g}\binom{g}{i}\left(\frac{\theta}{\theta+\beta}\right)^{i}\left(\frac{\beta}{\theta+\beta}\right)^{g-i}\alpha_i\gamma_{\alpha_i+1}(\theta x)}{\sum_{i=0}^{g}\binom{g}{i}\left(\frac{\theta}{\theta+\beta}\right)^{i}\left(\frac{\beta}{\theta+\beta}\right)^{g-i}\gamma_{\alpha_i}(\theta x)} \times \frac{\left(1-\sum_{i=0}^{g}\binom{g}{i}\left(\frac{\theta}{\theta+\beta}\right)^{i}\left(\frac{\beta}{\theta+\beta}\right)^{g-i}\gamma_{\alpha_i}(\theta x)\right)}{\sum_{i=0}^{g}\binom{g}{i}\left(\frac{\theta}{\theta+\beta}\right)^{i}\left(\frac{\beta}{\theta+\beta}\right)^{g-i}\alpha_i\Gamma_{\alpha_i+1}(\theta x)}\right\}. \quad (3.16)$$

Proof

(a) By definition

$$L_F(x) = \frac{\int_0^x tf(t)dt}{E(X)}. \quad (3.17)$$

Now

$$\begin{aligned}\int_0^x tf(t)dt &= \frac{1}{\theta}\sum_{i=0}^{g}\frac{p_i}{\Gamma(\alpha_i)}\gamma(\alpha_i+1,\theta x)\\ &= \frac{1}{\theta}\sum_{i=0}^{g}\binom{g}{i}\left(\frac{\theta}{\theta+\beta}\right)^{i}\left(\frac{\beta}{\theta+\beta}\right)^{g-i}\alpha_i\gamma_{\alpha_i+1}(\theta x).\end{aligned} \quad (3.18)$$

By using (3.18) and (2.14) in (3.17), we get (3.14)

(b) By definition

$$B_F(x) = \frac{\int_0^x tf(t)dt}{F(X)E(X)}.$$

By using (3.18), (2.14) and (2.11), we get (3.15)

(c) By definition

$$A(x) = 1 - \frac{\mu^-(x)}{\mu^+(x)}. \tag{3.19}$$

By using (3.18) and (2.11), we get $\mu^-(x)$ and by definition $\mu^+(x) = \frac{\int_x^\infty tf(t)dt}{\overline{F}(X)} = v(x)$. which is given in (3.8). Substituting $\mu^-(x)$ and $\mu^+(x)$ in (3.19), we get (3.16). □

3.3 Entropy

Here we derive the expressions for Rényi Entropy and Havrda-Charvát-Tsallis (HCT) entropy. We are also deriving the expression for a recently developed uncertainty measure, namely extropy and its residual version. For mathematical simplicity these results are derived for $g = 2$.

The concept of entropy was introduced and extensively studied by Shannon (1948). Let X be a non-negative random variable admitting an absolutely continuous cdf $F(x)$ and with pdf $f(x)$. Then the Shannon's entropy associated with X is defined as $H(X) = -\int_0^\infty f(x)\ \log f(x)\ dx$. It gives the expected uncertainty contained in $f(x)$ about the predictability of an outcome of X.

Several generalizations of Shannon's entropy have been put forward by researchers. A generalization which has received much attention subsequently is due to Rényi (1959). The Rényi's entropy of order v is defined as

$$H^v(X) = \frac{1}{1-v}\log\int_0^\infty f^v(x)\ dx, \text{ for } v > 0, v \neq 1.$$

Another important generalization of Shannon's entropy is the Havrda-Charvát-Tsallis (HCT) entropy. It was introduced by Havrda and Charvát (1967) and further developed by Tsallis (1988) and is given by,

$$H^\xi(X) = \frac{1}{\xi - 1}\left(1 - \int_0^\infty f^\xi(x)\ dx\right), \text{ for } \xi > 0, \xi \neq 1.$$

Result 3.5 *The R ényi entropy function for BMLD has the following* form,

$$H^{v}(x)=\frac{1}{1-v}\log\left\{\left(\frac{\beta^{2}\theta^{\alpha_0}}{(\theta+\beta)^{2}\Gamma(\alpha_0)}\right)^{v}\sum_{j=0}^{v}\binom{v}{j}\sum_{k=0}^{j}\binom{j}{k}\left(\frac{2\theta^{\alpha_1-\alpha_2-1}\Gamma(\alpha_2)}{\Gamma(\alpha_1)}\right)^{k}\right.$$
$$\left(\frac{\theta^{\alpha_2-\alpha_0+2}\Gamma(\alpha_0)}{\beta^{2}\Gamma(\alpha_2)}\right)^{j}$$
$$\left.\times\frac{\Gamma\big((\alpha_1-\alpha_2)k+(\alpha_2-\alpha_0)j+(\alpha_0-1)v+1\big)}{(v\theta)^{(\alpha_1-\alpha_2)k+(\alpha_2-\alpha_0)j+(\alpha_0-1)v+1}}\right\}. \tag{3.20}$$

Proof Using the definition of Rényi entropy, we have

$$\begin{aligned}H^{v}(x)=&\frac{1}{(1-v)}\log\int_0^\infty\left(\frac{1}{(\theta+\beta)^{2}}\right)^{v}\left\{\frac{\beta^{2}\theta^{\alpha_0}x^{\alpha_0-1}}{\Gamma(\alpha_0)}\right.\\&\left.+\frac{2\theta^{\alpha_1+1}\beta x^{\alpha_1-1}\Gamma(\alpha_2)+\theta^{\alpha_2+2}x^{\alpha_2-1}\Gamma(\alpha_1)}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\right\}^{v}e^{-v\theta x}dx\\=&\frac{1}{(1-v)}\log\left\{\frac{1}{(\theta+\beta)^{2v}}\left(\frac{\beta^{2}\theta^{\alpha_0}}{\Gamma(\alpha_0)}\right)^{v}\sum_{j=0}^{v}\binom{v}{j}\right.\\&\left.\int_0^\infty\left\{\frac{2\theta^{\alpha_1-\alpha_0+1}\Gamma(\alpha_0)x^{\alpha_1-\alpha_0}}{\beta\Gamma(\alpha_1)}+\frac{\theta^{\alpha_2-\alpha_0+2}\Gamma(\alpha_0)x^{\alpha_2-\alpha_0}}{\beta^{2}\Gamma(\alpha_2)}\right\}^{j}x^{(\alpha_0-1)v}e^{-v\theta x}dx\right\}\end{aligned}$$

$$=\frac{1}{(1-v)}\log\left\{\left(\frac{\beta^{2}\theta^{\alpha_0}}{(\theta+\beta)^{2}\Gamma(\alpha_0)}\right)^{v}\sum_{j=0}^{v}\binom{v}{j}\sum_{k=0}^{j}\binom{j}{k}\left(\frac{2\theta^{\alpha_1-\alpha_2-1}\Gamma(\alpha_2)}{\Gamma(\alpha_1)}\right)^{k}\right.$$
$$\left.\left(\frac{\theta^{\alpha_2-\alpha_0+2}\Gamma(\alpha_0)}{\beta^{2}\Gamma(\alpha_2)}\right)^{j}\int_0^\infty x^{(\alpha_1-\alpha_2)k+(\alpha_2-\alpha_0)j+(\alpha_0-1)v}e^{-v\theta x}dx\right\}$$
$$=\frac{1}{1-v}\log\left\{\left(\frac{\beta^{2}\theta^{\alpha_0}}{(\theta+\beta)^{2}\Gamma(\alpha_0)}\right)^{v}\sum_{j=0}^{v}\binom{v}{j}\sum_{k=0}^{j}\binom{j}{k}\left(\frac{2\theta^{\alpha_1-\alpha_2-1}\Gamma(\alpha_2)}{\Gamma(\alpha_1)}\right)^{k}\left(\frac{\theta^{\alpha_2-\alpha_0+2}\Gamma(\alpha_0)}{\beta^{2}\Gamma(\alpha_2)}\right)^{j}\right.$$
$$\left.\frac{\Gamma((\alpha_1-\alpha_2)k+(\alpha_2-\alpha_0)j+(\alpha_0-1)v+1)}{(v\theta)^{(\alpha_1-\alpha_2)k+(\alpha_2-\alpha_0)j+(\alpha_0-1)v+1}}\right\}.$$

□

Remark 3.1 When $v\to 1$ in (3.20), it reduces to Shannon entropy.

Result 3.6 The Havrda-Charv*á*t-Tsallis entropy of order ρ, for *BMLD* has the following form,

$$H^{\xi}(x) = \frac{1}{(\xi-1)}\left\{1 - \left(\frac{\beta^2\theta^{\alpha_0}}{(\theta+\beta)^2\Gamma(\alpha_0)}\right)^{\xi}\sum_{j=0}^{\xi}\binom{\xi}{j}\sum_{k=0}^{j}\binom{j}{k}\left(\frac{2\theta^{\alpha_1-\alpha_2-1}\Gamma(\alpha_2)}{\Gamma(\alpha_1)}\right)^{k}\right.$$
$$\left.\left(\frac{\theta^{\alpha_2-\alpha_0+2}\Gamma(\alpha_0)}{\beta^2\Gamma(\alpha_2)}\right)^{j}\frac{\Gamma((\alpha_1-\alpha_2)k+(\alpha_2-\alpha_0)j+(\alpha_0-1)\xi+1)}{(\xi\theta)^{(\alpha_1-\alpha_2)k+(\alpha_2-\alpha_0)j+(\alpha_0-1)\xi+1}}\right\}. \tag{3.21}$$

Proof Proof is similar to that of Result 3.5 and hence omitted. □

3.4 Extropy

Recently, Lad et al. (2015) defined statistically the term extropy as a potential measure of uncertainty, an alternative measure of Shannon entropy. For a random variable X, its extropy is defined as

$$J(X) = -\frac{1}{2}\int_{0}^{\infty} f^2(x)dx. \tag{3.22}$$

In statistical point of view, the term extropy is used to score the forecasting distributions under the total log scoring rule.

A serious difficulty involved in the application of Shannon's entropy is that, it is not applicable to a system which has survived for some units of time. In this situation, Ebrahimi (1996) proposed the concept of residual entropy. As in the scenario of introducing the concept of residual entropy, Qiu and Jia (2018) introduced residual extropy to measure the residual uncertainty of a random variable. For a random variable X, its residual extropy is defined as (see, Qiu and Jia 2018)

$$J(f;t) = \frac{-1}{2\overline{F}^2(t)}\int_{t}^{\infty} f^2(x)dx, \tag{3.23}$$

Result 3.7 *The extropy function for BMLD has the following form,*

$$J(X) = \frac{-1}{2(\theta+\beta)^4}\left\{\frac{\Gamma(2\alpha_0-1)}{\Gamma^2(\alpha_0)}\beta^4\theta 2^{1-2\alpha_0} + \frac{\Gamma(2\alpha_1-1)}{\Gamma^2(\alpha_1)}\beta^2\theta^3 2^{3-2\alpha_1}\right.$$
$$+\frac{\Gamma(2\alpha_2-1)}{\Gamma^2(\alpha_2)}\theta^5 2^{1-2\alpha_2} + \frac{\Gamma(\alpha_0+\alpha_1-1)}{\Gamma(\alpha_0)\Gamma(\alpha_1)}\beta^3\theta^2 2^{3-\alpha_0-\alpha_1} \tag{3.24}$$
$$\left.+\frac{\Gamma(\alpha_1+\alpha_2-1)}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\beta\theta^4 2^{3-\alpha_1-\alpha_2} + \frac{\Gamma(\alpha_0+\alpha_2-1)}{\Gamma(\alpha_0)\Gamma(\alpha_2)}\beta^2\theta^3 2^{2-\alpha_0-\alpha_2}\right\}.$$

Proof By definition of $J(X)$,

$$\begin{aligned} J(X) =& \frac{-1}{2(\theta+\beta)^4}\int_0^\infty \left\{\frac{\beta^2\theta^{\alpha_0}x^{\alpha_0-1}}{\Gamma(\alpha_0)}+\frac{2\beta\theta^{\alpha_1+1}x^{\alpha_1-1}}{\Gamma(\alpha_1)}+\frac{\theta^{\alpha_2+2}x^{\alpha_2-1}}{\Gamma(\alpha_2)}\right\}^2 e^{-2\theta x}dx \\ =& \frac{-1}{2(\theta+\beta)^4}\left\{\frac{\beta^4\theta^{2\alpha_0}}{\Gamma^2(\alpha_0)}\int_0^\infty x^{2\alpha_0-2}e^{-2\theta x}dx+\frac{4\beta^2\theta^{2\alpha_1+2}}{\Gamma^2(\alpha_1)}\int_0^\infty x^{2\alpha_1-2}e^{-2\theta x}dx\right. \\ &+\frac{\theta^{2\alpha_2+4}}{\Gamma^2(\alpha_2)}\int_0^\infty x^{2\alpha_2-2}e^{-2\theta x}dx+\frac{4\beta^3\theta^{\alpha_0+\alpha_1+1}}{\Gamma(\alpha_0)\Gamma(\alpha_1)}\int_0^\infty x^{\alpha_0+\alpha_1-2}e^{-2\theta x}dx \\ &\left.+\frac{4\beta\theta^{\alpha_1+\alpha_2+3}}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_0^\infty x^{\alpha_1+\alpha_2-2}e^{-2\theta x}dx+\frac{2\beta^2\theta^{\alpha_0+\alpha_2+2}}{\Gamma(\alpha_0)\Gamma(\alpha_2)}\int_0^\infty x^{\alpha_0+\alpha_2-2}e^{-2\theta x}dx\right\}. \end{aligned} \tag{3.25}$$

By simplifying (3.25), we get (3.24). □

Result 3.8 *The residual extropy function for BMLD has the following form,*

$$\begin{aligned} J(f;t) =& \frac{-1}{2\left(1-\sum_{i=0}^{2}\binom{2}{i}\left(\frac{\theta}{\theta+\beta}\right)^i\left(\frac{\beta}{\theta+\beta}\right)^{2-i}\gamma_{\alpha_i}(\theta t)\right)^2(\theta+\beta)^4}\left\{\frac{\Gamma(2\alpha_0-1,2\theta t)}{\Gamma^2(\alpha_0)}\beta^4\theta 2^{1-2\alpha_0}\right. \\ &+\frac{\Gamma(2\alpha_1-1,2\theta t)}{\Gamma^2(\alpha_1)}\beta^2\theta^3 2^{3-2\alpha_1}+\frac{\Gamma(2\alpha_2-1,2\theta t)}{\Gamma^2(\alpha_2)}\theta^5 2^{1-2\alpha_2} \\ &+\frac{\Gamma(\alpha_0+\alpha_1-1,2\theta t)}{\Gamma(\alpha_0)\Gamma(\alpha_1)}\beta^3\theta^2 2^{3-\alpha_0-\alpha_1}+\frac{\Gamma(\alpha_1+\alpha_2-1,2\theta t)}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\beta\theta^4 2^{3-\alpha_1-\alpha_2} \\ &\left.+\frac{\Gamma(\alpha_0+\alpha_2-1,2\theta t)}{\Gamma(\alpha_0)\Gamma(\alpha_2)}\beta^2\theta^3 2^{2-\alpha_0-\alpha_2}\right\}, \text{ for } g=2. \end{aligned} \tag{3.26}$$

Proof Proof is similar to that of Result 3.7 and hence omitted. □

4 Estimation and Inference

Estimation of unknown parameters of a distribution is essential in all areas of statistics. In this section, first we obtain the maximum likelihood estimates (MLEs) of the parameters of *BMLD* for a given random sample. The Fisher information matrix is also computed in this section for the interval estimation. For mathematical simplicity all these inferences are made for $g = 2$.

4.1 Maximum Likelihood Estimation

The method of maximum likelihood is the most frequently used technique for parameter estimation. It's success stems from its many desirable properties including consistency, asymptotic efficiency, invariance property as well as intuitive appeal.

Let $X_1, X_2, ..., X_n$ be observed values from the *BMLD* with unknown parameter vector $\Theta = \left(\theta, \beta, \alpha_0, \alpha_1, \alpha_2\right)$. The likelihood function is given by

$$
\begin{aligned}
l(\Theta) &= \prod_{i=1}^{n} f_i\left(x; \theta, \beta, \alpha_0, \alpha_1, \alpha_2\right) \\
&= \left(\frac{1}{(\theta+\beta)^2}\right)^n e^{-\theta \sum_{i=1}^{n} x_i} \prod_{i=1}^{n} \left(\frac{\beta^2 \theta^{\alpha_0} x_i^{\alpha_0 - 1}}{\Gamma(\alpha_0)} + \frac{2\beta\theta^{\alpha_1+1} x_i^{\alpha_1-1}}{\Gamma(\alpha_1)} + \frac{\theta^{\alpha_2+2} x_i^{\alpha_2-1}}{\Gamma(\alpha_2)}\right).
\end{aligned}
$$

The partial derivatives of $\log l(\Theta)$ with respect to the parameters are given by

$$
\begin{aligned}
\frac{\partial \log l}{\partial \theta} &= -\frac{2n}{\theta+\beta} - \sum_{i=1}^{n} x_i + \frac{1}{\theta} \sum_{i=1}^{n} \left(\frac{\alpha_0 A_i + (\alpha_1+1) B_i + (\alpha_2+2) C_i}{A_i + B_i + C_i}\right), \\
\frac{\partial \log l}{\partial \beta} &= -\frac{2n}{\theta+\beta} + \frac{1}{\beta} \sum_{i=1}^{n} \left(\frac{2A_i + B_i}{A_i + B_i + C_i}\right), \\
\frac{\partial \log l}{\partial \alpha_0} &= \sum_{i=1}^{n} \left(\frac{A_i\left(\log(\theta x_i) - \psi(\alpha_0)\right)}{A_i + B_i + C_i}\right), \\
\frac{\partial \log l}{\partial \alpha_1} &= \sum_{i=1}^{n} \left(\frac{B_i\left(\log(\theta x_i) - \psi(\alpha_1)\right)}{A_i + B_i + C_i}\right)
\end{aligned} \tag{4.1}
$$

and

$$
\frac{\partial \log l}{\partial \alpha_2} = \sum_{i=1}^{n} \left(\frac{C_i\left(\log(\theta x_i) - \psi(\alpha_2)\right)}{A_i + B_i + C_i}\right), \tag{4.2}
$$

where $A_i = \frac{\beta^2 \theta^{\alpha_0} x_i^{\alpha_0-1}}{\Gamma(\alpha_0)}$, $B_i = \frac{2\beta\theta^{\alpha_1+1} x_i^{\alpha_1-1}}{\Gamma(\alpha_1)}$ and $C_i = \frac{\theta^{\alpha_2+2} x_i^{\alpha_2-1}}{\Gamma(\alpha_2)}$.

The MLE of the parameters $\Theta = \left(\theta, \beta, \alpha_0, \alpha_1, \alpha_2\right)$ are obtained by solving the equations $\frac{\partial \log l}{\partial \theta} = 0$, $\frac{\partial \log l}{\partial \beta} = 0$, $\frac{\partial \log l}{\partial \alpha_0} = 0$, $\frac{\partial \log l}{\partial \alpha_1} = 0$, $\frac{\partial \log l}{\partial \alpha_2} = 0$ simultaneously. This can only be achieved by numerical optimization technique such as the Newton-Raphson method and Fisher's scoring algorithm using mathematical packages like R, Mathematica etc. To avoid local minima problem, we first obtain the moment estimators of the parameters of *BMLD* and setting these estimators as the initial values to obtain MLEs of the parameters of *BMLD*.

4.2 Fisher Information Matrix

In order to determine the confidence interval for the parameters of *BMLD*, we need to find the expected Fisher information matrix $I(\Theta)$. The expected Fisher information matrix of *BMLD* is given by,

$$I(\Theta)=\begin{pmatrix} -E\left(\frac{\partial^2\log l}{\partial\theta^2}\right) & -E\left(\frac{\partial^2\log l}{\partial\theta\partial\beta}\right) & -E\left(\frac{\partial^2\log l}{\partial\theta\partial\alpha_0}\right) & -E\left(\frac{\partial^2\log l}{\partial\theta\partial\alpha_1}\right) & -E\left(\frac{\partial^2\log l}{\partial\theta\partial\alpha_2}\right)\\ -E\left(\frac{\partial^2\log l}{\partial\beta\partial\theta}\right) & -E\left(\frac{\partial^2\log l}{\partial\beta^2}\right) & -E\left(\frac{\partial^2\log l}{\partial\beta\partial\alpha_0}\right) & -E\left(\frac{\partial^2\log l}{\partial\beta\partial\alpha_1}\right) & -E\left(\frac{\partial^2\log l}{\partial\beta\partial\alpha_2}\right)\\ -E\left(\frac{\partial^2\log l}{\partial\alpha_0\partial\theta}\right) & -E\left(\frac{\partial^2\log l}{\partial\alpha_0\partial\beta}\right) & -E\left(\frac{\partial^2\log l}{\partial\alpha_0^2}\right) & -E\left(\frac{\partial^2\log l}{\partial\alpha_0\partial\alpha_1}\right) & -E\left(\frac{\partial^2\log l}{\partial\alpha_0\partial\alpha_2}\right)\\ -E\left(\frac{\partial^2\log l}{\partial\alpha_1\partial\theta}\right) & -E\left(\frac{\partial^2\log l}{\partial\alpha_1\partial\beta}\right) & -E\left(\frac{\partial^2\log l}{\partial\alpha_1\partial\alpha_0}\right) & -E\left(\frac{\partial^2\log l}{\partial\alpha_1^2}\right) & -E\left(\frac{\partial^2\log l}{\partial\alpha_1\partial\alpha_2}\right)\\ -E\left(\frac{\partial^2\log l}{\partial\alpha_2\partial\theta}\right) & -E\left(\frac{\partial^2\log l}{\partial\alpha_2\partial\beta}\right) & -E\left(\frac{\partial^2\log l}{\partial\alpha_2\partial\alpha_0}\right) & -E\left(\frac{\partial^2\log l}{\partial\alpha_2\partial\alpha_1}\right) & -E\left(\frac{\partial^2\log l}{\partial\alpha_2^2}\right) \end{pmatrix}$$

The expected Fisher information can be approximated by the observed Fisher information matrix $J(\widehat{\Theta})$ given by,

$$J(\widehat{\Theta})=\begin{pmatrix} -\frac{\partial^2\log l}{\partial\theta^2} & -\frac{\partial^2\log l}{\partial\theta\partial\beta} & -\frac{\partial^2\log l}{\partial\theta\partial\alpha_0} & -\frac{\partial^2\log l}{\partial\theta\partial\alpha_1} & -\frac{\partial^2\log l}{\partial\theta\partial\alpha_2}\\ -\frac{\partial^2\log l}{\partial\beta\partial\theta} & -\frac{\partial^2\log l}{\partial\beta^2} & -\frac{\partial^2\log l}{\partial\beta\partial\alpha_0} & -\frac{\partial^2\log l}{\partial\beta\partial\alpha_1} & -\frac{\partial^2\log l}{\partial\beta\partial\alpha_2}\\ -\frac{\partial^2\log l}{\partial\alpha_0\partial\theta} & -\frac{\partial^2\log l}{\partial\alpha_0\partial\beta} & -\frac{\partial^2\log l}{\partial\alpha_0^2} & -\frac{\partial^2\log l}{\partial\alpha_0\partial\alpha_1} & -\frac{\partial^2\log l}{\partial\alpha_0\partial\alpha_2}\\ -\frac{\partial^2\log l}{\partial\alpha_1\partial\theta} & -\frac{\partial^2\log l}{\partial\alpha_1\partial\beta} & -\frac{\partial^2\log l}{\partial\alpha_1\partial\alpha_0} & -\frac{\partial^2\log l}{\partial\alpha_1^2} & -\frac{\partial^2\log l}{\partial\alpha_1\partial\alpha_2}\\ -\frac{\partial^2\log l}{\partial\alpha_2\partial\theta} & -\frac{\partial^2\log l}{\partial\alpha_2\partial\beta} & -\frac{\partial^2\log l}{\partial\alpha_2\partial\alpha_0} & -\frac{\partial^2\log l}{\partial\alpha_2\partial\alpha_1} & -\frac{\partial^2\log l}{\partial\alpha_2^2} \end{pmatrix}$$

That is,

$$\lim_{n\to\infty}\frac{1}{n}J(\widehat{\Theta})=I(\Theta).$$

For large n, the following approximation can be used,

$$J(\widehat{\Theta})=nI(\Theta)$$

The elements of $J(\widehat{\Theta})$ are given in APPENDIX.

4.3 Asymptotic Confidence Interval

Here we present the asymptotic confidence intervals for the parameters of *BMLD*. Let $\widehat{\Theta}=\left(\widehat{\theta},\widehat{\beta},\widehat{\alpha_0},\widehat{\alpha_1},\widehat{\alpha_2}\right)$ be the maximum likelihood estimator of

$\Theta = \left(\theta, \beta, \alpha_0, \alpha_1, \alpha_2\right)$. Under the usual regularity conditions and that the parameters are in the interior of the parameter space, but not on the boundary, we have $\sqrt{n}(\Theta - \widehat{\Theta}) \xrightarrow{d} N_2(\underline{0}, I^{-1}(\Theta))$, where $I(\Theta)$ is the expected Fisher information matrix. The asymptotic behaviour is still valid if $I(\Theta)$ is replaced by the observed Fisher information matrix $J(\widehat{\Theta})$. The multivariate normal distribution, $N_5\left(\underline{0}, I^{-1}(\Theta)\right)$ with mean vector $\underline{0} = \left(0,0,0,0,0\right)^{\tau}$ can be used to construct confidence interval for the parameters. The approximate $100(1-\varphi)\%$ two-sided confidence intervals for $\theta, \beta, \alpha_0, \alpha_1,$ and α_2 are respectively given by, $\widehat{\theta} \pm Z_{\frac{\varphi}{2}}\sqrt{I_{\theta\theta}^{-1}(\hat{\theta})}$, $\widehat{\beta} \pm Z_{\frac{\varphi}{2}}\sqrt{I_{\beta\beta}^{-1}(\hat{\beta})}$, $\widehat{\alpha_0} \pm Z_{\frac{\varphi}{2}}\sqrt{I_{\alpha_0\alpha_0}^{-1}(\hat{\alpha}_0)}$, $\widehat{\alpha_1} \pm Z_{\frac{\varphi}{2}}\sqrt{I_{\alpha_1\alpha_1}^{-1}(\hat{\alpha}_1)}$ and $\widehat{\alpha_2} \pm Z_{\frac{\varphi}{2}}\sqrt{I_{\alpha_2\alpha_2}^{-1}(\hat{\alpha}_2)}$, where $I_{\theta\theta}^{-1}(\hat{\theta})$, $I_{\beta\beta}^{-1}(\hat{\beta})$, $I_{\alpha_0\alpha_0}^{-1}(\hat{\alpha}_0)$, $I_{\alpha_1\alpha_1}^{-1}(\hat{\alpha}_1)$, $I_{\alpha_2\alpha_2}^{-1}(\hat{\alpha}_2)$ are diagonal elements of $J^{-1}(\widehat{\Theta})$ and $Z_{\frac{\varphi}{2}}$ is the upper $\frac{\varphi^{th}}{2}$ percentile of a standard normal distribution.

5 Simulation Study

Here we perform a simulation study to investigate the performance of maximum likelihood estimators of parameters of *BMLD*. As the model is a general model, we take $g = 2$ in (2.1) and do the Monte Carlo Simulation. The estimates were calculated for true values of parameters ($\theta = 1.5$, $\beta = 3$, $\alpha_0 = 0.6$, $\alpha_1 = 1.9$ and $\alpha_2 = 1.7$) and ($\theta = 0.5$, $\beta = 0.01$, $\alpha_0 = 1.5$, $\alpha_1 = 1.3$ and $\alpha_2 = 1$) for N = 1000 samples of sizes 25,50,100,200,400 and 800 and the following quantities are computed.

1. Mean of the MLEs, $\widehat{\Theta}$ of parameters $\Theta = \left(\theta, \beta, \alpha_0, \alpha_1, \alpha_2\right)$,
$$\widehat{\Theta} = \frac{1}{N}\sum_{i=1}^{N}\widehat{\Theta}_i.$$
2. Average absolute bias of MLEs of parameters,
$$Bias(\Theta) = \frac{1}{N}\sum_{i=1}^{N}(\widehat{\Theta}_i - \Theta).$$
3. Root Mean Square Error (RMSE) of MLEs of parameters:
$$RMSE(\Theta) = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(\widehat{\Theta}_i - \Theta)^2}.$$

The simulation results are presented in Table 1. From Table 1, one can infer that estimates are quite stable and more precisely close to the true parameter values.

Table 1 The simulation results for the corresponding values of the parameters

$\theta = 1.5,\ \beta = 3,\ \alpha_0 = 0.6,\ \alpha_1 = 1.9, \alpha_2 = 1.7$						$\theta = 0.5,\ \beta = 0.01,\ \alpha_0 = 1.5,\ \alpha_1 = 1.3,\ \alpha_2 = 1$					
Parameter	n	Mean	Bias	MSE	RMSE	Parameter	n	Mean	Bias	MSE	RMSE
θ	25	3.5678	2.0678	12.586	3.5477	θ	25	1.0523	0.5523	1.5212	1.2334
	50	2.5944	1.0944	3.775	1.9429		50	0.7745	0.2745	0.3957	0.6291
	100	2.0427	0.5427	0.9675	0.9836		100	0.6482	0.1482	0.1574	0.3967
	200	1.8168	0.3168	0.4144	0.6437		200	0.5993	0.0993	0.0838	0.2894
	400	1.7195	0.2194	0.2157	0.4645		400	0.5622	0.0622	0.0426	0.2063
	800	1.6273	0.1273	0.0927	0.3045		800	0.5512	0.0512	0.0266	0.1538
β	25	4.7197	1.7197	20.1292	4.4866	β	25	0.3412	0.3312	3.122	1.7669
	50	3.8153	0.8153	8.1403	2.8531		50	0.2327	0.2227	0.8919	0.9444
	100	3.4461	0.4461	6.9499	2.6446		100	0.1827	0.1727	0.5446	0.738
	200	3.0629	0.0629	2.7046	1.6446		200	0.1856	0.1756	0.4152	0.6444
	400	2.8998	0.1002	1.5622	1.2499		400	0.1267	0.1167	0.2748	0.5242
	800	2.8321	0.1679	0.9892	0.9946		800	0.0915	0.0815	0.1439	0.3795
α_0	25	0.989	0.389	2.3613	1.5366	α_0	25	2.6739	1.1739	19.707	4.4393
	50	0.7040	0.1040	0.234	0.4837		50	2.161	0.6609	5.7592	2.3998
	100	0.6309	0.0309	0.0716	0.2676		100	1.9174	0.4174	4.1005	2.025
	200	0.5934	0.0066	0.0224	0.143		200	1.8169	0.3168	2.353	1.5339
	400	0.5817	0.0018	0.0119	0.1094		400	1.7047	0.2047	1.2168	1.1031
	800	0.5822	0.0017	0.0068	0.0827		800	1.6966	0.1966	1.0443	1.0219
α_1	25	3.9737	2.0737	18.3713	4.2862	α_1	25	2.3612	1.0616	10.3053	3.2102
	50	3.0134	1.1124	5.9254	2.4342		50	1.9681	0.6681	3.9167	1.9791
	100	2.4548	0.5548	1.5252	1.2349		100	1.7994	0.4994	1.474	1.2141
	200	2.2077	0.3077	0.7493	0.8656		200	1.6905	0.3905	0.8841	0.9403
	400	2.0876	0.1876	0.3876	0.6226		400	1.6179	0.3179	0.561	0.749

Table 1 continued

$\theta = 1.5,\ \beta = 3,\ \alpha_0 = 0.6,\ \alpha_1 = 1.9, \alpha_2 = 1.7$						$\theta = 0.5,\ \beta = 0.01,\ \alpha_0 = 1.5,\ \alpha_1 = 1.3,\ \alpha_2 = 1$					
Parameter	n	Mean	Bias	MSE	RMSE	Parameter	n	Mean	Bias	MSE	RMSE
	800	1.9883	0.0883	0.1811	0.4256		800	1.5899	0.2899	0.4218	0.6495
α_2	25	3.9861	2.2861	28.4488	5.3337	α_2	25	1.524	0.524	4.0315	2.0079
	50	3.1087	1.4087	13.4599	3.6688		50	1.2458	0.2458	1.0205	1.0102
	100	2.3204	0.6204	4.8695	2.2067		100	1.1325	0.1325	0.2406	0.4905
	200	2.0024	0.3024	2.2631	1.5043		200	1.0589	0.0589	0.1276	0.3572
	400	1.9337	0.2337	1.5199	1.2329		400	1.0353	0.0353	0.0568	0.2383
	800	1.8505	0.1505	0.8497	0.9218		800	1.0191	0.0191	0.0302	0.1737

Also the estimated biases, MSEs and RMSEs are decreasing when the sample size n is increasing. These results reveal the consistency property of the MLEs.

6 Data Analysis

In this section we illustrate the superiority of *BMLD* as compared to some other distributions using three real data sets. The first one is the lifetimes of 50 devices provided by Aarset (1987). Second one is the strength of glass fibres of length 1.5 cm from the National Physical Laboratory in England (see, Smith and Naylor 1987). And the final one is the survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli, observed and reported by Bjerkedal (1960). A graphical method based on Total Time on Test (TTT) (see, Aarset 1987) is used here to determine the shape of hazard rate function of the datasets we considered. The empirical TTT plot is,

$$G\left(\frac{r}{n}\right) = \frac{\sum_{i=1}^{r} X_{(i)} + (n-r)X_{(r)}}{\sum_{i=1}^{n} X_{(i)}}, \quad r = 1, 2, ..., n,$$

where $X_{(i)}$ denote the ith order statistic of the sample. Figure 3 depicts the empirical TTT plots of the three data sets that we have considered here.

For the data set, lifetimes of 50 devices provided by Aarset (1987), the empirical TTT transform is convex then concave, so the hazard function is bathtub shaped. For the other two data sets, the empirical TTT transform is concave, therefore both have increasing hazard function.

For the three data sets we compute model adequacy measures and goodness of fit statistic of *BMLD*, and compare it with that of classical distributions such as Modified Weibull (*MW*) (see, Lai et al. 2003), Additive Weibull (*AW*) (see, Lemonte et al. 2014), Exponentiated Lindley (*EL*) (see, Nadarajah et al. (2011), Weighted Lindley (*WL*) (see, Ghitany et al. 2011), Generalized Lindley (*GL*) (see, Zakerzadeh and Dolati 2009), Lindley Exponential (*LE*) (see, Bhati et al. 2015), New Generalized Lindley (*NGL*) (see, Abouammoh et al. 2015), Extended Generalized Lindley (*EGL*) (see,Ranjbar et al. 2019) and Exponentiated Weibull (*EW*) (see, Pal et al. 2006).

The estimates of the parameters, -Log Likelihood ($-\log$ L), Akaike information criterion (AIC), Bayesian information criterion (BIC), Corrected Akaike information criterion (AICc), Kolmogorov Smirnov (KS) statistic values along with the p value are calculated for these datasets and are given in Tables 2, 3 and 4 respectively. The plots of fitted densities and cumulative densities with respective to the given data sets are also plotted.

The best model is the one with lowest AIC, BIC, AICc and KS statistic with largest p value. From the Tables 2,3 and 4 we can clearly observe that *BMLD* has the smallest value for its model adequacy measures such as AIC, BIC and AICc. Thus one can conclude that *BMLD* has the better performance compared to the other competing models. Further the Kolmogorv Smirnov (KS) statistic is computed to check the goodness of fit for the data set to *BMLD* as well as the other models. The

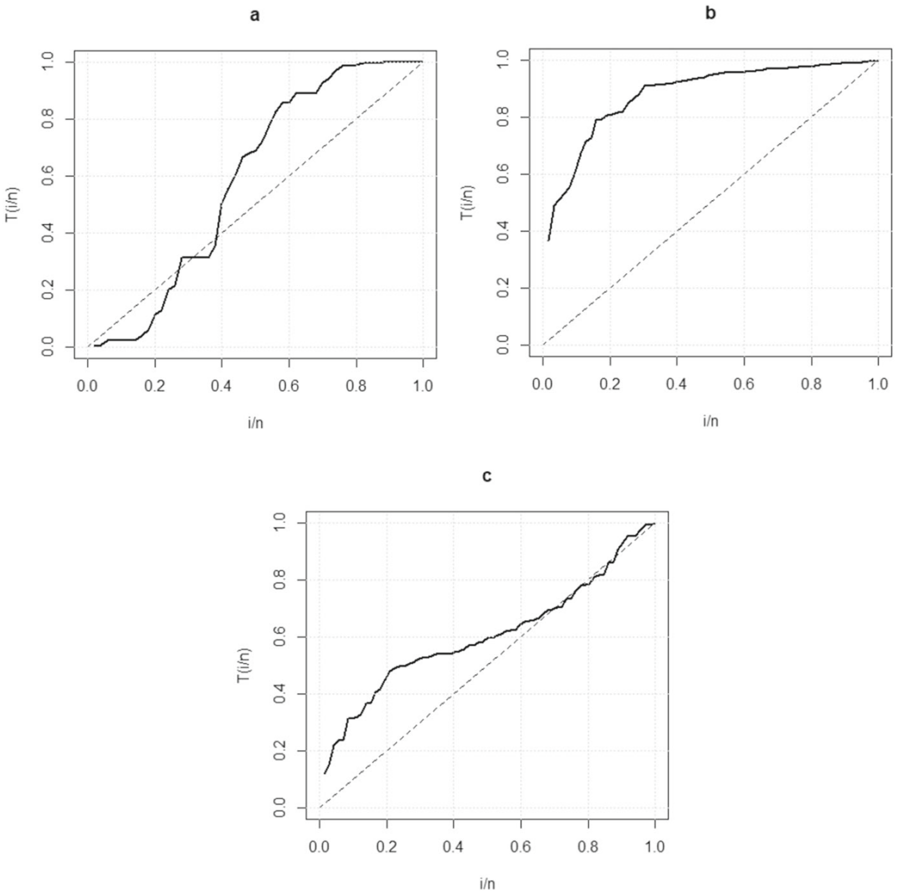

Fig. 3 Empirical TTT plots of datas of **a** Lifetimes of 50 devices, **b** Strength of glass fibres and **c** Survival times of 72 guinea pigs

value of KS statistic indicates that the *BMLD* has high fitting ability compared to other models considered here.

The plots of fitted densities and cumulative densities with respective to the datasets are given in Figs. 3, 4 and 5 respectively.

Figures 4a, 5a and 6a depicts the empirical histograms of the real data and the fitted densities of the *BMLD* and other distributions considered here. The fit of *BMLD* seems to be closer to the histogram of real data sets than other distributions. Also Figs. 4b, 5b and 6b shows the empirical and fitted cumulative density functions of *BMLD* and other distributions with the real data set. From these plots it is clear that *BMLD* will give consistently better fits than other competitive models.

Table 2 Estimates, model adequacy measures and KS statistic for the data of lifetimes of 50 devices

Model	Estimates	− log L	AIC	BIC	AICc	KS(*p* value)
BMLD	$\hat{\theta} = 0.1989$	224.5299	459.0598	468.6199	460.4235	0.1433 (0.2561)
	$\hat{\beta} = 0.5635$					
	$\hat{\alpha_0} = 14.0108$					
	$\hat{\alpha_1} = 0.8606$					
	$\hat{\alpha_2} = 4.5295$					
AW	$\hat{\alpha} = 0.1045$	235.5743	479.1486	486.7966	480.0374	0.1987 (0.0386)
	$\hat{\beta} = 0.3433$					
	$\hat{\gamma} = 0.0030$					
	$\hat{\delta} = 1.3966$					
EGL	$\hat{\theta} = 0.0003$	229.0514	466.1028	473.7509	466.9917	0.1585 (0.1623)
	$\hat{\beta} = 1.9759$					
	$\hat{\alpha} = 0.2254$					
	$\hat{\gamma} = 0.0005$					
MW	$\hat{\theta} = 0.0186$	239.4842	484.9684	490.7045	485.4902	0.1943 (0.0459)
	$\hat{\alpha} = 0.0404$					
	$\hat{\beta} = 0.3730$					
GL	$\hat{\theta} = 0.0263$	236.9578	479.9156	485.6517	480.4373	0.1793 (0.0804)
	$\hat{\alpha} = 0.5282$					
	$\hat{\beta} = 0.0534$					
NGL	$\hat{\theta} = 1.1463$	241.399	486.798	490.6221	487.0534	0.1888 (0.0566)
	$\hat{\alpha} = 0.02458$					
EW	$\hat{\theta} = 6.8267$	250.7883	505.5765	509.4006	505.8319	0.2287 (0.0107)
	$\hat{\alpha} = 0.2761$					
LE	$\hat{\theta} = 1.0409$	242.0492	488.0983	491.9224	488.3537	0.2109 (0.0233)
	$\hat{\alpha} = 0.5282$					
WL	$\hat{\theta} = 0.0254$	239.4157	482.8314	486.6554	483.0867	0.1846 (0.0662)
	$\hat{\alpha} = 0.2500$					
EL	$\hat{\theta} = 0.02785$	238.9909	481.9817	485.8058	482.2371	0.1937 (0.0469)
	$\hat{\alpha} = 0.4548$					

7 Conclusion

In this article, we proposed a wider class of Lindley distribution called the binomial mixture Lindley distribution (*BMLD*), which generalizes *ED*, *GD*, *LD*, LD_2, *WLD*, *GLD*, *NGLD* and $NGLD_1$. Its flexibility allows increasing, decreasing, bathtub shaped and upside-down bathtub shaped hazard rates. Owing to the attractive

Table 3 Estimates, model adequacy measures and KS statistic for the data of strength of glass fibres

Model	Estimates	− log L	AIC	BIC	AICc	KS(p value)
BMLD	$\hat{\theta} = 34.5379$	10.5395	31.0789	41.7946	32.1316	0.0997 (0.5586)
	$\hat{\beta} = 64.0653$					
	$\hat{\alpha_0} = 55.2549$					
	$\hat{\alpha_1} = 55.2558$					
	$\hat{\alpha_2} = 29.1012$					
AW	$\hat{\alpha} = 0.0621$	13.7401	35.4802	44.0528	36.1699	0.1213 (0.3117)
	$\hat{\beta} = 2.6426$					
	$\hat{\gamma} = 0.0193$					
	$\hat{\delta} = 7.3790$					
EGL	$\hat{\theta} = 0.0585$	13.0681	34.1362	42.7087	34.8258	0.1135 (0.3918)
	$\hat{\beta} = 6.2133$					
	$\hat{\alpha} = 0.495$					
	$\hat{\delta} = 0.1768$					
MW	$\hat{\theta} = 0.0309$	14.8947	35.7894	42.2188	36.1962	0.1333 (0.2131)
	$\hat{\alpha} = 0.0408$					
	$\hat{\beta} = 6.3768$					
GL	$\hat{\theta} = 11.7214$	23.8833	53.7665	60.1959	54.1733	0.2161 (0.0056)
	$\hat{\alpha} = 16.9727$					
	$\hat{\beta} = 26.0137$					
NGL	$\hat{\theta} = 18.4314$	23.9494	51.8987	56.185	52.0987	0.2164 (0.0055)
	$\hat{\alpha} = 11.6209$					
EW	$\hat{\theta} = 5.8269$	23.8711	51.7421	56.0284	51.9421	0.2313 (0.0024)
	$\hat{\alpha} = 2.051$					
LE	$\hat{\theta} = 32.2974$	31.4079	66.8159	71.1022	67.0159	0.2293 (0.0027)
	$\hat{\alpha} = 2.6118$					
WL	$\hat{\theta} = 11.7389$	23.8878	51.7756	56.0619	51.9756	0.2161 (0.0056)
	$\hat{\alpha} = 17.0957$					
EL	$\hat{\theta} = 2.9900$	30.6199	65.2397	69.5259	65.4397	0.2264 (0.0031)
	$\hat{\alpha} = 26.1719$					

feature of hazard rate function of *BMLD* it can be used to model any type of failure data sets. The estimation of parameters was explored by MLE method and the statistical properties of the estimators are investigated using a simulation study. Finally to establish the potentiality of this model, we use three real data sets in which one among them has bathtub shaped hazard rate and the other two have increasing hazard rate. For all these data sets *BMLD* performs better when compared

Table 4 Estimates, model adequacy measures and KS statistic for the data of survival times of 72 guinea pigs

Model	Estimates	− log L	AIC	BIC	AICc	KS(p value)
BMLD	$\hat{\theta} = 0.0658$	385.8537	781.7074	793.0907	782.6165	0.0843 (0.6851)
	$\hat{\beta} = 0.0386$					
	$\hat{\alpha_0} = 18.1939$					
	$\hat{\alpha_1} = 5.5223$					
	$\hat{\alpha_2} = 3.8266$					
AW	$\hat{\alpha} = 0.01$	399.057	806.114	815.2207	806.711	0.1627 (0.0443)
	$\hat{\beta} = 0.1032$					
	$\hat{\gamma} = 0.0032$					
	$\hat{\delta} = 1.2354$					
EGL	$\hat{\theta} = 1.05 \times 10^{-5}$	402.1303	812.2606	821.3673	812.8576	0.1964 (0.0078)
	$\hat{\beta} = 2.1984$					
	$\hat{\alpha} = 0.5006$					
	$\hat{\gamma} = 1.67 \times 10^{-9}$					
MW	$\hat{\theta} = 0.0101$	404.1817	814.3634	821.1934	814.7163	0.2178 (0.0022)
	$\hat{\alpha} = 0.01$					
	$\hat{\beta} = 0.0099$					
GL	$\hat{\theta} = 0.021$	394.3466	794.6931	801.5231	795.0461	0.1388 (0.1246)
	$\hat{\alpha} = 1.1058$					
	$\hat{\beta} = 2.0004$					
EW	$\hat{\theta} = 65.5435$	390.3485	784.697	789.2504	784.871	0.109 (0.3599)
	$\hat{\alpha} = 0.3518$					
NGL	$\hat{\theta} = 0.0212$	394.4382	792.8765	797.4298	793.0504	0.1401 (0.1185)
	$\hat{\alpha} = 2.1353$					
LE	$\hat{\theta} = 2.9562$	392.567	789.134	793.6873	789.3079	0.127 (0.1956)
	$\hat{\alpha} = 0.0163$					
WL	$\hat{\theta} = 0.0213$	394.4176	792.8351	797.3885	793.009	0.1403 (0.1176)
	$\hat{\alpha} = 1.1447$					
EL	$\hat{\theta} = 0.0212$	394.2822	792.5644	797.1178	792.7383	0.1431 (0.1047)
	$\hat{\alpha} = 1.1389$					

to other competing models. Summing up, the *BMLD* provides a better model for fitting the wide spectrum of positive data sets arising in engineering, survival analysis, hydrology, economics, physics as well as numerous other fields of scientific investigation.

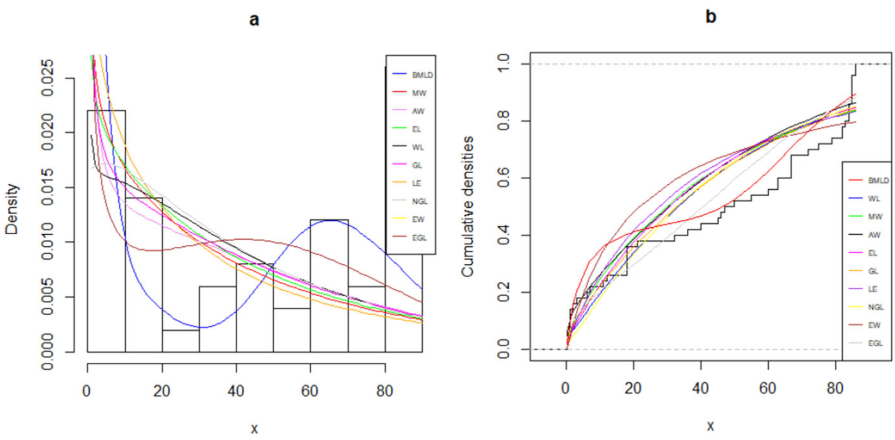

Fig. 4 Fitted densities (**a**) and cumulative densities (**b**) of data of lifetimes of 50 devices

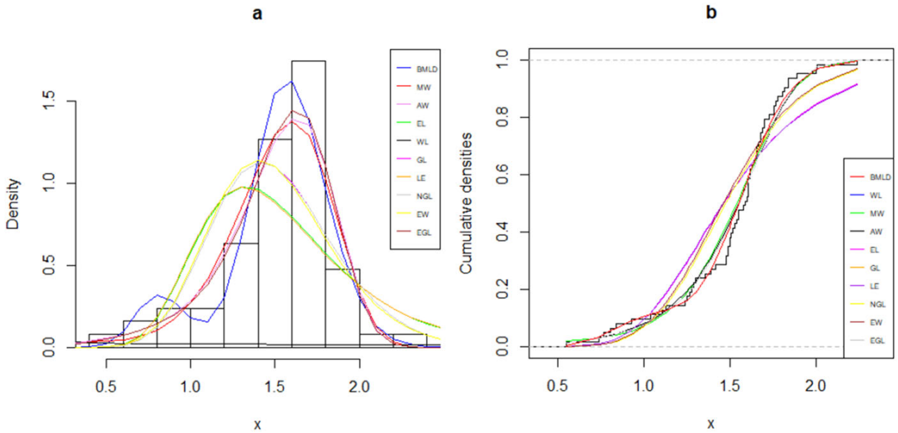

Fig. 5 Fitted densities (**a**) and cumulative densities (**b**) of data of strength of glass fibres

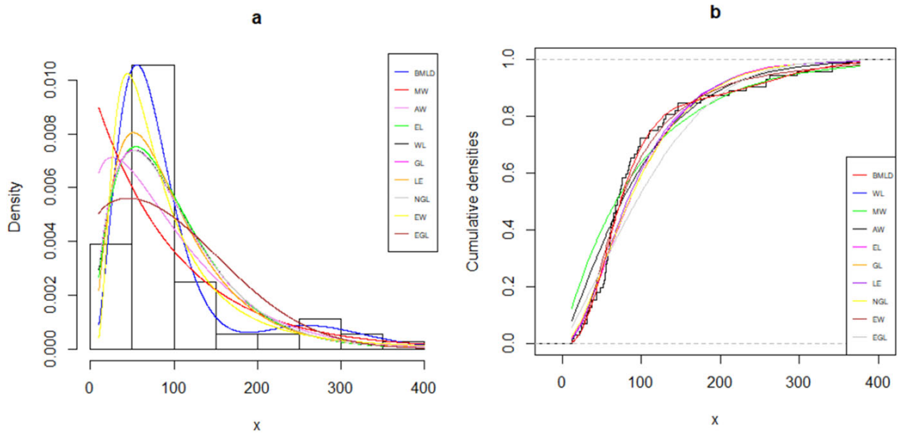

Fig. 6 Fitted densities (**a**) and cumulative densities (**b**) of data of survival times of 72 guinea pigs

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Appendix

The second partial and cross derivatives with respect to the parameters are derived as,

$$\frac{\partial^2 \log l}{\partial \theta^2} = \frac{2n}{(\theta+\beta)^2} + \frac{1}{\theta^2}\sum_{i=1}^{n}\frac{1}{(A_i+B_i+C_i)^2}\left\{\Big(A_i+B_i+C_i\Big)\Big(\alpha_0(\alpha_0-1)A_i + (\alpha_1+1)\alpha_1 B_i + (\alpha_2+2)(\alpha_2+1)C_i\Big) - \Big(\alpha_0 A_i + (\alpha_1+1)B_i + (\alpha_2+2)C_i\Big)^2\right\},$$

$$\frac{\partial^2 \log l}{\partial \theta \partial \beta} = \frac{2n}{(\theta+\beta)^2} + \frac{1}{\theta\beta}\sum_{i=1}^{n}\frac{1}{(A_i+B_i+C_i)^2}\left\{(\alpha_0-\alpha_1-1)A_iB_i + 2(\alpha_0-\alpha_2-2)A_iC_i + (\alpha_1-\alpha_2-1)B_iC_i\right\},$$

$$\frac{\partial^2 \log l}{\partial \theta \partial \alpha_0} = \frac{1}{\theta}\sum_{i=1}^{n}\frac{A_i}{(A_i+B_i+C_i)^2}\left\{\Big(A_i+B_i+C_i\Big)\Big(1+\alpha_0\Big(\log(\theta x_i) - \psi(\alpha_0)\Big)\Big) - \Big(\alpha_0 A_i + (\alpha_1+1)B_i + (\alpha_2+2)C_i\Big)\Big(\log(\theta x_i) - \psi(\alpha_0)\Big)\right\},$$

$$\frac{\partial^2 \log l}{\partial \theta \partial \alpha_1} = \frac{1}{\theta}\sum_{i=1}^{n}\frac{B_i}{(A_i+B_i+C_i)^2}\left\{\Big(A_i+B_i+C_i\Big)\Big(1+(\alpha_1+1)\Big(\log(\theta x_i) - \psi(\alpha_1)\Big)\Big) - \Big(\alpha_0 A_i + (\alpha_1+1)B_i + (\alpha_2+2)C_i\Big)\Big(\log(\theta x_i) - \psi(\alpha_1)\Big)\right\},$$

$$\frac{\partial^2 \log l}{\partial \theta \partial \alpha_2} = \frac{1}{\theta}\sum_{i=1}^{n}\frac{C_i}{(A_i+B_i+C_i)^2}\left\{\Big(A_i+B_i+C_i\Big)\Big(1+(\alpha_2+2)\Big(\log(\theta x_i) - \psi(\alpha_2)\Big)\Big) - \Big(\alpha_0 A_i + (\alpha_1+1)B_i + (\alpha_2+2)C_i\Big)\Big(\log(\theta x_i) - \psi(\alpha_2)\Big)\right\},$$

$$\frac{\partial^2 \log l}{\partial \beta^2} = \frac{2n}{(\theta+\beta)^2} + \frac{1}{\beta^2}\sum_{i=1}^{n}\frac{1}{(A_i+B_i+C_i)^2}\left\{2A_i(A_i+B_i+C_i) - (2A_i+B_i)^2\right\},$$

$$\frac{\partial^2 \log l}{\partial \beta \partial \alpha_0} = \frac{1}{\beta}\sum_{i=1}^{n}\frac{A_i}{(A_i+B_i+C_i)^2}\left\{\Big(\log(\theta x_i) - \psi(\alpha_0)\Big)\Big(B_i + 2C_i\Big)\right\},$$

$$\frac{\partial^2 \log l}{\partial\beta\partial\alpha_1} = \frac{1}{\beta}\sum_{i=1}^{n}\frac{B_i}{(A_i+B_i+C_i)^2}\left\{\Big(\log(\theta x_i)-\psi(\alpha_1)\Big)\Big(C_i-A_i\Big)\right\},$$

$$\frac{\partial^2 \log l}{\partial\beta\partial\alpha_2} = \frac{1}{\beta}\sum_{i=1}^{n}\frac{C_i}{(A_i+B_i+C_i)^2}\left\{-\Big(\log(\theta x_i)-\psi(\alpha_2)\Big)\Big(2A_i+B_i\Big)\right\},$$

$$\frac{\partial^2 \log l}{\partial\alpha_0^2} = \sum_{i=1}^{n}\frac{A_i}{(A_i+B_i+C_i)^2}\left\{\Big(A_i+B_i+C_i\Big)\bigg(\Big(\log(\theta x_i)-\psi(\alpha_0)\Big)^2 - \psi^{'}(\alpha_0)\bigg) - A_i\Big(\log(\theta x_i)-\psi(\alpha_0)\Big)^2\right\},$$

$$\frac{\partial^2 \log l}{\partial\alpha_0\partial\alpha_1} = \sum_{i=1}^{n}\frac{1}{(A_i+B_i+C_i)^2}\left\{-A_iB_i\Big(\log(\theta x_i)-\psi(\alpha_0)\Big)\Big(\log(\theta x_i)-\psi(\alpha_1)\Big)\right\},$$

$$\frac{\partial^2 \log l}{\partial\alpha_0\partial\alpha_2} = \sum_{i=1}^{n}\frac{1}{(A_i+B_i+C_i)^2}\left\{-A_iC_i\Big(\log(\theta x_i)-\psi(\alpha_0)\Big)\Big(\log(\theta x_i) - \psi(\alpha_2)\Big)\right\},$$

$$\frac{\partial^2 \log l}{\partial\alpha_1^2} = \sum_{i=1}^{n}\frac{B_i}{(A_i+B_i+C_i)^2}\left\{\Big(A_i+B_i+C_i\Big)\bigg(\Big(\log(\theta x_i)-\psi(\alpha_1)\Big)^2 - \psi^{'}(\alpha_1)\bigg) - B_i\Big(\log(\theta x_i)-\psi(\alpha_1)\Big)^2\right\},$$

$$\frac{\partial^2 \log l}{\partial\alpha_1\partial\alpha_2} = \sum_{i=1}^{n}\frac{1}{(A_i+B_i+C_i)^2}\left\{-B_iC_i\Big(\log(\theta x_i)-\psi(\alpha_1)\Big)\Big(\log(\theta x_i) - \psi(\alpha_2)\Big)\right\}$$

and

$$\frac{\partial^2 \log l}{\partial\alpha_2^2} = \sum_{i=1}^{n}\frac{C_i}{(A_i+B_i+C_i)^2}\left\{\Big(A_i+B_i+C_i\Big)\bigg(\Big(\log(\theta x_i)-\psi(\alpha_2)\Big)^2 - \psi^{'}(\alpha_2)\bigg) - C_i\Big(\log(\theta x_i)-\psi(\alpha_2)\Big)^2\right\}.$$

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