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ORIGINAL PAPER

Some operations on intuitionistic fuzzy graphs via novel versions of the Sombor index for internet routing

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Abstract

Topological indices (TIs) are numerical structures that are associated with a graph to identify its topology. TIs are highly popular in the literature with a wide range of applications from chemistry to economics. However, TIs have limitations in representating complex relations within the graphs creating some uncertainities. Fuzzy graph (FG) and intuitionistic fuzzy graph (IFG) are introduced to overcome these uncertainities. While a FG a describes degree of membership of an object in a graph, IFG delineate information on membership or nonmembership under uncertainity. This study aims to introduce novel TIs such as the general second Zagreb index, the Sombor index of the third version, and the Sombor index of the fourth version in the IFG framework in order to improve practicality of FG and IFG applications. Some properties of the proposed indices and their upper bounds are provided as well. Proposed TIs are applied to an internet routing network as a case study. Results of the study show that adding more internet routers in the network can increase internet speed and the strength of the entire system. Finally, comparative studies for the Sombor index of the third version and the Sombor index of the fourth version are also revealed.

Keywords Intuitionistic fuzzy set · Fuzzy graph · Intuitionistic fuzzy graph · Second Zagreb index · Sombor index of third version · Sombor index of fourth version · Internet routing

1 Introduction

In 1965, Zadeh introduced the fuzzy set (FS) concept in which a values between 0 and 1 are used to describe the membership degree of the elements of an ordinary (crisp) set (Zadeh 1965). Crisp set theory operations like union, intersection, convexity, and many other operations were extended to the FS which is more general than a crisp set. Zadeh (1978) later introduced the possibility theory by using the concept of fuzzy restriction on possibility distribution (Zadeh 1978). Literature is rich with various applications of FSs, including computer sciences (Zimmermann 2010), applications in medical sciences or medical diagnosis (Schuh 2005; Horng et al. 2005; De et al. 2001), psychology (Kochen 1975), social sciences (Smithson 1988; Treadwell 1995; Akram and Bibi 2023), and in many other fields (Flores and Srirama 2013; Chen and Wang 2010; Karwowski and Mital 1986; Zimmermann 2012; Chen et al. 2009; Arya and Kumar 2020; Chen and Jian 2017).

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In 1975, Rosenfeld explored the fuzzy relation on the structure of the graph and fuzzy set to create a fuzzy graph (Rosenfeld 1975). He also analyzed the features of several basic terms such as forest, bridge, trees, etc. Mordeson and Chang-Shyh (1994) proposed operations like a Cartesian product, join, and union of the fuzzy graphs. Yeh and Bang (1975), and Chen et al. (2019), developed the basic properties of fuzzy relation on fuzzy graphs and proposed networking, graph modeling, and clustering in applications.

Line, complete, and complement of fuzzy graphs were proposed by Massadeh and Gharaibeh (2011) and Mordeson (1993). Hawary (2011) introduced some product operations of FSs such as strong products, and direct products, and also analyzed the conditions for balanced fuzzy graphs. The classification of arcs was provided by Mathew and Sunitha (2009). Bhutani and Rosenfeld (2003) presented the concept of geodesic in fuzzy graphs and their application in networks. New fuzzy graphs with novel properties called complex Pythagorean Dombi fuzzy operations were proposed by Khan and Akram (2021). These features of fuzzy graphs applied in many fields including human trafficking, decision-making, and social networking (Ali et al. 2021; Akram et al. 2020; Hassanpour 2020; Jan et al. 2019; Alahmadi et al. 2023).

Atanassov (1986) and Atanassov and Atanassov (1999) generalized and extended FSs to the intuitionistic fuzzy sets (IFSs). Atanassov (2012), and Atanassov and Shannon (1994) proposed a new value on vertices and edges of fuzzy graphs, which is known as negative membership value or non-membership value. With its more detailed descriptions, IFSs attracted more attention than FSs (Ejegwa et al. 2014; Lin et al. 2007; Sotoudeh 2022; Akram et al. 2023; Bukhari et al. 2023; Dagistanli 2023; Diner et al. 2023; Al-Zibaree and Konur 2023; Ghoushchi and Sarvi 2023).

Intuitionistic fuzzy graph (IFG) which is an extended version of IFS was developed by Parvathi et al. (2009). Later, Akram and Davvaz (2012) introduced the concept of strong IFG and their properties. Davvaz et al. (2019) presented the nth type of IFG and Dinar et al. (2023) analyzed its applications. Naeem et al. (2022), Chen and Wang (1995) investigated the bounds on fuzzy Wiener index of IF-cycles, and IF-trees. Parvathi and Thamizhendhi (2010) proposed the independent set, dominating number, and total dominating number in IFGs. Akram and Dudek (2013) developed the idea of a modeling system by using the concept of hyperfuzzy graphs. Ahmad et al.(2023) investigates the upper bounds of fuzzy harmonic index and their application in the cybercrime problem.

Intuitionistic fuzzy graph theory is also used to model ambiguous networks, such as social and financial networks. An important factor in studying the network properties of such networks is their connectedness. IFG modelling can work for modelling behavioral choices and decision-making. Intuitionistic fuzzy graphs (IFGs) are mathematical structures that extend traditional graphs to handle uncertainty and vagueness. In the context of IFGs, various characteristics and properties can be analyzed and reflected in different indexes. For example, topological indices can describe the physical and chemical properties of a chemical network. Similarly, fuzzy topological indices explore the chemical and physical characteristics of a fuzzy graph network that help to analyze these properties for decision-making on a fuzzy chemical network. These characteristics and corresponding topological indices provide a comprehensive view of the structural properties of an IFG, reflecting the uncertainty and vagueness inherent in the representation. The choice of specific indices depends on the aspects of the IFG that researchers aim to analyze.

Researchers can choose a combination of indices that best suit their research objectives, with a comprehensive approach provided by intuitionistic fuzzy graphs' structural and connectivity characteristics. The systematic integration of these indices can enhance the coherence of the analysis.

FS and IFG are extensions of the topological indices (TI), which is a numeric value that describes the structural properties of a graph. TIs have numerous characteristics such as that the values of all the isomorphic graphs are the same (Kalathian et al. 2020; Liu et al. 2022; Ismail et al. 2023; Nadeem et al. 2021). TI is described as the first Zagreb index by Balaban (1983); as the Wiener index by Graovac and Pisanski (1991); and as different versions of the Sombor index by Gutman (2022a, b) and Imran et al. (2023). Imran et al. (2022) determined the values of the supramolecular chain of the third and fourth versions of the Sombor index from a geometric perspective.

An IFS describes two types of information: membership and non-membership. On the other hand, FG is limited to one degree only and does not cover the non-membership facility in decision-making. Therefore, an IFS is very important for modelling real-world problems. Similarly, after-sets, fuzzy graphs, and IFG are also very helpful tools in decision-making and expert systems. Along with the properties of complex networks, topological indices provide efficient tools to simulate lab research-heavy experiments. Combination of fuzzy set environments, fuzzy graph theory, and topological indices deliver multidisciplinary approaches to problems in chemistry, computer science, mathematics and even social sciences.

To our knowledge, there is no study in the literature analyzing IFG concerning graph operations in coordination with Sombor indices. In this paper, first, we discuss the general second Zagreb index and the Sombor index of the third and fourth versions, and we provide some of the basic operations and bounds of these indices as well. Moreover, we explore applications of the general second Zagreb index and the Sombor index of the third and fourth versions. In order to fulfill the gap in the literature, we design Sombor indices for IFG, and then we apply these newly developed Sombor indices concerning IFG and some graph operations. There are many articles on Sombor indices (Das et al. 2021; Cruz et al. 2021; Gutman 2022a), but they are computed only for the crisp graph. In this study, we introduced novel versions of Sombor indices for IFG.

The rest of the paper is organized as follows: Basic definitions of the fuzzy set and the fuzzy graph, the proposed upper bounds of the third and fourth versions of fuzzy Sombor indices are given in Sect. 2. In Sect. 2, proposed results on the general second Zagreb index are revealed. In Sect. 3, the algorithm and an application of fuzzy graph parameters that are under consideration are provided. Section 4 is devoted to the conclusion and future direction.

2 Preliminary

Definition 2.1 Let $G = (V, E)$ be a graph, with vertex set V and E edge set satisfies.

(1) $\Gamma_v : V \rightarrow [0,1]$ and $\Omega_v : V \rightarrow [0,1]$ are functions called membership and non-membership values which we assign to the vertex $\zeta \in V$, respectively, having condition $0 \leq \Gamma_v(\zeta) + \Omega_v(\zeta) \leq 1 \forall \in V$.

(2) $\Gamma_e : V \rightarrow [0,1]$ and $\Omega_e : V \rightarrow [0,1]$ are functions called membership and non-membership values which we assign to every edge $(\kappa, \zeta) \in E$, such that

$\Gamma_e(\kappa,\zeta) \leq \min\{\Gamma_e(\kappa), \Gamma_e(\zeta)\}$ $\Omega_e(\kappa,\zeta) \leq \max\{\Omega_e(\kappa), \Omega_e(\zeta)\}$, with the condition $0 \leq \Gamma_e(\kappa,\zeta) + \Omega_e(\kappa,\zeta) \leq 1$ for all edges $(\kappa,\zeta) \in E$.

Definition 2.2 The order of IFG is represented by $O(\text{IFG}) = (V_m, V_n)$ are defined as $V_m = \sum_{\kappa_i \in V} \Gamma_v(\kappa_i)$, $V_n = \sum_{\kappa_i \in V} \Omega_v(\kappa_i)$, where V_m and V_n is the membership and non-membership order of IFG, respectively.

Definition 2.3 Let $G = (V, E)$ be an IFG and $\kappa_i \in V(G)$ and the degree of κ_i is a pair consisting of membership and non-membership values of κ_i, which is

$$d(\kappa_i) = (d_\Gamma(\kappa_i), d_\Omega(\kappa_i)) = \left(\sum_{\substack{\kappa \in V, \\ \kappa_i \neq \kappa_j}} \Gamma_v(\kappa_i, \kappa_j), \sum_{\substack{\kappa \in V, \\ \kappa_i \neq \kappa_j}} \Gamma_v(\kappa_i, \kappa_j) \right).$$

Definition 2.4 The size of IFG is represented by $S(\text{IFG}) = (E_\psi, E_\omega)$ are defined as $E_\Gamma = \sum_{\kappa \neq \zeta} \Gamma_e(\kappa,\zeta)$, $E_\Omega = \sum_{\kappa \neq \zeta} \Omega_e(\kappa,\zeta)$, where E_ψ and E_ω is the membership and non-membership size of IFG, respectively.

Definition 2.5 Let $G = (V, E)$ be an IFG is a complete IFG, with order and size is V_c and E_c. If $\Gamma_v(\kappa,\zeta) = \min\{\Gamma_v(\kappa), \Gamma_v(\zeta)\}$ and $\Omega_v(\kappa,\zeta) = \max\{\Omega_v(\kappa), \Omega_v(\zeta)\}$, for every edge $(\kappa,\zeta) \in E$

Definition 2.6 Let $G = (V, E)$ be an IFG. The V_G-order of G is formulated as $V_G = (v_\Gamma, v_\Omega) = \left(\sum_{\kappa \in V} \Gamma_v(\kappa_i), \sum_{\kappa \in V} \Omega_v(\kappa_i)\right)$

Definition 2.7 The degree of any vertex κ_i of an IFG, is denoted by $d(\kappa_i)$, and formulated as:

$$d(\kappa_i) = \left(\sum_{\substack{\kappa_i \in V \\ \kappa_i \neq \kappa_j}} \Gamma_v(\kappa_i, \kappa_j), \sum_{\substack{\kappa_i \in V \\ \kappa_i \neq \kappa_j}} \Omega_v(\kappa_i, \kappa_j) \right).$$

Definition 2.8 An IFG is known as a regular IFG. If $\sum_{\substack{\kappa_i \in V \\ \kappa_i \neq \kappa_j}} \Gamma_v(\kappa_i, \kappa_j) = \beta$, and $\sum_{\substack{\kappa_i \in V \\ \kappa_i \neq \kappa_j}} \Omega_v(\kappa_i, \kappa_j) = \beta$, for any constant value β.

Definition 2.9 If we delete a vertex v from the vertex set of graph G with the order n and the size m, then we get a new graph $G - v$, with order $n - v$, and it's size is $m - d(v)$.

If we delete an edge e from the edge set of graph G, then we obtain a new graph $G - e$, with the same order and its size is $m - 1$.

Definition 2.10 If we delete a vertex u from the vertex set of an IFG, then, we get a new graph $(\text{IFG} - u)$, with order graph $= O(\text{IFG}) = \left(\sum_{\kappa \in V} \Gamma_u(\kappa_i) - (\Gamma_u), \sum_{\kappa \in V} \Omega_u(\kappa_i) - (\Omega_u)\right)$, where u_Γ, u_Ω shows membership value and the non-membership value of u respectively, and the size of the graph will be $S(\text{IFG}) = \left(\sum_{\kappa \neq \zeta} \Gamma_u(\kappa, v) - d(\Gamma_u), \sum_{\kappa \neq \zeta} u_\Omega(\kappa, v) - d(\Omega_u)\right)$, where $d(\Omega_u)$, $d(\Gamma_u)$ shows the degree of non-membership and membership value of the deleted vertex respectively.

If we remove an edge from the IFG then we get a new graph $(\text{IFG} - e)$ that has the same order but the size of the new graph is $S(\text{IFG}) = \left(\sum_{\kappa \neq \zeta} \Gamma_u(\kappa, v) - d(\Gamma_u), \sum_{\kappa \neq \zeta} \Omega_v(\kappa, v) - d(\Omega_u)\right)$, where $d(\Gamma_u)$, $d(\Omega_u)$ represents the degree of non-membership and membership values of deleted edge respectively.

Definition 2.11 (Gutman 2022a, b). The third version of the Sombor index for IFG is defined as:

$$\mathrm{IFSO}_3(G) = \sum_{u_iu_j\in E(G)} \sqrt{2}\left[\frac{(t_\Gamma(u_i),f_\Omega(u_i))d^2(u_i)+(t_\Gamma(u_j),f_\Omega(u_j))d^2(u_j)}{(t_\Gamma(u_i),f_\Omega(u_i))d(u_i)+(t_\Gamma(u_j),f_\Omega(u_j))d(u_j)}\right]\pi. \quad (1)$$

Definition 2.12 (Imran et al. 2022). The fourth version of the Sombor index for IFG is defined as:

$$\mathrm{IFSO}_4(G) = \sum_{u_iu_j\in E(G)} \frac{1}{2}\left[\frac{(t_\Gamma(u_i),f_\Omega(u_i))d^2(u_i)+(t_\Gamma(u_j),f_\Omega(u_j))d^2(u_j)}{(t_\Gamma(u_i),f_\Omega(u_i))d(u_i)+(t_\Gamma(u_j),f_\Omega(u_j))d(u_j)}\right]^2\pi. \quad (2)$$

Definition 2.13 General second Zagreb index for IFG is defined as:

$$\mathrm{IFM}_2^{\gamma}(G) = \sum_{u_iu_j\in E(G)} \left[(t_\Gamma(u_i),f_\Omega(u_i))d^{\gamma}(u_i)\left(t_\Gamma(u_j),f_\Omega(u_j)\right)d^{\gamma}(u_j)\right]. \quad (3)$$

Theorem 2.1 *Let G be an IFG having vertex κ and edge $\kappa\nu$. Then*

$$(i) IFSO_3(G) > IFSO_3(G-\kappa). \quad (4)$$

$$(ii) IFSO_3(G) > IFSO_3(G-e). \quad (5)$$

Proof

(i) Let G be an IFG having vertex κ and edge $\kappa\nu$. Then, we have

$$\begin{aligned}
&= \sum_{\substack{\mu\zeta\in E(G)\\ \nu\mu,\kappa\zeta\notin E(G)\\ \mu,\zeta\neq\kappa}} \sqrt{2}\left[\frac{(t_\Gamma(\mu),f_\Omega(\mu))d^2(\mu)+(t_\Gamma(\zeta),f_\Omega(\zeta))d^2(\zeta)}{(t_\Gamma(\mu),f_\Omega(\mu))d(\mu)+(t_\Gamma(\zeta),f_\Omega(\zeta))d(\zeta)}\right]\pi\\
&+ \sum_{\substack{\mu'\zeta'\in E(G)\\ \nu\mu'\notin E(G),\kappa\zeta'\in E(G)\\ \mu',\zeta'\neq\kappa}} \sqrt{2}\left[\frac{(t_\Gamma(\mu'),f_\Omega(\mu'))d^2(\mu')+(t_\Gamma(\zeta'),f_\Omega(\zeta'))d^2(\zeta')}{(t_\Gamma(\mu'),f_\Omega(\mu'))d(\mu')+(t_\Gamma(\zeta'),f_\Omega(\zeta'))d(\zeta')}\right]\pi\\
&+ \sum_{\substack{\kappa\kappa'\in E(G)\\ \kappa'\neq\kappa}} \sqrt{2}\left[\frac{(t_\Gamma(\kappa),f_\Omega(\kappa))d^2(\kappa)+(t_\Gamma(\kappa'),f_\Omega(\kappa'))d^2(\kappa')}{(t_\Gamma(\kappa),f_\Omega(\kappa))d(\kappa)+(t_\Gamma(\kappa'),f_\Omega(\kappa'))d(\kappa')}\right]\pi\\
&> \sum_{\substack{\mu\zeta\in E(G)\\ \nu\mu,\kappa\zeta\notin E(G)\\ \mu,\zeta\neq\kappa}} \sqrt{2}\left[\frac{(t_\Gamma(\mu),f_\Omega(\mu))d^2(\mu)+(t_\Gamma(\zeta),f_\Omega(\zeta))d^2(\zeta)}{(t_\Gamma(\mu),f_\Omega(\mu))d(\mu)+(t_\Gamma(\zeta),f_\Omega(\zeta))d(\zeta)}\right]\pi\\
&+ \sum_{\substack{\mu''\zeta''\in E(G)\\ \nu\mu''\in E(G),\kappa\zeta''\in E(G)\\ \mu'',\zeta''\neq\kappa}} \sqrt{2}\left[\frac{\lambda_1\left(d^2(\mu'')-(\lambda_2)\right)+\lambda_3\left(d^2(\zeta'')-\left(t_\Gamma^2(\kappa,\zeta''),f_\Omega^2(\kappa,\zeta'')\right)\right)}{\lambda_1(d(\mu'')-\lambda_4)+\lambda_3\left(d(\zeta'')-\left(t_\Gamma^2(\kappa,\zeta''),f_\Omega^2(\kappa,\zeta'')\right)\right)}\right]\pi\\
&+ \sum_{\substack{\mu'\zeta'\in E(G)\\ \nu\mu'\notin E(G),\kappa\zeta'\in E(G)\\ \mu',\zeta'\neq\kappa}} \sqrt{2}\left[\frac{(t_\Gamma(\mu'),f_\Omega(\mu'))d^2(\mu')-(t_\Gamma(\zeta'),f_\Omega(\zeta'))\left(d^2(\zeta')+\left(t_\Gamma^2(\kappa,\zeta'),f_\Omega^2(\kappa,\zeta')\right)\right)}{(t_\Gamma(\mu'),f_\Omega(\mu'))d(\mu')-(t_\Gamma(\zeta'),f_\Omega(\zeta'))(d(\zeta')+(t_\Gamma(\kappa,\zeta'),f_\Omega(\kappa,\zeta')))}\right]\pi\\
&= \mathrm{IFSO}_3(G-\kappa),
\end{aligned} \quad (6)$$

where $\lambda_1 = (t_\Gamma(\mu''),f_\Omega(\mu''))$, $\lambda_2 = \left(t^2_\Gamma(\kappa,\mu''),f^2_\Omega(\kappa,\mu'')\right)$, $\lambda_3 = (t_\Gamma(\zeta''),f_\Omega(\zeta''))$, and $\lambda_4 = (t_\Gamma(\kappa,\mu''),f_\Omega(\kappa,\mu''))$.

(ii) Let G be an IFG has an edge $e = \kappa v$, which is deleted from G. Then, we have

$$\mathrm{IFSO}_3(G) = \sum_{u_iu_j \in E(G)} \sqrt{2}\left[\frac{(t_\Gamma(u_i),f_\Omega(u_i))d^2(u_i) + \left(t_\Gamma(u_j),f_\Omega(u_j)\right)d^2(u_j)}{(t_\Gamma(u_i),f_\Omega(u_i))d(u_i) + \left(t_\Gamma(u_j),f_\Omega(u_j)\right)d(u_j)}\right]\pi \tag{7}$$

$$\begin{aligned}
&= \sum_{\substack{\rho\psi \in E(G)\\ v\rho,\kappa\rho \notin E(G)\\ v\psi,\kappa\psi \notin E(G)}} \sqrt{2}\left[\frac{(t_\Gamma(\rho),f_\Omega(\rho))d^2(\rho) + (t_\Gamma(\psi),f_\Omega(\psi))d^2(\psi)}{(t_\Gamma(\rho),f_\Omega(\rho))d(\rho) + (t_\Gamma(\psi),f_\Omega(\psi))d(\psi)}\right]\pi \\
&\quad + \sum_{\substack{\gamma\varsigma \in E(G)\\ \varsigma \neq v}} \sqrt{2}\left[\frac{(t_\Gamma(\gamma),f_\Omega(\gamma))d^2(\gamma) + (t_\Gamma(\varsigma),f_\Omega(\varsigma))d^2(\varsigma)}{(t_\Gamma(\gamma),f_\Omega(\gamma))d(\gamma) + (t_\Gamma(\varsigma),f_\Omega(\varsigma))d(\varsigma)}\right]\pi \\
&\quad + \sum_{\substack{\chi\kappa \in E(G)\\ \chi \neq \kappa}} \sqrt{2}\left[\frac{(t_\Gamma(\chi),f_\Omega(\chi))d^2(\chi) + (t_\Gamma(\kappa),f_\Omega(\kappa))d^2(\kappa)}{(t_\Gamma(\chi),f_\Omega(\chi))d(\chi) + (t_\Gamma(\kappa),f_\Omega(\kappa))d(\kappa)}\right]\pi \\
&\quad + \sqrt{2}\left[\frac{(t_\Gamma(\kappa),f_\Omega(\kappa))d^2(\kappa) + (t_\Gamma(v),f_\Omega(v))d^2(v)}{(t_\Gamma(\kappa),f_\Omega(\kappa))d(\kappa) + (t_\Gamma(v),f_\Omega(v))d(v)}\right]\pi \\
&> \sum_{\substack{\rho\psi \in E(G)\\ v\rho,\kappa\rho \notin E(G)\\ v\psi,\kappa\psi \notin E(G)}} \sqrt{2}\left[\frac{(t_\Gamma(\rho),f_\Omega(\rho))d^2(\rho) + (t_\Gamma(\psi),f_\Omega(\psi))d^2(\psi)}{(t_\Gamma(\rho),f_\Omega(\rho))d(\rho) + (t_\Gamma(\psi),f_\Omega(\psi))d(\psi)}\right]\pi \\
&\quad + \sum_{\substack{\gamma\varsigma \in E(G)\\ \varsigma \neq v}} \sqrt{2}\left[\frac{(t_\Gamma(\gamma),f_\Omega(\gamma))\left(d^2(\gamma) - \left(t^2_\Gamma(\kappa,v),f^2_\Omega(\kappa,v)\right)\right) + (t_\Gamma(\varsigma),f_\Omega(\varsigma))d^2(\varsigma)}{(t_\Gamma(\gamma),f_\Omega(\gamma))(d(\gamma) - (t_\Gamma(\kappa,v),f_\Omega(\kappa,v))) + (t_\Gamma(\varsigma),f_\Omega(\varsigma))d(\varsigma)}\right]\pi \\
&\quad + \sum_{\substack{\chi\kappa \in E(G)\\ \chi \neq \kappa}} \sqrt{2}\left[\frac{(t_\Gamma(\chi),f_\Omega(\chi))\left(d^2(\chi) - \left(t^2_\Gamma(\kappa,v),f^2_\Omega(\kappa,v)\right)\right) + (t_\Gamma(\kappa),f_\Omega(\kappa))d^2(\kappa)}{(t_\Gamma(\chi),f_\Omega(\chi))(d(\chi) - (t_\Gamma(\kappa,v),f_\Omega(\kappa,v))) + (t_\Gamma(\kappa),f_\Omega(\kappa))d(\kappa)}\right]\pi \\
&= \mathrm{IFSO}_3(G - e).
\end{aligned} \tag{8}$$

Corollary 2.1 *Suppose G be an IFG and K_n be a complete IFG, such that $G \neq K_n$. Then, we have*

$$IFSO_3(G) < IFSO_3(K_n). \tag{9}$$

Theorem 2.2 *The bounds for the third version of the Sombor index, for an IFG, is given as*

$$IFSO_3(G) < \frac{1}{2\sqrt{2}}\left[\frac{(\Gamma_t^2+\Omega_t^2)+(\Gamma_f^2+\Omega_f^2)}{(\Gamma_t+\Gamma_f)+(\Omega_t+\Omega_f)}\right]\pi. \tag{10}$$

Proof Suppose a graph G with size (Ω_t,Ω_f) and order is (Γ_t,Γ_f). Each edge has two intuitionistic fuzzy vertices. The membership value of every vertex $\kappa_i \in V(G)$ is defined as $(t_\Gamma(\kappa_i), f_\Omega(\kappa_i))$, and its degree is $d(\kappa_i) = (d_t(\kappa_i), d_f(\kappa_i))$. It is also the summation of truth-membership values of all edges whose incident with a vertex κ_i. The Γ_t donates the summation of truth-membership values of all the vertices of the considerable graph G. So, the truth value of each vertex verifies the $\Gamma_t > d_t(\kappa_i)$, where $\{i = 1,2,3,...,n\}$, we have to conclude the $\Gamma_t^2 > d_t^2(\kappa_i)$ for every vertex $\kappa_i \in V(G)$, whereas $d^2(\kappa_i) = \sum_{\kappa_j\in V,\kappa_j\neq\kappa_i} t_\Gamma^2(\kappa_i,\kappa_j)$ and $\Gamma_t^2 = \sum_{i=1}^{\Omega} t_\Omega^2(\kappa_i)$. Therefore, $\Gamma_t^2 > \sum_{i=1}^{\Omega} t_\Gamma(\kappa_i)d_t^2(\kappa_i)$, and also $\Omega_t^2 > \sum_{i=1}^{\Omega} t_\Gamma(\kappa_i)d_t^2(\kappa_i)$. By utilizing both inequalities and getting the results as $\frac{\Gamma_t^2+\Omega_t^2}{2} > \sum_{i=1}^{\Omega} t_\Gamma(\kappa_i)d_t^2(\kappa_i)$. Similarly, we have $\frac{\Gamma_f^2+\Omega_f^2}{2} > \sum_{i=1}^{\Omega} f_\Omega(\kappa_i)d_f^2(\kappa_i)$.

Now, $\Gamma_t > t_\Gamma(\kappa_i)$ and $\Gamma_t > d_t(\kappa_i)$ both inequalities can be written as $\Gamma_t > d_t(\kappa_i)t_\Gamma(\kappa_i)$ similarly, we have got the other inequality as $\Gamma_f > d_f(\kappa_i)f_\Omega(\kappa_i)$. Using the above inequalities, we have $\Gamma_f + \Gamma_t > d_f(\kappa_i)f_\Omega(\kappa_i) + d_t(\kappa_i)t_\Gamma(\kappa_i)$. Similarly, we have $\Omega_f + \Omega_t > d_f(\kappa_i)t_\Gamma(\kappa_i) + d_t(\kappa_i)t_\Gamma(\kappa_i)$, and add the both inequalities as $(\Gamma_f+\Gamma_t)+(\Omega_f+\Omega_t) > \left(d_f(\kappa_i)f_\Omega(\kappa_i)+d_t(\kappa_i)t_\Gamma(\kappa_i)\right)+\left(d_f(\kappa_i)t_\Gamma(\kappa_i)+d_t(\kappa_i)t_\Gamma(\kappa_i)\right)$. Thus, we have $2\left[(\Gamma_f+\Gamma_t)+(\Omega_f+\Omega_t)\right] > \left(d_f(\kappa_i)f_\Omega(\kappa_i)+d_t(\kappa_i)t_\Gamma(\kappa_i)\right)+\left(d_f(\kappa_i)t_\Gamma(\kappa_i)+d_t(\kappa_i)t_\Gamma(\kappa_i)\right)$.

The third version of the Sombor index can be defined as:

$$\mathrm{SO}_3(G) = \sum_{\kappa_i\kappa_j\in E(G)} \sqrt{2}\left[\frac{(t_\Gamma(\kappa_i),f_\Omega(\kappa_i))d^2(\kappa_i) + \left(t_\Gamma(\kappa_j),f_\Omega(\kappa_j)\right)d^2(\kappa_j)}{(t_\Gamma(\kappa_i),f_\Omega(\kappa_i))d(\kappa_i) + \left(t_\Gamma(\kappa_j),f_\Omega(\kappa_j)\right)d(\kappa_j)}\right]\pi. \tag{11}$$

$$\begin{aligned}
&= \sqrt{2}\left[\frac{(t_\Gamma(\kappa_i),f_\Omega(\kappa_i))\left(d_t^2(\kappa_i),d_f^2(\kappa_i)\right) + \left(t_\Gamma(\kappa_j),f_\Omega(\kappa_j)\right)\left(d_t^2(\kappa_j),d_f^2(\kappa_j)\right)}{(t_\Gamma(\kappa_i),f_\Omega(\kappa_i))\left(d_t(\kappa_i),d_f(\kappa_i)\right) + \left(t_\Gamma(\kappa_j),f_\Omega(\kappa_j)\right)\left(d_t(\kappa_j),d_f(\kappa_j)\right)}\right]\pi \\
&= \sqrt{2}\left[\frac{\left(t_\Gamma(\kappa_i)d_t^2(\kappa_i)\right) + \left(f_\Omega(\kappa_i)d_f^2(\kappa_i)\right) + \left(t_\Gamma(\kappa_j)d_t^2(\kappa_j)\right) + \left(f_\Omega(\kappa_j)d_f^2(\kappa_j)\right)}{\psi_1 d_t(\kappa_i) + \psi_1 d_f(\kappa_i) + \psi_2 d_t(\kappa_j) + \psi_2 d_f(\kappa_j)}\right]\pi \\
&= \sqrt{2}\left[\frac{\frac{\Gamma_t^2+\Omega_t^2}{2} + \frac{\Gamma_f^2+\Omega_f^2}{2}}{2[(\Gamma_t+\Gamma_f)+(\Omega_t+\Omega_f)]}\right]\pi = \sqrt{2}\left[\frac{(\Gamma_t^2+\Omega_t^2)+(\Gamma_f^2+\Omega_f^2)}{4[(\Gamma_t+\Gamma_f)+(\Omega_t+\Omega_f)]}\right]\pi \\
&= \frac{1}{2\sqrt{2}}\left[\frac{(\Gamma_t^2+\Omega_t^2)+(\Gamma_f^2+\Omega_f^2)}{(\Gamma_t+\Gamma_f)+(\Omega_t+\Omega_f)}\right]\pi,
\end{aligned} \tag{12}$$

where $\psi_1 = (t_\Gamma(\kappa_i), f_\Omega(\kappa_i))$ and $\psi_2 = \left(t_\Gamma(\kappa_j), f_\Omega(\kappa_j)\right)$.

Theorem 2.3 *Let G be an IFG having vertex κ and edge $\kappa\nu$. Then*

$$(i) IFSO_4(G) > IFSO_4(G-\kappa). \tag{13}$$

$$(ii) IFSO_4(G) > IFSO_4(G-\zeta). \tag{14}$$

Proof

(i) Let G be an IFG having vertex κ and edge $\kappa\nu$. Then, we have

$$\mathrm{IFSO}_4(G) = \sum_{u_iu_j\in E(G)} \frac{1}{2}\left[\frac{(t_\Gamma(u_i),f_\Omega(u_i))d^2(u_i) + \left(t_\Gamma(u_j),f_\Omega(u_j)\right)d^2(u_j)}{(t_\Gamma(u_i),f_\Omega(u_i))d(u_i) + \left(t_\Gamma(u_j),f_\Omega(u_j)\right)d(u_j)}\right]^2\pi. \tag{15}$$

$$\begin{aligned}
&= \sum_{\substack{\mu\zeta\in E(G)\\ v\mu,\kappa\zeta\notin E(G)\\ \mu,\zeta\neq\kappa}} \frac{1}{2}\left[\frac{(t_\Gamma(\mu),f_\Omega(\mu))d^2(\mu)+(t_\Gamma(\zeta),f_\Omega(\zeta))d^2(\zeta)}{(t_\Gamma(\mu),f_\Omega(\mu))d(\mu)+(t_\Gamma(\zeta),f_\Omega(\zeta))d(\zeta)}\right]^2\pi\\
&+ \sum_{\substack{\mu'\zeta'\in E(G)\\ v\mu'\notin E(G),\kappa\zeta'\in E(G)\\ \mu',\zeta'\neq\kappa}} \frac{1}{2}\left[\frac{(t_\Gamma(\mu'),f_\Omega(\mu'))d^2(\mu')+(t_\Gamma(\zeta'),f_\Omega(\zeta'))d^2(\zeta')}{(t_\Gamma(\mu'),f_\Omega(\mu'))d(\mu')+(t_\Gamma(\zeta'),f_\Omega(\zeta'))d(\zeta')}\right]^2\pi\\
&+ \sum_{\substack{\kappa\kappa'\in E(G)\\ \kappa'\neq\kappa}} \frac{1}{2}\left[\frac{(t_\Gamma(\kappa),f_\Omega(\kappa))d^2(\kappa)+(t_\Gamma(\kappa'),f_\Omega(\kappa'))d^2(\kappa')}{(t_\Gamma(\kappa),f_\Omega(\kappa))d(\kappa)+(t_\Gamma(\kappa'),f_\Omega(\kappa'))d(\kappa')}\right]^2\pi\\
&> \sum_{\substack{\mu\zeta\in E(G)\\ v\mu,\kappa\zeta\notin E(G)\\ \mu,\zeta\neq\kappa}} \frac{1}{2}\left[\frac{(t_\Gamma(\mu),f_\Omega(\mu))d^2(\mu)+(t_\Gamma(\zeta),f_\Omega(\zeta))d^2(\zeta)}{(t_\Gamma(\mu),f_\Omega(\mu))d(\mu)+(t_\Gamma(\zeta),f_\Omega(\zeta))d(\zeta)}\right]^2\pi\\
&+ \sum_{\substack{\mu''\zeta''\in E(G)\\ v\mu''\in E(G),\kappa\zeta''\in E(G)\\ \mu'',\zeta''\neq\kappa}} \frac{1}{2}\left[\frac{\lambda_1\left(d^2(\mu'')-\lambda_2\right)+\lambda_3\left(d^2(\zeta'')-\left(t_\Gamma^2(\kappa,\mu''),f_\Omega(\kappa,\zeta'')\right)\right)}{\lambda_1(d(\mu'')-\lambda_4)+\lambda_3(d(\zeta'')-(t_\Gamma(\kappa,\zeta''),f_\Omega(\kappa,\zeta'')))}\right]^2\pi\\
&+ \sum_{\substack{\mu'\zeta'\in E(G)\\ v\mu'\notin E(G),\kappa\zeta'\in E(G)\\ \mu',\zeta'\neq\kappa}} \frac{1}{2}\left[\frac{(t_\Gamma(\mu'),f_\Omega(\mu'))d^2(\mu')+(t_\Gamma(\zeta'),f_\Omega(\zeta'))\left(d^2(\zeta')-\left(t_\Gamma^2(\kappa,\zeta'),f_\Omega^2(\kappa,\zeta')\right)\right)}{(t_\Gamma(\mu'),f_\Omega(\mu'))d(\mu')+(t_\Gamma(\zeta'),f_\Omega(\zeta'))(d(\zeta')-(t_\Gamma(\kappa,\zeta'),f_\Omega(\kappa,\zeta')))}\right]^2\pi\\
&= \mathrm{IFSO}_4(G-\kappa),
\end{aligned} \tag{16}$$

where $\lambda_1 = (t_\Gamma(\mu''),f_\Omega(\mu''))$, $\lambda_2 = \left(t_\Gamma^2(\kappa,\mu''),f_\Omega^2(\kappa,\mu'')\right)$, $\lambda_3 = (t_\Gamma(\zeta''),f_\Omega(\zeta''))$, and $\lambda_4 = (t_\Gamma(\kappa,\mu''),f_\Omega(\kappa,\mu''))$.

(ii) Let G be an IFG has an edge $e = \kappa v$, which is deleted from G. Then, we have

$$\mathrm{IFSO}_4(G)=\sum_{u_iu_j\in E(G)}\frac{1}{2}\left[\frac{(t_\Gamma(u_i),f_\Omega(u_i))d^2(u_i)+\big(t_\Gamma(u_j),f_\Omega(u_j)\big)d^2(u_j)}{(t_\Gamma(u_i),f_\Omega(u_i))d(u_i)+\big(t_\Gamma(u_j),f_\Omega(u_j)\big)d(u_j)}\right]^2\pi \tag{17}$$

$$\begin{aligned}
&=\sum_{\substack{\rho\psi\in E(G)\\ v\rho,\kappa\rho\notin E(G)\\ v\psi,\kappa\psi\notin E(G)}}\frac{1}{2}\left[\frac{(t_\Gamma(\rho),f_\Omega(\rho))d^2(\rho)+(t_\Gamma(\psi),f_\Omega(\psi))d^2(\psi)}{(t_\Gamma(\rho),f_\Omega(\rho))d(\rho)+(t_\Gamma(\psi),f_\Omega(\psi))d(\psi)}\right]^2\pi\\
&+\sum_{\substack{\gamma\varsigma\in E(G)\\ \varsigma\neq v}}\frac{1}{2}\left[\frac{(t_\Gamma(\gamma),f_\Omega(\gamma))d^2(\gamma)+(t_\Gamma(\varsigma),f_\Omega(\varsigma))d^2(\varsigma)}{(t_\Gamma(\gamma),f_\Omega(\gamma))d(\gamma)+(t_\Gamma(\varsigma),f_\Omega(\varsigma))d(\varsigma)}\right]^2\pi\\
&+\sum_{\substack{\chi\kappa\in E(G)\\ \chi\neq\kappa}}\frac{1}{2}\left[\frac{(t_\Gamma(\chi),f_\Omega(\chi))d^2(\chi)+(t_\Gamma(\kappa),f_\Omega(\kappa))d^2(\kappa)}{(t_\Gamma(\chi),f_\Omega(\chi))d(\chi)+(t_\Gamma(\kappa),f_\Omega(\kappa))d(\kappa)}\right]^2\pi\\
&+\frac{1}{2}\left[\frac{(t_\Gamma(\kappa),f_\Omega(\kappa))d^2(\kappa)+(t_\Gamma(v),f_\Omega(v))d^2(v)}{(t_\Gamma(\kappa),f_\Omega(\kappa))d(\kappa)+(t_\Gamma(v),f_\Omega(v))d(v)}\right]^2\pi\\
&>\sum_{\substack{\rho\psi\in E(G)\\ v\rho,\kappa\rho\notin E(G)\\ v\psi,\kappa\psi\notin E(G)}}\frac{1}{2}\left[\frac{(t_\Gamma(\rho),f_\Omega(\rho))d^2(\rho)+(t_\Gamma(\psi),f_\Omega(\psi))d^2(\psi)}{(t_\Gamma(\rho),f_\Omega(\rho))d(\rho)+(t_\Gamma(\psi),f_\Omega(\psi))d(\psi)}\right]^2\pi\\
&+\sum_{\substack{\gamma\varsigma\in E(G)\\ \varsigma\neq v}}\frac{1}{2}\left[\frac{(t_\Gamma(\gamma),f_\Omega(\gamma))\big(d^2(\gamma)-\big(t_\Gamma^2(\kappa,v),f_\Gamma^2(\kappa,v)\big)\big)+(t_\Gamma(\varsigma),f_\Omega(\varsigma))d^2(\varsigma)}{(t_\Gamma(\gamma),f_\Omega(\gamma))(d(\gamma)-(t_\Gamma(\kappa,v),f_\Gamma(\kappa,v)))+(t_\Gamma(\varsigma),f_\Omega(\varsigma))d(\varsigma)}\right]^2\pi\\
&+\sum_{\substack{\chi\kappa\in E(G)\\ \chi\neq\kappa}}\frac{1}{2}\left[\frac{(t_\Gamma(\chi),f_\Omega(\chi))\big(d^2(\chi)-\big(t_\Gamma^2(\kappa,v),f_\Gamma^2(\kappa,v)\big)\big)+(t_\Gamma(\kappa),f_\Omega(\kappa))d^2(\kappa)}{(t_\Gamma(\chi),f_\Omega(\chi))(d(\chi)-(t_\Gamma(\kappa,v),f_\Gamma(\kappa,v)))+(t_\Gamma(\kappa),f_\Omega(\kappa))d(\kappa)}\right]^2\pi\\
&=\mathrm{IFSO}_4(G-e).
\end{aligned}\tag{18}$$

Corollary 2.2 *Suppose G be an IFG and K_n be a complete IFG, such that $G\neq K_n$. Then, we have*

$$IFSO_4(G)<IFSO_4(K_n). \tag{19}$$

Theorem 2.4 *The bounds for the fourth version of the Sombor index, for an IFG, is given as*

$$IFSO_4(G)<\frac{1}{32}\left[\frac{(\Gamma_t^2+\Omega_t^2)+(\Gamma_f^2+\Omega_f^2)}{(\Gamma_t+\Gamma_f)+(\Omega_t+\Omega_f)}\right]^2\pi. \tag{20}$$

Proof Suppose a graph G with size (Ω_t,Ω_f) and order is (Γ_t,Γ_f). Each edge has two intuitionistic Fuzzy vertices. The membership value of every vertex $\kappa_i\in V(G)$ is defined as $(t_\Gamma(\kappa_i),f_\Omega(\kappa_i),)$ and its degree is $d(\kappa_i)=\big(d_t(\kappa_i),d_f(\kappa_i)\big)$. It is also the summation of truth-membership values of all edges whose incident with a vertex κ_i. The Γ_t donates the summation of truth-membership values of all the vertices of the considerable graph G. So, the truth value of each vertex verifies the $\Gamma_t > d_t(\kappa_i)$, where $\{i = 1, 2, 3, ..., n\}$, we have to conclude the $\Gamma_t^2 > d_t^2(\kappa_i)$ for every vertex $\kappa_i \in V(G)$, whereas $d^2(\kappa_i) = \sum_{\kappa_j \in V, \kappa_j \neq \kappa_i} t_\Gamma^2(\kappa_i, \kappa_j)$ and $\Gamma_t^2 = \sum_{i=1}^{\Omega} t_\Omega^2(\kappa_i)$. Therefore, $\Gamma_t^2 > \sum_{i=1}^{\Omega} t_\Gamma(\kappa_i) d_t^2(\kappa_i)$, and also $\Omega_t^2 > \sum_{i=1}^{\Omega} t_\Gamma(\kappa_i) d_t^2(\kappa_i)$. By utilizing both inequalities and getting the results as $\frac{\Gamma_t^2 + \Omega_t^2}{2} > \sum_{i=1}^{\Omega} t_\Gamma(\kappa_i) d_t^2(\kappa_i)$. Similarly, we have $\frac{\Gamma_f^2 + \Omega_f^2}{2} > \sum_{i=1}^{\Omega} f_\Omega(\kappa_i) d_f^2(\kappa_i)$.

Now, $\Gamma_t > t_\Gamma(\kappa_i)$ and $\Gamma_t > d_t(\kappa_i)$ both inequalities can be written as $\Gamma_t > d_t(\kappa_i) t_\Gamma(\kappa_i)$ similarly, we have got the other inequality as $\Gamma_f > d_f(\kappa_i) f_\Omega(\kappa_i)$. Using the above inequalities, we have $\Gamma_f + \Gamma_t > d_f(\kappa_i) f_\Omega(\kappa_i) + d_t(\kappa_i) t_\Gamma(\kappa_i)$. Similarly, we have $\Omega_f + \Omega_t > d_f(\kappa_i) t_\Gamma(\kappa_i) + d_t(\kappa_i) t_\Gamma(\kappa_i)$, and add the both inequalities as $(\Gamma_f + \Gamma_t) + (\Omega_f + \Omega_t) > \left(d_f(\kappa_i) f_\Omega(\kappa_i) + d_t(\kappa_i) t_\Gamma(\kappa_i)\right) + \left(d_f(\kappa_i) t_\Gamma(\kappa_i) + d_t(\kappa_i) t_\Gamma(\kappa_i)\right)$. Thus, we have $2\left[(\Gamma_f + \Gamma_t) + (\Omega_f + \Omega_t)\right] > \left(d_f(\kappa_i) f_\Omega(\kappa_i) + d_t(\kappa_i) t_\Gamma(\kappa_i)\right) + \left(d_f(\kappa_i) t_\Gamma(\kappa_i) + d_t(\kappa_i) t_\Gamma(\kappa_i)\right)$. The fourth version of the Sombor index can be defined as:

$$\begin{aligned}
\mathrm{IFSO}_4(G) &= \sum_{\kappa_i \kappa_j \in E(G)} \frac{1}{2} \left[\frac{(t_\Gamma(\kappa_i), f_\Omega(\kappa_i)) d^2(\kappa_i) + \left(t_\Gamma(\kappa_j), f_\Omega(\kappa_j)\right) d^2(\kappa_j)}{(t_\Gamma(\kappa_i), f_\Omega(\kappa_i)) d(\kappa_i) + \left(t_\Gamma(\kappa_j), f_\Omega(\kappa_j)\right) d(\kappa_j)} \right]^2 \pi \\
&= \frac{1}{2} \left[\frac{(t_\Gamma(\kappa_i), f_\Omega(\kappa_i)) \left(d_t^2(\kappa_i), d_f^2(\kappa_i)\right) + \left(t_\Gamma(\kappa_j), f_\Omega(\kappa_j)\right) \left(d_t^2(\kappa_{}), d_f^2(\kappa_j)\right)}{(t_\Gamma(\kappa_i), f_\Omega(\kappa_i)) \left(d_t(\kappa_i), d_f(\kappa_i)\right) + \left(t_\Gamma(\kappa_j), f_\Omega(\kappa_j)\right) \left(d_t(\kappa_j), d_f(\kappa_j)\right)} \right]^2 \pi \\
&= \frac{1}{2} \left[\frac{\left(t_\Gamma(\kappa_i) d_t^2(\kappa_i)\right) + \left(f_\Omega(\kappa_i) d_f^2(\kappa_i)\right) + \left(t_\Gamma(\kappa_j) d_t^2(\kappa_j)\right) + \left(f_\Omega(\kappa_j) d_f^2(\kappa_j)\right)}{\psi_1 d_t(\kappa_i) + \psi_1 d_f(\kappa_i) + \psi_2 d_t(\kappa_j) + \psi_2 d_f(\kappa_j)} \right]^2 \pi \\
&= \frac{1}{2} \left[\frac{\frac{\Gamma_t^2 + \Omega_t^2}{2} + \frac{\Gamma_f^2 + \Omega_f^2}{2}}{2[(\Gamma_t + \Gamma_f) + (\Omega_t + \Omega_f)]} \right]^2 \pi = \frac{1}{2} \left[\frac{(\Gamma_t^2 + \Omega_t^2) + (\Gamma_f^2 + \Omega_f^2)}{4[(\Gamma_t + \Gamma_f) + (\Omega_t + \Omega_f)]} \right]^2 \pi \\
&= \frac{1}{32} \left[\frac{(\Gamma_t^2 + \Omega_t^2) + (\Gamma_f^2 + \Omega_f^2)}{(\Gamma_t + \Gamma_f) + (\Omega_t + \Omega_f)} \right]^2 \pi,
\end{aligned} \tag{21}$$

where $\psi_1 = (t_\Gamma(\kappa_i), f_\Omega(\kappa_i))$ and $\psi_2 = \left(t_\Gamma(\kappa_j), f_\Omega(\kappa_j)\right)$.

Theorem 3.1 *Let G be an IFG having vertex κ and edge κv. Then, we have*

$$(i) IFM_2^\gamma(G - \kappa) < IFM_2^\gamma(G). \tag{22}$$

$$(ii) IFM_2^\gamma(G - e) < IFM_2^\gamma(G). \tag{23}$$

Proof

(i) Let G be an IFG having vertex κ and edge κv. Then, we have

$$\mathrm{IFM}_2^\gamma(G) = \sum_{u_i u_j \in E(G)} \left[(t_\Gamma(u_i), f_\Omega(u_i)) d^\gamma(u_i) \left(t_\Gamma(u_j), f_\Omega(u_j)\right) d^\gamma(u_j) \right]. \tag{24}$$

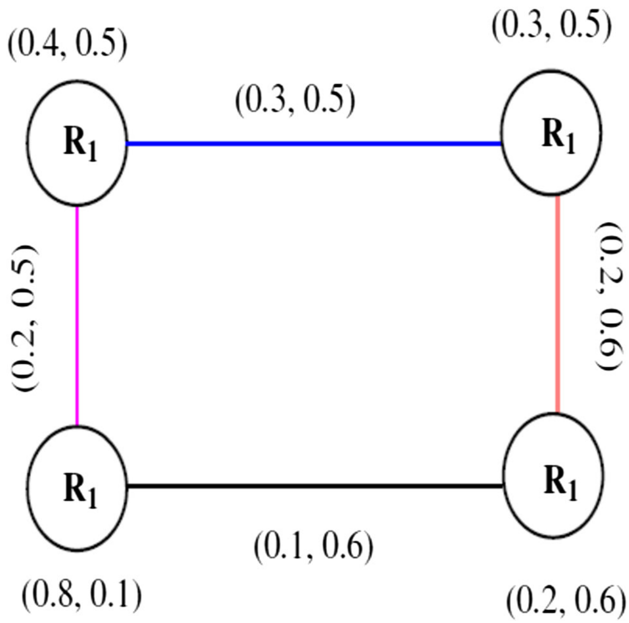

Fig. 1 Graph of internet routing services

(ii) Let G be an IFG having vertex κ and edge $\kappa\nu$. Then, we have

$$\mathrm{IFM}_2^{\gamma}(G) = \sum_{u_i u_j \in E(G)} \left[(t_\Gamma(u_i), f_\Omega(u_i)) d^{\gamma}(u_i) \left(t_\Gamma(u_j), f_\Omega(u_j)\right) d^{\gamma}(u_j) \right]. \tag{26}$$

$$\begin{aligned}
&= \sum_{\substack{\varsigma\psi \in E(G) \\ \nu\varsigma, \kappa\psi \notin E(G) \\ \varsigma, \psi \neq \kappa}} [(t_\Gamma(\varsigma), f_\Omega(\varsigma)) d^{\gamma}(\varsigma)(t_\Gamma(\psi), f_\Omega(\psi)) d^{\gamma}(\psi)] \\
&\quad + \sum_{\substack{\varsigma'\psi' \in E(G) \\ \nu\varsigma' \notin E(G), \kappa\psi' \in E(G) \\ \varsigma', \psi' \neq \kappa}} [(t_\Gamma(\varsigma'), f_\Omega(\varsigma')) d^{\gamma}(\varsigma')(t_\Gamma(\psi'), f_\Omega(\psi')) d^{\gamma}(\psi')] \\
&\quad + \sum_{\substack{\kappa'\kappa \in E(G) \\ \kappa' \neq \kappa}} [(t_\Gamma(\kappa'), f_\Omega(\kappa')) d^{\gamma}(\kappa')(t_\Gamma(\psi'), f_\Omega(\psi')) d^{\gamma}(\psi')] \\
&\quad > \sum_{\substack{\varsigma\psi \in E(G) \\ \nu\varsigma, \kappa\psi \notin E(G) \\ \varsigma, \psi \neq \kappa}} [(t_\Gamma(\varsigma), f_\Omega(\varsigma)) d^{\gamma}(\varsigma)(t_\Gamma(\psi), f_\Omega(\psi)) d^{\gamma}(\psi)] \\
&\quad + \sum_{\substack{\varsigma'\psi' \in E(G) \\ \nu\varsigma' \notin E(G), \kappa\psi' \in E(G) \\ \varsigma', \psi' \neq \kappa}} [(t_\Gamma(\varsigma'), f_\Omega(\varsigma')) d^{\gamma}(\varsigma')(t_\Gamma(\psi'), f_\Omega(\psi')) \left(d^{\gamma}(\psi') - \left(t_\Gamma^{\gamma}(\kappa, \psi'), f_\Omega^{\gamma}(\kappa, \psi')\right)\right)] \\
&\quad + \sum_{\substack{\varsigma''\psi'' \in E(G) \\ \nu\varsigma'' \in E(G), \kappa\psi'' \in E(G) \\ \varsigma'', \psi'' \neq \kappa}} \left[(t_\Gamma(\varsigma''), f_\Omega(\varsigma''))\left(d^{\gamma}(\varsigma'') - \left(t_\Gamma^{\gamma}(\kappa, \varsigma''), f_\Omega^{\gamma}(\kappa, \varsigma'')\right)\right)\right. \\
&\quad \left.(t_\Gamma(\psi''), f_\Omega(\psi''))\left(d^{\gamma}(\psi'') - \left(t_\Gamma^{\gamma}(\kappa, \psi''), f_\Omega^{\gamma}(\kappa, \psi'')\right)\right)\right] \\
&\quad = \mathrm{IFM}_2^{\gamma}(G - U).
\end{aligned} \tag{25}$$

$$\begin{aligned}
&= \sum_{\substack{\kappa'\nu' \in E(G) \\ \kappa\kappa', \kappa'\nu \in E(G) \\ \kappa\nu', \nu\nu' \notin E(G)}} (t_\Gamma(\kappa'), f_\Omega(\kappa'))d^\gamma(\kappa')(t_\Gamma(\nu'), f_\Omega(\nu'))d^\gamma(\nu') \\
&\quad + \sum_{\substack{\kappa\psi \in E(G) \\ \psi \neq \kappa}} (t_\Gamma(\kappa), f_\Omega(\kappa))d^\gamma(\kappa)(t_\Gamma(\psi), f_\Omega(\psi))d^\gamma(\psi) \\
&\quad + \sum_{\substack{\nu\psi \in E(G) \\ \psi \neq \nu}} (t_\Gamma(\nu), f_\Omega(\nu))d^\gamma(\nu)(t_\Gamma(\psi), f_\Omega(\psi))d^\gamma(\psi) \\
&> \sum_{\substack{\kappa'\nu' \in E(G) \\ \kappa\kappa', \kappa'\nu \in E(G) \\ \kappa\nu', \nu\nu' \notin E(G)}} (t_\Gamma(\kappa'), f_\Omega(\kappa'))d^\gamma(\kappa')(t_\Gamma(\nu'), f_\Omega(\nu'))d^\gamma(\nu') \\
&\quad + \sum_{\substack{\kappa\psi \in E(G) \\ \psi \neq \kappa}} (t_\Gamma(\kappa), f_\Omega(\kappa))\left(d^\gamma(\kappa) - \left(t_\Gamma^\gamma(\kappa, \nu), f_\Gamma^\gamma(\kappa, \nu)\right)\right)(t_\Gamma(\psi), f_\Omega(\psi))d^\gamma(\psi) \\
&\quad + \sum_{\substack{\nu\psi \in E(G) \\ \psi \neq \nu}} (t_\Gamma(\nu), f_\Omega(\nu))\left(d^\gamma(\nu) - \left(t_\Gamma^\gamma(\kappa, \nu), f_\Omega^\gamma(\kappa, \nu)\right)\right)(t_\Gamma(\psi), f_\Omega(\psi))d^\gamma(\psi) \\
&= \mathrm{IFM}_2^\gamma(G - e).
\end{aligned} \tag{27}$$

Corollary 3.1 *Suppose G be an IFG and K_n be a complete IFG, such that $G \neq K_n$. For $\gamma \geq 1$, then we have*

$$IFM_2^\gamma(G) < IFM_2^\gamma(K_n). \tag{28}$$

Theorem 3.2 *The bounds on the general second Zagreb index for connected IFG are given as*:

$$IFM_2^\gamma(G) < \left(m_t + m_f\right)\left(n_t + n_f\right). \tag{29}$$

Proof. Let G be a finite connected IFG, having the order (m_t, m_f), and the size is (n_t, n_f), respectively. Suppose that, G has finite edges and two IF vertices of each edge. The membership value of each $\kappa_i \in V(G)$ is represented by $(t_\Gamma(\kappa_i), t_\Omega(\kappa_i))$, and the degree of the vertices of the G is shown as $d\kappa_i = (d_\Gamma(\kappa_i), d_\Omega(\kappa_i))$. By the formulation of the general second Zagreb index as:

$$\begin{aligned}
IFM_2^\gamma(G) &= \sum_{\kappa_i\kappa_j \in E(G)} (t_\Gamma(\kappa_i), t_\Omega(\kappa_i))d^\gamma(\kappa_i)\left(t_\Gamma(\kappa_j), t_\Omega(\kappa_j)\right)d^\gamma(\kappa_j), \\
&= \sum_{\kappa_i\kappa_j \in E(G)} (t_\Gamma(\kappa_i), t_\Omega(\kappa_i))\left(d_\Gamma^\gamma(\kappa_i), d_\Omega^\gamma(\kappa_i)\right)\left(t_\Gamma(\kappa_j), t_\Omega(\kappa_j)\right)\left(d_\Gamma^\gamma(\kappa_j), d_\Omega^\gamma(\kappa_j)\right).
\end{aligned} \tag{30}$$

By applying the basic dot product operation as follows.

$$= \sum_{\kappa_i\kappa_j \in E(G)} \left(t_\Gamma(\kappa_i)d_\Gamma^\gamma(\kappa_i) + t_\Omega(\kappa_i)d_\Omega^\gamma(\kappa_i)\right)\left(t_\Gamma(\kappa_j)d_\Gamma^\gamma(\kappa_j) + t_\Omega(\kappa_j)d_\Omega^\gamma(\kappa_j)\right). \tag{31}$$

As $t_\Gamma(\kappa_i) < m_t$, $d_\Gamma(\kappa_i) < m_t$, and $d_\Gamma^\gamma(\kappa_i) < m_t$ $t_\Gamma(\kappa_i)d_\Gamma^\gamma(\kappa_i) < m_t$. Similarly, $t_\Omega(\kappa_i)d_\Omega^\gamma(\kappa_i) < m_f$. Now, Add the above inequalities, $\left(t_\Gamma(\kappa_i)d_\Gamma^\gamma(\kappa_i) + t_\Omega(\kappa_i)d_\Omega^\gamma(\kappa_i)\right) < \left(m_t + m_f\right)$. Similarly, we get the other inequality $\left(t_\Gamma(\kappa_j)d_\Gamma^\gamma(\kappa_j) + t_\Omega(\kappa_j)d_\Omega^\gamma(\kappa_j)\right) < \left(n_t + n_f\right)$. Here, we have used both new inequalities: $\sum_{\kappa_i\kappa_j \in E(G)} \left(t_\Gamma(\kappa_i)d_\Gamma^\gamma(\kappa_i) + t_\Omega(\kappa_i)d_\Omega^\gamma(\kappa_i)\right)\left(t_\Gamma(\kappa_j)d_\Gamma^\gamma(\kappa_j) +\right.$

$t_\Omega(\kappa_j)d^{\gamma}_{\Omega}(\kappa_j)) < (m_t + m_f)(n_t + n_f)$.

This, $M_2^{\gamma}(G) < (m_t + m_f)(n_t + n_f)$.

3 Algorithm and application

Let IFG_1 and IFG_2 as two graphs, one with a membership value and the other one with a non-membership value. To obtain the desired result, we compute the non-membership value and membership value individually and then add them. Consider a graph $\mathrm{IFG} = (V, E)$. To determine the value of the general second Zagreb index. We developed the following algorithm.

Step 1: Compute the membership and non-membership values for each vertex (router).

Step 2: Compute the membership and non-membership value for each edge.

Step 3: Compute the membership and non-membership $d^2(v)-$ degrees for each vertex (router).

Step 4: Compute the membership and non-membership $d^{\gamma}(v)-$ degrees for each vertex (router).

Step 5: Compute the second general Zagreb index for membership and non-membership values.

This algorithm can be used to compute any topological number. So the third and fourth versions of Sombor number can be computed by using this algorithm.

In this section, we study an internet routing in which the general second Zagreb index, third and fourth version intuitionistic fuzzy Sombor graph parameters are used. In this study, we consider a network of internet routers that is active in a particular area. Using the algorithm and the novel indices we design for IFG, we show that adding more routers to the network increases the strength of every router in the whole system. By the general second Zagreb index and Sombor index of the third and fourth versions, strength of the system is increased after adding more routers. In addition, energy consumption decrease due to the improvement of the system.

Let's assume a fuzzified internet router in particular areas as shown in Fig. 1. The vertices of graphs are known as routers in the internet network.

First, we calculate the degree of every router. Step 1 of the algorithm is exercised here like $(0.2 + 0.3)$ are the membership values and $(0.5 + 0.5)$ are non-membership values. Meanwhile, the Step 2 of the algorithm can also be presented here for the edges instead of vertices.

$$d(R_1) = [(0.2 + 0.3), (0.5 + 0.5)] = (0.5, 1),$$
$$d(R_2) = [(0.2 + 0.3), (0.5 + 0.6)] = (0.5, 1.1),$$
$$d(R_3) = [(0.1 + 0.2), (0.6 + 0.6)] = (0.3, 1.2),$$
$$d(R_4) = [(0.1 + 0.2), (0.6 + 0.5)] = (0.3, 1.1).$$

Now, we calculate the square degree of every router. Step 3 of the algorithm is presented here. In which squared of the degrees of the vertices are computed.

$$d^2(R_1) = \left[((0.2)^2 + (0.3)^2), ((0.5)^2 + (0.5)^2)\right] = (0.13, 0.5),$$
$$d^2(R_2) = \left[((0.2)^2 + (0.3)^2), ((0.5)^2 + (0.6)^2)\right] = (0.13, 0.61),$$
$$d^2(R_3) = \left[((0.1)^2 + (0.2)^2), ((0.6)^2 + (0.6)^2)\right] = (0.05, 0.72),$$
$$d^2(R_4) = \left[((0.1)^2 + (0.2)^2), ((0.6)^2 + (0.5)^2)\right] = (0.05, 0.61).$$

Here, we working on the general second Zagreb index. Step 4 of the algorithm is presented here. In which $\gamma-$ power of the degrees of the vertices are computed.

$$d^{\gamma}(R_1) = [((0.2)^{\gamma} + (0.3)^{\gamma}), ((0.5)^{\gamma} + (0.5)^{\gamma})],$$
$$d^{\gamma}(R_2) = [((0.2)^{\gamma} + (0.3)^{\gamma}), ((0.5)^{\gamma} + (0.6)^{\gamma})],$$
$$d^{\gamma}(R_3) = [((0.1)^{\gamma} + (0.2)^{\gamma}), ((0.6)^{\gamma} + (0.6)^{\gamma})],$$
$$d^{\gamma}(R_4) = [((0.1)^{\gamma} + (0.2)^{\gamma}), ((0.6)^{\gamma} + (0.5)^{\gamma})].$$

We calculate the degree of every router when we choose $\gamma = 4$.

$$d^4(R_1) = \left[((0.2)^4 + (0.3)^4), ((0.5)^4 + (0.5)^4)\right] = (0.0097, 0.125),$$

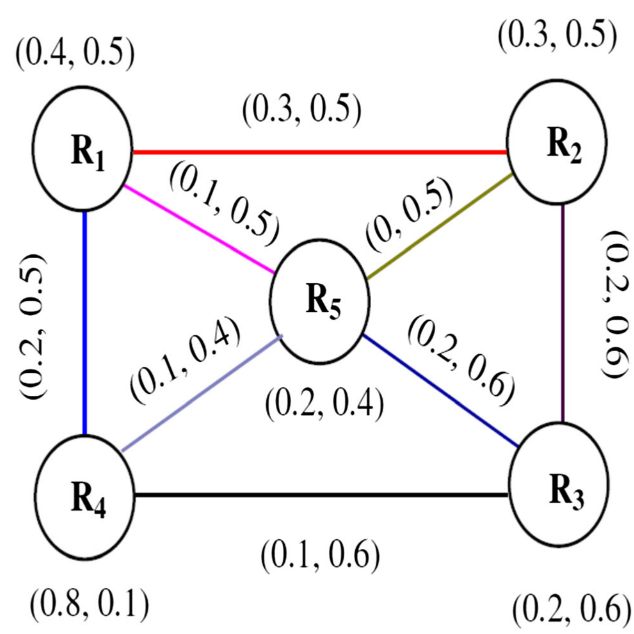

Fig. 2 $(G + u)$: add a router in internet router services

$$d^4(R_2) = \left[((0.2)^4 + (0.3)^4), ((0.5)^4 + (0.6)^4)\right] = (0.0097, 0.1921),$$

$$d^4(R_3) = \left[((0.1)^4 + (0.2)^4), ((0.6)^4 + (0.6)^4)\right] = (0.0017, 0.2592),$$

$$d^4(R_4) = \left[((0.1)^4 + (0.2)^4), ((0.6)^4 + (0.5)^4)\right] = (0.0017, 0.1921).$$

We compute the value of $\mathrm{IFM}_2^{\gamma}(G)$ for choosing $\gamma = 4$.

$$\mathrm{IFM}_2^{\gamma}(G) = \sum_{\psi_i\psi_j \in E(G)} \left[(t_\Gamma(\psi_i), f_\Omega(\psi_i))d^{\gamma}(\psi_i)(t_\Gamma(\psi_j), f_\Omega(\psi_j))d^{\gamma}(\psi_j)\right].$$

$$= (0.4, 0.5)(0.0097, 0.125) + (0.3, 0.5)(0.0097, 0.1921) + (0.2, 0.6)(0.0017, 0.2592) + (0.8, 0.1)(0.0017, 0.1921),$$

$$= 0.00388 + 0.125 + 0.00291 + 0.09605 + 0.00034 + 0.15552 + 0.00136 + 0.01921$$

$$= 0.40427. \tag{32}$$

In the Eq. (32), Step 5 of the algorithm is presented.

Compute the value of the Sombor index of the third version:

By applying the formulation of the third version of the Sombor index.

$$\mathrm{IFSO}_3(G) = \sum_{u_iu_j \in E(G)} \sqrt{2}\left[\frac{(t_\Gamma(u_i), f_\Omega(u_i))d^2(u_i) + (t_\Gamma(u_j), f_\Omega(u_j))d^2(u_j)}{(t_\Gamma(u_i), f_\Omega(u_i))d(u_i) + (t_\Gamma(u_j), f_\Omega(u_j))d(u_j)}\right]\pi$$

$$= \sqrt{2}\left[\frac{(0.4, 0.5)(0.13, 0.5) + (0.3, 0.5)(0.13, 0.61)}{(0.4, 0.5)(0.5, 1) + (0.3, 0.5)(0.5, 1.1)} + \frac{(0.3, 0.5)(0.13, 0.61) + (0.2, 0.6)(0.05, 0.72)}{(0.3, 0.5)(0.5, 1.1) + (0.2, 0.6)(0.3, 1.2)} + \frac{(0.2, 0.6)(0.05, 0.72) + (0.8, 0.1)(0.05, 0.61)}{(0.2, 0.6)(0.3, 1.2) + (0.8, 0.1)(0.3, 1.1)} + \frac{(0.8, 0.1)(0.05, 0.61) + (0.4, 0.5)(0.13, 0.5)}{(0.8, 0.1)(0.3, 1.1) + (0.4, 0.5)(0.5, 1)}\right]\pi$$

$$= \sqrt{2}\left[\frac{0.052 + 0.25 + 0.039 + 0.305}{0.2 + 0.5 + 0.15 + 0.55} + \frac{0.039 + 0.305 + 0.01 + 0.432}{0.15 + 0.55 + 0.2 + 0.5} + \frac{0.01 + 0.432 + 0.04 + 0.061}{0.06 + 0.72 + 0.24 + 0.11} + \frac{0.04 + 0.061 + 0.052 + 0.25}{(0.8, 0.1)(0.3, 1.1) + (0.4, 0.5)(0.5, 1)}\right]\pi$$

$$= \sqrt{2}\left[\frac{0.646}{1.4} + \frac{0.786}{1.4} + \frac{0.543}{1.13} + \frac{0.403}{1.05}\right]\pi$$

$$= \sqrt{2}(0.864340497)\pi = 1.222362\pi \tag{33}$$

Also, compute the value of the Sombor index of the fourth version:

By applying the formulation of the fourth version of the Sombor index.

$$\begin{aligned}
\mathrm{IFS}O_4(G) &= \sum_{u_iu_j \in E(G)} \frac{1}{2}\left[\frac{(t_\Gamma(u_i), f_\Omega(u_i))d^2(u_i) + (t_\Gamma(u_j), f_\Omega(u_j))d^2(u_j)}{(t_\Gamma(u_i), f_\Omega(u_i))d(u_i) + (t_\Gamma(u_j), f_\Omega(u_j))d(u_j)}\right]^2\pi \\
&= \frac{1}{2}\left[\left(\frac{(0.4, 0.5)(0.13, 0.5) + (0.3, 0.5)(0.13, 0.61)}{(0.4, 0.5)(0.5, 1) + (0.3, 0.5)(0.5, 1.1)}\right)^2 + \left(\frac{(0.3, 0.5)(0.13, 0.61) + (0.2, 0.6)(0.05, 0.72)}{(0.3, 0.5)(0.5, 1.1) + (0.2, 0.6)(0.3, 1.2)}\right)^2 + \left(\frac{(0.2, 0.6)(0.05, 0.72) + (0.8, 0.1)(0.05, 0.61)}{(0.2, 0.6)(0.3, 1.2) + (0.8, 0.1)(0.3, 1.1)}\right)^2 + \left(\frac{(0.8, 0.1)(0.05, 0.61) + (0.4, 0.5)(0.13, 0.5)}{(0.8, 0.1)(0.3, 1.1) + (0.4, 0.5)(0.5, 1)}\right)^2\right]\pi \\
&= \frac{1}{2}\left[\left(\frac{0.052 + 0.25 + 0.039 + 0.305}{0.2 + 0.5 + 0.15 + 0.55}\right)^2 + \left(\frac{0.039 + 0.305 + 0.01 + 0.432}{0.15 + 0.55 + 0.2 + 0.5}\right)^2 + \left(\frac{0.01 + 0.432 + 0.04 + 0.061}{0.06 + 0.72 + 0.24 + 0.11}\right)^2 + \left(\frac{0.04 + 0.061 + 0.052 + 0.25}{(0.8, 0.1)(0.3, 1.1) + (0.4, 0.5)(0.5, 1)}\right)^2\right]\pi \\
&= \frac{1}{2}\left[\left(\frac{0.646}{1.4}\right)^2 + \left(\frac{0.786}{1.4}\right)^2 + \left(\frac{0.543}{1.13}\right)^2 + \left(\frac{0.403}{1.05}\right)^2\right]\pi. = \frac{1}{2}(0.9087178)\pi \\
&= 0.4531690\pi
\end{aligned} \tag{34}$$

Table 1 Comparasion between different indices

FTIs	G_1	G_2	$G_2 - G_1$
FSO_3	1.222362π	5.30147π	$5.30147\pi - 1.222362\pi$=12.81493
FSO_4	0.4531690π	0.888455π	$0.888455\pi - 0.4531690\pi$=1.36749
FM_2^γ	0.40427	0.68747	0.68747–0.40427 = 0.2832

By adding a router in the network, then check the signal impact (Fig. 2):

First, we calculate the degree of every router. When we added a router:

$$d(R_1) = [(0.1 + 0.2 + 0.3), (0.5 + 0.5 + 0.5)] = (0.6, 1.5),$$

$$d(R_2) = [(0 + 0.2 + 0.3), (0.5 + 0.5 + 0.6)] = (0.5, 1.6),$$

$$d(R_3) = [(0.1 + 0.2 + 0.2), (0.6 + 0.6 + 0.6)] = (0.5, 1.8),$$

$$d(R_4) = [(0.1 + 0.1 + 0.2), (0.6 + 0.4 + 0.5)] = (0.4, 1.5),$$

$$d(R_5) = [(0 + (0.1) + (0.1) + (0.2)), ((0.6) + (0.5) + (0.5) + (0.4))] = (0.4, 2).$$

Now, we calculate the square degree of every router:

$$d^2(R_1) = \left[((0.1)^2 + (0.2)^2 + (0.3)^2), ((0.5)^2 + (0.5)^2 + (0.5)^2)\right] = (0.14, 0.75),$$

$$d^2(R_2) = \left[(0^2 + (0.2)^2 + (0.3)^2), ((0.5)^2 + (0.5)^2 + (0.6)^2)\right] = (0.13, 0.86),$$

$$d^2(R_3) = \left[((0.1)^2 + (0.2)^2 + (0.2)^2), ((0.6)^2 + (0.6)^2 + (0.6)^2)\right] = (0.09, 1.08),$$

$$d^2(R_4) = \left[((0.1)^2 + (0.1)^2 + (0.2)^2), ((0.6)^2 + (0.5)^2 + (0.4)^2)\right] = (0.06, 0.77),$$

$$d^2(R_5) = \left[(0^2 + (0.1)^2 + (0.1)^2 + (0.2)^2), ((0.6)^2 + (0.5)^2 + (0.5)^2 + (0.4)^2)\right], = (0.06, 1.02).$$

Note that the performance of each router increased by adding the new router R_5. Also, router R_1 still has minimum strength, and R_3 has maximum strength. The strength of the whole system by all three indices is given below.

Here, we working for the general second Zagreb index:

$$d^\gamma(R_1) = [((0.1)^\gamma + (0.2)^\gamma + (0.3)^\gamma), ((0.5)^\gamma + (0.5)^\gamma + (0.5)^\gamma)],$$

$$d^\gamma(R_2) = [(0^\gamma + (0.2)^\gamma + (0.3)^\gamma), ((0.5)^\gamma + (0.5)^\gamma + (0.6)^\gamma)],$$

$$d^\gamma(R_3) = [((0.1)^\gamma + (0.2)^\gamma + (0.2)^\gamma), ((0.6)^\gamma + (0.6)^\gamma + (0.6)^\gamma)],$$

$$d^\gamma(R_4) = [((0.1)^\gamma + (0.1)^\gamma + (0.2)^\gamma), ((0.6)^\gamma + (0.5)^\gamma + (0.4)^\gamma)],$$

$$d^\gamma(R_5) = [(0^\gamma + (0.1)^\gamma + (0.1)^\gamma + (0.2)^\gamma), ((0.6)^\gamma + (0.5)^\gamma + (0.5)^\gamma + (0.4)^\gamma)].$$

We calculate the degree of every router when we choose $\gamma = 4$.

$$d^4(R_1) = \left[((0.1)^4 + (0.2)^4 + (0.3)^4), ((0.5)^4 + (0.5)^4 + (0.5)^4)\right] = (0.0098, 0.1875),$$

$$d^4(R_2) = \left[(0^4 + (0.2)^4 + (0.3)^4), ((0.5)^4 + (0.5)^4 + (0.6)^4)\right] = (0.0097, 0.2546),$$

$$d^4(R_3) = \left[((0.1)^4 + (0.2)^4 + (0.2)^4), ((0.6)^4 + (0.6)^4 + (0.6)^4)\right] = (0.0033, 0.3888),$$

$$d^4(R_4) = \left[((0.1)^4 + (0.1)^4 + (0.2)^4), ((0.6)^4 + (0.5)^4 + (0.4)^4)\right] = (0.0018, 0.2177),$$

$$d^4(R_5) = \left[(0^4 + (0.1)^4 + (0.1)^4 + (0.2)^4), ((0.6)^4 + (0.5)^4 + (0.5)^4 + (0.4)^4)\right],$$

$$= (0.0018, 0.2802).$$

We compute the value of $M_2^\gamma(G)$ for choosing $\gamma = 4$.

$$IFM_2^\gamma(G) = \sum_{R_iR_j \in E(G)} \left[(t_\Gamma(R_i), f_\Omega(R_i))d^\gamma(R_i)\big(t_\Gamma(R_j), f_\Omega(R_j)\big)d^\gamma(R_j)\right].$$

$$\begin{aligned} &= (0.4, 0.5)(0.0098, 0.1875) + (0.3, 0.5)(0.0097, 0.2546) \\ &\quad + (0.2, 0.6)(0.0033 + (0.8, 0.1)(0.0018, 0.2177) \\ &\quad + (0.2, 0.4)(0.0018, 0.2802), \\ &= 0.00392 + 0.09375 + 0.00291 + 0.1273 + 0.00066 \\ &\quad + 0.23328 + 0.00144 + 0.02177 + 0.00036 + 0.11208, \\ &\quad = 0.68747\pi. \end{aligned} \tag{35}$$

Compute the value of the Sombor index of the third version:

By applying the formulation of the third version of the Sombor index.

$$
\begin{aligned}
&IFSO_3(G)\\
&= \sum_{u_iu_j \in E(G)} \sqrt{2}\left[\frac{(t_\Gamma(u_i),f_\Omega(u_i))d^2(u_i) + \big(t_\Gamma(u_j),f_\Omega(u_j)\big)d^2(u_j)}{(t_\Gamma(u_i),f_\Omega(u_i))d(u_i) + \big(t_\Gamma(u_j),f_\Omega(u_j)\big)d(u_j)}\right]\pi\\
&= \sqrt{2}\left[\frac{(0.4,0.5)(0.14,0.75)+(0.3,0.5)(0.13,0.86)}{(0.4,0.5)(0.6,1.5)+(0.3,0.5)(0.5,1.6)}\right.\\
&+\frac{(0.3,0.5)(0.13,0.86)+(0.2,0.6)(0.09,1.08)}{(0.3,0.5)(0.5,1.6)+(0.2,0.6)(0.5,1.8)}\\
&+\frac{(0.2,0.6)(0.09,1.08)+(0.8,0.1)(0.06,0.77)}{(0.2,0.6)(0.5,1.8)+(0.8,0.1)(0.4,1.5)}\\
&+\frac{(0.4,0.5)(0.14,0.75)+(0.8,0.1)(0.06,0.77)}{(0.4,0.5)(0.6,1.5)+(0.8,0.1)(0.4,1.5)}\\
&+\frac{(0.4,0.5)(0.14,0.75)+(0.2,0.4)(0.06,1.02)}{(0.4,0.5)(0.6,1.5)+(0.2,0.4)(0.4,2)}\\
&+\frac{(0.3,0.5)(0.13,0.86)+(0.2,0.4)(0.06,1.02)}{(0.3,0.5)(0.5,1.5)+(0.2,0.4)(0.4,2)}\\
&+\frac{(0.2,0.6)(0.09,1.08)+(0.2,0.4)(0.06,1.02)}{(0.2,0.6)(0.5,1.8)+(0.2,0.4)(0.4,2)}\\
&\left.+\frac{(0.8,0.1)(0.06,0.77)+(0.2,0.4)(0.06,1.02)}{(0.8,0.1)(0.4,1.5)+(0.2,0.4)(0.4,2)}\right]\pi\\
&= \sqrt{2}\left[\frac{45}{93}+\frac{227}{426}+\frac{791}{1650}+\frac{139}{365}+\frac{851}{1870}+\frac{889}{1830}+\frac{543}{1030}+\frac{109}{270}\right]\pi\\
&= 5.301476\pi
\end{aligned} \tag{36}
$$

Also, compute the value of the Sombor index of the fourth version:

By applying the formulation of the fourth version of the Sombor index.

$$
\begin{aligned}
&\mathrm{IFSO}_4(G)\\
&= \sum_{u_iu_j \in E(G)} \frac{1}{2}\left[\frac{(t_\Gamma(u_i),f_\Omega(u_i))d^2(u_i) + \big(t_\Gamma(u_j),f_\Omega(u_j)\big)d^2(u_j)}{(t_\Gamma(u_i),f_\Omega(u_i))d(u_i) + \big(t_\Gamma(u_j),f_\Omega(u_j)\big)d(u_j)}\right]^2\pi\\
&= \frac{1}{2}\left[\left(\frac{(0.4,0.5)(0.14,0.75)+(0.3,0.5)(0.13,0.86)}{(0.4,0.5)(0.6,1.5)+(0.3,0.5)(0.5,1.6)}\right)^2\right.\\
&+\left(\frac{(0.3,0.5)(0.13,0.86)+(0.2,0.6)(0.09,1.08)}{(0.3,0.5)(0.5,1.6)+(0.2,0.6)(0.5,1.8)}\right)^2\\
&+\left(\frac{(0.2,0.6)(0.09,1.08)+(0.8,0.1)(0.06,0.77)}{(0.2,0.6)(0.5,1.8)+(0.8,0.1)(0.4,1.5)}\right)^2\\
&+\left(\frac{(0.4,0.5)(0.14,0.75)+(0.8,0.1)(0.06,0.77)}{(0.4,0.5)(0.6,1.5)+(0.8,0.1)(0.4,1.5)}\right)^2\\
&+\left(\frac{(0.3,0.5)(0.13,0.86)+(0.2,0.4)(0.06,1.02)}{(0.3,0.5)(0.5,1.5)+(0.2,0.4)(0.4,2)}\right)^2\\
&+\left(\frac{(0.3,0.5)(0.13,0.86)+(0.2,0.4)(0.06,1.02)}{(0.3,0.5)(0.5,1.5)+(0.2,0.4)(0.4,2)}\right)^2\\
&+\left(\frac{(0.8,0.1)(0.06,0.77)+(0.2,0.4)(0.06,1.02)}{(0.8,0.1)(0.4,1.5)+(0.2,0.4)(0.4,2)}\right)^2\\
&\left.+\left(\frac{(0.8,0.1)(0.06,0.77)+(0.2,0.4)(0.06,1.02)}{(0.8,0.1)(0.4,1.5)+(0.2,0.4)(0.4,2)}\right)^2\right]\pi\\
&= \frac{1}{2}\left[\left(\frac{45}{93}\right)^2+\left(\frac{227}{426}\right)^2+\left(\frac{791}{1650}\right)^2+\left(\frac{139}{365}\right)^2+\left(\frac{851}{1870}\right)^2+\left(\frac{889}{1930}\right)^2\right.\\
&\left.+\left(\frac{543}{1030}\right)^2+\left(\frac{109}{270}\right)^2\right]\pi = \frac{1}{2}(1.77691)\pi = 0.888455\pi
\end{aligned} \tag{37}
$$

By including a new router (vertex) in the network/graph in the previous network. This time, a specific neighborhood has better internet access with less consumption. We also observe that the network's maximum strength grew by the general Zagreb index and the third and fourth versions of the Sombor indices. On behalf of human opinion when adding a router their performance will be increased in a particular area. So, our method is more liable than other existing methods or fuzzy topological graph parameters.

Comparative analysis: Because graph theory has so many application fields, there is a surge in studies introducing different kinds of fuzzy graphs that can be used to model imprecise and uncertain data.

Intuitionistic fuzzy sets: Since intuitionistic fuzzy sets incorporate the idea of non-membership degrees, uncertainty can be better modeled than it can be in pure fuzzy graphs, which only take into account membership degrees. An enhanced foundation for simulating intricate interactions is provided by intuitionistic fuzzy graphs, which embrace the duality of membership and non-membership. *Graph operations*: in order to investigate the effects of uncertainty on network dynamics, graph operations specifically, vertex and edge deletion are carried out on intuitionistic fuzzy graphs rather than pure fuzzy graphs. Nodes and edges may have different degrees of reluctance or non-membership in addition to variable degrees of membership in real-world circumstances. Through the use of graph operations in this setting, we can capture the complex interactions between these uncertainties and offer insights into the network's resilience and flexibility. *Sombor index for internet routing*: novel internet routing-specific versions of the Sombor index have been introduced to meet the demand for a complete measure that takes intuitionistic fuzzy graph uncertainty into account in addition to connectivity. Adjusted for reluctance and non-membership, the Sombor index is more in line with the dynamic character of network settings, where uncertainties are critical to routing decisions.

Although we continue to concentrate on intuitionistic fuzzy graphs, a comparison with other fuzzy parameters, including pure fuzzy graphs or hybrid models, might provide a more comprehensive viewpoint. It is deliberate, therefore, that these comparisons are not included in our study. We argue that the special features of intuitionistic fuzzy graphs deserve special consideration and that before making any direct comparisons with other models, a thorough investigation of their properties is necessary. *Mathematics and numerics*: consider a vertex deletion operation on an intuitionistic fuzzy graph $G = (V, E, \mu, \nu)$ to demonstrate the mathematical foundations. With ν as a vertex in V, we can create a new graph $G = (V', E', \mu', \nu')$ by deleting ν'. The modified μ' and ν' membership and non-membership functions would take into account how the loss of a vertex would affect the remaining nodes. A quantitative indicator of the shift in the network's routing efficiency is the Sombor index, which can be calculated numerically for internet routing both before and after the graph operation.

Essentially, to concentrate on intuitionistic fuzzy graphs and conduct graph operations within this context is a strategic decision to decipher the intricacies of uncertainty in network architectures. In this way, we develop a comprehensive model taking connectivity, uncertainty, and routing efficiency into account, that is consistent with the dynamic environment of contemporary network systems.

Here we compare the results of intuitionistic fuzzy Sombor third and fourth versions of Graphs 1 and 2 listed below in Table 1. This table shows that when we add a vertex between all the vertices, the performance of each vertex (router) increases. Studied the behavior of both graphs among the third invariant of intuitionistic fuzzy Sombor graph parameter is more reliable than other fuzzy graph parameters. In Table 1, FTIs is fuzzy topological indices, FSO_3 is the fuzzy third version of the Sombor topological index and similarly FSO_4 are the fuzzy fourth version of the Sombor topological index, while FM_2^{γ} is the fuzzy second generalize index.

4 Conclusion

Multidisciplinary approches can provide a lot of insights in providing effcient solutions to exsiting problems in different fields. In this study, we combine mathematical tools namely topological indices and intiutionistic fuzzy graphs used in various disciplines such as chemistry, biology, economics, and computer sciences. We introduce a novel indices and offer new algorithm to solve problems in networks that can model different complex relations analyzed under different disciplines. Then, we have studied an application of our algorithm in a computer science problem.

Specifically, we proposed some new TIs for the frame of IFG. IFG has comparatively less loss of information than FG. The main outcomes of our research work are below.

(1) We introduced the idea of the general second Zagreb index for IFG and determined its bounds and their properties.
(2) We defined the concept of the Sombor index in the IFGs framework and also, their bounds and their properties are determined.
(3) An application of TIs to the internet routing network is explored. In this application, we have shown that adding more internet routers in the network can increase internet speed and the strength of the entire system.

A few particular advantages are: topological indices are very useful in practical approaches in mathematical chemistry, in economics of social networks, even in the engineering side of applied mathematics. Interlinking topological indices with mathematical tools such as fuzzy sets are providing solutions to modelling and problem solving in network studies. Fuzzy topological indices or intuitionistic fuzzy topological indices are more generalized versions. That would be more practical in various fields. So, in this study, we developed new fuzzy topological indices and provided a practical approach to studying these indices. Moreover, we also studied some bounds in this study.

In the future, fulfilling the gap in the literature regarding fuzzy graph theory is an highly important issue due to the increased of need to study networks in many fields and disciplines such as economics, chemistry, biology, and computer science. Different algorithms can be designed ith different modelling techniques. We are particularly, working on topological graph parameters for various fuzzy environments, such as picture soft fuzzy graph, hyper soft fuzzy graph, picture fuzzy graph, soft fuzzy graph, neutrosophic fuzzy graph, and interval-valued fuzzy graph.

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Declarations

Conflict of interest The author declares that he has no conflict of interest.

References

Ahmad U, Khan NK, Saeid AB (2023) Fuzzy topological indices with application to cybercrime problem. Granul Comput 8:967–980

Akram M, Bibi R (2023) Multi-criteria group decision making based on an integrated PROMETHEE approach with 2 tuple linguistic Fermatean fuzzy sets. Granul Comput 8(5):917–941

Akram M, Davvaz M (2012) Strong intuitionistic fuzzy graphs. Filomat 26(1):177–196

Akram M, Dudek WA (2013) Intuitionistic fuzzy hypergraphs with applications. Inf Sci 218:182–193

Akram M, Saleem D, Al-Hawary T (2020) Spherical fuzzy graphs with application to decision-making. Math Comput Appl 25(1):8

Akram M, Shahzadi S, Shah SMU, Allahviranloo T (2023) A fully Fermatean fuzzy multi-objective transportation model using an extended DEA technique. Granul Comput 8(6):1173–1204

Alahmadi RA, Ganie AH, Al-Qudah Y, Khalaf M, Ganie AH (2023) Multi-attribute decision-making based on novel Fermatean fuzzy similarity measure and entropy measure. Granul Comput 8(6):1385–1405

Ali S, Mathew S, Mordeson JN (2021) Hamiltonian fuzzy graphs with application to human trafficking. Inf Sci 550:268–284

Al-Zibaree HKY, Konur M (2023) Fuzzy analytic hierarchal process for sustainable public transport system. J Oper Intell 1(1):1–10

Arya V, Kumar S (2020) Knowledge measure and entropy: a complementary concept in fuzzy theory. Granul Comput 6(3):631–643

Atanassov KT (1986) Intuitionistic fuzzy sets. Fuzzy Sets Syst 20(1):87–96

Atanassov KT (2012) Intuitionistic fuzzy sets: theory and applications, studies in fuzziness and soft computing. Phys Heidelberg. https://doi.org/10.1007/978-3-7908-1870-3

Atanassov KT (1999) Intuitionistic fuzzy sets. Physica-Verlag HD, pp 1–137

Atanassov KT, Shannon A (1994) A first step to a theory of the intuitionistic fuzzy graphs. In: Proceedings of the first workshop on fuzzy based expert systems. Sofia, pp 59–61

Balaban AT (1983) Topological indices based on topological distances in molecular graphs. Pure Appl Chem 55(2):199–206

Bhutani KR, Rosenfeld A (2003) Geodesics in fuzzy graphs. Electron Notes Discrete Math 15:49–52

Bukhari S, Mk J, Azeem M, Swaray S (2023) Patched network and its vertex-edge metric-based dimension. IEEE Access 11:4478–4485

Chen SM, Jian WS (2017) Fuzzy forecasting based on two-factors second-order fuzzy-trend logical relationship groups, similarity measures and PSO techniques. Inf Sci 391:65–79

Chen SM, Wang JY (1995) Document retrieval using knowledge-based fuzzy information retrieval techniques. IEEE Trans Syst Man Cybern 25(5):793–803

Chen SM, Wang NY (2010) Fuzzy forecasting based on fuzzy-trend logical relationship groups. IEEE Trans Syst Man Cybern 40(5):1343–1358

Chen SM, Ko YK, Chang YC, Pan JS (2009) Weighted fuzzy interpolative reasoning based on weighted increment transformation and weighted ratio transformation techniques. IEEE Trans Fuzzy Syst 17(6):1412–1427

Chen SM, Zou XY, Gunawan GC (2019) Fuzzy time series forecasting based on proportions of intervals and particle swarm optimization techniques. Inf Sci 500:127–139

Cruz R, Gutman I, Rada J (2021) Sombor index of chemical graphs. Appl Math Comput 399:126018

Dagistanli HA (2023) An integrated fuzzy MCDM and trend analysis approach for financial performance evaluation of energy companies in Borsa Istanbul Sustainability Index. J Soft Comput DecisAnal 1(1):39–49

Das KC, Cevik AS, Cangul IC, Shang Y (2021) On Sombor Index. Symmetry 13(1):140

Davvaz B, Jan N, Mahmood T, Ullah K (2019) Intuitionistic fuzzy graphs of nth type with applications. J Intell Fuzzy Syst 36(4):3923–3932

De SK, Biswas R, Roy AR (2001) An application of intuitionistic fuzzy sets in medical diagnosis. Fuzzy Sets Syst 117(2):209–213

Dinar J, Hussain Z, Zaman S, Rehman SU (2023) Wiener index for an intuitionistic fuzzy graph and its application in the water pipeline network. Ain Shams Eng J 14(1):101826

Diner H, Yuksel S, Eti S (2023) Identifying the right policies for increasing the efficiency of the renewable energy transition with a novel fuzzy decision-making model. J Soft Comput Decis Anal 1(1):50–62

Ejegwa PA, Akowe SO, Otene PM, Ikyule JM (2014) An overview on intuitionistic fuzzy sets. Int J Sci Technol Res 3(3):142–145

Flores H, Srirama S (2013) Adaptive code offloading for mobile cloud applications: exploiting fuzzy sets and evidence-based learning. In: Proceeding of the fourth ACM workshop on mobile cloud computing and services, pp 9–16

Fuzziness and soft computing. Physica New York

Ghoushchi SJ, Sarvi S (2023) Prioritizing and evaluating risks of ordering and prescribing in the chemotherapy process using an extended SWARA and MOORA under fuzzy Z-numbers. J Oper Intell 1(1):44–66

Graovac A, Pisanski T (1991) On the Wiener index of a graph. J Math Chem 8(1):53–62

Gutman I (2022a) Sombor indices-back to geometry. Open J Discrete Appl Math 5(2):1–5. https://doi.org/10.30538/psrp-odam2022.0072

Gutman I (2022b) TEMO theorem for Sombor index. Open J Discrete Appl Math 5(1):25–28

Hassanpour M (2020) Classification of seven Iranian recycling industries using MCDM models. Ann Optim Theory Pract 3(4):37–52

Hawary A (2011) Complete graphs. Int J Math Comb 4:26–34

Horng YJ, Chen SM, Chang YC, Lee CH (2005) A new method for fuzzy information retrieval based on fuzzy hierarchical clustering and fuzzy inference techniques. IEEE Trans Fuzzy Syst 13(2):216–228

Imran M, Luo R, Jamil MK, Azeem M, Fahd KM (2022) Geometric perspective to a degree “Based topological indices of the supramolecular chain.” Results Eng 16:100716

Imran M, Ismail R, Azeem M, JamilM K, Al-Sabri EHA (2023) Sombor topological indices for different nanostructures. Heliyon 9(10):e20600

Ismail R, Azeem M, Shang Y, Imran M, Ahmad A (2023) A unified approach for extremal general exponential multiplicative Zagreb indices. Axioms 12(7):675

Jan N, Zedam L, Mahmood T, Ullah K (2019) Cubic bipolar fuzzy graphs with applications. J Intell Fuzzy Syst 37(2):2289–2307

Kalathian S, Ramalingam S, Raman S, Srinivasan N (2020) Some topological indices in fuzzy graphs. J Intell Fuzzy Syst 39(5):6033–6046

Karwowski W, Mital A (1986) Potential applications of fuzzy sets in industrial safety engineering. Fuzzy Sets Syst 19(2):105–120

Khan A, Akram M (2021) Complex Pythagorean Dombi fuzzy graphs for decision making. Granul Comput 6:645–669

Kochen M (1975) Application of fuzzy set in psychology. Fuzzy sets and their applications to cognitive and decision processes. Elsevier, Amsterdam, pp 395–408

Lin L, Yuan XH, Xia ZO (2007) Multicriteria fuzzy decision-making methods based on intuitionistic fuzzy sets. J Comput Syst Sci 73(1):84–88

Liu JB, Nadeem MF, Azeem M (2022) Bounds on the partition dimension of convex polytopes. Comb Chem High Throughput Screen 25(3):547–553

Massadeh MO, Gharaibeh NK (2011) Some properties on fuzzy graphs. Adv Fuzzy Math 6(2):245–252

Mathew S, Sunitha MS (2009) Types of arcs in a fuzzy graph. Inf Sci 179(11):1760–1768

Mordeson JN (1993) Fuzzy line graphs. Pattern Recognit Lett 14(5):381–384

Mordeson JN, Chang-Shyh P (1994) Operations on fuzzy graphs. Inf Sci 79(3–4):159–170

Nadeem MF, Azeem M, Siddiqui HMA (2021) Comparative study of Zagreb indices for capped, semi-capped, and uncapped carbon nanotubes. Polycycl Aromat Compd 42(6):3545–3562

Naeem T, Jamil MK, Fahd KM, AlAmeri A (2022) Wiener index of intuitionistic fuzzy graphs with an application to transport network flow. Complexity 2022:1–14

Parvathi R, Thamizhendhi G (2010) Domination in intuitionistic fuzzy graphs. Notes Intuit Fuzzy Sets 16(2):39–49

Parvathi R, Karunambigai MG, Atanassov KT (2009) Operations on intuitionistic fuzzy graphs. In: 2009 IEEE international conference on fuzzy systems, pp 1396–1401

Rosenfeld A (1975) Fuzzy graphs. In: Fuzzy sets and their applications to cognitive and decision processes, Proceedings of the US-Japan Seminar on Fuzzy Sets and their Applications, Held at the University of California, Berkeley, California, July 1–4, pp 77-95.

Schuh C (2005) Fuzzy sets and their application in medicine. In: NAFIPS 2005–2005. Annual meeting of the North American Fuzzy Information Processing Society. IEEE, pp 86–91

Smithson M (1988) Fuzzy set theory and the social sciences: the scope for applications. Fuzzy Sets Syst 26(1):1–21

Sotoudeh AA (2022) The applications of MCDM methods in COVID-19 pandemic: a state of the art review. Appl Soft Comput 126:109238

Treadwell WA (1995) Fuzzy set theory movement in the social sciences. Public Adm Rev 55(1):91

Yeh RT, Bang SY (1975) Fuzzy relations, fuzzy graphs, and their applications to clustering analysis. Fuzzy sets and their applications to cognitive and decision processes. Academic Press, New York, pp 125–149

Zadeh LA (1965) Fuzzy sets. Inf Control 8(3):338–353

Zadeh LA (1978) Fuzzy sets as a basis for a theory of possibility. Fuzzy Sets Syst 1(1):3–28

Zimmermann HJ (2010) Fuzzy set theory. Wiley Interdiscipl Rev Comput Stat 2(3):317–332

Zimmermann HJ (2012) Practical applications of fuzzy technologies, vol 6. Springer Science and Business Media, Berlin