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RESEARCH ARTICLE

Normal Approximation of Posterior Distribution in $GI/G/1$ Queue

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Abstract
The paper deals with the asymptotic joint posterior distribution of (θ, ϕ) in a $GI/G/1$ queueing system over a continuous time interval $(0, T]$ where θ and ϕ are unknown parameters of arrival process and departure process respectively and T is a suitable stopping time.

Keywords $GI/G/1$ queue · Exponential families · Maximum likelihood estimator · Posterior distribution · Asymptotic normality

Mathematics Subject Classification 60K25 · 68M20 · 62F12

1 Introduction

Though statistical inference plays a major role in any use of queueing models, study of asymptotic inference problems for queueing system can be hardly traced back to the works by Basawa and Prabhu (1981, 1988) where they have discussed about the maximum likelihood (ML) estimators of the parameters in single server queues. Basawa et al. (1996) have studied the consistency and asymptotic normality of the parameters in a $GI/G/1$ queue based on information on waiting times. Acharya (1999) has studied the rate of convergence of the distribution of the maximum likelihood estimators of the arrival and the service rates from a single server queue. Acharya and Mishra (2007) have proved the Bernstein–von Mises theorem for the arrival process in a $M/M/1$ queue.

From a Bayesian outlook, inferences about the parameter are based on its posterior distribution. The study of asymptotic posterior normality can be traced back to the

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time of Laplace and it has attracted the attention of many authors. A conventional approach to such problems starts from a Taylor series expansion of the log-likelihood function around the maximum likelihood estimator (MLE) and proceeds from there to develop expansions that have standard normal as a leading term and hold in probability or almost surely, given the data. This type of study have not been done in queueing system. For the general set up in this direction the previous work seems to be those by Walker (1969), Johnston (1970) for i.i.d observations; Hyde and Johnston (1979), Basawa and Prakasa Rao (1980), Chen (1985) and Sweeting and Adekola (1987) for stochastic process. The most recent work was done by Kim (1998) in which he provided a set of conditions to prove the asymptotic normality under quite general situations of possible non-stationary time series model and Weng and Tsai (2008) where they studied asymptotic normality for multiparameter problems.

In this paper, our aim is to prove that the joint posterior distribution of (θ, ϕ) is asymptotically normal for $GI/G/1$ queueing model in the context of exponential families. In Sect. 2 we introduce the model of our interest and explain some elements of maximum likelihood estimator (MLE) as well as Bayesian procedure. In Sect. 3 we prove our main result. For the illustration purpose we provide an example Sect. 4. Section 5 deals with the simulation study while in Sect. 6 concluding remarks are given.

2 *GI*/*G*/1 Queueing Model

Consider a single server queueing system in which the interarrival times $\{u_k, k \geq 1\}$ and the service times $\{v_k, k \geq 1\}$ are two independent sequences of independent and identically distributed nonnegative random variables with densities $f(u;\theta)$ and $g(v;\phi)$, respectively, where θ and ϕ are unknown parameters. Let us assume that f and g belong to the continuous exponential families given by

$$f(u;\theta) = a_1(u)\exp\{\theta h_1(u) - k_1(\theta)\}, \tag{2.1}$$

$$g(v;\phi) = a_2(v)\exp\{\phi h_2(v) - k_2(\phi)\}. \tag{2.2}$$

and

$$f(u;\theta) = g(v;\phi) = 0 \quad \text{on} \quad (-\infty, 0)$$

where $\Theta_1 = \{\theta > 0 : \ k_1(\theta) < \infty\}$ and $\Theta_2 = \{\phi > 0 : \ k_2(\phi) < \infty\}$ are open subsets of $\mathbb{R}$. It is easy to see that, $E_\theta(h_1(u)) = k_1'(\theta)$, $var_\theta(h_1(u)) = k_1''(\theta)$, $E_\phi(h_2(v)) = k_2'(\phi)$, $var_\phi(h_2(v)) = k_2''(\phi)$, are supposed to be finite.

For simplicity we assume that the initial customer arrives at time $t = 0$. Our sampling scheme is to observe the system over a continuous time interval $(0, T]$, where T is a suitable stopping time. The sample data consist of

$$\{A(T), D(T), u_1, u_2, u_3, \ldots, u_{A(T)}, v_1, v_2, \ldots, v_{D(T)}\}, \tag{2.3}$$

where $A(T)$ is the number of arrivals and $D(T)$ is the number of departures during $(0, T]$. Obviously no arrivals occur during $[\sum_{i=1}^{A(T)} u_i, T]$ and no departures during $[\gamma(T) + \sum_{i=1}^{D(T)} v_i, T]$, where $\gamma(T)$ is the total idle period in $(0, T]$.

The likelihood function based on data (2.3) is given by

$$L_T(\theta, \phi) = \prod_{i=1}^{A(T)} f(u_i, \theta) \prod_{i=1}^{D(T)} f(v_i, \phi) \times \left[1 - F_\theta\left[T - \sum_{i=1}^{A(T)} u_i\right]\right]\left[1 - G_\phi\left[T - \gamma(T) - \sum_{i=1}^{D(T)} v_i\right]\right], \tag{2.4}$$

where F and G are distribution functions corresponding to the densities f and g respectively.

The approximate likelihood $L_T^{(a)}(\theta, \phi)$ is defined as

$$L_T^{(a)}(\theta, \phi) = \prod_{i=1}^{A(T)} f(u_i, \theta) \prod_{i=1}^{D(T)} f(v_i, \phi) = L_T^{(a)}(\theta) L_T^{(a)}(\phi), \tag{2.5}$$

where

$$L_T^{(a)}(\theta) = \left[\prod_{i=1}^{A(T)} a_1(u_i)\right] \exp\left\{\sum_{i=1}^{A(T)} [\theta h_1(u_i) - k_1(\theta)]\right\} \tag{2.6}$$

and

$$L_T^{(a)}(\phi) = \left[\prod_{i=1}^{D(T)} a_2(v_i)\right] \exp\left\{\sum_{i=1}^{D(T)} [\phi h_2(v_i) - k_2(\phi)]\right\}. \tag{2.7}$$

The maximum likelihood estimates obtained from (2.5) are asymptotically equivalent to those obtained from (2.4) provided that the following two conditions are satisfied for $T \to \infty$:

$$(A(T))^{-1/2} \frac{\partial}{\partial \theta} \log\left[1 - F_\theta\left(T - \sum_{i=1}^{A(T)} u_i\right)\right] \xrightarrow{p} 0 \tag{2.8}$$

and

$$(D(T))^{-1/2} \frac{\partial}{\partial \phi} \log\left[1 - G_\phi\left(T - \gamma(T) - \sum_{i=1}^{D(T)} v_i\right)\right] \xrightarrow{p} 0. \tag{2.9}$$

The implications of these conditions have been explained by Basawa and Prabhu (1988).

Basawa and Prabhu (1988) have shown that the maximum likelihood estimator of θ and ϕ are given by

$$\hat{\theta}_T = \eta_1^{-1}\Big[(A(T))^{-1}\sum_{i=1}^{A(T)} h_1(u_i)\Big], \tag{2.10}$$

$$\hat{\phi}_T = \eta_2^{-1}\Big[(D(T))^{-1}\sum_{i=1}^{D(T)} h_2(v_i)\Big] \tag{2.11}$$

where $\eta_i^{-1}(.)$ denotes the inverse functions of $\eta_i(.)$ for $i = 1, 2$ and

$$\eta_1(\theta) = E_\theta(h_1(u)) = k_1^{'}(\theta)$$

and

$$\eta_2(\phi) = E_\phi(h_2(v)) = k_2^{'}(\phi).$$

The Fisher information matrix is given by

$$I(\theta,\phi) = \begin{bmatrix} k_1^{''}(\theta)E(A(T)) & 0 \\ 0 & k_2^{''}(\phi)E(D(T)) \end{bmatrix} = \begin{bmatrix} I(\theta) & 0 \\ 0 & I(\phi) \end{bmatrix}. \tag{2.12}$$

Under suitable stability conditions on stopping times, Basawa and Prabhu (1988) have proved that the estimators $\hat{\theta}_T$ and $\hat{\phi}_T$ are consistent, i.e,

$$\hat{\theta}_T \xrightarrow{a.s.} \theta_0 \quad \text{and} \quad \hat{\phi}_T \xrightarrow{a.s.} \phi_0 \quad \text{as} \quad T \to \infty \tag{2.13}$$

and

$$I^{\frac{1}{2}}(\theta_0,\phi_0)\begin{bmatrix} \hat{\theta}_T - \theta_0 \\ \hat{\phi}_T - \phi_0 \end{bmatrix} \Rightarrow N\left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right], \tag{2.14}$$

where θ_0 and ϕ_0 denote the true value of θ and ϕ respectively, and the symbol $\Rightarrow$ denotes the convergence in distribution.

From Eq. (2.5) we have the loglikelihood function

$$\ell_T(\theta,\phi) = \log L_T^{(a)}(\theta,\phi) = \ell_T(\theta) + \ell_T(\phi), \tag{2.15}$$

where

$$\ell_T(\theta) = \log L_T^{(a)}(\theta) = \sum_{i=1}^{A(T)} a_1(u_i) + \theta\sum_{i=1}^{A(T)} h_1(u_i) - A(T)k_1(\theta) \tag{2.16}$$

and

$$\ell_T(\phi) = \log L_T^{(a)}(\phi) = \sum_{i=1}^{D(T)} a_2(v_i) + \phi \sum_{i=1}^{D(T)} h_2(v_i) - D(T)k_2(\phi). \tag{2.17}$$

Let

$$\ell_T^{'}(\theta_0) = \frac{\partial}{\partial\theta}\ell_T(\theta,\phi)\bigg|_{\theta=\theta_0} = \frac{\partial}{\partial\theta}\ell_T(\theta)\bigg|_{\theta=\theta_0},$$
$$\ell_T^{''}(\theta_0) = \frac{\partial^2}{\partial\theta^2}\ell_T(\theta,\phi)\bigg|_{\theta=\theta_0} = \frac{\partial^2}{\partial\theta^2}\ell_T(\theta)\bigg|_{\theta=\theta_0}.$$

Similarly $\ell_T^{'}(\hat{\theta}_T)$, $\ell_T^{'}(\hat{\phi}_T)$, $\ell_T^{'}(\phi_0)$, $\ell_T^{''}(\phi_0)$, $\ell_T^{''}(\hat{\theta}_T)$ and $\ell_T^{''}(\hat{\phi}_T)$ are defined.

Let $\pi_1(\theta)$ and $\pi_2(\phi)$ be the prior distributions of θ and ϕ respectively. Let the joint prior distribution θ and ϕ be $\pi(\theta,\phi)$. Since the interarrival time and service time distributions are independent, so we have $\pi(\theta,\phi) = \pi_1(\theta)\pi_2(\phi)$. Then the joint posterior density of (θ,ϕ) is

$$\pi(\theta,\phi|(u_i,v_i);\ i \geq 1) = \pi_1(\theta|u_i;\ i=1,\ldots,A(T))\pi_2(\phi|v_i;\ i=1,\ldots,D(T)) \tag{2.18}$$

with

$$\pi_1(\theta|u_i;\ i=1,\ldots,A(T)) = \frac{L_T^{(a)}(\theta)\pi_1(\theta)}{\int_{\Theta_1} L_T^{(a)}(\theta)\pi_1(\theta)d\theta}$$
$$= \frac{\exp\left\{\sum_{i=1}^{A(T)}[\theta h_1(u_i) - k_1(\theta)]\right\}\pi_1(\theta)}{\int_{\Theta_1}\exp\left\{\sum_{i=1}^{A(T)}[\theta h_1(u_i) - k_1(\theta)]\right\}\pi_1(\theta)d\theta} \tag{2.19}$$

and

$$\pi_2(\phi|v_i;\ i=1,\ldots,D(T)) = \frac{\exp\left\{\sum_{i=1}^{D(T)}[\phi h_2(v_i) - k_2(\phi)]\right\}\pi_2(\phi)}{\int_{\Theta_2}\exp\left\{\sum_{i=1}^{D(T)}[\phi h_2(v_i) - k_2(\phi)]\right\}\pi_2(\phi)d\phi} \tag{2.20}$$

the marginal posterior densities of θ and ϕ, respectively. Let $\tilde{\theta}_T$ and $\tilde{\phi}_T$ be Bayes estimator of θ and ϕ respectively.

In the next section we will state and prove our main result.

3 Main Result

Theorem 3.1 *Let $(\theta_0, \phi_0) \in \Theta_1 \times \Theta_2$. If the prior densities $\pi_1(\theta)$ and $\pi_2(\phi)$ are continuous and positive at θ_0 and ϕ_0 respectively then, for any α_i, β_i such that $-\infty \leq \alpha_i \leq \beta_i \leq \infty$, $i = 1, 2$, the posterior probability that $(\hat{\theta}_T + \alpha_1\sigma_T \leq \theta \leq \hat{\theta}_T + \beta_1\sigma_T, \hat{\phi}_T + \alpha_2\tau_T \leq \phi \leq \hat{\phi}_T + \beta_2\tau_T)$, namely*

$$\int_{\hat{\theta}_T+\alpha_1\sigma_T}^{\hat{\theta}_T+\beta_1\sigma_T} \int_{\hat{\phi}_T+\tau_T\sigma_2}^{\hat{\phi}_T+\beta_2\tau_T} \pi(\theta, \phi|(u_i, v_i), i \geq 1) d\theta d\phi$$

tends in $[P_{(\theta_0,\phi_0)}]$ probability to

$$(2\pi)^{-1} \int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} e^{-\frac{1}{2}(x^2+y^2)} dx dy$$

as $T \to \infty$, where σ_T and τ_T are the positive square roots of $[-\ell_T''(\hat{\theta}_T)]^{-1}$ and $[-\ell_T''(\hat{\phi}_T)]^{-1}$ respectively.

Proof of Theorem 3.1 The former integral of the above theorem can be written as the product of the integrals of the marginal posterior densities, i.e.,

$$\begin{aligned} &\int_{\hat{\theta}_T+\alpha_1\sigma_T}^{\hat{\theta}_T+\beta_1\sigma_T} \int_{\hat{\phi}_T+\alpha_2\tau_T}^{\hat{\phi}_T+\beta_2\tau_T} \pi(\theta, \phi|(u_i, v_i), i \geq 1) d\theta d\phi \\ &= \int_{\hat{\theta}_T+\alpha_1\sigma_T}^{\hat{\theta}_T+\beta_1\sigma_T} \frac{\exp\left\{\sum_{i=1}^{A(T)} [\theta h_1(u_i) - k_1(\theta)]\right\} \pi_1(\theta)}{\int_{\Theta_1} \exp\left\{\sum_{i=1}^{A(T)} [\theta h_1(u_i) - k_1(\theta)]\right\} \pi_1(\theta) d\theta} d\theta \\ &\times \int_{\hat{\phi}_T+\alpha_2\tau_T}^{\hat{\phi}_T+\beta_2\tau_T} \frac{\exp\left\{\sum_{i=1}^{D(T)} [\phi h_2(v_i) - k_2(\phi)]\right\} \pi_2(\phi)}{\int_{\Theta_2} \exp\left\{\sum_{i=1}^{D(T)} [\phi h_2(v_i) - k_2(\phi)]\right\} \pi_2(\phi) d\phi} d\phi \end{aligned} \tag{3.1}$$

and the convergence of both can be established separately.

For any $\delta > 0$, let us write $\mathcal{N}(\varrho, \delta) = (\varrho - \delta, \varrho + \delta)$ with $\varrho \in \Theta_1$ and $\mathcal{J}_B = \int_B L_T^{(a)}(\theta)\pi_1(\theta) d\theta$ where $B \subseteq \Theta_1$. Hence,

$$\int_{\hat{\theta}_T+\alpha_1\sigma_T}^{\hat{\theta}_T+\beta_1\sigma_T} \frac{\exp\left\{\sum_{i=1}^{A(T)} [\theta h_1(u_i) - k_1(\theta)]\right\} \pi_1(\theta)}{\int_{\Theta_1} \exp\left\{\sum_{i=1}^{A(T)} [\theta h_1(u_i) - k_1(\theta)]\right\} \pi_1(\theta) d\theta} d\theta = (\mathcal{J}_{\Theta_1})^{-1} \mathcal{J}_{\mathcal{N}(\theta_T, \delta_T)} \tag{3.2}$$

with $\delta_T = \frac{\sigma_T(\beta_1-\alpha_1)}{2}$ and $\theta_T = \hat{\theta}_T + \frac{\sigma_T(\alpha_1+\beta_1)}{2}$. Then, we want to prove that

$$(\mathcal{J}_{\Theta_1})^{-1}\mathcal{J}_{\mathcal{N}(\theta_T,\delta_T)} \to \Phi(\beta_1) - \Phi(\alpha_1) = \frac{1}{\sqrt{2\pi}}\int_{\alpha_1}^{\beta_1} e^{-\frac{x^2}{2}}dx \tag{3.3}$$

in probability $[P_{\theta_0}]$, where $\Phi(z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} e^{-\frac{s^2}{2}}ds$.

Let us split $\mathcal{J}_{\Theta_1}$ into $J_{\Theta_1\setminus\mathcal{J}_{\mathcal{N}(\theta_0,\delta)}}$ and $\mathcal{J}_{\mathcal{N}(\theta_0,\delta)}$. Then, to obtain the above result it is sufficient to prove that the following statements holds in probability $[P_{\theta_0}]$: For some $\delta > 0$,

(a) $\lim_{T\to\infty}[L_T^{(a)}(\hat{\theta}_T)\sigma_T]^{-1}J_{\Theta_1\setminus\mathcal{J}_{\mathcal{N}(\theta_0,\delta)}} = 0$

(b) $\lim_{T\to\infty}[L_T^{(a)}(\hat{\theta}_T)\sigma_T]^{-1}\mathcal{J}_{\mathcal{N}(\theta_0,\delta)} = (2\pi)^{\frac{1}{2}}\pi_1(\theta_0)$

(c) $\lim_{T\to\infty}[L_T^{(a)}(\hat{\theta}_T)\sigma_T]^{-1}\mathcal{J}_{\mathcal{N}(\theta_T,\delta_T)} = (2\pi)^{\frac{1}{2}}\pi_1(\theta_0)(\Phi(\beta_1) - \Phi(\alpha_1))$

Define

$$r_T(\theta) = -\frac{\ell_T''(\theta) - \ell_T''(\hat{\theta}_T)}{\ell_T''(\hat{\theta}_T)} = 1 - \frac{\ell_T''(\theta)/\ell_T''(\theta_0)}{\ell_T''(\hat{\theta}_T)/\ell_T''(\theta_0)}. \tag{3.4}$$

If θ belongs to $\mathcal{N}(\theta_0,\delta)$ for some $\delta > 0$, $\ell_T''(\theta)/\ell_T''(\theta_0)$ is close enough to 1 and, since $\hat{\theta}_T \to \theta_0$ almost surely, $\ell_T''(\hat{\theta}_T)/\ell_T''(\theta_0)$ is almost surely close to 1 for T sufficiently large. Therefore we can deduce that for given $\varepsilon > 0$, we can take δ such that, if T is large enough,

$$\sup_{\theta\in\mathcal{N}(\theta_0,\delta)} |r_T(\theta)| < \varepsilon \ \ [P_{\theta_0}]. \tag{3.5}$$

Consider also

$$q_T(\theta) = -\frac{\ell_T(\theta) - \ell_T(\hat{\theta}_T)}{\ell_T''(\theta_0)} = \frac{(\theta - \hat{\theta}_T)\sum_{i=1}^{A(T)} h_1(u_i) - A(T)(k_1(\theta) - k_1(\hat{\theta}_T))}{A(T)k_1''(\theta_0)}.$$

Since $\ell_T(.)$ has a strict maximum at $\hat{\theta}_T$, it is obvious that $q_T(.)$ is negative on $\Theta_1 \setminus \mathcal{N}(\theta_0,\delta)$ for T large enough. Moreover, since $\hat{\theta}_T \to \theta_0$ almost surely, it can be shown that there exists a positive constant $\kappa(\delta)$ such that

$$\sup_{\theta\in\Theta_1\setminus\mathcal{N}(\theta_0,\delta)} q_T < -\kappa(\delta) \ \ [P_{\theta_0}]. \tag{3.6}$$

Now,

$$\begin{aligned}
&[L_T^{(a)}(\hat{\theta}_T)\sigma_T]^{-1}\mathcal{J}_{\Theta_1\setminus\mathcal{N}(\theta_0,\delta)} \\
&= [L_T^{(a)}(\hat{\theta}_T)\sigma_T]^{-1}\int_{\Theta_1\setminus\mathcal{N}(\theta_0,\delta)} L_T^{(a)}(\theta)\pi_1(\theta)d\theta \\
&= [L_T^{(a)}(\hat{\theta}_T)\sigma_T]^{-1}L_T^{(a)}(\hat{\theta}_T)\int_{\Theta_1\setminus\mathcal{N}(\theta_0,\delta)} \pi_1(\theta)\exp\{\ell_T(\theta) - \ell_T(\hat{\theta}_T)\}d\theta
\end{aligned}$$

$$= (-\ell_T''(\hat{\theta}_T))^{\frac{1}{2}} \int_{\Theta_1 \setminus \mathcal{N}(\theta_0,\delta)} \pi_1(\theta) \exp\{q_T(\theta)(-\ell_T''(\theta_0))\} d\theta$$
$$\leq (-\ell_T''(\hat{\theta}_T))^{\frac{1}{2}} \exp\{-\kappa(\delta)(-\ell_T''(\theta_0))\} \qquad \text{(using Eq. 3.6)}$$
$$= \frac{(-\ell_T''(\hat{\theta}_T))^{\frac{1}{2}}}{(-\ell_T''(\theta_0))^{\frac{1}{2}}} (-\ell_T''(\theta_0))^{\frac{1}{2}} \exp\{-\kappa(\delta)(-\ell_T''(\theta_0))\} \ [P_{\theta_0}].$$

We have $-\ell_T''(\theta_0) = A(T)\sigma^2(\theta_0)$ diverges to ∞ almost surely as $T \to \infty$. So, in the above expression

$$(-\ell_T''(\theta_0))^{\frac{1}{2}} \exp\{-\kappa(\delta)(-\ell_T''(\theta_0))\} \to 0$$

in probability and, using Eq. (3.5), for some constant M and T large enough

$$\frac{(-\ell_T''(\hat{\theta}_T))^{\frac{1}{2}}}{(-\ell_T''(\theta_0))^{\frac{1}{2}}} = \left(\frac{1}{1 - r_T(\theta_0)}\right)^{\frac{1}{2}} < M$$

in probability and, consequently (a) holds.

Let us prove (b). Write

$$L_T^{(a)}(\theta) = L_T^{(a)}(\hat{\theta}_T) \exp\{\ell_T(\theta) - \ell_T(\hat{\theta}_T)\}. \tag{3.7}$$

Using Taylor expansion around $\hat{\theta}_T$,

$$\ell_T(\theta) = \ell_T(\hat{\theta}_T) + \frac{1}{2}(\theta - \hat{\theta}_T)^2 \ell_T''(\bar{\theta}_T) \tag{3.8}$$

for $\bar{\theta}_T = \theta + \xi(\hat{\theta}_T - \theta)$ with $0 < \xi < 1$. Thus letting

$$R_T = R_T(\theta) = \sigma_T^2\{\ell_T''(\bar{\theta}_T) - \ell_T''(\hat{\theta}_T)\},$$

we have

$$-\frac{1 - R_T}{\sigma_T^2} = \ell_T''(\bar{\theta}_T). \tag{3.9}$$

Using Eqs. (3.8) and (3.9) in Eq. (3.7) and, for some $\delta > 0$ and T large enough such that $\hat{\theta}_T \in \mathcal{N}(\theta_0, \delta)$, we have, for every $\theta \in \mathcal{N}(\theta_0, \delta)$

$$L_T^{(a)}(\theta) = L_T^{(a)}(\hat{\theta}_T) \exp\left\{ -\frac{(\theta - \hat{\theta}_T)^2}{2\sigma_T^2}(1 - R_T)\right\} \ [P_{\theta_0}] \tag{3.10}$$

and consequently,

$$[L_T^{(a)}(\hat{\theta}_T)\pi_1(\theta_0)]^{-1} \mathcal{J}_{\mathcal{N}(\theta_0,\delta)} = \int_{\mathcal{N}(\theta_0,\delta)} \frac{\pi_1(\theta)}{\pi_1(\theta_0)} \exp\left\{ -\frac{(\theta - \hat{\theta}_T)^2}{2\sigma_T^2}(1 - R_T)\right\} d\theta \ [P_{\theta_0}] \tag{3.11}$$

Since $\pi_1(\theta)$ is continuous and positive at $\theta = \theta_0$, then for given $0 < \varepsilon < 1$, we can choose δ small enough so that

$$1 - \varepsilon < \inf_{\theta \in \mathcal{N}(\theta_0,\delta)} \frac{\pi_1(\theta)}{\pi_1(\theta_0)} < \sup_{\theta \in \mathcal{N}(\theta_0,\delta)} \frac{\pi_1(\theta)}{\pi_1(\theta_0)} < 1 + \varepsilon. \tag{3.12}$$

Denote

$$\tilde{\mathcal{J}}_B = \int_B \exp\left\{ - \frac{(\theta - \hat{\theta}_T)^2}{2\sigma_T^2}(1 - R_T) \right\} d\theta, \quad B \subseteq \Theta_1.$$

Then from Eq. (3.12) we get that

$$(1 - \varepsilon)\tilde{\mathcal{J}}_{\mathcal{N}(\theta_0,\delta)} < [L_T^{(a)}(\hat{\theta}_T)\pi_1(\theta_0)]^{-1}\mathcal{J}_{\mathcal{N}(\theta_0,\delta)} < (1 + \varepsilon)\tilde{\mathcal{J}}_{\mathcal{N}(\theta_0,\delta)}. \tag{3.13}$$

If $\sup_{\theta \in \mathcal{N}(\theta_0,\delta)} |R_T| < \varepsilon < 1$, then

$$\int_{\mathcal{N}(\theta_0,\delta)} \exp\left\{ - \frac{(\theta - \hat{\theta}_T)^2}{2\sigma_T^2}(1 + \varepsilon) \right\} d\theta$$
$$< \tilde{\mathcal{J}}_{\mathcal{N}(\theta_0,\delta)} < \int_{\mathcal{N}(\theta_0,\delta)} \exp\left\{ - \frac{(\theta - \hat{\theta}_T)^2}{2\sigma_T^2}(1 - \varepsilon) \right\} d\theta$$

and for $\eta = +\varepsilon$ or $-\varepsilon$, making a change of variable,

$$\begin{aligned} &\int_{\mathcal{N}(\theta_0,\delta)} \exp\left\{ - \frac{(\theta - \hat{\theta}_T)^2}{2\sigma_T^2}(1 + \eta) \right\} d\theta \\ &= \frac{\sigma_T}{(1 + \eta)^{\frac{1}{2}}} \int_{(\theta_0 - \delta - \hat{\theta}_T)(1+\eta)^{\frac{1}{2}}\sigma_T^{-1}}^{(\theta_0 + \delta - \hat{\theta}_T)(1+\eta)^{\frac{1}{2}}\sigma_T^{-1}} e^{-\frac{x^2}{2}} dx \\ &= (2\pi)^{\frac{1}{2}}\sigma_T(1 + \eta)^{-\frac{1}{2}} \Big[\Phi\big\{\sigma_T^{-1}(\theta_0 + \delta - \hat{\theta}_T)(1 + \eta)^{\frac{1}{2}}\big\} \\ &\quad - \Phi\big\{\sigma_T^{-1}(\theta_0 - \delta - \hat{\theta}_T)(1 + \eta)^{\frac{1}{2}}\big\}\Big]. \end{aligned} \tag{3.14}$$

Since $\sigma_T^{-1} \to \infty$ and $\hat{\theta}_T \to \theta_0$ almost surely, it is deduced that the limits $(\theta_0 - \delta - \hat{\theta}_T)(1 + \eta)^{\frac{1}{2}}\sigma_T^{-1}$ and $(\theta_0 + \delta - \hat{\theta}_T)(1 + \eta)^{\frac{1}{2}}\sigma_T^{-1}$ of the integrals in the above equation converges to $-\infty$ and ∞ respectively. Therefore, the term in square brackets in Eq. (3.14) converges to 1. Thus, using an appropriate bound on R_T it follows that,

$$(2\pi)^{\frac{1}{2}}(1 + \varepsilon)^{-\frac{1}{2}} < \sigma_T^{-1}\tilde{\mathcal{J}}_{\mathcal{N}(\theta_0,\delta)} < (2\pi)^{\frac{1}{2}}(1 - \varepsilon)^{-\frac{1}{2}}$$

in probability as $T \to \infty$ and, using the above expression with the Eq. (3.13) we have the following bounds for $\mathcal{J}_{\mathcal{N}(\theta_0,\delta)}$:

$$(1+\varepsilon)^{-\frac{1}{2}}(1-\varepsilon) < \left[L_T(\hat{\theta}_T)\pi_1(\theta_0)(2\pi)^{\frac{1}{2}}\sigma_T\right]^{-1}\mathcal{J}_{\mathcal{N}(\theta_0,\delta)} < (1-\varepsilon)^{-\frac{1}{2}}(1+\varepsilon) \ \ [P_{\theta_0}]$$

Hence (b) holds.

Finally, let us show (c). Using the same arguments and notations above, given $\varepsilon > 0$, there exists δ such that if $\mathcal{N}(\theta_T, \delta_T) \subseteq \mathcal{N}(\theta_0, \delta)$ for T large enough then

$$(1-\varepsilon)\tilde{\mathcal{J}}_{\mathcal{N}(\theta_T,\delta_T)} < [L_T^{(a)}(\hat{\theta}_T)\pi_1(\theta_0)]^{-1}\mathcal{J}_{\mathcal{N}(\theta_T,\delta_T)} < (1+\varepsilon)\tilde{\mathcal{J}}_{\mathcal{N}(\theta_T,\delta_T)} \ \ [P_{\theta_0}]$$

While the last term in Eq. (3.14) becomes

$$(2\pi)^{\frac{1}{2}}\sigma_T(1+\eta)^{-\frac{1}{2}}\left[\Phi(\beta_1(1+\eta)^{\frac{1}{2}}) - \Phi(\alpha_1(1+\eta)^{\frac{1}{2}})\right].$$

Therefore, we obtain that

$$[L_T^{(a)}(\hat{\theta}_T)\pi_1(\theta_0)]^{-1}\mathcal{J}_{\mathcal{N}(\theta_T,\delta_T)} \to (2\pi)^{\frac{1}{2}}\pi_1(\theta_0)[\Phi(\beta_1) - \Phi(\alpha_1)] \ \ [P_{\theta_0}]$$

and now (3.3) is established.

Similarly, using the same arguments as in the above, it can be shown that

$$\int_{\hat{\phi}_T+\alpha_2\tau_T}^{\hat{\phi}_T+\beta_2\tau_T} \frac{\exp\left\{\sum_{i=1}^{D(T)}[\phi h_2(v_i) - k_2(\phi)]\right\}\pi_2(\phi)}{\int_{\Theta_2}\exp\left\{\sum_{i=1}^{D(T)}[\phi h_2(v_i) - k_2(\phi)]\right\}\pi_2(\phi)d\phi}d\phi \to \frac{1}{\sqrt{2\pi}}\int_{\alpha_2}^{\beta_2} e^{-\frac{y^2}{2}}$$

in probability $[P_{\phi_0}]$ and the proof is completed. □

4 Example

Let us consider a $M/M/1$ queueing system. Under the Markovian set-up we have

$$f(u;\theta) = \theta e^{-\theta u} \quad \text{and} \quad g(v;\phi) = \phi e^{-\phi v}.$$

So, the loglikelihood function is written as

$$\ell_T(\theta,\phi) = A(T)\log\theta - \theta\sum_{i=1}^{A(T)} u_i + D(T)\log\phi - \phi\sum_{i=1}^{D(T)} v_i$$

and the MLEs are given by

$$\hat{\theta}_T = \left[\frac{\sum_{i=1}^{A(T)} u_i}{A(T)}\right]^{-1} \quad \text{and} \quad \hat{\phi}_T = \left[\frac{\sum_{i=1}^{D(T)} v_i}{D(T)}\right]^{-1}.$$

Here $\sigma_T = \left[-\ell''_T(\hat{\theta}_T)\right]^{-\frac{1}{2}} = \frac{\sum_{i=1}^{A(T)} u_i}{\sqrt{A(T)}}$ and $\tau_T = \left[-\ell''_T(\hat{\phi}_T)\right]^{-\frac{1}{2}} = \frac{\sum_{i=1}^{D(T)} v_i}{\sqrt{D(T)}}$.

Let us assume that the conjugate prior distributions of θ and ϕ are gamma distributions with hyper-parameters (a_1, b_1) and (a_2, b_2), that is

$$\pi_1(\theta) = \frac{b_1^{a_1}}{\Gamma(a_1)}\theta^{a_1-1}e^{-b_1\theta} \quad \text{and} \quad \pi_2(\phi) = \frac{b_2^{a_2}}{\Gamma(a_2)}\phi^{a_2-1}e^{-b_2\phi}$$

where $a_i, b_i > 0$ for $i = 1, 2$.

Then, the posterior distribution of θ can be computed as:

$$\begin{aligned}
&\pi_1(\theta|u_i;\ i = 1, 2, \ldots, A(T)) \\
&\quad = \frac{L_T^a(\theta)\pi_1(\theta)}{\int_{\Theta_1} L_T^a \pi_1(\theta)d\theta} \\
&\quad = \frac{\theta^{A(T)+a_1-1}e^{-\left(\sum_{i=1}^{A(T)} u_i + b_1\right)\theta}}{\int_0^\infty \theta^{A(T)+a_1-1}e^{-\left(\sum_{i=1}^{A(T)} u_i + b_1\right)\theta}d\theta} \\
&\quad = \frac{\left(\sum_{i=1}^{A(T)} u_i + b_1\right)^{A(T)+a_1}}{\Gamma\left(A(T)+a_1\right)}\theta^{A(T)+a_1-1}e^{-\left(\sum_{i=1}^{A(T)} u_i + b_1\right)\theta}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\pi_2(\phi|v_i;\ i = 1, 2, \ldots, D(T)) \\
&\quad = \frac{\left(\sum_{i=1}^{D(T)} v_i + b_2\right)^{D(T)+a_2}}{\Gamma\left(D(T)+a_2\right)}\phi^{D(T)+a_2-1}e^{-\left(\sum_{i=1}^{D(T)} v_i + b_2\right)\phi}.
\end{aligned}$$

It is easy to see that

$$\tilde{\theta}_T = \frac{A(T)+a_1}{\sum_{i=1}^{A(T)} u_i + b_1} \quad \text{and} \quad \tilde{\phi}_T = \frac{D(T)+a_2}{\sum_{i=1}^{D(T)} v_i + b_2}.$$

Here, the posterior distributions of θ and ϕ are seen to be gamma distributions [Gamma$(A(T)+a_1, \sum_{i=1}^{A(T)} u_i + b_1)$ and Gamma$(D(T)+a_2, \sum_{i=1}^{D(T)} v_i + b_2)$]. Hence, by Central Limit Theorem (CLT), the joint posterior distribution converges to normal distribution as $T \to \infty$.

5 Simulation

For the feasibility of the main result discussed in Sect. 3, simulation was conducted for $M/M/1$ queueing system. For given values of true parameters θ_0 and ϕ_0 MLEs ($\hat{\theta}_T$ and $\hat{\phi}_T$) are computed at different time interval $(0, T]$. Also by choosing different values of hyper-parameters of gamma distribution we compute the Bayes estimators ($\tilde{\theta}_T$ and $\tilde{\phi}_T$)

Table 1 For $(\theta_0, \phi_0) = (1, 2)$, $(a_1, b_1) = (1.5, 2.5)$ and $(a_2, b_2) = (3, 3.5)$ calculation of MLEs, Bayes estimators and their statndard errors

$(0, T]$	$\hat{\theta}_T, \hat{\phi}_T$	$\tilde{\theta}_T, \tilde{\phi}_T$
(0, 10]	1.1093, 2.1080	1.0392, 2.0640
	(0.0119, 0.0116)	(0.0015, 0.0041)
(0, 20]	1.0507, 2.0584	1.0917, 2.1175
	(0.0025, 0.0034)	(0.0084, 0.0138)
(0, 30]	1.0414, 2.0149	1.0938, 1.9102
	(0.0017, 0.0002)	(0.0087, 0.0081)
(0, 40]	1.0148, 2.0304	1.0502, 1.9866
	(0.0087, 0.0002)	(0.0025, 0.0002)
(0, 50]	1.0144, 2.0239	1.0722, 2.0243
	(0.0002, 0.0006)	(0.0052, 0.0006)
(0, 60]	1.0146, 2.0199	1.1026, 2.0856
	(0.0002, 0.0004)	(0.0111, 0.0073)
(0, 70]	1.0081, 2.0181	1.0563, 2.0965
	(0.0001, 0.0003)	(0.0031, 0.0093)
(0, 80]	1.0090, 2.0173	1.1066, 2.0206
	(0.0001, 0.0003)	(0.0001, 0.0004)

Table 2 For $(\theta_0, \phi_0) = (2, 3)$, $(a_1, b_1) = (1.5, 2.5)$ and $(a_2, b_2) = (3, 3.5)$ calculation of MLEs, and their standard errors

$(0, T]$	$\hat{\theta}_T, \hat{\phi}_T$	$\tilde{\theta}_T, \tilde{\phi}_T$
(0, 10]	2.1267, 3.0669	1.9663, 3.0188
	(0.0161, 0.0044)	(0.0012, 0.0004)
(0, 20]	2.0663, 3.0271	2.0351, 2.8563
	(0.0043, 0.0007)	(0.0012, 0.0207)
(0, 30]	2.0513, 3.0262	1.9825, 3.0906
	(0.0026, 0.0007)	(0.0003, 0.0082)
(0, 40]	2.0158, 3.0152	1.9966, 3.1323
	(0.0003, 0.0002)	(0.0001, 0.0175)
(0, 50]	2.0200, 3.0240	1.9671, 2.9936
	(0.0004, 0.0006)	(0.0010, 0.0001)
(0, 60]	2.0072, 3.0104	1.9643, 2.9299
	(0.0001, 0.0002)	(0.0013, 0.0049)
(0, 70]	2.0119, 2.9944	2.0410, 2.9256
	(0.0002, 0.0001)	(0.0017, 0.0055)
(0, 80]	2.0198, 3.0088	2.0941, 3.0678
	(0.0004, 0.0001)	(0.0088, 0.0046)

of θ and ϕ. Here, we consider two pair of true value of parameters θ_0 and ϕ_0 as $(1, 2)$ and $(2, 3)$. For the hyper-parameters we have taken as: $(a_1, b_1) = (1.5, 2.5)$, $(a_2, b_2) = (3, 3.5)$ and $(a_1, b_1) = (3, 5)$, $(a_2, b_2) = (4, 5.5)$. The simulation procedure are repeated 10000 time to estimate the parameters. The computed values of estimators

Table 3 For $(\theta_0, \phi_0) = (1, 2)$, $(a_1, b_1) = (3, 5)$ and $(a_2, b_2) = (4, 5.5)$ calculation of MLEs, Bayes estimators and standard errors

$(0, T]$	$\hat{\theta}_T, \hat{\phi}_T$	$\tilde{\theta}_T, \tilde{\phi}_T$
(0, 10]	1.0171, 2.1420	1.0164, 2.0274
	(0.0002, 0.0201)	(0.0003, 0.0007)
(0, 20]	1.0616, 2.0595	1.0566, 1.9658
	(0.0037, 0.0035)	(0.0032, 0.0012)
(0, 30]	1.0462, 2.0415	1.1018, 2.0134
	(0.0021, 0.0017)	(0.0104, 0.0002)
(0, 40]	1.0134, 2.0216	1.0536, 2.0560
	(0.0002, 0.0005)	(0.0029, 0.0031)
(0, 50]	1.0162, 2.0220	0.9989, 2.1331
	(0.0002, 0.0005)	(0.0029, 0.0031)
(0, 60]	1.0153, 2.0163	1.0809, 2.0087
	(0.0002, 0.0002)	(0.0065, 0.0001)
(0, 70]	1.0135, 2.0172	0.9642, 1.9908
	(0.0001, 0.0003)	(0.0012, 0.0001)
(0, 80]	1.0182, 2.0120	1.0099 1.9962
	(0.0117, 0.0001)	(0.0001, 0.0001)

and their respective standard errors are presented in Tables 1, 2 and 3. The values in the parenthesis indicate the standard errors.

6 Concluding Remarks

In simulation study we present the estimates by proposed methods. It is clear that the estimators are quite closer to the true parameter values and their standard errors are negligible.

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References

Acharya SK (1999) On normal approximation for maximum likelihood estimation from single server queues. Queueing Syst 31:207–216

Acharya SK, Mishra MN (2007) Bernstein–von Mises theorem for $M|M|1$ queue. Commun Stat Theory Methods 36:207–209

Basawa IV, Prabhu NU (1981) Estimation in single server queues. Naval Res Logist Q 28:475–487

Basawa IV, Prabhu NU (1988) Large sample inference from single server queues. Queueing Syst 3(4):289–304

Basawa IV, Prakasa Rao BLS (1980) Statistical inference for stochastic processes. Academic, London

Basawa IV, Prabhu NU, Lund R (1996) Maximum likelihood estimation for single server queues from waiting time data. Queueing Syst 24:155–167

Chen CF (1985) On asymptotic normality of limiting density functions with Bayesian implications. J R Stat Soc Ser B 47:540–546

Hyde CC, Johnston IM (1979) On asymptotic posterior normality for stochastic process. J R Stat Soc Ser B 41:184–189
Johnston R (1970) Asymptotic expansion associated with posterior distributions. Ann Math Stat 41:851–864
Kim JY (1998) Large sample properties of posterior densities, Bayesian information criterion and the likelihood principle in nonstationary time series models. Econometrica 66(2):359–380
Sweeting TJ, Adekola AO (1987) Asymptotic posterior normality for stochastic process revisted. J R Stat Soc Ser B 49:215–222
Walker AM (1969) On the asymptotic behaviour of posterior distributions. J R Stat Soc Ser B 31:80–88
Weng RC, Tsai WC (2008) Asymptotic posterior normality for multiparameter problems. J Stat Plan Inference 138:4068–4080