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RESEARCH ARTICLE

A Unified Approach for Developing Laplace-Type Distributions

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Abstract We present a unified approach for the development and the study of discrete and continuous Laplace-type distributions. As illustrations, we used the proposed approach to develop and study Laplace-type versions of the generalized Pareto, the Geometric, the Poisson and the Negative Binomial distributions.

Keywords The Laplace distribution · The random sign · Random sign mixture transformations

Mathematics Subject Classification Primary 62E15; Secondary 62F12

1 Introduction and Preliminaries

The Laplace distribution (LD) has been studied extensively in the literature. A number of important representations of the LD are given and studied in Kotz et al. [11]. In this paper we present and study in detail two of these representations (transformations) and use them to present a unified approach for the development and the study of discrete and continuous Laplace-type distributions. These transformations are defined next.

Definition 1 The Random Sign Transformation (RST). Assume X and Y are independent rv and Y is $Bernoulli(\beta)$. The RST transformation of X is given by

$$Z_1 = (2Y - 1)\,X.$$

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Definition 2 The Random Sign Mixture Transformation (RSMT). Assume X_1, X_2, Y are independent rv and Y is $Bernoulli(\beta)$. The RSMT of X_1 and X_2 is given by

$$Z_2 = Y X_1 - (1 - Y)\, X_2.$$

Note that the RST is a special case of the RSMT.

The cumulative distribution function (CDF), the quantile function (QF), the probability density function (pdf) and the moment generating function (MGF) of the random variable V will be denoted, respectively, by $F_V(\cdot)$, $F_V^{-1}(\cdot)$, $f_V(\cdot)$ and $M_V(\cdot)$. The entropy (Shannon [19]) of V is defined as $H(V) = -E\{\ln(f_V(V)\}$. The reliability of V relative to the random variable U is defined as $R(V, U) = P(U < V)$. For any $0 \le \alpha \le 1$, we write $\overline{\alpha} = 1 - \alpha$.

Lemma 1 *The following results hold for Z_1 of the RST*:

$$M_{Z_1}(t) = \beta M_X(t) + \overline{\beta} M_X(-t), \tag{1}$$
$$F_{Z_1}(x) = \beta F_X(x) + \overline{\beta}(1 - F_X(-x)) \text{ and } f_{Z_1}(x) = \beta f_X(x) + \overline{\beta} f_X(-x), \tag{2}$$
$$Z_1^+ = Y X^+ + (1 - Y)\, X^-, Z_1^- = Y X^- + (1 - Y)\, X^+ \text{ and } |Z_1| \overset{d}{=} |X|, \tag{3}$$
$$\text{For } r = 2, 4, \ldots, Z_1^r \overset{d}{=} X^r \text{ and, obviously } E(Z_1^r) = E(X^r), \tag{4}$$
$$\text{For } r = 1, 3, 5, \ldots, Z_1^r \overset{d}{=} (2Y - 1)\, X^r \text{ and } E(Z_1^r) = (2\beta - 1)\, E(X^r), \tag{5}$$
$$Var(Z_1) = Var(X) + 4\beta\overline{\beta}\,(E(X))^2, \tag{6}$$
$$E(Z_1 - E(Z_1))^3 = (2\beta - 1)\, E(X - E(X))^3 - 8\beta\overline{\beta}\,(2\beta - 1)\, E(X^3) \tag{7}$$

and

$$E(Z_1 - E(Z_1))^4 = E(X - E(X))^4 + 16\beta\overline{\beta} E(X^3) E(X) - 24\beta\overline{\beta} E(X^2)\,(E(X))^2 + 24\beta\overline{\beta}\,(E(X))^4 \left(1 - 2\beta\overline{\beta}\right). \tag{8}$$

Proof By

$$M_{Z_1}(t) = E\left\{e^{t(2Y-1)X}\right\} = E\left\{E\left\{e^{t(2Y-1)X}\middle| Y\right\}\right\} = E\{M_X(t(2Y-1))\} = \beta M_X(t) + \overline{\beta} M_X(-t)$$

we get (1). Similarly we get (2). For (3) we note that Z_1^+ equals X^+ if $Y = 1$ and $(-X)^+ = X^-$ if $Y = 0$. Similarly, Z_1^- equals X^- if $Y = 1$ and $(-X)^- = X^+$ if $Y = 0$. The proofs of (4) and (5) are straight forward. By (4) and (5) we obtain

$$E(Z_1) = (2\beta - 1)\, E(X),$$
$$E(Z_1^2) = E(X^2),$$
$$E(Z_1^3) = (2\beta - 1)\, E(X^3)$$

and

$$E(Z_1^4) = E(X^4),$$

Hence (6)–(8) are obtained by straightforward computation.

Lemma 2 *The following results hold for Z_2 of the RSMT:*

$$M_{Z_2}(t) = \beta M_{X_1}(t) + \overline{\beta} M_{X_2}(-t), \tag{9}$$

$$F_{Z_2}(x) = \beta F_{X_1}(x) + \overline{\beta}(1 - F_{X_2}(-x)) \text{ and } f_{Z_2}(x) = \beta f_{X_1}(x) + \overline{\beta} f_{X_2}(-x) \tag{10}$$

$$Z_2^+ = YX_1^+ + (1-Y)\,X_2^-, Z_2^- = YX_1^- + (1-Y)\,X_2^+ \text{ and } |Z_2| \stackrel{d}{=} Y\,|X_1| + (1-Y)\,|X_2| \tag{11}$$

$$Z_2^r \stackrel{d}{=} YX_1^r + (-1)^r\,(1-Y)\,X_2^r \text{ and } E(Z_2^r) = \beta E(X_1^r) + (-1)^r \overline{\beta} E(X_2^r), \quad r = 1, 2, \ldots \tag{12}$$

and

$$Var(Z_2) = \beta Var(X_1) + \overline{\beta} Var(X_2) + \beta\overline{\beta}\left\{E(X_1) + E(X_2)\right\}^2. \tag{13}$$

Proof By

$$\begin{aligned} M_{Z_2}(t) &= E\left\{e^{t(YX_1-(1-Y)X_2)}\right\} = E\left\{E\left\{e^{t(YX_1-(1-Y)X_2)}\middle| Y\right\}\right\} \\ &= E\left\{M_{X_1}(tY)M_{X_2}(t(1-Y))\right\} \\ &= \beta M_{X_1}(t) + \overline{\beta} M_{X_2}(-t) \end{aligned}$$

we get (9). Similarly we get (10). For (11) we note that Z_2^+ equals X_1^+ if $Y = 1$ and $(-X_2)^+ = X_2^-$ if $Y = 0$. Similarly, Z_2^- equals X_1^- if $Y = 1$ and $(-X_2)^- = X_2^+$ if $Y = 0$. The proof of (12) is straight forward. By (12) we obtain (13) using

$$E(Z_2) = \beta E(X_1) - \overline{\beta} E(X_2) \text{ and } E(Z_2^2) = \beta E(X_1^2) + \overline{\beta} E(X_2^2).$$

Definition 3 The difference transformation (DT). Assume X_1, X_2 are independent rv. The DT of X_1 and X_2 is given by

$$Z_3 = X_1 - X_2.$$

Different versions of the LD are obtained using the DT, the RST and the RSMT as explained next (see, for example, Sections 2.2 and 3.2 of Kotz et al. [11]). The RST when X has the exponential distribution with mean λ ($X \sim EXP(\lambda)$), results in the LD,

$$f_{Z_1}(x) = \begin{cases} \frac{\overline{\beta}}{\lambda} e^{\frac{x}{\lambda}}, & x < 0 \\ \frac{\beta}{\lambda} e^{-\frac{x}{\lambda}}, & x \geq 0 \end{cases}.$$

The RSMT, $Z_2 = YX_1 - (1-Y)X_2$, with $X_i \sim EXP(\lambda_i)$, $i = 1, 2$ and X_1, X_2 and Y are independent, results in the LD,

$$f_{Z_2}(x) = \begin{cases} \frac{\overline{\beta}}{\lambda_2} e^{\frac{x}{\lambda_2}}, & x < 0 \\ \frac{\beta}{\lambda_1} e^{-\frac{x}{\lambda_1}}, & x \geq 0 \end{cases}. \tag{14}$$

The DT, $Z_3 = X_1 - X_2$, with $X_i \sim EXP(\lambda_i)$, $i = 1, 2$ and X_1 and X_2 are independent results in the LD,

$$f_{Z_3}(x) = \frac{1}{\lambda_1 + \lambda_2} \begin{cases} e^{\frac{x}{\lambda_2}}, & x < 0 \\ e^{-\frac{x}{\lambda_1}}, & x \geq 0 \end{cases}.$$

Note that taking $\beta = \frac{\lambda_1}{\lambda_1 + \lambda_2}$ in (14) results in $Z_2 \stackrel{d}{=} Z_3$.

In general, the RSMT and the DT result in two different distributions. For example, when X_1 and X_2 are iid uniform rv on (0, 1), the DT results in

$$f_{Z_3}(z) = 1 - |z|, -1 \leq z \leq 1$$

while the RSMT results in

$$f_{Z_2}(z) = \begin{cases} \overline{\beta}, & -1 \leq z < 0 \\ \beta, & 0 \leq z \leq 1 \end{cases}.$$

Another more general example is as follows. Let X_1 and X_2 be independent rv with $E(X_i) = \mu_i$ and $Var(X_i) = \sigma_i^2, i = 1, 2$. In this case the DT results in $E(Z_3) = \mu_1 - \mu_2$ and $Var(Z_3) = \sigma_1^2 + \sigma_2^2$, whereas the RSMT with $\beta = \frac{\mu_1}{\mu_1 + \mu_2}$ results in $E(Z_2) = \mu_1 - \mu_2$ and $Var(Z_2) = \frac{1}{\mu_1 + \mu_2} \left\{ \mu_1 \sigma_1^2 + \mu_2 \sigma_2^2 + \mu_1 \mu_2 (\mu_1 + \mu_2) \right\}$.

The RST, RSMT and DT are most useful when applied to nonnegative rv to create new rv with negative and positive values. In this paper we will focus our attention on the RST and RSMT when applied to nonnegative rv

In Sects. 2 and 3 we study in details the RST and the RSMT of nonnegative rv. In Sect. 4 we introduce and study the Double Generalized Pareto distributions. In Sect. 5 we introduce and study new double discrete distributions based on the discrete Generalized Pareto, the Geometric, the Poisson, the Binomial and the Negative Binomial distributions. In Sect. 6 we consider the distributions of sums of independent rv obtained using the RST and the RSMT of nonnegative rv. In Sect. 7 we apply the Double Poisson distribution to two real data sets.

2 The RST of a Nonnegative *r v*

We assume in this section that $X \geq 0$ in the RST.

Lemma 3 *In addition to Lemma 1, the following results hold*

1. $Z_1^+ = YX$, $Z_1^- = (1-Y)\,X$ *and* $|Z_1| \stackrel{d}{=} X$
2. $E\left(|Z_1|^r\right) = E(X^r)$, *for* $r = 1, 2, 3, \ldots$
3.

$$F_{Z_1}(x) = \begin{cases} \overline{\beta}(1 - F_X(|x|)), & x < 0 \\ \overline{\beta} + \beta F_X(x), & x \geq 0 \end{cases}, \tag{15}$$

$$f_{Z_1}(x) = \begin{cases} \overline{\beta} f_X(|x|), & x < 0 \\ \beta f_X(x), & x \geq 0 \end{cases} \tag{16}$$

and

$$F_{Z_1}^{-1}(t) = \begin{cases} -F_X^{-1}(1 - \frac{t}{\overline{\beta}}), & 0 < t \leq \overline{\beta} \\ F_X^{-1}\left(\frac{t - \overline{\beta}}{\beta}\right), & \overline{\beta} \leq t < 1 \end{cases}.$$

Lemma 4 *The entropy of* Z_1 *is given by*

$$H(Z_1) = H(Y) + H(X),$$

if X *is continuous and*

$$H(Z_1) = (1 - f_X(0))\, H(Y) + H(X)$$

if X *is discrete, where*

$$H(Y) = -\beta \ln \beta - \overline{\beta} \ln \overline{\beta}. \tag{17}$$

We will not give a proof of Lemma 4 because it follows from Lemma 7 as a special case.

Lemma 5 *For* $i = 1, 2$, *assume that* $Y_i \sim Bernoulli(\beta_i)$, $X_i > 0$ *is continuous and* $Z_{1,i} = (2Y_i - 1)X_i$, *where* Y_1, Y_2, X_1 *and* X_2 *are independent. Then,*

$$R(Z_{1,1}, Z_{1,2}) = \overline{\beta}_2 + (\beta_1 + \beta_2 - 1)\, R(X_1, X_2). \tag{18}$$

Proof Note that (18) follows from (15), (16) and

$$\begin{aligned} R(Z_{1,1}, Z_{1,2}) &= \int_{-\infty}^{\infty} F_{Z_{1,2}}(x) f_{Z_{1,1}}(x) dx \\ &= \int_{-\infty}^{0} \overline{\beta}_2 \left(1 - F_{X_2}(-x)\right) \overline{\beta}_1 f_{X_1}(-x) dx + \int_0^{\infty} \big(\overline{\beta}_2 \end{aligned}$$

$$
\begin{aligned}
&+\beta_2 F_{X_2}(x)\big)\,\beta_1 f_{X_1}(x)dx\\
&=\overline{\beta}_1\overline{\beta}_2\int_{-\infty}^{0} f_{X_1}(-x)dx-\overline{\beta}_1\overline{\beta}_2\int_{-\infty}^{0} F_{X_2}(-x)f_{X_1}(-x)dx\\
&+\beta_1\overline{\beta}_2\int_{0}^{\infty} f_{X_1}(x)dx+\beta_1\beta_2\int_{0}^{\infty} F_{X_2}(x)f_{X_1}(x)dx\\
&=\overline{\beta}_1\overline{\beta}_2-\overline{\beta}_1\overline{\beta}_2 R(X_1,X_2)+\beta_1\overline{\beta}_2+\beta_1\beta_2 R(X_1,X_2).
\end{aligned}
$$

Theorem 1 *Assume that $\underline{T}(\underline{X})$ is the MLE of $\underline{\theta}$ based on a random sample from $f_X(x;\underline{\theta})$ and let $I_X(\underline{\theta})$ be the corresponding Fisher information Matrix. Let $Z_{1,1}, Z_{1,2}, \ldots, Z_{1,n}$ be a random sample from*

$$
f_{Z_1}(z;\underline{\theta},\beta)=\begin{cases}\overline{\beta} f_X\left(|z|\,;\underline{\theta}\right) & ,z<0\\ \beta f_X(z;\underline{\theta}), & z\ge 0\end{cases}.
$$

Assume that $0<n_1=\sum I(z_{1,i}\ge 0)<n$. *Let* $\widehat{\beta}$ *and* $\widehat{\underline{\theta}}$ *be the MLE of* β *and* $\underline{\theta}$. *Then,*

1.

$$
\widehat{\underline{\theta}}=\underline{T}(\left|Z_{1,1}\right|,\left|Z_{1,2}\right|,\ldots,\left|Z_{1,n}\right|) \tag{19}
$$

2. *If X is continuous*

$$
\widehat{\beta}=\frac{n_1}{n}. \tag{20}
$$

3. *If X is discrete with* $f_X(0;\underline{\theta})>0$

$$
\widehat{\beta}=\frac{\sum I(z_{1,i}>0)}{\sum I(z_{1,i}>0)+\sum I(z_{1,i}<0)}. \tag{21}
$$

4.

$$
\sqrt{n}\begin{pmatrix}\widehat{\beta}-\beta\\ \widehat{\underline{\theta}}-\underline{\theta}\end{pmatrix}\xrightarrow{d} N\left(\underline{0},\begin{bmatrix}\beta(1-\beta) & 0\\ 0 & I_X^{-1}(\underline{\theta})\end{bmatrix}\right). \tag{22}
$$

Proof For simplicity, we will prove the results when X is continuous and $\underline{\theta}$ has dimension 1. The LF of the sample is given by

$$
L(\theta,\beta;\underline{z}_1)=\beta^{n_1}(1-\beta)^{n-n_1}\prod_{i=1}^{n} f_X(\left|z_{1,i}\right|;\theta).
$$

Hence we obtain (19) and (20). Note that

$$
\frac{\partial^2\ln f_{Z_1}(x;\theta,\beta)}{\partial\beta^2}=\begin{cases}-\frac{1}{(1-\beta)^2}, & x<0\\ -\frac{1}{\beta^2}, & x\ge 0\end{cases},
$$

$$\frac{\partial^2 \ln f_{Z_1}(x;\theta)}{\partial\theta^2} = \begin{cases} \frac{\partial^2 \ln f_X(|x|;\theta)}{\partial\theta^2} & , x < 0 \\ \frac{\partial^2 \ln f_X(x;\theta)}{\partial\theta^2} & , x \geq 0 \end{cases}$$

and

$$\frac{\partial^2 \ln f_{Z_1}(x;\theta,\beta)}{\partial\beta\partial\theta} = 0.$$

Hence

$$-E\left\{\frac{\partial^2 \ln f_{Z_1}(Z_1;\theta,\beta)}{\partial\beta^2}\right\} = -\beta\left(-\frac{1}{\beta^2}\right) - \overline{\beta}\left(-\frac{1}{(1-\beta)^2}\right) = \frac{1}{\beta\,(1-\beta)}$$

and

$$\begin{aligned} -E\left\{\frac{\partial^2 \ln f_{Z_1}(Z_1;\theta,\beta)}{\partial\theta^2}\right\} &= -\beta E\left\{\frac{\partial^2 \ln f_X(|Z_1|\,;\theta)}{\partial\theta^2}\right\} - \overline{\beta}E\left\{\frac{\partial^2 \ln f_X(|Z_1|\,;\theta)}{\partial\theta^2}\right\} \\ &= -E\left\{\frac{\partial^2 \ln f_X(X;\theta)}{\partial\theta^2}\right\} = I_X(\theta). \end{aligned}$$

This proves (22).

3 The RSMT of Nonnegative *rv*

We assume in this section that $X_1 \geq 0$ and $X_2 \geq 0$ in the RSMT.

Lemma 6 *In addition to the results of Lemma 2 we have*

1. $|Z_2| \overset{d}{=} YX_1 + (1-Y)\,X_2$
2. $|Z_2|^r \overset{d}{=} YX_1^r + (1-Y)\,X_2^r$ *and* $E\left(|Z_2|^r\right) \overset{d}{=} \beta E\left(X_1^r\right) + \overline{\beta}E\left(X_2^r\right)$
3.

$$F_{Z_2}(x) = \begin{cases} \overline{\beta}(1 - F_{X_2}(|x|)), & x < 0 \\ \overline{\beta} + \beta F_{X_1}(x), & x \geq 0 \end{cases},$$

$$f_{Z_2}(x) = \begin{cases} \overline{\beta} f_{X_2}(|x|), & x < 0 \\ \beta f_{X_1}(x), & x \geq 0 \end{cases}$$

and

$$F_{Z_2}^{-1}(t) = \begin{cases} -F_{X_2}^{-1}(1 - \frac{t}{\overline{\beta}}), & 0 < t \leq \overline{\beta} \\ F_{X_1}^{-1}\left(\frac{t-\overline{\beta}}{\beta}\right), & \overline{\beta} \leq t < 1 \end{cases}.$$

Lemma 7 *Let $H(Y)$ be as in (17). For the entropy of Z_2 we have*

1. *If X_1 and X_2 are continuous or X_1 and X_2 are discrete with $f_{X_1}(0)\cdot f_{X_2}(0) = 0$, then*

$$H(Z_2) = H(Y) + \beta H(X_1) + \overline{\beta} H(X_2). \tag{23}$$

2. *If X_1 and X_2 are discrete with $f_{X_1}(0)\cdot f_{X_2}(0) > 0$, then*

$$\begin{aligned} H(Z_2) &= \beta H(X_1) + \overline{\beta} H(X_2) + H(Y) + \beta f_{X_1}(0) \ln\left(\frac{\beta f_{X_1}(0)}{\beta f_{X_1}(0) + \overline{\beta} f_{X_2}(0)}\right) \\ &\quad + \overline{\beta} f_{X_2}(0) \ln\left(\frac{\overline{\beta} f_{X_2}(0)}{\beta f_{X_1}(0) + \overline{\beta} f_{X_2}(0)}\right). \end{aligned} \tag{24}$$

Proof We will prove only (24). Assume that X_1 and X_2 are discrete with $f_{X_1}(0)\cdot f_{X_2}(0) > 0$. Then,

$$f_{Z_2}(x) = \begin{cases} \overline{\beta} f_{X_2}(|x|) & , x = -1, -2, \ldots \\ \beta f_{X_1}(0) + \overline{\beta} f_{X_2}(0) & , x = 0 \\ \beta f_{X_1}(x) & , x = 1, 2, \ldots \end{cases}.$$

Note that (24) follows from

$$\begin{aligned} H(Z_2) =& -\overline{\beta} \sum_{-1}^{-\infty} f_{X_2}(|x|) \ln(\overline{\beta} f_{X_2}(|x|)) - \left(\beta f_{X_1}(0) + \overline{\beta} f_{X_2}(0)\right) \ln\left(\beta f_{X_1}(0) + \overline{\beta} f_{X_2}(0)\right) \\ & - \beta \sum_{1}^{\infty} f_{X_1}(x) \ln(\beta f_{X_1}(x)), \\ & - \overline{\beta} \sum_{-1}^{-\infty} f_{X_2}(|x|) \ln(\overline{\beta} f_{X_2}(|x|)) = -\overline{\beta} \sum_{0}^{\infty} f_{X_2}(x) \ln(\overline{\beta} f_{X_2}(x)) + \overline{\beta} f_{X_2}(0) \ln(\overline{\beta} f_{X_2}(0)) \\ =& \overline{\beta} H(X_2) - \overline{\beta} \ln \overline{\beta} + \overline{\beta} f_{X_2}(0) \ln(\overline{\beta} f_{X_2}(0)) \end{aligned}$$

and

$$-\beta \sum_{1}^{\infty} f_{X_1}(x) \ln(\beta f_{X_1}(x)) = \beta H(X_1) - \beta \ln \beta + \beta f_{X_1}(0) \ln(\beta f_{X_1}(0)).$$

Remarks 1. If $f_{X_1}(0) = f_{X_2}(0) = p$ in (24), then

$$H(Z_2) = \beta H(X_1) + \overline{\beta} H(X_2) + (1 - p)\, H(Y).$$

2. Lemma 4 is the special case of Lemma 7 when $X_1 \overset{d}{=} X_2$.

Lemma 8 *For $i = 1, 2$, assume that $Y_i \sim Bernoulli(\beta_i)$, $X_{i,j} > 0$, $j = 1, 2$ are continuous and $Z_{2,i} = Y_i X_{i,1} - (1 - Y_i)\, X_{i,2}$, where Y_1, Y_2 and the X's are independent. Then,*

$$R(Z_{2,1}, Z_{2,2}) = \overline{\beta}_2 - \overline{\beta}_1 \overline{\beta}_2 R(X_{1,2}, X_{2,2}) + \beta_1 \beta_2 R(X_{1,1}, X_{2,1}).$$

The proof of Lemma 8 is parallel to that of Lemma 5.

Remark In the rest of this paper when the X's of the RSMT are discrete we will assume that $f_{X_2}(0; \underline{\theta}_2) = 0$.

Theorem 2 *Assume, for $j = 1, 2$, that $\underline{T}_{j,m}(\underline{X}_j)$ is the MLE of $\underline{\theta}_j$ based on a random sample of size m from $f_{X_j}(x; \underline{\theta}_j)$ and let $I_{X_j}(\underline{\theta}_j)$ be the corresponding Fisher information Matrix. Let $Z_{2,1}, Z_{2,2}, \ldots, Z_{2,n}$ be a random sample from*

$$f_{Z_2}(z; \underline{\theta}_1, \underline{\theta}_2, \beta) = \begin{cases} \overline{\beta} f_{X_2}(|z|\,; \underline{\theta}_2), & z < 0 \\ \beta f_{X_1}(z; \underline{\theta}_1), & z \geq 0 \end{cases}.$$

Assume, without any loss of generality that $z_{2,i} \geq 0$, $i = 1, 2, \ldots, n_1$, $0 < n_1 < n$ and $z_{2,i} < 0$, $i = n_1 + 1, \ldots, n$. Let $\widehat{\beta}$, $\widehat{\underline{\theta}}_1$ and $\widehat{\underline{\theta}}_2$ be the MLE of β, $\underline{\theta}_1$ and $\underline{\theta}_2$. Then,

$$\widehat{\beta} = \frac{n_1}{n}, \tag{25}$$

$$\widehat{\underline{\theta}}_1 = \underline{T}_{1,n_1}(z_{2,1}, z_{2,2}, \ldots, z_{2,n_1}), \tag{26}$$

$$\widehat{\underline{\theta}}_2 = \underline{T}_{2,n-n_1}(-z_{2,n_1+1}, -z_{2,n_1+2}, \ldots, -z_{2,n}) \tag{27}$$

and

$$\sqrt{n} \begin{pmatrix} \widehat{\beta} - \beta \\ \widehat{\underline{\theta}}_1 - \underline{\theta}_1 \\ \widehat{\underline{\theta}}_2 - \underline{\theta}_2 \end{pmatrix} \xrightarrow{d} N \left(\underline{0}, \begin{bmatrix} \beta\,(1-\beta) & 0 & 0 \\ 0 & \frac{1}{\beta} I_{X_1}^{-1}(\underline{\theta}_1) & 0 \\ 0 & 0 & \frac{1}{\overline{\beta}} I_{X_2}^{-1}(\underline{\theta}_2) \end{bmatrix} \right). \tag{28}$$

Proof For simplicity, we will prove the results when both $\underline{\theta}_1$ and $\underline{\theta}_2$ have dimension 1. It is clear that (25)–(27) follow from the result that the LF of the sample is given by

$$L(\theta_1, \theta_2, \beta; \underline{z}_2) = \beta^{n_1}\,(1-\beta)^{n-n_1} \prod_{i=1}^{n_1} f_{X_1}(z_{2,i}; \theta_1) \prod_{i=n_1+1}^{n} f_{X_2}(\left|z_{2,i}\right|; \theta_2).$$

Note that

$$-E\left\{\frac{\partial^2 \ln f_{Z_2}(Z_2; \theta_1, \theta_2, \beta)}{\partial \beta^2}\right\} = -\beta\left(-\frac{1}{\beta^2}\right) - \overline{\beta}\left(-\frac{1}{(1-\beta)^2}\right) = \frac{1}{\beta\,(1-\beta)},$$

$$-E\left\{\frac{\partial^2 \ln f_{Z_2}(Z_2; \theta_1, \theta_2, \beta)}{\partial \theta_1^2}\right\} = -\beta E\left\{\frac{\partial^2 \ln f_{X_1}(X_1; \theta_1)}{\partial \theta_1^2}\right\} = \beta I_{X_1}(\theta_1),$$

$$-E\left\{\frac{\partial^2 \ln f_{Z_2}(Z_2;\theta_1,\theta_2,\beta)}{\partial\theta_2^2}\right\} = -\beta E\left\{\frac{\partial^2 \ln f_{X_2}(X_2;\theta_2)}{\partial\theta_2^2}\right\} = \overline{\beta} I_{X_2}(\theta_2)$$

and

$$E\left\{\frac{\partial^2 \ln f_{Z_2}(Z_2;\theta_1,\theta_2,\beta)}{\partial\beta\partial\theta_1}\right\} = E\left\{\frac{\partial^2 \ln f_{Z_2}(Z_2;\theta_1,\theta_2,\beta)}{\partial\beta\partial\theta_2}\right\}$$
$$= E\left\{\frac{\partial^2 \ln f_{Z_2}(Z_2;\theta_1,\theta_2,\beta)}{\partial\theta_1\partial\theta_2}\right\} = 0.$$

This completes the proof of (28).

Next we consider the special case when X_1 and X_2 are from the same family but with some common parameters. In this case

$$f_{Z_2}(z;\underline{\delta},\underline{\theta}_1,\underline{\theta}_2,\beta) = \begin{cases} \overline{\beta} f(|z|;\underline{\delta},\underline{\theta}_2), & z < 0 \\ \beta f(z;\underline{\delta},\underline{\theta}_1), & z \geq 0 \end{cases}.$$

Theorem 3 *Let $\widehat{\beta}, \widehat{\underline{\delta}}, \widehat{\underline{\theta}}_1$ and $\widehat{\underline{\theta}}_2$ be the MLE of $\beta, \underline{\delta}, \underline{\theta}_1$ and $\underline{\theta}_2$ based on a random sample $Z_{2,1}, Z_{2,2}, \ldots, Z_{2,n}$ from*

$$f_{Z_2}(z;\underline{\delta},\underline{\theta}_1,\underline{\theta}_2,\beta) = \begin{cases} \overline{\beta} f(|z|;\underline{\delta},\underline{\theta}_2) & , z < 0 \\ \beta f(z;\underline{\delta},\underline{\theta}_1) & , z \geq 0 \end{cases}.$$

Assume that Fisher information Matrix associated with $f(x;\underline{\delta},\underline{\theta})$ is given by

$$I(\underline{\delta},\underline{\theta}) = \left[I_{ij}(\underline{\delta},\underline{\theta})\right]_{i,j=1,2}$$

and $0 < \sum_{i=1}^n I(Z_{2,i} > 0) < n$. Then,

$$\widehat{\beta} = \frac{\sum_{i=1}^n I(Z_{2,i} > 0)}{n},$$

and $\widehat{\underline{\delta}}, \widehat{\underline{\theta}}_1$ and $\widehat{\underline{\theta}}_2$ are obtained by solving the normal equations

$$\sum_{i=1}^n \frac{\partial \ln f(z_{2,i};\underline{\delta},\underline{\theta}_1)}{\partial\underline{\delta}} I(z_{2,i} > 0) + \sum_{i=1}^n \frac{\partial \ln f(\left|z_{2,i}\right|;\underline{\delta},\underline{\theta}_2)}{\partial\underline{\delta}} I(z_{2,i} < 0) = 0,$$
$$\sum_{i=1}^n \frac{\partial \ln f(z_{2,i};\underline{\delta},\underline{\theta}_1)}{\partial\underline{\theta}_1} I(z_{2,i} > 0) = 0$$

and

$$\sum_{i=1}^n \frac{\partial \ln f(\left|z_{2,i}\right|;\underline{\delta},\underline{\theta}_2)}{\partial\underline{\theta}_2} I(z_{2,i} < 0) = 0.$$

In addition, we have

$$\sqrt{n}\begin{pmatrix}\widehat{\beta}-\beta\\ \widehat{\underline{\delta}}-\underline{\delta}\\ \widehat{\underline{\theta}}_1-\underline{\theta}_1\\ \widehat{\underline{\theta}}_2-\underline{\theta}_2\end{pmatrix}\xrightarrow{d}N\left(\underline{0},\begin{bmatrix}\beta(1-\beta) & \underline{0}\\ \underline{0} & I^{-1}(\underline{\delta},\underline{\theta}_1,\underline{\theta}_2)\end{bmatrix}\right),$$

where

$$I(\underline{\delta},\underline{\theta}_1,\underline{\theta}_2)=\begin{bmatrix}\beta I_{11}(\underline{\delta},\underline{\theta}_1)+\overline{\beta}I_{11}(\underline{\delta},\underline{\theta}_2) & \beta I_{12}(\underline{\delta},\underline{\theta}_1) & \overline{\beta}I_{12}(\underline{\delta},\underline{\theta}_2)\\ \beta I_{12}(\underline{\delta},\underline{\theta}_1) & \beta I_{22}(\underline{\delta},\underline{\theta}_1) & 0\\ \overline{\beta}I_{12}(\underline{\delta},\underline{\theta}_2) & 0 & \overline{\beta}I_{22}(\underline{\delta},\underline{\theta}_2)\end{bmatrix}.$$

4 The Double Generalized Pareto distributions

Following the notation of de Zea Bermudez and Kotz [6], the rv X has the two parameter Generalized Pareto distribution $GP(\kappa,\sigma)$ if its CDF and pdf are given by

$$F(x)=1-\left(1-\frac{\kappa x}{\sigma}\right)^{\frac{1}{\kappa}},\kappa\neq 0$$

and

$$f(x)=\frac{1}{\sigma}\left(1-\frac{\kappa x}{\sigma}\right)^{\frac{1}{\kappa}-1},\kappa\neq 0,$$

with $0\leq x\leq\frac{\sigma}{\kappa}$ if $\kappa>0$ and $0\leq x<\infty$ if $\kappa<0$. Smith [22] proved the asymptotic Normality of the MLE of κ and σ, for $\kappa<0.5$.

The first Double Generalized Pareto distribution, denoted by $DGP(\beta,\kappa,\sigma)$, is obtained using the RST when X has the $GP(\kappa,\sigma)$. By Lemma 1, the CDF, QF and pdf of the $DGP(\beta,\kappa,\sigma)$ are given by

$$F_{Z_1}(x)=\begin{cases}1-\beta\left(1-\frac{\kappa x}{\sigma}\right)^{\frac{1}{\kappa}}, & x\geq 0\\ \overline{\beta}\left(1-\frac{\kappa|x|}{\sigma}\right)^{\frac{1}{\kappa}}, & x<0\end{cases},$$

$$F_{Z_1}^{-1}(t)=\begin{cases}-\frac{\sigma}{\kappa}\left(1-\left(\frac{t}{\overline{\beta}}\right)^{\kappa}\right), & 0<t\leq\overline{\beta}\\ \frac{\sigma}{\kappa}\left(1-\left(\frac{1-t}{\beta}\right)^{\kappa}\right), & \overline{\beta}\leq t<1\end{cases}$$

and

$$f_{Z_1}(x)=\begin{cases}\frac{\beta}{\sigma}\left(1-\frac{\kappa x}{\sigma}\right)^{\frac{1}{\kappa}-1}, & x\geq 0\\ \frac{\overline{\beta}}{\sigma}\left(1-\frac{\kappa|x|}{\sigma}\right)^{\frac{1}{\kappa}-1}, & x<0\end{cases}.$$

The MLE of the parameters of $DGP(\beta, \kappa, \sigma)$ and the corresponding asymptotic theory are obtained using Theorem 1 and the results of Smith [22].

Note that $DGP(\frac{1}{2}, \kappa, \sigma)$ appeared in the work of Armagan et al. [2] and Wang [24]. Nadarajah et al. [13] studied in details the distribution of $DGP(\frac{1}{2}, \kappa, \sigma)$ and obtained the maximum likelihood estimators of its parameters.

The second distribution, denoted by $DGP(\beta, \kappa_1, \sigma_1; \kappa_2, \sigma_2)$, is obtained by assuming in the RSMT that X_i has the $GP(\kappa_i, \sigma_i)$, $i = 1, 2$. For this distribution, by Lemma 2, we have

$$F_{Z_2}(x) = \begin{cases} 1 - \beta\left(1 - \frac{\kappa_1 x}{\sigma_1}\right)^{\frac{1}{\kappa_1}}, & x \geq 0 \\ \overline{\beta}\left(1 - \frac{\kappa_2 |x|}{\sigma_2}\right)^{\frac{1}{\kappa_2}}, & x < 0 \end{cases},$$

$$F_{Z_2}^{-1}(t) = \begin{cases} -\frac{\sigma_2}{\kappa_2}\left(1 - \left(\frac{t}{\overline{\beta}}\right)^{\kappa_2}\right), & 0 < t \leq \overline{\beta} \\ \frac{\sigma_1}{\kappa_1}\left(1 - \left(\frac{1-t}{\beta}\right)^{\kappa_1}\right), & \overline{\beta} \leq t < 1 \end{cases}$$

and

$$f_{Z_2}(x) = \begin{cases} \frac{\beta}{\sigma_1}\left(1 - \frac{\kappa_1 x}{\sigma_1}\right)^{\frac{1}{\kappa_1} - 1}, & x \geq 0 \\ \frac{\overline{\beta}}{\sigma_2}\left(1 - \frac{\kappa_2 |x|}{\sigma_2}\right)^{\frac{1}{\kappa_1} - 1}, & x < 0 \end{cases}.$$

Other important special cases of $DGP(\beta, \kappa_1, \sigma_1; \kappa_2, \sigma_2)$ are $DGP(\beta, \kappa, \sigma_1; \kappa, \sigma_2)$, $DGP(\frac{1}{2}, \kappa_1, \sigma_1; \kappa_2, \sigma_2)$, $DGP(\frac{1}{2}, \kappa, \sigma_1; \kappa, \sigma_2)$, $DGPD(\beta, \kappa_1, \sigma; \kappa_2, \sigma)$ and $DGP(\frac{1}{2}, \kappa_1, \sigma; \kappa_2, \sigma)$. In any of these special cases the MLE of the parameters and the corresponding asymptotic theory are obtained using Theorem 2 or 3 and the results of Smith [22].

The rv X has the Generalized Pareto(IV) distribution $GP\text{-}IV(\kappa, \sigma, \gamma)$ if its CDF and pdf are given by

$$F(x) = 1 - \left(1 - \kappa\left(\frac{x}{\sigma}\right)^{\frac{1}{\gamma}}\right)^{\frac{1}{\kappa}}, \kappa \neq 0$$

and

$$f(x) = \frac{x^{\frac{1}{\gamma} - 1}}{\gamma \sigma^{\frac{1}{\gamma}}}\left(1 - \kappa\left(\frac{x}{\sigma}\right)^{\frac{1}{\gamma}}\right)^{\frac{1}{\kappa} - 1}, \kappa \neq 0,$$

where $\gamma > 0$ and $0 \leq x \leq \frac{\sigma}{\kappa^\gamma}$ if $\kappa > 0$ and $0 \leq x < \infty$ if $\kappa < 0$. The information Matrix for the parameters of $GP\text{-}IV(\kappa, \sigma, \gamma)$ is given in Barazauskas [4].

Similar to $DGP(\beta, \kappa, \sigma)$ we can use the RST to obtain the first Double Generalized Pareto (IV) distribution, denoted by $DGP\text{-}IV(\beta, \kappa, \sigma, \gamma)$. In addition, similar to

$DGP(\beta, \kappa_1, \sigma_1; \kappa_2, \sigma_2)$ we can use the RSMT to obtain the second Double Generalized Pareto (IV) distribution, denoted by $DGPD\text{-}IV(\beta, \kappa_1, \sigma_1, \gamma_1; \kappa_2, \sigma_2, \gamma_2)$.

5 Double Discrete Distributions

The development of integer-valued rv with negative and positive support has received increased attention in the past decade, see for example, Skellam [21], Kozubowski and Inusah [12], Alzaid and Omair [1], Barbiero [5], Seetha Lekshmi and Sebastian [18] and Bakouch et al. [3]. Integer-valued rv with negative and positive support have recently been used in the developement of stationary integer-valued Time Series with negative and positive support. Some exmples of theses models are given in Freeland [8] and Nastić et al. [14].

In this section we introduce and study a number of double discrete distributions using the RST and the RSMT. To avoid facing the issue of identifiably at zero when using the RSMT for discrete rv we only use distributions for X_1 and X_2 such that $f_{X_1}(0) \cdot f_{X_2}(0) = 0$. In the following, without any loss of generality, we will assume that $f_{X_2}(0) = 0$.

5.1 The Double Discrete Generalized Pareto Distributions

The rv X has the two parameter Discrete Generalized Pareto distribution $DGP(\kappa, \sigma)$ (See Buddana and Kozubowski [7] for an alternative definition) if its CDF and pdf are given by

$$F(x) = 1 - \left(1 - \frac{\kappa\left(\lfloor x \rfloor + 1\right)}{\sigma}\right)^{\frac{1}{\kappa}}, \kappa \neq 0$$

and

$$f(x) = \left(1 - \frac{\kappa x}{\sigma}\right)^{\frac{1}{\kappa}} - \left(1 - \frac{\kappa\left(x + 1\right)}{\sigma}\right)^{\frac{1}{\kappa}}, \kappa \neq 0$$

where $\lfloor \cdot \rfloor$ is the floor function and $x = 0, 1, \ldots, \left\lfloor \frac{\sigma}{\kappa} \right\rfloor - 1$ if $\kappa > 0$ and $x = 0, 1, 2, \ldots$ if $\kappa < 0$.

The first Double Discrete Generalized Pareto distribution, denoted by $DDGP\text{-}I(\beta, \kappa, \sigma)$, is obtained using the RST when X has the $DGP(\kappa, \sigma)$. By Lemma 1, the CDF and pdf of the $DDGP(\beta, \kappa, \sigma)$ are given by

$$F_{Z_1}(x) = \begin{cases} 1 - \beta\left(1 - \frac{\kappa(\lfloor x \rfloor + 1)}{\sigma}\right)^{\frac{1}{\kappa}}, & x \geq 0 \\ \overline{\beta}\left(1 - \frac{\kappa(-\lfloor x \rfloor)}{\sigma}\right)^{\frac{1}{\kappa}}, & x < 0 \end{cases},$$

and

$$f_{Z_1}(x) = \begin{cases} \beta\left\{\left(1-\frac{\kappa x}{\sigma}\right)^{\frac{1}{\kappa}} - \left(1-\frac{\kappa(x+1)}{\sigma}\right)^{\frac{1}{\kappa}}\right\}, & x = 0, 1, \ldots \\ \overline{\beta}\left\{\left(1-\frac{\kappa|x|}{\sigma}\right)^{\frac{1}{\kappa}} - \left(1-\frac{\kappa(|x|+1)}{\sigma}\right)^{\frac{1}{\kappa}}\right\}, & x = -1, -2, \ldots \end{cases}.$$

The second distribution, denoted by $DDGP\text{-}II(\beta, \kappa_1, \sigma_1, \kappa_2, \sigma_2)$, is obtained by assuming in the RSMT that $X_1 \sim DGP(\kappa_1, \sigma_1)$ and $X_2 \sim (DGP(\kappa_2, \sigma_2) + 1)$. For this distribution, by Lemma 2, we have

$$F_{Z_2}(x) = \begin{cases} 1-\beta\left(1-\frac{\kappa_1(\lfloor x\rfloor+1)}{\sigma_1}\right)^{\frac{1}{\kappa_1}}, & x \geq 0 \\ \overline{\beta}\left(1-\frac{\kappa_2(-\lfloor x\rfloor-1)}{\sigma_2}\right)^{\frac{1}{\kappa_2}}, & x < 0 \end{cases}$$

and

$$f_{Z_2}(x) = \begin{cases} \beta\left\{\left(1-\frac{\kappa_1 x}{\sigma_1}\right)^{\frac{1}{\kappa_1}} - \left(1-\frac{\kappa_1(x+1)}{\sigma_1}\right)^{\frac{1}{\kappa_1}}\right\}, & x = 0, 1, 2, \ldots \\ \overline{\beta}\left\{\left(1-\frac{\kappa_2(|x|-1)}{\sigma_2}\right)^{\frac{1}{\kappa_2}} - \left(1-\frac{\kappa_2|x|}{\sigma_2}\right)^{\frac{1}{\kappa_2}}\right\}, & x = -1, -2, \ldots \end{cases}.$$

Important special cases of $DDGP\text{-}II(\beta, \kappa_1, \sigma_1; \kappa_2, \sigma_2)$ are $DDGP\text{-}II(\beta, \kappa, \sigma_1, \sigma_2) \overset{d}{=} DDGP\text{-}II(\beta, \kappa, \sigma_1, \kappa, \sigma_2)$ and $DDGP\text{-}II(\beta, \kappa_1, \kappa_2, \sigma) \overset{d}{=} DDGP\text{-}II(\beta, \kappa_1, \sigma, \kappa_2, \sigma)$.

5.2 The Double Geometric Distributions Using the RST

The rv U is said to have the Geometric $Geo_1(\theta)$ (resp. $Geo_0(\theta)$) if its pdf is given by $f_U(x) = \overline{\theta}\theta^{x-1}$, $x = 1, 2, \ldots$ (resp., $f_U(x) = \overline{\theta}\theta^x$, $x = 0, 1, 2, \ldots$). Note that for $U \sim Geo_1(\theta)$ we have $F_U(x) = 1 - \theta^{\lfloor x\rfloor}$, $x \geq 0$, $E(U) = \frac{1}{\overline{\theta}}$, $Var(U) = \frac{\theta}{\overline{\theta}^2}$ and

$$M_U(t) = \frac{\overline{\theta}e^t}{1-\theta e^t}.$$

For both $Geo_1(\theta)$ and $Geo_0(\theta)$, $I(\theta) = \frac{1}{\theta\overline{\theta}^2}$.

Consider the RST $Z_1 = (2Y - 1)X$, where $Y \sim Bernoulli(\beta)$ and X is Geometric rv and X and Y are independent. Depending on the distribution of X we have two Double Geometric distributions. The MLE of the parameters and the corresponding asymptotic theory are obtained using Theorem 1 and the well known results for the MLE of the Geometric distribution.

5.2.1 $DG - I(\beta, \theta)$

Using $X \sim Geo_1(\theta)$ in the RST we obtain $DG - I(\beta, \theta)$. For this distribution we have

1. $f_{Z_1}(x) = \begin{cases} \overline{\beta\theta}\theta^{-x-1} & , x = -1, -2, \ldots \\ 0 & , x = 0 \\ \beta\overline{\theta}\theta^{x-1} & , x = 1, 2, \ldots \end{cases}$
2. $F_{Z_1}(x) = \begin{cases} \overline{\beta}\theta^{-\lfloor x \rfloor - 1} & , x < -1 \\ \overline{\beta} & , -1 \leq x < 1 \\ 1 - \beta\theta^{\lfloor x \rfloor} & , x \geq 1 \end{cases}$
3. $M_{Z_1}(t) = \frac{\overline{\theta}^2 + \beta\overline{\theta}\xi(t) + \overline{\beta\theta}\xi(-t)}{\overline{\theta}^2 - \theta\xi(t) - \theta\xi(-t)}, \xi(t) = e^t - 1$
4. $\mu_{Z_1} = (2\beta - 1)\frac{1}{\overline{\theta}}$
5. $\sigma^2_{Z_1} = \frac{\theta}{\overline{\theta}^2} + 4\beta\overline{\beta}\left(\frac{1}{\overline{\theta}}\right)^2.$

Note that this distribution can be useful in modelling data with no zeros. Note also that $DG - I(\frac{1}{2}, \theta)$ is symmetric about zero.

5.2.2 $DG - II(\beta, \theta)$

Using $X \sim Geo_0(\theta)$ in the RST we obtain $DG - II(\beta, \theta)$. For this distribution we have

1. $f_{Z_1}(x) = \begin{cases} \overline{\beta\theta}\theta^{-x}, & x = -1, -2, \ldots \\ \overline{\theta} & , x = 0 \\ \beta\overline{\theta}\theta^{x}, & x = 1, 2, \ldots \end{cases}$
2. $F_{Z_1}(x) = \begin{cases} \overline{\beta}\theta^{-\lfloor x \rfloor}, & x < 0 \\ \overline{\beta} + \beta\overline{\theta}, & 0 \leq x < 1 \\ 1 - \beta\theta^{\lfloor x \rfloor + 1}, & x \geq 1 \end{cases}$
3. $M_{Z_1}(t) = \frac{\overline{\theta}^2 - \beta\theta\overline{\theta}\xi(t) - \overline{\beta}\theta\overline{\theta}\xi(-t)}{\overline{\theta}^2 - \theta\xi(t) - \theta\xi(-t)}$
4. $\mu_{Z_1} = (2\beta - 1)\frac{\theta}{\overline{\theta}}$
5. $\sigma^2_{Z_1} = \frac{\theta}{\overline{\theta}^2} + 4\beta\overline{\beta}\left(\frac{\theta}{\overline{\theta}}\right)^2.$

Note that $DG - II(\frac{1}{2}, \theta)$ is symmetric about zero.

5.3 The Double Geometric Distributions Using the RSMT

In this case $Z_2 = YX_1 - (1 - Y)X_2$, where $Y \sim Bernoulli(\beta)$, X_1 and X_2 are Geometric rv and X_1, X_2 and Y are independent. We will consider the following three Double Geometric distributions.

	$X_2 \sim Geo_0(\theta_2)$	$X_2 \sim Geo_1(\theta_2)$
$X_1 \sim Geo_0(\theta_1)$		$DG-III(\beta,\theta_1,\theta_2)$
$X_1 \sim Geo_1(\theta_1)$	$DG-IV(\beta,\theta_1,\theta_2)$	$DG-V(\beta,\theta_1,\theta_2)$

The MLE of the parameters and the corresponding asymptotic theory are obtained using Theorem 2 and the well known results for the MLE of the Geometric distribution.

5.3.1 $DG-III(\beta,\theta_1,\theta_2)$

In this case $X_1 \sim Geo_0(\theta_1)$ and $X_2 \sim Geo_1(\theta_2)$. The following results hold:

1. $f_{Z_2}(x) = \begin{cases} \overline{\beta\theta}_2\theta_2^{-x-1}, & x=-1,-2,\ldots \\ \beta\overline{\theta}_1, & x=0 \\ \beta\overline{\theta}_1\theta_1^x, & x=1,2,\ldots \end{cases}$
2. $F_{Z_2}(x) = \begin{cases} \overline{\beta}\theta_2^{-\lfloor x\rfloor-1}, & x<0 \\ \overline{\beta}+\beta\overline{\theta}_1, & 0\le x<1 \\ 1-\beta\theta_1^{\lfloor x\rfloor+1}, & x\ge 1 \end{cases}$
3. $M_{Z_2}(t) = \frac{\overline{\theta}_1\overline{\theta}_2+\left(\overline{\beta\theta}_2-\beta\overline{\theta}_1\theta_2\right)\xi(-t)}{\overline{\theta}_1\overline{\theta}_2-\theta_1\xi(t)-\theta_2\xi(-t)}$
4. $\mu_{Z_2} = \beta\frac{\theta_1}{\overline{\theta}_1} - \overline{\beta}\frac{1}{\overline{\theta}_2}$
5. $\sigma^2_{Z_2} = \beta\frac{\theta_1}{\overline{\theta}_1^2} + \overline{\beta}\frac{\theta_2}{\overline{\theta}_2^2} + \beta\overline{\beta}\left(\frac{\theta_1}{\overline{\theta}_1}+\frac{1}{\overline{\theta}_2}\right)^2.$

Remarks 1. Kozubowski and Inusah [12] introduced and studied the Skew Discrete Laplace distribution ($SDL(\theta_1,\theta_2)$) with parameters $0<\theta_1,\theta_2<1$ as $Z_3 = X_1 - X_2$, where X_1 and X_2 are independent rv such that $X_1 \sim Geo_0(\theta_1)$ and $X_2 \sim Geo_0(\theta_2)$ (or $X_1 \sim Geo_1(\theta_1)$ and $X_2 \sim Geo_1(\theta_2)$). Inusah and Kozubowski [9] introduced and studied the Discrete Laplace distribution ($DL(\theta)$) which is the special case of $SDL(\theta_1,\theta_2)$ when $\theta_1=\theta_2=\theta$.

2. By Proposition 3.3 of Kozubowski and Inusah [12], $DG-III(\frac{\overline{\theta}_2}{1-\theta_1\theta_2},\theta_1,\theta_2) \overset{d}{=} SDL(\theta_1,\theta_2)$.
3. $DG-III(\frac{\ln\theta_2}{\ln(\theta_1\theta_2)},\theta_1,\theta_2)$ is the same as the Skew discrete Laplace distribution of Barbiero [5] who derived it as a discretization of a certain parametrization of the LD.

5.3.2 $DG-IV(\beta,\theta_1,\theta_2)$

In this case $X_1 \sim Geo_1(\theta_1)$ and $X_2 \sim Geo_0(\theta_2)$. Note that $DG-IV(\beta,\theta_1,\theta_2) = -DG-III(\overline{\beta},\theta_2,\theta_1)$.

5.3.3 $DG-V(\beta,\theta_1,\theta_2)$

In this case $X_1 \sim Geo_1(\theta_1)$ and $X_2 \sim Geo_1(\theta_2)$. The following results hold:

1. $f_{Z_2}(x) = \begin{cases} \overline{\beta}\overline{\theta}_2\theta_2^{-x-1}, & x=-1,-2,\ldots \\ 0, & x=0 \\ \beta\overline{\theta}_1\theta_1^{x-1}, & x=1,2,\ldots \end{cases}$
2. $F_{Z_2}(x) = \begin{cases} \overline{\beta}\theta_2^{-\lfloor x\rfloor-1}, & x<0 \\ \overline{\beta}, & 0\le x<1 \\ 1-\beta\theta_1^{\lfloor x\rfloor}, & x\ge 1 \end{cases}$
3. $M_{Z_2}(t) = \frac{\overline{\theta}_1\overline{\theta}_2-\beta\overline{\theta}_1\theta_2\xi(-t)-\overline{\beta}\theta_1\overline{\theta}_2\xi(t)}{\overline{\theta}_1\overline{\theta}_2-\theta_1\xi(t)-\theta_2\xi(-t)}$
4. $\mu_{Z_2} = \beta\frac{1}{\overline{\theta}_1} - \overline{\beta}\frac{1}{\overline{\theta}_2}$
5. $\sigma^2_{Z_2} = \beta\frac{\theta_1}{\overline{\theta}_1^2} + \overline{\beta}\frac{\theta_2}{\overline{\theta}_2^2} + \beta\overline{\beta}\left(\frac{1}{\overline{\theta}_1}+\frac{1}{\overline{\theta}_2}\right)^2.$

Note that this distribution is useful in modelling data with no zeros. Note also that $DG-V(\beta,\theta,\theta) = DG-I(\beta,\theta)$.

5.4 The Double Poisson Distributions

The first distribution, $DP-I(\beta,\theta)$, is obtained using $X\sim P(\theta)$ in the RST. This distribution is the same as the Extended Poisson distribution of Bakouch et al. [3].

The second distribution, $DP-II(\beta,\theta_{1,}\theta_2)$ is obtained using $X_1 \backsim P(\theta_1)$ and $X_2 \backsim (P(\theta_2)+1)$ in the RSMT. For this distribution we have

1.
$$f_{Z_2}(x) = \begin{cases} \overline{\beta}\frac{\theta_2^{-x-1}e^{-\theta_2}}{(|x|-1)!}, & x=-1,-2,\ldots \\ \beta\frac{\theta_1^x e^{-\theta_1}}{x!}, & x=0,1,2,\ldots \end{cases}$$
2. $M_{Z_2}(t) = \beta e^{\theta_1(e^t-1)} + \overline{\beta}e^{\theta_2(e^{-t}-1)+t}$
3. $\mu_{Z_2} = \beta\theta_1 - \overline{\beta}(\theta_2+1)$
4. $\sigma^2_{Z_2} = \beta\theta_1 + \overline{\beta}\theta_2 + \beta\overline{\beta}(\theta_1+\theta_2+1)^2$.

The third distribution is the well known Skellam distribution ($SK(\theta_1,\theta_2)$ developed by Skellam [21] by using the DT $Z_3 = X_1 - X_2$, where $X_i \sim P(\theta_i), i=1,2$ and are independent. This distribution was studied in details in Alzaid and Omair [1]. For this distribution

$$f_{Z_3}(x) = e^{-\theta_1-\theta_2}\left(\frac{\theta_1}{\theta_2}\right)^{\frac{x}{2}} I_{|x|}\left(2\sqrt{\theta_1\theta_2}\right), x=\ldots,-1,0,1,\ldots, \tag{29}$$

where

$$I_y(t) = \left(\frac{t}{2}\right)^y \sum_{k=0}^{\infty}\frac{\left(\frac{t^2}{4}\right)^k}{k!(y+k)!}$$

is the modified Bessel function of the first kind.

5.5 The Double Negative Binomial Distributions

The first DNBD, denoted by $DNBD-I(\beta,\nu,\theta)$, is developed using the RST when $X\sim NB(\upsilon,\theta)$ with

$$f_X(x,\theta)=\binom{x+\nu-1}{x}\overline{\theta}^{\nu}\theta^{x},x=0,1,2,\ldots$$

For this distribution we obtain

1. $f_{Z_1}(x)=\begin{cases}\overline{\beta}\binom{|x|+\nu-1}{|x|}\overline{\theta}^{\nu}\theta^{|x|}, & x=-1,-2,\ldots\\ \overline{\theta}^{\nu}, & x=0\\ \beta\binom{x+\nu-1}{x}\overline{\theta}^{\nu}\theta^{x}, & x=1,2,\ldots\end{cases}$
2. $M_{Z_1}(t)=\beta\left(\frac{\overline{\theta}}{1-\theta e^{t}}\right)^{\nu}+\overline{\beta}\left(\frac{\overline{\theta}}{1-\theta e^{-t}}\right)^{\nu}$
3. $\mu_{Z_1}=(2\beta-1)\frac{\nu\theta}{\overline{\theta}}$
4. $\sigma^2_{Z_1}=\frac{\nu\theta}{\overline{\theta}^2}+4\beta\overline{\beta}\left(\frac{\nu\theta}{\overline{\theta}}\right)^2.$

The second DNBD, denoted by $DNBD-II(\beta,\nu_1,\nu_2,\theta_1,\theta_2)$, is developed using $X_1\sim NB(\upsilon_1,\theta_1)$ and $X_2\sim(NB(\upsilon_2,\theta_2)+1)$ in the RSMT. For this distribution we obtain

1. $f_{Z_2}(x)=\begin{cases}\overline{\beta}\binom{|x|+\nu_2-2}{|x|-1}\overline{\theta}_2^{\nu_2}\theta_2^{|x|-1}, & x=-1,-2,\ldots\\ \beta\binom{x+\nu_1-1}{x}\overline{\theta}_1^{\nu_1}\theta_1^{x}, & x=0,1,2,\ldots\end{cases}$
2. $M_{Z_2}(t)=\beta\left(\frac{\overline{\theta}_1}{1-\theta_1 e^{t}}\right)^{\nu_1}+\overline{\beta}e^{t}\left(\frac{\overline{\theta}_2}{1-\theta_2 e^{-t}}\right)^{\nu_2}$
3. $\mu_{Z_2}=\beta\frac{\nu_1\theta_1}{\overline{\theta}_1}-\overline{\beta}\left(\frac{\nu_2\theta_2}{\overline{\theta}_2}+1\right)$
4. $\sigma^2_{Z_2}=\beta\frac{\nu_1\theta_1}{\overline{\theta}_1^2}+\overline{\beta}\frac{\nu_2\theta_2}{\overline{\theta}_2}+\beta\overline{\beta}\left(\frac{\nu_1\theta_1}{\overline{\theta}_1}+\frac{\nu_2\theta_2}{\overline{\theta}_2}+1\right)^2.$

Remark Note that $DNBD-II(\beta,\nu_1,\nu_2,\theta_1,\theta_2)$ has the following important special cases

1. $DNBD-II(\beta,\nu,\theta_1,\theta_2)\overset{d}{=}DNBD-II(\beta,\nu,\nu,\theta_1,\theta_2)$
2. $DNBD-II(\beta,\nu,\theta)\overset{d}{=}DNBD-II(\beta,\nu,\nu,\theta,\theta)$
3. $DNBD-II(\beta,1,\nu,\theta_1,\theta_2)$
4. $DNBD-II(\beta,\nu,1,\theta_1,\theta_2)$
5. $DNBD-II(\beta,1,\nu,\theta)\overset{d}{=}DNBD-II(\beta,1,\nu,\theta,\theta)$

Using the DT, Seetha Lekshmi and Sebastian [18] introduced and studied the DNBD, $Z_3=X_1-X_2$, where $X_i\sim NB(\upsilon,\theta_i), i=1,2$ and are independent. Let $DNBD-III(\nu_1,\nu_2,\theta_1,\theta_2)$ denotes the distribution of Z_3 when $X_i\sim NB(\upsilon_i,\theta_i), i=1,2$ and are independent. For this distribution we have

1.

$$f_{Z_3}(x) = \begin{cases} \overline{\theta}_2^{\nu_2}\overline{\theta}_1^{\nu_1} \sum_{k=-x}^{\infty} \binom{\nu_2+k-1}{k}\binom{\nu_1+k+x-1}{k+x}\theta_2^k\theta_1^{k+x}, & x=-1,-2,\ldots \\ \overline{\theta}_2^{\nu_2}\overline{\theta}_1^{\nu_1} \sum_{k=x}^{\infty} \binom{\nu_1+k-1}{k}\binom{\nu_2+k+x-1}{k+x}\theta_1^k\theta_2^{k-x}, & x=0,1,2,\ldots \end{cases} \tag{30}$$

2. $M_{Z_3}(t) = \left(\frac{\overline{\theta}_1}{1-\theta_1 e^t}\right)^{\nu_1}\left(\frac{\overline{\theta}_2}{1-\theta_2 e^{-t}}\right)^{\nu_2}$
3. $\mu_{Z_3} = \frac{\nu_1\theta_1}{\overline{\theta}_1} - \frac{\nu_2\theta_2}{\overline{\theta}_2}$
4. $\sigma_{Z_3}^2 = \frac{\nu_1\theta_1}{\overline{\theta}_1^2} + \frac{\nu_2\theta_2}{\overline{\theta}_2}$.

Remarks 1. Note that $DNBD-III(\nu_1,\nu_2,\theta_1,\theta_2)$ has the following important special cases
   (a) $DNBD-III(\nu,\theta_1,\theta_2) \overset{d}{=} DNBD-III(\nu,\nu,\theta_1,\theta_2)$ which has been introduced and studied in Seetha Lekshmi and Sebastian [18]
   (b) $DNBD-III(1,\nu,\theta_1,\theta_2)$
   (c) $DNBD-III(1,\nu,\theta,\theta)$
2. Ong et al. [16] proved recurrence relations and gave some distributional properties of the rv resulting from the DT when X_1 and X_2 are discrete rv belonging to Panjer's [17] family of discrete distributions. Sundt and Jewell [23] proved that the only non-degenerate members of this family are the Binomial, Poisson and Negative Binomial distributions.

5.6 The RSMT of a Binomial and a Poisson Distributions

The RSMT of a Binomial and a Poisson rv, $DPB(\theta_1,\theta_2,\beta)$, is developed using $X_1 \sim Binomial(n,\theta_1)$ and $X_2 \sim (P(\theta_2)+1)$ in the RSMT. For this distribution we obtain

1. $f_{Z_2}(x) = \begin{cases} \overline{\beta}\frac{\theta_2^{|x|-1}e^{-\theta_2}}{(|x|-1)!}, & x=-1,-2,\ldots \\ \beta\binom{n}{|x|}\theta_1^x(1-\theta_1)^{n-x}, & x=0,1,2,\ldots \end{cases}$

2. $M_{Z_2}(t) = \beta\left(1-\theta_1+\theta_1 e^{-t}\right)^n + \overline{\beta}e^{\theta_2(e^t-1)+t}$

3. $\mu_{Z_2} = n\beta\theta_1 - \overline{\beta}(\theta_2+1)$

4. $\sigma_{Z_2}^2 = n\beta\theta_1(1-\theta_1) + \overline{\beta}\theta_2 + \beta\overline{\beta}(n\theta_1+\theta_2+1)^2$.

5.7 The RSMT of a NB and a Poisson Distributions

The $DPNBD-(\beta,\nu,\theta_1,\theta_2)$ is obtained using $X_1 \sim NB(\upsilon,\theta_1)$ and $X_2 \sim (P(\theta_2)+1)$ in the RSMT. For this distribution we obtain

1. $f_{Z_2}(x) = \begin{cases} \overline{\beta}\frac{\theta_2^{|x|-1}e^{-\theta_2}}{(|x|-1)!}, & x=-1,-2,\ldots \\ \beta\binom{x+\nu-1}{x}\overline{\theta}_1^{\nu}\theta_1^x, & x=0,1,2,\ldots \end{cases}$

2. $M_{Z_2}(t) = \beta\left(\frac{\overline{\theta}_1}{1-\theta_1 e^{-t}}\right)^{\nu} + \overline{\beta}e^{\theta_2(e^t-1)+t}$
3. $\mu_{Z_2} = \beta\frac{\nu\theta_1}{\overline{\theta}_1} - \overline{\beta}(\theta_2+1)$
4. $\sigma^2_{Z_2} = \beta\frac{\nu\theta_1}{\overline{\theta}_1^2} + \overline{\beta}\theta_1 + \beta\overline{\beta}\left(\theta_2+1+\frac{\nu\theta_1}{\overline{\theta}_1}\right)^2$.

Note that $DPNBD-(\beta,1,\theta_1,\theta_2)$ is the RSMT of $X_1 \sim Geo_0(\theta_1)$ and $X_2 \sim (P(\theta_2)+1)$ and will be denoted by $DPG-I(\theta_1,\theta_2,\beta)$. The RSMT of $X_1 \sim P(\theta_1)$ and $X_2 \sim Geo_1(\theta_2)$ will be denoted by $DPG-II(\theta_1,\theta_2,\beta)$. For this distribution

1. $f_{Z_2}(x) = \begin{cases} \overline{\beta\theta}_2\theta^{-x-1}, & x=-1,-2,\ldots,-n \\ \beta e^{-\theta_1}, & x=0 \\ \beta\frac{\theta_1^x e^{-\theta_1}}{x!}, & x=1,2,\ldots \end{cases}$
2. $M_{Z_2}(t) = \beta e^{\theta_1(e^t-1)} + \overline{\beta}\frac{\overline{\theta}_2 e^{-t}}{1-\theta_2 e^{-t}}$
3. $\mu_{Z_2} = \beta\theta_1 - \overline{\beta}\frac{1}{\overline{\theta}_2}$
4. $\sigma^2_{Z_2} = \beta\theta_1 + \overline{\beta}\frac{\theta_2}{\overline{\theta}_2^2} + \beta\overline{\beta}\left(\theta_1+\frac{1}{\overline{\theta}_2}\right)^2$.

6 The Distribution of Sums

In this section we use the notation that $\varphi \neq J \subsetneq \{1,2,\ldots,n\}$ and $n_J = \#$ of elements in J.

Lemma 9 *Assume that Y_i and X_i, $i=1,2,\ldots,n$ are such that $Y_i \sim Bernoulli(\beta_i)$, $X_i \sim F_i(\cdot)$ and are all independent. Define*

$$Z_{1,i} = (2Y_i-1)X_i, i=1,2,\ldots,n \text{ and } T_n = \sum_{i=1}^{n} Z_{1,i}.$$

Then,

$$P(T_n \le x) \stackrel{d}{=} \prod_{i=1}^{n}\beta_i P\left\{\sum_{i=1}^{n} X_i \le x\right\} + \prod_{i=1}^{n}\overline{\beta}_i P\left\{\sum_{i=1}^{n} X_i \ge -x\right\} + \sum_J \prod_{i\in J}\beta_i \prod_{i\in J^c}\overline{\beta}_i P\left\{\sum_{i\in J} X_i - \sum_{i\in J^c} X_i \le x\right\}.$$

For discrete rv

$$P(T_n = x) \stackrel{d}{=} \prod_{i=1}^{n}\beta_i P\left\{\sum_{i=1}^{n} X_i = x\right\} + \prod_{i=1}^{n}\overline{\beta}_i P\left\{\sum_{i=1}^{n} X_i = -x\right\} + \sum_J \prod_{i\in J}\beta_i \prod_{i\in J^c}\overline{\beta}_i P\left\{\sum_{i\in J} X_i - \sum_{i\in J^c} X_i = x\right\}.$$

The proof follows from the result that

$$T_n \stackrel{d}{=} \begin{cases} \sum_{i=1}^{n} X_i, & \textit{with probability} & \prod_{i=1}^{n} \beta_i \\ -\sum_{i=1}^{n} X_i, & \textit{with probability} & \prod_{i=1}^{n} \overline{\beta}_i \\ \sum_{i \in J} X_i - \sum_{i \in J^c} X_i, & \textit{for each } J \textit{ with probability} & \prod_{i \in J} \beta_i \prod_{i \in J^c} \overline{\beta}_i \end{cases}.$$

Example 1 Assume $X_i \sim P(\theta_i), i = 1, 2, \ldots, n$. Define $\theta_J = \sum_{i \in J} \theta_i$. Then,

$$\begin{aligned} P(T_n = x) \stackrel{d}{=} & \prod_{i=1}^{n} \beta_i \frac{\left(\sum_{i=1}^{n} \theta_i\right)^x e^{-\sum_{i=1}^{n} \theta_i}}{x!} I(x \geq 0) \\ & + \prod_{i=1}^{n} \overline{\beta}_i \frac{\left(\sum_{i=1}^{n} \theta_i\right)^{-x} e^{-\sum_{i=1}^{n} \theta_i}}{|x|!} I(x \leq 0) \\ & + \sum_{J} \prod_{i \in J} \beta_i \prod_{i \in J^c} \overline{\beta}_i P\{SK(\theta_J, \theta_{J^c}) = x\}, \end{aligned} \tag{31}$$

where $P\{SK(\theta_1, \theta_2) = x\}$ is as given in (29). In particular, when $\beta_i = \beta$ and $\theta_i = \theta, i = 1, 2, \ldots, n$ we have

$$\begin{aligned} P(T_n = x) \stackrel{d}{=} & \beta^n \frac{(n\theta)^x e^{-n\theta}}{x!} I(x \geq 0) + \overline{\beta}^n \frac{(n\theta)^{-x} e^{-n\theta}}{|x|!} I(x \leq 0) \\ & + \sum_{r=1}^{n-1} \binom{n}{r} \beta^{n-r} \overline{\beta}^r P\{SK((n-r)\theta, r\theta) = x\}. \end{aligned}$$

The special case of (31) when $n = 2$ is given in (15) - (17) of Bakouch et al. [3].

Example 2 Assume $X_i \sim Geo_o(\theta), i = 1, 2, \ldots, n$. Then,

$$\begin{aligned} P(T_n = x) \stackrel{d}{=} & \prod_{i=1}^{n} \beta_i P\{NB(n, \theta) = x\} I(x \geq 0) + \prod_{i=1}^{n} \overline{\beta}_i P\{NB(n, \theta) \\ & = -x\} I(x \leq 0) + \sum_{J} \prod_{i \in J} \beta_i \prod_{i \in J^c} \overline{\beta}_i P\{NB(n_J, \theta) - NB(n_{J^c}, \theta) = x\}, \end{aligned}$$

where $P\{NB(\nu_1, \theta_1) - NB(\nu_2, \theta_2) = x\}$ can be computed using (30). When $\beta_i = \beta$,

$$\begin{aligned} P(T_n = x) \stackrel{d}{=} & \beta^n P\{NB(n, \theta) = x\} I(x \geq 0) + \overline{\beta}^n P\{NB(n, \theta) = -x\} I(x \leq 0) \\ & + \sum_{r=1}^{n-1} \binom{n}{r} \beta^{n-r} \overline{\beta}^r P\{NB(n-r, \theta) - NB(r, \theta) = x\}. \end{aligned}$$

Lemma 10 *Assume that* $Y_i, X_{i,1}$ *and* $X_{i,2}, i = 1, 2, \ldots, n$ *are such that* $Y_i \sim Bernoulli(\beta_i), X_{i,j} \sim F_{i,j}(\cdot), j = 1, 2$ *and are all independent. Define*

$$Z_{2,i} = Y_i X_{i,1} - (1 - Y_i) X_{i,2}, i = 1, 2, \ldots, n \text{ and } S_n = \sum_{i=1}^{n} Z_{2,i}.$$

Then,

$$P(S_n \le x) \stackrel{d}{=} \prod_{i=1}^{n} \beta_i P\left\{\sum_{i=1}^{n} X_{i,1} \le x\right\} + \prod_{i=1}^{n} \overline{\beta}_i P\left\{\sum_{i=1}^{n} X_{i,2} \ge -x\right\} + \sum_J \prod_{i \in J} \beta_i \prod_{i \in J^c} \overline{\beta}_i P\left\{\sum_{i \in J} X_{i,1} - \sum_{i \in J^c} X_{i,2} \le x\right\}.$$

For discrete rv

$$P(S_n = x) \stackrel{d}{=} \prod_{i=1}^{n} \beta_i P\left\{\sum_{i=1}^{n} X_{i,1} = x\right\} + \prod_{i=1}^{n} \overline{\beta}_i P\left\{\sum_{i=1}^{n} X_{i,2} = -x\right\} + \sum_J \prod_{i \in J} \beta_i \prod_{i \in J^c} \overline{\beta}_i P\left\{\sum_{i \in J} X_{i,1} - \sum_{i \in J^c} X_{i,2} = x\right\}.$$

Example 3 For $i = 1, 2, \ldots, n$,assume $X_{i,1} \sim P(\theta_{i,1})$ and $X_{i,2} \sim \big(P(\theta_{i,2}) + 1\big)$. Then,

$$\begin{aligned} P(S_n = x) \stackrel{d}{=} &\prod_{i=1}^{n} \beta_i \frac{\left(\sum_{i=1}^{n} \theta_{i,1}\right)^x e^{-\sum_{i=1}^{n} \theta_{i,1}}}{x!} I(x \ge 0) \\ &+ \prod_{i=1}^{n} \overline{\beta}_i \frac{\left(\sum_{i=1}^{n} \theta_{i,2}\right)^{-x-n} e^{-\sum_{i=1}^{n} \theta_{i,2}}}{(|x| - n)!} I(x \le -n) \\ &+ \sum_J \prod_{i \in J} \beta_i \prod_{i \in J^c} \overline{\beta}_i P\left\{SK(\theta_{J,1}, \theta_{J^c,2}) = x - n_J\right\}. \end{aligned}$$

In particular, when $\beta_i = \beta, \theta_{i,1} = \theta_1$ and $\theta_{i,2} = \theta_2, i = 1, 2, \ldots, n$ we have

$$\begin{aligned} P(S_n = x) \stackrel{d}{=} &\beta^n \frac{(n\theta_1)^x e^{-n\theta_1}}{x!} I(x \ge 0) + \overline{\beta}^n \frac{(n\theta_2)^{-x-n} e^{-n\theta_2}}{(|x| - n)!} I(x \le -n) \\ &+ \sum_{r=1}^{n-1} \binom{n}{r} \beta^{n-r} \overline{\beta}^r P\left\{SK((n-r)\theta_1, r\theta_2) = x - r\right\}. \end{aligned}$$

Example 4 Assume $X_{i,j} \sim EXP(\theta_j), i = 1, 2, \ldots, n, j = 1, 2$. Then,

$$P(S_n \le x) \stackrel{d}{=} \prod_{i=1}^{n} \beta_i P\{G(n, \theta_1) \le x\} + \prod_{i=1}^{n} \overline{\beta}_i P\{G(n, \theta_2) \ge -x\}$$
$$+ \sum_{J \subset \{1,2,\ldots,n\}} \prod_{i \in J} \beta_i \prod_{i \in J^c} \overline{\beta}_i P\{G(n_J, \theta_1) - G(n_{J^c}, \theta_2) \le x\},$$

where $J \underset{\neq}{\subset} \{1, 2, \ldots, n\}$. When $\beta_i = \beta$,

$$P(S_n \le x) \stackrel{d}{=} \beta^n P\{G(n, \theta_1) \le x\} + \overline{\beta}^n P\{G(n, \theta_2) \ge -x\}$$
$$+ \sum_{r=1}^{n-1} \binom{n}{r} \beta^{n-r} \overline{\beta}^r P\{G(n-r, \theta_1) - G(r, \theta_2) \le x\}.$$

For the computation of $P\{G(n_J, \theta_1) - G(n_{J^c}, \theta_2) \le x\}$ we refer to Klar [10], Omura and Kailath ([15], p. 25) and Simon ([20], p.28).

Example 5 For $i = 1, 2, \ldots, n$,assume $X_{i,1} \sim Geo_o(\theta_1)$ and $X_{i,2} \sim Geo_1(\theta_2)$. Then,

$$P(S_n = x) \stackrel{d}{=} \prod_{i=1}^{n} \beta_i P\{NB(n, \theta_1) = x\} I(x \ge 0) + \prod_{i=1}^{n} \overline{\beta}_i P\{NB(n, \theta_2)$$
$$= -x - n\} I(x \le -n)$$
$$+ \sum_{J} \prod_{i \in J} \beta_i \prod_{i \in J^c} \overline{\beta}_i P\{NB(n_J, \theta_1) - NB(n_{J^c}, \theta_2) = x - n_{J^c}\},$$

where $P\{NB(\nu_1, \theta_1) - NB(\nu_2, \theta_2) = x\}$ can be computed using (30). When $\beta_i = \beta$,

$$P(T_n = sx) \stackrel{d}{=} \beta^n P\{NB(n, \theta_1) = x\} I(x \ge 0) + \overline{\beta}^n P\{NB(n, \theta_2)$$
$$= -x - n\} I(x \le -n)$$
$$+ \sum_{r=1}^{n-1} \binom{n}{r} \beta^{n-r} \overline{\beta}^r P\{NB(n-r, \theta_1) - NB(r, \theta_2) = x - r\}.$$

7 Applications

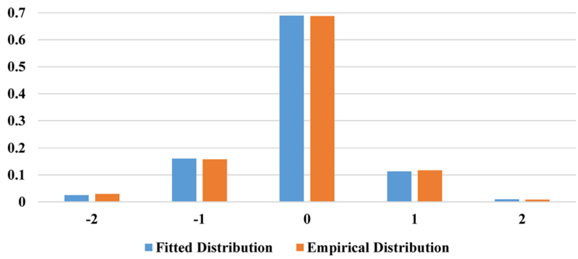
Consider the $DP - II(\beta, \theta_{1,} \theta_2)$ for which $Z_2 = YX_1 - (1 - Y)X_2$ with $X_1 \backsim P(\theta_1)$ and $X_2 \backsim (P(\theta_2) + 1))$ as a competitor to the Skellam (Poisson Difference) distribution.

Let $Z_{2,i}, i = 1, 2, \ldots, n$ be a random sample from $DP - II(\beta, \theta_{1,} \theta_2)$. Let $n_1 = \sum I(Z_{2,i} \ge 0)$, $Sum_+ = \sum Z_{2,i} I(Z_{2,i} \ge 0)$ and $Sum_- = -\sum Z_{2,i} I(Z_{2,i} < 0)$

Table 1 Goodness-of-fit results for SABIC and Arabian Shield

Stock	Fitted distribution	p value
SABIC	$Skellam(0.1682, 0.2516)$	0.449862
	$DP-II(0.8125, 0.1641, 0.1556)$	0.71918
Arabian Shield	$Skellam(0.451, 0.5551)$	0.137931
	$DP-II(0.721, 0.399, 0.403)$	0.25818

Fig. 1 The fitted and emprical distributions of SABIC data



and assume that $0 < n_1 < n$. By Theorem 2, the MLE of β, θ_1 and θ_2 are given by

$$\widehat{\beta} = \frac{n_1}{n}, \widehat{\theta}_1 = \frac{Sum_+}{n_1} \text{ and } \widehat{\theta}_2 = \frac{Sum_-}{n-n_1} - 1.$$

In addition,

$$\sqrt{n}\begin{pmatrix} \widehat{\beta} - \beta \\ \widehat{\theta}_1 - \theta_1 \\ \widehat{\theta}_2 - \theta_2 \end{pmatrix} \xrightarrow{d} N\left(\underline{0}, \begin{bmatrix} \beta(1-\beta) & 0 & 0 \\ 0 & \frac{\theta_1}{\beta} & 0 \\ 0 & 0 & \frac{\theta_2}{\beta} \end{bmatrix}\right).$$

Alzaid and Omair (2010) considered the following two real data sets from the Saudi Stock Exchange (TASI). Trading in Saudi Basic Industry (SABIC) and Arabian Shield from TASI recorded every minute of June 30, 2007. The price can move up and down by a multiple of SAR 0.25. The two data sets consist of $4 \times (close\ price - open\ price)$ in every minute. They used the runs test on each sample to show that the samples are random. They used the Skellam distribution to fit each of the two data sets. We used the $DP-II(\beta, \theta_1, \theta_2)$ to fit each of the two data sets Their results together with ours are summarized in Table 1. The given p values of Table 1 are obtained by using Pearson Chi-square goodness-of-fit test.

Figure 1 gives a plot of the empirical distribution and the fitted $DP-II(0.8125, 0.1641, 0.1556)$ distribution for SABIC data. Figure 2 gives a plot of the empirical distribution and the fitted $DP-II(0.721, 0.399, 0.403)$ distribution for Arabian Shield data.

Bakouch et al. [3] used the $DP-I(\beta, \theta)$ to fit a data set based on the number of students from the Bachelor program at the IDRAC International Management School (Lyon, France) in 60 consecutive Sessions of courses in Marketing.

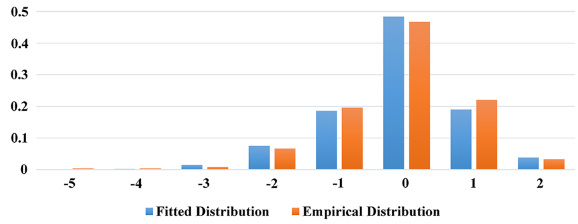


Fig. 2 The fitted and emprical distributions of Arabian Shield data

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