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ORIGINAL PAPER

A probabilistic dual hesitant fuzzy multi-attribute decision-making method based on entropy and cross-entropy

Pingping Wang[1] · Jiahua Chen[1,2]

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Abstract

The probabilistic dual hesitant fuzzy set (PDHFS), as an extension of generalization of the dual hesitant fuzzy set, contains not only the hesitation values of membership degree (MD) and non-membership degree (NMD), but also considers the probabilities corresponding to MDs and NMDs, which are the degree of support and confidence of the decision makers in the evaluation value given by them. Distance measures, entropy measures, and cross-entropy measures are important tools in multi-attribute decision-making. In the PDHFS environment, distance and entropy measures are improved, and cross-entropy is proposed, and multi-attribute decision-making methods based on distance and entropy and cross-entropy are given. First, in order to effectively compare the distances between different PDHFSs, we improve the existing distance measures. Second, we review the existing formulations of probabilistic dual hesitant fuzzy entropy and find that they could not effectively distinguish the uncertainty of different PDHFSs due to ignoring the uncertainty caused by the differences between different MDs and between different NMDs, so we improve the existing entropy measure. Additionally, the formulas and properties of the cross-entropy of PDHFS and the axiomatic definition of the generalized cross-entropy of PDHFS are given. Finally, depending on the distance and entropy and cross-entropy built, we propose a new multi-attribute decision method to solve the multi-attribute decision problem with completely unknown attribute weights. We apply the proposed method to the protective decision-making for the release of radioactive substances, and the feasibility of the method is verified by comparative analysis.

Keywords Multi-attribute group decision-making · Probabilistic dual hesitant fuzzy set · Entropy measure · Cross-entropy measure · Distance measures

1 Introduction

As a crucial component of decision theory, multi-attribute decision-making has been widely applied in economics, management, industry, and other fields (Chen et al.2009, 2017; Shahzaib et al. 2020; Dong et al. 2021; She et al. 2021; Dhankhar et al.2022; Zeb et al. 2022; Wang et al. 2023). However, in the actual decision-making process, the variability of the decision environment and decision information leads to decision makers frequently being unable to give the exact values. Zadeh (1965) proposed fuzzy set (FS). In order to better express decision makers' viewpoints and attitudes toward the decision, Torra et al. (2010) proposed the definition of hesitating fuzzy set (HFS), which allows decision makers to hesitate between several different decision evaluation values, and HFS is an important extension in fuzzy theory. Zhu et al. (2012) proposed dual hesitating fuzzy set (DHFS) based on HFS, which increases decision makers' hesitation for NMD. In more cases, decision makers have different knowledge experiences and tendencies, and their preferences between MDs may be different and uphold different support for different MDs. Xu et al. (2017) proposed probabilistic hesitant fuzzy sets (PHFS) to achieve a more accurate representation of decision makers' mental states, and PHFS gives the probability of occurrence of each MD,

✉ Jiahua Chen
chen.jiahua@usc.edu.cn

1 School of Economics, Management and Law, University of South China, Hengyang 421001, China

2 Hunan Provincial Key Laboratory of Emergency Safety Technology and Equipment for Nuclear Facilities, Hengyang 421001, China

which also solves the case of inconsistent attitudes of decision makers. Since NMD is an important information in decision problems, PHFS does not take NMD into account. Hao et al. (2017) combined the information of MD, NMD and probability to propose probabilistic dual hesitant fuzzy set (PDHFS), which contains more comprehensive information.

In these years, the research of PDHFS has gradually attracted academic attention, and the existing research on PDHFS has focused on multi-attribute decision methods and multi-attribute group decision methods (Ren et al. 2019; Harish et al. 2019; Liu et al. 2019; Shao et al. 2021; Song et al. 2021a, b; Li et al. 2022; Ning et al. 2022a, b, c; Noor et al. 2022; Kumar et al. 2022). For example, Noor et al. (2022) proposed multi-attribute group decision-making based on probabilistic pairwise hesitant fuzzy Maclaurin symmetric mean operator, and developed two new methods to deal with multi-attribute group decision-making problems based on improved agglomerative operator with the help of COPRAS technique. The distance measure and information measure are important tools in solving multi-attribute decision problems. There are abundant studies on the distance measures of FS, HFS, DHFS and PHFS (Biplab et al. 2019; Ali et al. 2021; Zeeshan et al. 2022; Yan et al. 2022; Zeng et al. 2022), but the existing distance measures of PDHFS are relatively few and do not effectively compare the distances between different PDHFSs (Garg et al. 2018; Ning et al.2022a, b and c), so this paper improves on the original distance measures and proposes a formula that can effectively compare the distances between PDHFSs.

Information measure is a measure of information content in fuzzy decision preferences, and entropy and cross-entropy are the two most common ways, among which there are many studies on fuzzy theoretical entropy and cross-entropy (Chen et al. 2001; Zarandi 2010; Zeng 2011; Jun 2016; Tiantian et al. 2017; Zhang 2020; Zhang et al. 2021; Erdal et al. 2023; Du et al. 2023; Boffa et al. 2023), and fewer studies on the entropy of PDHFS. Hao et al. (2017) gave the axiomatic definition and calculation formula of probabilistic dual hesitant fuzzy entropy while defining PDHFS, but ignored the existence of probabilistic dual hesitant fuzzy elements with incomplete probability situation. To remedy this deficiency, Su et al. (2022) took the probabilistic incompleteness of MDs and NMDs into the calculation of probabilistic dual hesitant fuzzy entropy on this basis and defined the correlation coefficient. Similarly, Ning et al. 2022a, b and c) proposed a decision-making method based on probability dual hesitant fuzzy entropy. However, the existing formulas for calculating the probabilistic dual hesitant fuzzy entropy all have certain defects, ignoring the hesitation degree brought about by the differences between different MDs or between different NMDs. Cross-Entropy (CE) is an important concept in Shannon information theory, which is mainly used to measure the discrepancy information between two probability distributions (Khalaj et al. 2020; Gao et al. 2021; Rogulj et al. 2021). There is no clear definition of probabilistic dual hesitant fuzzy cross-entropy in the current study.

In this study, the concepts of distance measure and entropy measure in a fuzzy environment are reviewed, and some problems were identified in the review process. The motivations for this study are put forward as follows: (1) PDHFS can more comprehensively express decision makers' preference information in an increasingly complex decision environment. (2) Existing distance formulas for PDHFSs cannot effectively compare the distances between different PDHFSs. (3) The existing probabilistic dual hesitation fuzzy entropy measure cannot effectively distinguish the uncertainty of different PDHFEs, ignoring the uncertainty caused by the differences between different MDs or between different NMDs. (4) The probabilistic dual hesitation cross-entropy has not been clearly defined, and the existing studies lack knowledge information related to the cross-entropy of PDHFS. (5) The existing probabilistic dual hesitant fuzzy multi-attribute decision models are obtained based on flawed distance formulas or entropy formulas. For the multi-attribute decision problem with completely unknown attribute weights, the method of solving attribute weights needs to be updated. (6) The release of radioactive substances from nuclear power plants is a matter of public, social, and environmental issues, and the emergency decision of radioactive substance release is in urgent need of comprehensive decision information and new decision methods.

To effectively compare the distances between different PDHFSs, improve the probability dual hesitant fuzzy entropy measure, accurately distinguish the uncertainty of different PDHFEs, and supplement the cross-entropy definition of PDHFS. This study improves the existing axiomatic definition and formula of probabilistic dual hesitant fuzzy distance and entropy, and proposes the axiomatic definition and formula of probabilistic dual hesitant fuzzy cross-entropy and generalized cross-entropy. This paper uses entropy and cross-entropy to determine the attribute weights when the attributes are completely unknown, and constructs a new multi-attribute decision model by combining the proposed distance measure. Finally, this paper validates the method model with an example of protective decision-making for the release of radioactive substances and comparative analysis. The main contributions of this study are as follows: (1) The probabilistic dual hesitant fuzzy distance measure is improved, and the new distance formula can effectively compare the distance between different PDHFSs. (2) The probabilistic dual hesitant fuzzy entropy measure is improved. The new entropy measure formula adds the consideration of the difference between different MDs or between different NMDs, and the new entropy measure expresses a more comprehensive uncertainty. (3) The formula and properties of the cross-entropy of PDHFS and the axiomatic definition of generalized cross-entropy of PDHFS are given, which fills the gap in the research on the cross-entropy of PDHFS. (4) A multi-attribute decision method based on entropy and cross-entropy is proposed and applied to the example of protective decision-making for the release of radioactive substances. The attribute weight determination in the proposed method is obtained based on the combination of entropy and cross-entropy, and the combination of entropy and cross-entropy considers both fuzziness and deviation, providing an idea of the attribute weight determination method for the multi-attribute decision-making problem with completely unknown attribute weights.

The main work of this paper is as follows: Sect. 2 reviews the definition and formula of PDHFSs, points out the shortcomings of the existing distance formula and proposes an improved axiomatic definition and calculation formula of distance; Sect. 3 reviews the existing formulas of probabilistic dual hesitant fuzzy entropy and points out the defects in the existing entropy formulas and proposes a novel definition and calculation formula of probabilistic dual hesitant fuzzy entropy. Section 4 gives the calculation formula and properties of probabilistic dual hesitant fuzzy cross-entropy and proposes the axiomatic definition of generalized cross-entropy. Section 5 determines the attribute weights using entropy and cross-entropy, and constructs a probabilistic dual hesitant fuzzy multi-attribute decision model by combining the improved distance formula. Section 6 applies the constructed model to the protective decision-making for the release of radioactive substances and verifies the feasibility of the method by comparative analysis. Section 7 describes the conclusions of this study and future research directions.

2 Preliminaries

2.1 Probabilistic dual hesitant fuzzy sets

1 (Hao et al. 2017) Let Z be a fixed set, a PDHFS on Z is recorded as:

$$A = \{\langle z_i, h_i(z)|p_i(z), g_i(z)|q_i(z)\rangle | z_i \in Z \tag{1}$$

The components $h_i(z)|p_i(z) = \{\gamma_i^\lambda|p_i^\lambda\}(\lambda = 1,2,\ldots,\#h_i, \sum_{\lambda=1}^{\#h_i} p_i^\lambda = 1)$ and $g_i(z)|q_i(z) = \{\eta_i^\lambda|q_i^\lambda\}(\lambda = 1,2,\ldots,\#g_i, \sum_{\lambda=1}^{\#g_i} q_i^\lambda = 1)$ are some possible elements of membership degree (MD) and non-membership degree (NMD), where $h_i(z)$ and $g_i(z)$ are the MD and NMD of $z_i \in Z$, respectively. $p_i(z)$ and $q_i(z)$ are the probabilistic information for $h_i(z)$ and $g_i(z)$. Also,

$$0 \le \gamma_i^\lambda, \eta_i^\lambda \le 1; 0 \le \max_\lambda\{\gamma_i^\lambda\} + \max_\lambda\{\eta_i^\lambda\} \le 1; 0 \le p_i^\lambda, q_i^\lambda \le 1.$$

The element of PDHES $\xi = \langle h(z)|p(z), g(z)|q(z)\rangle$ is called as the PDHFE, recorded $\xi = \langle h|p, g|q\rangle$, and the elements in $h|p$ and $g|q$ are arranged in ascending order of MD γ^λ and NMD η^λ, respectively. The complementary set of α is $\xi^c = \langle g|q, h|p\rangle$.

If $\sum_{\lambda=1}^{\#h_i} p_i^\lambda \le 1, \sum_{\lambda=1}^{\#g_i} q_i^\lambda \le 1$, then A is called a generalized PDHFS.

Definition 2 (Hao et al. 2017) Let $\xi = \langle h|p, g|q\rangle$ be a PDHFE, then the score function of the PDHFE is built as:

$$s(\xi) = \sum_{\lambda=1}^{\#h} \gamma^\lambda p^\lambda - \sum_{\lambda=1}^{\#g} \eta^\lambda q^\lambda. \tag{2}$$

Definition 3 (Hao et al. 2017) Let $\xi = \langle h|p, g|q$ be a PDHFE, then the deviation degree of the PDHFE is built as:

$$\sigma(\xi) = \left(\sum_{\lambda=1}^{\#h} \left(\gamma^\lambda - s(\xi)\right)^2 p^\lambda + \sum_{\lambda=1}^{\#g} (\eta^\lambda - s(\xi))^2 q^\lambda\right)^{\frac{1}{2}} \tag{3}$$

Definition 4 (Hao et al. 2017) Let $\xi_i (i = 1,2)$ be two PDHFEs, $s(\xi)$ and $\sigma(\xi)$ are the score function and the deviation degree, respectively. Then,

(1) If $s(\xi_1) > s(\xi_2)$, then the PDHFE ξ_1 is superior to ξ_2, denoted by $\xi_1 > \xi_2$; On the contrary, there is $\xi_1 < \xi_2$;

(2) If $s(\xi_1) = s(\xi_2)$, then, $\begin{cases} \sigma(\xi_1) > \sigma(\xi_2) & \rightarrow & \xi_1 > \xi_2 \\ \sigma(\xi_1) = \sigma(\xi_2) & \rightarrow & \xi_1 = \xi_2 \\ \sigma(\xi_1) < \sigma(\xi_2) & \rightarrow & \xi_1 < \xi_2 \end{cases}$.

Definition 5 (Hao et al. 2017) Let $\xi_i (i = 1,2)$ be two PDHFEs, then.

$$\xi_1 \oplus \xi_2 = \bigcup_{\substack{\gamma_1 \in h_1, \gamma_2 \in h_2, \\ \eta_1 \in g_1, \eta_2 \in g_2}} \{\{(\gamma_1 + \gamma_2 - \gamma_1\gamma_2)|p_{\gamma_1}p_{\gamma_2}\}, \{\eta_1\eta_2|q_{\eta_1}q_{\eta_2}\}\} \tag{4}$$

$$\theta\xi = \bigcup_{\gamma \in h, \eta \in g} \{\{1 - (1-\gamma)^\theta|p_\gamma\}, \{\eta^\theta|q_\eta\}\} \tag{5}$$

Definition 6 (Hao et al. 2017) Let $\xi_i = \langle h_i|p_i, g_i|q_i\rangle(i = 1,2,\ldots,n)$ be n PDHFEs and $\omega_i(i = 1,2,\ldots,n)$ be the weight with $\omega_i \in [0,1]$ and $\sum_{i=1}^{n}\omega_i = 1$. Then PDHFWA operator is defined as:

$$PDHFWA(\xi_1,\xi_2,\ldots,\xi_n) = \oplus_{j=1}^{n}\omega_k\alpha_i = \bigcup_{\substack{\gamma_1\in h_1,\gamma_2\in h_2,\ldots,\gamma_n\in h_n,\\ \eta_1\in g_1,\eta_2\in g_2,..,\eta_n\in g_n}} \left\{\left\{\left(1-\prod_{i=1}^{n}(1-\gamma_i)^{\omega_i}\right)|\prod_{i=1}^{n}p_{\gamma_i}\right\},\left\{\prod_{i=1}^{n}\eta_i^{\omega_i}|\prod_{i=1}^{n}q_{\eta_i}\right\}\right\} \quad (6)$$

if $\mu = 2$, $d(\xi_1,\xi_2)$ becomes the Euclidean distance $d_{ED}(\xi_1,\xi_2)$.

Definition 10 (Garg et al. 2018) Let $\xi_1 = \langle h_1|p_1, g_1|q_1\rangle$ and $\xi_2 = \langle h_2|p_2, g_2|q_2\rangle$ be 2 PDHFEs, then the distance between the two is:

Definition 7 (Hao et al. 2017) Let $\xi = \langle h|p, g|q\rangle$.If $\#h \geq 2$, $\#g \geq 2$, $\sum_{\lambda=1}^{\#h} p^{\lambda} \leq 1, \sum_{\lambda=1}^{\#g} q^{\lambda} \leq 1$, then the normalized form of a generalized PDHFS is denoted by:

$$\widetilde{\xi} = \langle h|\widetilde{p}, g|\widetilde{q}\rangle = \langle\{\gamma^{\lambda}|\widetilde{p}^{\lambda}\},\{\eta^{\lambda}|\widetilde{q}^{\lambda}\}\rangle \quad (7)$$

where $\tilde{p}^{\lambda} = p^{\lambda} + \frac{p^{\lambda}}{\sum_{\lambda=1}^{\#h} p^{\lambda}}\left(1-\sum_{\lambda=1}^{\#h} p^{\lambda}\right)$, $\tilde{q}^{\lambda} = q^{\lambda} + \frac{q^{\lambda}}{\sum_{\lambda=1}^{\#g} q^{\lambda}}(1-\sum_{\lambda=1}^{\#h} q^{\lambda})$.

2.2 Distance measures between PDHFSs

There are relatively few existing studies on the distance measure of a PDHFE. The definition and calculation formulas of the distance measure in the literature (Garg et al. 2018) are as follows.

Definition 8 (Garg et al. 2018) Let $\xi_1 = \langle h_1|p_1, g_1|q_1\rangle$ and $\xi_2 = \langle h_2|p_2, g_2|q_2\rangle$ be 2 PDHFEs, the distance $d(\xi_1,\xi_2)$ measure of PDHFEs needs to satisfy the following three axiomatic conditions:

(1) $0 \leq d(\xi_1,\xi_2) \leq 1$;
(2) $d(\xi_1,\xi_2) = d(\xi_2,\xi_1)$;
(3) $d(\xi_1,\xi_2) = 0$, if $\xi_1 = \xi_2$.

Definition 9 (Garg et al. 2018) Let $\xi_1 = \langle h_1|p_1, g_1|q_1\rangle$ and $\xi_2 = \langle h_2|p_2, g_2|q_2\rangle$ be 2 PDHFEs, then the distance between the two is:

$$d(\xi_1,\xi_2) = \left[\frac{1}{L+K}\left(\sum_{\lambda=1}^{L}\left|\gamma_1^{\lambda}p_1^{\lambda}-\gamma_2^{\lambda}p_2^{\lambda}\right|^{\mu}+\sum_{\lambda=1}^{K}\left|\eta_1^{\lambda}q_1^{\lambda}-\eta_2^{\lambda}q_2^{\lambda}\right|^{\mu}\right)\right]^{\frac{1}{\mu}} \quad (8)$$

where $L = \max(\#h_1,\#h_2), K = \max(\#g_1,\#g_2)$. If $\mu = 1$, $d(\xi_1,\xi_2)$ becomes the Hamming distance $d_{HD}(\xi_1,\xi_2)$, and

$$d(\xi_1,\xi_2) = \left(\left|\frac{1}{\#h_1}\sum_{\lambda=1}^{\#h_1}\gamma_1^{\lambda}p_1^{\lambda}-\frac{1}{\#h_2}\sum_{\lambda=1}^{\#h_2}\gamma_2^{\lambda}p_2^{\lambda}\right|^{\mu}+\left|\frac{1}{\#g_1}\sum_{\lambda=1}^{\#g_1}\eta_1^{\lambda}q_1^{\lambda}-\frac{1}{\#g_2}\sum_{\lambda=1}^{\#g_2}\eta_2^{\lambda}q_2^{\lambda}\right|^{\mu}\right)^{\frac{1}{\mu}} \quad (9)$$

If $\mu = 1$, $d(\xi_1,\xi_2)$ becomes the Hamming distance $d_{HD}(\xi_1,\xi_2)$, and if $\mu = 2$, $d(\xi_1,\xi_2)$ becomes the Euclidean distance $d_{ED}(\xi_1,\xi_2)$.

***Example* 1** There are 2 PDHFEs $\xi_1 = \langle\{0.2|0.3, 0.4|0.5\},\{0.1|0.2, 0.4|0.3\}\rangle$ and $\xi_2 = \langle\{0.3|0.2, 0.5|0.4\},\{0.2|0.1, 0.3|0.4\}\rangle$, and the distance $d(\xi_1,\xi_2) = 0$ between ξ_1 and ξ_2 was obtained according to the distance formulas in the literature (Garg et al. 2018). However, ξ_1 and ξ_2 are two different PDHFEs. Existing distance formulas ignore the specific distributions of MDs and NMDs, focusing only on the overall distribution of MDs with their corresponding probabilities and the overall distribution of NMDs with their corresponding probabilities. Therefore, it is necessary to improve existing distance formulas.

Definition 11 Let $\xi_1 = \langle h_1|p_1, g_1|q_1\rangle$ and $\xi_2 = \langle h_2|p_2, g_2|q_2\rangle$ be 2 PDHFEs, then the distance $d(\xi_1,\xi_2)$ of PDHFEs needs to satisfy the following three axiomatic conditions:

(1) $0 \leq d(\xi_1,\xi_2) \leq 1$;
(2) $d(\xi_1,\xi_2) = d(\xi_2,\xi_1)$;
(3) $d(\xi_1,\xi_2) = 0$, if and only if $\xi_1 = \xi_2$.

Definition 12 Let $\xi_1 = \langle h_1|p_1, g_1|q_1\rangle$ and $\xi_2 = \langle h_2|p_2, g_2|q_2\rangle$ be 2 PDHFEs, then the distance between the two is:

$$\mathrm{d}(\xi_1,\xi_2)=\left[\frac{1}{2}\left(\frac{1}{2}\sum_{\lambda=1}^{L_1}\left(\left|\gamma_1^\lambda p_1^\lambda-\gamma_2^\lambda p_2^\lambda\right|^\mu+\left|\gamma_1^\lambda-\gamma_2^\lambda\right|^\mu p_1^\lambda p_2^\lambda\right)\right.\right.$$
$$\left.\left.+\frac{1}{2}\sum_{\lambda=1}^{L_2}(\left|\eta_1^\lambda q_1^\lambda-\eta_2^\lambda q_2^\lambda\right|^\mu+\left|\eta_1^\lambda-\eta_2^\lambda\right|^\mu q_1^\lambda q_2^\lambda\right)\right]^{\frac{1}{\mu}}, \tag{10}$$

where $L_1=\max\{\#h_1,\ \#h_2\}$, $L_2=\max\{\#g_1,\ \#g_2\}$.If $\#h_1<\#h_2$, then add $\gamma_1^*|0$ ($\gamma_1^*=\max_\lambda\{\gamma_1^\lambda\}$) to $h_1(p)$ until $\#h_1=\#h_2$;If $\#h_1>\#h_2$, then add $\gamma_2^*|0$ ($\gamma_2^*=\max_\lambda\{\gamma_2^\lambda\}$) to $h_2(p)$ until $\#h_1=\#h_2$;If $\#g_1<\#g_2$, then add $\eta_1^*|0$ ($\eta_1^*=\max_\lambda\{\eta_1^\lambda\}$) to $g_1(p)$ until $\#g_1=\#g_2$; If $\#g_1>\#g_2$, then add $\eta_2^*|0$ ($\eta_2^*=\max_\lambda\{\eta_2^\lambda\}$) to g_2 (p) until $\#g_1=\#g_2$.

If $\mu=1$, $d(\xi_1,\xi_2)$ becomes the Hamming distance $\mathrm{d}_{HD}(\xi_1,\xi_2)$, and if $\mu=2$, $\mathrm{d}(\xi_1,\xi_2)$ becomes the Euclidean distance $\mathrm{d}_{ED}(\xi_1,\xi_2)$.

Theorem 1 *In Definition 12, the novel distance measure determined by Eq. (10) satisfies the three axiomatic conditions of the distance measure of PDHFEs given in Definition 11.*

Proof 1 The improved distance formula clearly satisfies conditions (1) and (2) in Definition 11; therefore, we prove that the formula satisfies condition (3).

(1) If $d(\xi_1,\xi_2)=0$,then $\begin{cases}\left|\gamma_1^\lambda p_1^\lambda-\gamma_2^\lambda p_2^\lambda\right|^\mu+\left|\gamma_1^\lambda-\gamma_2^\lambda\right|^\mu p_1^\lambda p_2^\lambda=0\\ \left|\eta_1^\lambda q_1^\lambda-\eta_2^\lambda q_2^\lambda\right|^\mu+\left|\eta_1^\lambda-\eta_2^\lambda\right|^\mu q_1^\lambda q_2^\lambda=0\end{cases}$,then $\begin{cases}\gamma_1^\lambda p_1^\lambda-\gamma_2^\lambda p_2^\lambda=0\\ \gamma_1^\lambda-\gamma_2^\lambda=0\end{cases}$ and $\begin{cases}\eta_1^\lambda q_1^\lambda-\eta_2^\lambda q_2^\lambda=0\\ \eta_1^\lambda-\eta_2^\lambda=0\end{cases}$, then $\begin{cases}\gamma_1^\lambda=\gamma_2^\lambda\\ p_1^\lambda=p_2^\lambda\end{cases}$ and $\begin{cases}\eta_1^\lambda=\eta_2^\lambda\\ q_1^\lambda=q_2^\lambda\end{cases}$, then $\xi_1=\xi_2$.

(2) If $\xi_1=\xi_2$, the above arguments can be reversed to obtain $d(\xi_1,\xi_2)=0$.

To illustrate the validity of the distance measure proposed in this paper, the PDHFEs in Example 1 are selected for calculation; then, $\mathrm{d}(\xi_1,\xi_2)=0.0316$, and the distance between ξ_1 and ξ_2 can be clarified.

3 Entropy measure of PDHFEs

3.1 Classical probability dual hesitant fuzzy entropy

Definition 13 [14] Let a PDHFE $\xi=\langle h|p,g|q\rangle$, the normalized form of ξ is $\widetilde{\xi}=\langle h|\widetilde{p},g|\widetilde{q}\rangle$, then the entropy of ξ is defined as:

$$E(\xi)=\frac{1}{l}\sum_{\lambda=1}^{l}\left(1-\frac{\left|\gamma^\lambda p^\lambda-\eta^\lambda q^\lambda\right|^a+\left|\gamma^\lambda p^\lambda+\eta^\lambda q^\lambda\right|^b}{2}\right) \tag{11}$$

Example 2 There are two PDHFEs $\xi_1=\langle\{0.1|0.6,0.5|0.4\},\{0.2|0.4,0.4|0.6\}\rangle$ and $\xi_2=\langle\{0.1|0.3,0.5|0.2\},\{0.2|0.2,0.4|0.3\}\rangle$, ξ_2 is transformed into normalized PDHFE $\xi_2=\langle\{0.1|0.6,0.5|0.4\},\{0.2|0.4,0.4|0.6\}\rangle$ according to the normalization method proposed by Hao et al. (2017) (see Definition 7), taking $a=b=0.5$, we get: $E(\xi_1)=E(\xi_2)=0.6553$. The method proposed by Hao et al. (2017) cannot effectively compare the uncertainty of both. The reason for this is considered in the literature (Su et al. 2022), which ignores the uncertainty caused by incomplete probabilistic information, so the literature (Su et al. 2022) adds the consideration of probabilistic incompleteness to the literature (Hao et al. 2017), see definition 14.

Definition 14 (Su et al. 2022) Let a PDHFE $\xi=\langle h|p,g|q\rangle$ be transformed into an internally reconciled *PDHFE* $\ddot{\xi}\le h|\ddot{p},g|\ddot{q}>$, which is that its MD and NMD sets have the same number of elements and the same probability, then the entropy of ξ is defined as:

$$E(\xi)=f\left(\sum_{\lambda=1}^{l}\ddot{p}\,\lambda|\ddot{\gamma}\,\lambda-\ddot{\eta}\,\lambda|,\sum_{\lambda=1}^{l}\ddot{p}\,\lambda(\ddot{\gamma}\,\lambda+\ddot{\eta}\,\lambda),0.5\left(\sum_{\lambda=1}^{\#h}p^\lambda+\sum_{\lambda=1}^{\#g}q^\lambda\right)\right), \tag{12}$$

where $f{:}[0,1]^3\to f{:}[0,1]$, taking $f(x,y,z)=\frac{1}{3}(3-x-y-z)$.

Example 3 There are 2 PDHFEs $\xi_1=\langle\{0.4|0.4,0.6|0.6\},\{0.2\ |0.4,0.2|0.6\}\rangle$ and $\xi_2=\langle\{0.1|0.4,0.8\ |0.6\},\{0.1|0.4,0.2|0.6\}\rangle$. The number of elements of the MD and NMD sets and the probabilities in these 2 PDHFEs are the same, so they are simultaneously internally reconciled PDHFEs. The definition in the literature (Su et al. 2022) was used taken to calculate the entropy of ξ_1 and ξ_2 separately to obtain $E(\xi_1)=E(\xi_2)=0.32$, which can be seen using the entropy measure proposed by Su et al. (2022) and cannot effectively compare the uncertainty of different PDHFEs.

Definition 15 (Ning et al. 2022a, b, c) Let a PDHFE $\xi=\langle h|p,g|q\rangle$, then the entropy of ξ is defined as:

$$E(\xi)=\frac{\sum_{\lambda=1}^{\#h}\gamma^{\lambda}p^{\lambda}\sum_{\lambda=1}^{\#g}\eta^{\lambda}q^{\lambda}+\left(1-\sum_{\lambda=1}^{\#h}\gamma^{\lambda}p^{\lambda}-\sum_{\lambda=1}^{\#g}\eta^{\lambda}q^{\lambda}\right)}{\left(\sum_{\lambda=1}^{\#h}\gamma^{\lambda}p^{\lambda}\right)^{2}+\left(\sum_{\lambda=1}^{\#g}\eta^{\lambda}q^{\lambda}\right)^{2}+\left(1-\sum_{\lambda=1}^{\#h}\gamma^{\lambda}p^{\lambda}-\sum_{\lambda=1}^{\#g}\eta^{\lambda}q^{\lambda}\right)} \tag{13}$$

Example 4 There are 2 PDHFEs $\xi_1=\langle\{0.2|0.8,0.6|0.2\},\{0.2|0.5,0.4|0.5\}\rangle$ and $\xi_2=\langle\{0.1|0.4,0.4|0.6\},\{0.2|0.5,0.4|0.5\}\rangle$. The definition in the literature (Ning et al. 2022a, b, c) is taken to calculate the entropy of ξ_1 and ξ_2 separately to obtain $E(\xi_1)=E(\xi_2)=0.9993$, it can be seen the entropy measure proposed by Ning et al. (2022a; b, c) also cannot effectively compare the uncertainty of different PDHFEs.

The above analysis shows that the existing probabilistic dual hesitant fuzzy entropy measure formulas do not completely distinguish the uncertainty of PDHFE, mainly because the formula does not consider the difference between different MSs (or different NMDs). When there are multiple different MDs (or different NMDs) in a PDHFE, it characterizes the hesitation of the decision maker. The greater the difference between different MDs (or NMDs), the greater the degree of hesitation of the decision maker. Moreover, the smaller the difference between the probabilities of MD (or NMD), the less uniform the decision-maker's evaluation, and the greater the degree of hesitation. The effect of hesitation on the entropy value is that the greater the degree of hesitation, the higher the entropy value. Therefore, this study proposes a new axiomatic definition and calculation formula for probabilistic dual hesitant fuzzy entropy to improve the deficiencies of the existing calculation formulas.

3.2 Novel probability dual hesitant fuzzy entropy

Definition 16 Let a PDHFE $\xi=\langle h|p,g|q\rangle$, the normalized form of ξ is $\widetilde{\xi}=\langle h|\widetilde{p},g|\widetilde{q}\rangle$, a mapping $E:\xi\to[0,1]$ is named as entropy of PDHFE, if E satisfy the following four conditions:\

(1) $0\le E(\xi)\le 1$;
(2) $E(\xi)=0$ if and only if $\widetilde{\xi}=\langle\{0|1\},\{1|1\}\rangle$ or $\widetilde{\xi}=\langle\{1|1\},\{0|1\}\rangle$;
(3) $E(\xi_1,\xi_2)=1$, if and only if $\widetilde{\xi}=\langle\{0|0\},\{0|0\}\rangle$;
(4) $E(\xi)=E(\xi^c)$, where $\xi^c=\langle g|q,h|p\rangle$.

Definition 17 Let a PDHFE $\xi=\langle h|p,g|q\rangle$, the normalized form of ξ is $\widetilde{\xi}=\langle h|\widetilde{p},g|\widetilde{q}\rangle$, then the entropy of ξ is:

$$E(\xi)=f\left(\sum_{\lambda=1}^{L}|\gamma^{\lambda}\tilde{p}^{\lambda}-\eta^{\lambda}\tilde{q}^{\lambda}|\right),\sum_{\lambda=1}^{L}\left(\gamma^{\lambda}\tilde{p}^{\lambda}+\eta^{\lambda}\tilde{q}^{\lambda}\right),\frac{1}{2}\left(\sum_{\lambda=1}^{\#h}p^{\lambda}+\sum_{\lambda=1}^{\#g}q^{\lambda}\right),\\ \frac{1}{2}\left(2-H\left(h\left(\tilde{p}\right)\right)-H\left(g(\tilde{q})\right)\right) \tag{14}$$

$$H(h(\tilde{p}))=\begin{cases}\sum_{\lambda=1}^{\#h}\sum_{\varrho=\lambda+1}^{\#h}4\tilde{p}^{\lambda}\tilde{p}^{\varrho} & \#h>1\\ 0 & \#h=0\end{cases} \tag{15}$$

$$H(g(\tilde{q}))=\begin{cases}\sum_{\lambda=1}^{\#g}\sum_{\rangle=\lambda+1}^{\#g}4\tilde{q}^{\lambda}\tilde{q}^{\rangle} & \#g>1\\ 0 & \#g=0\end{cases} \tag{16}$$

where $L=\max\{\#h,\ \#g\}$.If $\#h>\#g$, then add $\eta^{*}|0$ ($\eta^{*}=\max_{\lambda}\{\eta^{\lambda}\}$) to $g(\widetilde{q})$ until $\#h=\#g$;If $\#h<\#g$, then add $\gamma^{*}|0$ ($\gamma^{*}=\max_{\lambda}\{\gamma^{\lambda}\}$) to $h(\widetilde{p})$ until $\#h=\#g$.

The function f: $[0,1]^4\to[0,1]$ in the entropy measure satisfies the following conditions:

(1) $x, y, z, k\in[0,1]$, $f(x, y, z, k)$ is a monotonically decreasing function with respect to x, y, z, k;
(2) $f(x, y, z, k)=0$ if and only if $x=y=z=k=1$;
(3) $f(x, y, z, k)=1$ if and only if $x=y=z=k=0$.

Theorem 2 *In Definition 17, the novel entropy measure determined by Eq. (14) satisfies the four axiomatic conditions of the entropy measure of PDHFEs given in Definition 16.*

The proposed probabilistic dual hesitant fuzzy entropy clearly satisfies conditions (1–5) in Definition 16, so this is not proven here. To illustrate the superiority of the proposed probabilistic dual hesitant fuzzy entropy proposed by this study, the PDHFEs in Examples 2, 3 and 4 are selected for the calculation.

(1) For the 2 PDHFEs $\xi_1=\langle\{0.1|0.6,0.5|0.4\},\{0.2|0.4,0.4|0.6\}\rangle$ and $\xi_2=\langle\{0.1|0.3,0.5|0.2\},\{0.2|0.2,0.4|0.3\}\rangle$ in Example 2, we get $E(\xi_1)=0.412<E(\xi_2)=0.537$.
(2) For the 2 PDHFEs $\xi_1=\langle\{0.4|0.4,0.6|0.6\},\{0.2|0.4,0.2|0.6\}\rangle$ and $\xi_2=\langle\{0.1|0.4,0.8|0.6\},\{0.1|0.4,0.2|0.6\}\rangle$ in Example 3, we get $E(\xi_1)=0.264<E(\xi_2)=0.336$
(3) For the 2 PDHFEs $\xi_1=\langle\{0.2|0.8,0.6|0.2\},\{0.2|0.5,0.4|0.5\}\rangle$ and $\xi_2=\langle\{0.1|0.4,0.4|0.6\},\{0.2|0.5,0.4|0.5\}\rangle$ in Example 4, we get $E(\xi_1)=0.337<E(\xi_2)=0.391$.

It can be seen that the entropy measure of PDHFEs proposed in this study can effectively distinguish the uncertainty of different PDHFEs, which measures both the closeness between MD and NMD in PDHFEs, the uncertainty caused by the lack of probabilistic information, and reflects the deviation between different MDs (or different MDs), the uncertainty caused by the closeness between the probabilities corresponding to the respective MD, and the entropy measure of PDHFE is more comprehensive.

4 Cross-entropy measure of PDHFEs

4.1 Cross-entropy of PDHFEs

Cross-entropy is a common tool used to measure the degree of difference between fuzzy information, and is widely used in fuzzy fields. In this study, we define probabilistic dual hesitant fuzzy cross-entropy.

Definition 18 Let $\xi_1 = \langle h_1|p_1, g_1|q_1\rangle$ and $\xi_2 = \langle h_2|p_2, g_2|q_2\rangle$ be 2 PDHFEs, then $CE(\xi_1, \xi_2)$ is named as cross-entropy measure between two PDHFEs, if $CE(\xi_1, \xi_2)$ satisfies:

(1) $CE(\xi_1, \xi_2) = CE(\xi_2, \xi_1)$;
(2) $CE(\xi_1, \xi_2) = 0$ if and only if $\xi_1 = \xi_2$;
(3) $0 \leq CE(\xi_1, \xi_2) \leq 1$;
(4) $CE(\xi_1, \xi_2) = 1$ if and only if $\xi_1 = \langle 1|1, 0|0\rangle$ and $\xi_2 = \langle 0|0, 1|1\rangle$ or $\xi_1 = \langle 0|0, 1|1\rangle$ and $\xi_2 = \langle 1|1, 0|0\rangle$.

Definition 19 Let $\xi_1 = \langle h_1|p_1, g_1|q_1\rangle$ and $\xi_2 = \langle h_2|p_2, g_2|q_2\rangle$ be 2 PDHFEs, the normalized form of ξ_1 and ξ_2 are $\widetilde{\xi}_1 = \langle h_1|\widetilde{p}_1, g_1|\widetilde{q}_1\rangle$ and $\widetilde{\xi}_2 = \langle h_2|\widetilde{p}_2, g_2|\widetilde{q}_2\rangle$, then the cross-entropy of ξ_1 and ξ_2 are defined as:

where $T_1 = 2((1+\beta)\ln(1+\beta) - (2+\beta)(\ln(2+\beta) - \ln 2))$, $\beta > 0$, $L_1 = \max\{\#h_1, \#h_2\}$, $L_2 = \max\{\#g_1, \#g_2\}$.

$$
\begin{aligned}
&CE_2(\xi_1,\xi_2)\\
&= \frac{1}{T_2}\Bigg[\frac{1}{L_1}\sum_{\lambda=1}^{L_1}\Bigg(\frac{\left(\gamma_1^\lambda\widetilde{p}_1^\lambda\right)^\delta + \left(\gamma_2^\lambda\widetilde{p}_2^\lambda\right)^\delta}{2} - \left(\frac{\gamma_1^\lambda\widetilde{p}_1^\lambda + \gamma_2^\lambda\widetilde{p}_2^\lambda}{2}\right)^\delta\\
&+ \frac{\left(1-\gamma_1^\lambda\widetilde{p}_1^\lambda\right)^\delta + \left(1-\gamma_2^\lambda\widetilde{p}_2^\lambda\right)^\delta}{2} - \left(\frac{2-\gamma_1^\lambda\widetilde{p}_1^\lambda - \gamma_2^\lambda\widetilde{p}_2^\lambda}{2}\right)^\delta\Bigg)\\
&+ \frac{1}{L_2}\sum_{\lambda=1}^{L_2}\Bigg(\frac{\left(\eta_1^\lambda\widetilde{q}_1^\lambda\right)^\delta + \left(\eta_2^\lambda\widetilde{q}_2^\lambda\right)^\delta}{2} - \left(\frac{\eta_1^\lambda\widetilde{q}_1^\lambda + \eta_2^\lambda\widetilde{q}_2^\lambda}{2}\right)^\delta\\
&+ \frac{\left(1-\eta_1^\lambda\widetilde{q}_1^\lambda\right)^\delta + \left(1-\eta_2^\lambda\widetilde{q}_2^\lambda\right)^\delta}{2} - \left(\frac{2-\eta_1^\lambda\widetilde{q}_1^\lambda - \eta_2^\lambda\widetilde{q}_2^\lambda}{2}\right)^\delta\Bigg)\Bigg]
\end{aligned}
\tag{18}
$$

where $T_2 = 2\left(1 - 2^{1-\delta}\right)$, $\delta > 1$, $L_1 = \max\ \{\#h_1, \#h_2\}$, $L_2 = \max\ \{\#g_1, \#g_2\}$.

Theorem 3 *In Definition 19, the cross-entropy measures determined by Eq. (17) and Eq. (18) satisfy the four axiomatic conditions of the cross-entropy measure of PDHFEs given in Definition 18.*

Proof 2 For $CE_1(\xi_1, \xi_2)$, $\frac{\mathrm{d}T_1}{\mathrm{d}\beta} = \ln\left(\frac{2+2\beta}{2+\beta}\right) > \ln 1 = 0$, then T_1 increases with the increase of β, if $\beta = 0$, then $T_1 = 0$, due to $\beta > 0$, then $\frac{1}{T_1} > 0$. For $CE_2(\xi_1, \xi_2)$, $\delta > 1$, then $T_2 = 2\left(1 - 2^{1-\delta}\right) > 0$, then $\frac{1}{T_2} > 0$. For $u_1(x) = (1+\beta x)\ln(1+\beta x)$ and $u_2(x) = x^\delta$, due to $x \in [0, 1]$ and $\beta > 0$, $\delta > 1$, then $\frac{\mathrm{d}u(x)}{\mathrm{d}x} > 0$, $\frac{\mathrm{d}^2u(x)}{\mathrm{d}x^2} > 0$, then $u_1(x)$ and $u_2(x)$ are concave functions about x. Therefore,

$$
\begin{aligned}
&CE_1(\xi_1,\xi_2)\\
&= \frac{1}{T_1}\Bigg[\frac{1}{L_1}\sum_{\lambda=1}^{L_1}\Bigg(\frac{\left(1+\beta\gamma_1^\lambda\widetilde{p}_1^\lambda\right)\ln\left(1+\beta\gamma_1^\lambda\widetilde{p}_1^\lambda\right)}{2} + \frac{\left(1+\beta\gamma_2^\lambda\widetilde{p}_2^\lambda\right)\ln\left(1+\beta\gamma_2^\lambda\widetilde{p}_2^\lambda\right)}{2} - \frac{\left(2+\beta\gamma_1^\lambda\widetilde{p}_1^\lambda+\beta\gamma_2^\lambda\widetilde{p}_2^\lambda\right)\ln\left(2+\beta\gamma_1^\lambda\widetilde{p}_1^\lambda+\beta\gamma_2^\lambda\widetilde{p}_2^\lambda\right)}{2}\\
&+ \frac{\left(1+\beta\left(1-\gamma_1^\lambda\widetilde{p}_1^\lambda\right)\right)\ln\left(1+\beta\left(1-\gamma_1^\lambda\widetilde{p}_1^\lambda\right)\right)}{2} + \frac{\left(1+\beta\left(1-\gamma_2^\lambda\widetilde{p}_2^\lambda\right)\right)\ln\left(1+\beta\left(1-\gamma_2^\lambda\widetilde{p}_2^\lambda\right)\right)}{2}\\
&- \frac{\left(2+\beta\left(2-\gamma_1^\lambda\widetilde{p}_1^\lambda-\gamma_2^\lambda\widetilde{p}_2^\lambda\right)\right)}{2}\ln\frac{\left(2+\beta\left(2-\gamma_1^\lambda\widetilde{p}_1^\lambda-\gamma_2^\lambda\widetilde{p}_2^\lambda\right)\right)}{2}\Bigg)\\
&+ \frac{1}{L_2}\sum_{\lambda=1}^{L_2}\Bigg(\frac{\left(1+\beta\eta_1^\lambda\widetilde{q}_1^\lambda\right)\ln\left(1+\beta\eta_1^\lambda\widetilde{q}_1^\lambda\right)}{2} + \frac{\left(1+\beta\eta_2^\lambda\widetilde{q}_2^\lambda\right)\ln\left(1+\beta\eta_2^\lambda\widetilde{q}_2^\lambda\right)}{2} - \frac{\left(2+\beta\eta_1^\lambda\widetilde{q}_1^\lambda+\beta\eta_2^\lambda\widetilde{q}_2^\lambda\right)\ln\left(2+\beta\eta_1^\lambda\widetilde{q}_1^\lambda+\beta\eta_2^\lambda\widetilde{q}_2^\lambda\right)}{2}\\
&+ \frac{\left(1+\beta\left(1-\eta_1^\lambda\widetilde{q}_1^\lambda\right)\right)\ln\left(1+\beta\left(1-\eta_1^\lambda\widetilde{q}_1^\lambda\right)\right)}{2} + \frac{\left(1+\beta\left(1-\eta_2^\lambda\widetilde{q}_2^\lambda\right)\right)\ln\left(1+\beta\left(1-\eta_2^\lambda\widetilde{q}_2^\lambda\right)\right)}{2}\\
&- \frac{\left(2+\beta\left(2-\eta_1^\lambda\widetilde{q}_1^\lambda-\eta_2^\lambda\widetilde{q}_2^\lambda\right)\right)}{2}\ln\frac{\left(2+\beta\left(2-\eta_1^\lambda\widetilde{q}_1^\lambda-\eta_2^\lambda\widetilde{q}_2^\lambda\right)\right)}{2}\Bigg)\Bigg]
\end{aligned}
\tag{17}
$$

$CE(\xi_1, \xi_2) \geq 0$. we can get the condition (1), condition (2) and condition (4) by combining the formulas of cross-entropy.

4.2 Generalized cross-entropy of PDHFEs

A definition of the generalized cross-entropy of PDHFEs is proposed based on the above cross-entropy of PDHFEs.

Definition 20 Let $\xi_1 = \langle h_1|p_1, g_1|q_1\rangle$ and $\xi_2 = \langle h_2|p_2, g_2|q_2\rangle$ be 2 PDHFEs, $CE_i(\xi_1, \xi_2)(i = 1, 2)$ are cross-entropy of ξ_1 and ξ_2, then the generalized cross-entropy of ξ_1 and ξ_2 is defined as:

$$DE(\xi_1, \xi_2) = \varphi(CE_1(\xi_1, \xi_2), CE_2(\xi_1, \xi_2)) \tag{19}$$

where $\varphi(x, y) : [0, 1]^2 \to [0, 1]$ satisfies the following conditions:

(1) $\varphi(x, y) = 0$ if and only if $x = y = 0$;
(2) $\varphi(x, y) = 1$ if and only if $x = y = 1$;
(3) $\varphi(x, y)$ increases with the increase of x, y.

At the same time $DE(\xi_1, \xi_2)$ satisfies all the properties of the cross-entropy $CE(\xi_1, \xi_2)$.

5 Probabilistic dual hesitant fuzzy multi-attribute decision model based on entropy and cross-entropy

The probabilistic dual hesitant fuzzy multi-attribute decision problem can be described as follows: The alternative set $X = \{x_1, x_2,\ldots, x_i,\ldots, x_m\}$, the attribute set $C = \{c_1, c_2,\ldots, c_j,\ldots, c_n\}$, the attribute weights are $W = [w_1,w_2,\ldots,w_j,\ldots,w_n]^T$, and the set of decision evaluation values $O = \{\xi_{11}, \xi_{12}, \ldots, \xi_{ij},\ldots, \xi_{mn}\}$.

5.1 Attribute determination method based on entropy and cross-entropy

The probabilistic dual hesitant fuzzy entropy of attribute c_j is $\sum_{i=1}^{m} E(\xi_{ij})$, and if the entropy value of an attribute is larger, the less useful information the attribute provides to the decision maker, the smaller the value assigned to weight of the attribute. Similarly, the average cross-entropy of alternative x_i with other alternatives under attribute c_j is $\frac{1}{m-1}\sum_{\tau=1,\tau\neq i}^{m} DE(\xi_{ij}, \xi_{\tau j})$,and the total cross-entropy of all alternatives under attribute c_j is $\sum_{i=1}^{m}\frac{1}{m-1}\sum_{\tau=1,\tau\neq i}^{m} DE(\xi_{ij}, \xi_{\tau j})$. The greater the difference between evaluations of alternatives under the same attribute, the greater the role the attribute plays in the ranking of alternatives, and the greater the value assigned to the attribute weight; conversely, the smaller the value assigned to the attribute weight.

Therefore, the weights of attribute c_j can be obtained as:

$$w_j = \frac{\sum_{i=1}^{m}\left(\left(1 - E(\xi_{ij})\right) + \frac{1}{m-1}\sum_{\tau=1,\tau\neq i}^{m} DE(\xi_{ij}, \xi_{\tau j})\right)}{\sum_{j=1}^{n}\sum_{i=1}^{m}\left(\left(1 - E(\xi_{ij})\right) + \frac{1}{m-1}\sum_{\tau=1,\tau\neq i}^{m} DE(\xi_{ij}, \xi_{\tau j})\right)}. \tag{20}$$

5.2 Process steps of the decision model

The steps of the constructed multi-attribute model are as follows:

Step1: Unifying the attribute types. The cost type should be converted into benefit type according to the method proposed by Zhao et al. (2020): the complementary set of cost type evaluation is its corresponding benefit type evaluation: $\xi_{ij} \to {\xi_{ij}}^c$.

Step2: Calculate attribute weights W based on the probability dual hesitant fuzzy entropy and cross-entropy in Sect. 5.2.

Step3: Selecting positive and negative ideal solutions based on the score function: $x^+ = \{{\xi_j}^+ | j = 1, 2, \ldots, n\}$ and $x^- = \{{\xi_j}^- | j = 1, 2, \ldots, n\}$, where ${\xi_j}^+ = \max_i\{\xi_{ij} | j = 1, 2, \ldots, n\}$, ${\xi_j}^- = \min_i\{\xi_{ij} | j = 1, 2, \ldots, n\}$. The larger the score function $s(\xi_{ij})$, the larger ξ_{ij}.

Step4: Calculate the distances $D(x_i,x^+)$ and $D(x_i, x^-)$ of each alternative from the positive and negative ideal solutions based on the proposed distance measure formula with attribute weights.

$$D(x_i, x^+) = \sum_{j=1}^{n} w_j d_{ED}\left(\xi_{ij}, \xi_j^+\right)(i = 1, 2, \ldots, m) \tag{21}$$

$$D(x_i, x^-) = \sum_{j=1}^{n} w_j d_{ED}\left(\xi_{ij}, \xi_j^-\right)(i = 1, 2, \ldots, m) \tag{22}$$

Step5: Calculate the closeness degree t_i of the alternative based on the distance between the alternative and positive and negative ideal solutions. A larger t_i value indicates a better alternative x_i.

$$t_i = \frac{D(x_i, x^-)}{D(x_i, x^+) + D(x_i, x^-)} \tag{23}$$

6 Analysis of calculation example

When a nuclear accident occurs, decision makers need to make a choice in a short period of time to select the optimal nuclear emergency response action to minimize the damage caused by the nuclear accident. It is a complex problem; that involves not only the evolution of nuclear accidents and the environment, but also public health, economic loss, and political impact, and is a multi-attribute risk decision problem.

In the event of an accident at a nuclear power plant, which results in the release of radioactive material to the outside, exposing staff and the public to radiation over time or at a level equivalent to the specified limit, it is known as a nuclear leak. The harmful pathway of radioactive material release is that the rays directly harm the human body, followed by a damaging effect on the human body through contaminated air, soil, water, and food. When a certain dose of radioactive material enters the human body, it has an ionizing effect on tissues, causing cell deformation, tissue damage, organ dysfunction, metabolic disorders, and other diseases. In the event of a serious release of radioactive material from a nuclear facility, certain forms of emergency protective measures, such as concealment, administration of stabilized iodine, evacuation, relocation, and food control, are required to protect public health and safety. When, where, and what measures should be taken are decisions that should be made by emergency commanders. Different decision-making behaviors often produce different effects, and choosing the right decision can help minimize the harm caused by radioactive materials to society, the public, the environment, and the economy. In this paper, the constructed model is presented as an example of protective decision-making for the release of radioactive substances.

6.1 Decision-making for the protection against releases of radiation substances

In this study, the release of radiation substances under the PWR5 accident source of nuclear power plant A was used as an example for the selection of multi-attribute decisions. The radiation prevention effect of a nuclear accident was evaluated under four attributes: negative psychosocial impact c_1, economic cost c_2, preventable maximum individual dose c_3, and preventable collective dose c_4, for which the attribute weights were completely unknown. To facilitate the calculation, the PDHFE evaluation values of the given alternatives under different attributes were normalized PDHFEs, as shown in Table 1.

Step1: The negative psychosocial impact c_1, and the economic cost c_2 belong to the cost type, so they were transformed into the benefit type, and the probabilistic dual hesitant fuzzy multi-attribute decision matrix was obtained, as shown in Table 2.

Step2: In this study, we chose $f(x,y,z,k) = \frac{1}{4}(4\text{-}x\text{–}y\text{-}z\text{-}k)$, $CE_1(\xi_1, \xi_2)$ to calculate the entropy and cross-entropy, respectively, took $\beta = 2$, and obtained the attribute weights $W = [0.2559, 0.2120, 0.2625, 0.2696]^{\mathrm{T}}$ based on Eq. (20).

Step3: The positive and negative ideal solutions were selected based on the score function.

Step4: The distances of each alternative from the positive and negative ideal solutions were obtained based on the distance measure formula and attribute weights: $D(x_1,x^+) = 0.0495$, $D(x_1,x^-) = 0.1621$; $D(x_2,x^+) = 0.0996$, $D(x_2,x^-) = 0.1381$; $D(x_3,x^+) = 0.0708$, $D(x_3,x^-) = 0.1705$; $D(x_2,x^+) = 0.1769$, $D(x_2,x^-) = 0.0421$.

Step5: The closeness degrees of the alternatives were calculated to obtain $t_1 = 7660$, $t_2 = 5810$, $t_3 = 7065$, and $t_4 = 1924$, so the ranking of the alternatives is: $x_1 > x_3 > x_2 > x_4$, and alternative x_1 is the best.

6.2 Comparative analysis

To further illustrate the effectiveness and feasibility of the multi-attribute decision scheme proposed in this study, the distance and method of determining attribute weights proposed in the literature (Garg et al. 2018) are used to calculate the decision evaluation values in the examples. The attribute weights in the decision-making method proposed in the literature (Garg et al. 2018) were obtained based on the maximum deviation method of the distance formulas, which are shown in Definitions 9 and 10. The ranking of the alternatives is: $x_3 > x_1 > x_2 > x_4$.The ranking of the alternatives obtained by the decision model in this study is compared with the ranking of the alternatives obtained using the distance measure in the literature (Garg et al. 2018), which shows that the optimal alternative is different and the ranking of the other alternatives is the same. The distance measure in the literature (Garg et al. 2018) ignores the specific distribution of MDs and NMDs, and the determination of attribute weights is obtained based on this distance formula; therefore, it is somewhat biased with the ranking of the solutions derived in this study.

The entropy formula proposed by Hao et al. (2017) was used to calculate the decision evaluation value in the example, as shown in Definition 13. The entropy formula proposed by Hao et al. (2017) was used to calculate attribute weights, and the ranking of the alternatives obtained by combining the distance formula proposed in this paper was $x_1 > x_3 > x_2 > x_4$, which is consistent with the ranking of the alternatives obtained by the method proposed in this paper. This indicates the effectiveness of the entropy formula and the attribute weight calculation method proposed in this study.

Table 1 Matrix of evaluation values

Alternatives/ attributes	c_1	c_2	c_3	c_4
x_1	$\langle\{0.3\mid 0.6,0.4\mid 0.4\},\{0.4\mid 0.5,0.5\mid 0.5\}\rangle$	$\langle\{0.1\mid 0.6,0.2\mid 0.4\},\{0.1\mid 0.6,0.2\mid 0.4\}\rangle$	$\langle\{0.4\mid 0.4,0.6\mid 0.6\},\{0.2\mid 0.6,0.3\mid 0.4\}\rangle$	$\langle\{0.4\mid 0.2,0.7\mid 0.8\},\{0.1\mid 0.5,0.3\mid 0.5\}\rangle$
x_2	$\langle\{0.2\mid 0.65,0.3\mid 0.35\},\{0.4\mid 0.5,0.5\mid 0.5\}\rangle$	$\langle\{0.1\mid 0.5,0.2\mid 0.5\},\{0.2\mid 0.5,0.3\mid 0.5\}\rangle$	$\langle\{0.3\mid 0.35,0.4\mid 0.65\},\{0.3\mid 0.7,0.4\mid 0.3\}\rangle$	$\langle\{0.3\mid 0.5,0.6\mid 0.5\},\{0.2\mid 0.35,0.3\mid 0.65\}\rangle$
x_3	$\langle\{0.2\mid 0.6,0.3\mid 0.4\},\{0.4\mid 0.4,0.5\mid 0.6\}\rangle$	$\langle\{0.1\mid 0.9,0.2\mid 0.1\},\{0.25\mid 0.6,0.4\mid 0.4\}\rangle$	$\langle\{0.4\mid 0.5,0.5\mid 0.5\},\{0.2\mid 0.5,0.3\mid 0.5\}\rangle$	$\langle\{0.4\mid 0.35,0.6\mid 0.65\},\{0.2\mid 0.5,0.3\mid 0.5\}\rangle$
x_4	$\langle\{0.1\mid 0.4,0.2\mid 0.6\},\{0.5\mid 0.6,0.6\mid 0.4\}\rangle$	$\langle\{0.1\mid 1\},\{0.25\mid 0.2,0.35\mid 0.2\}\rangle$	$\langle\{0.1\mid 0.6,0.2\mid 0.4\},\{0.5\mid 0.6,0.6\mid 0.4\}\rangle$	$\langle\{0.1\mid 0.2,0.2\mid 0.8\},\{0.3\mid 0.4,0.8\mid 0.6\}\rangle$

Table 2 Probabilistic dual hesitant fuzzy multi-attribute decision matrix

Alternatives/ Attributes	c_1	c_2	c_3	c_4
x_1	$\langle\{0.4\mid 0.5,0.5\mid 0.5\},\{0.3\mid 0.6,0.4\mid 0.4\}\rangle$	$\langle\{0.1\mid 0.6,0.2\mid 0.4\},\{0.1\mid 0.6,0.2\mid 0.4\}\rangle$	$\langle\{0.4\mid 0.4,0.6\mid 0.6\},\{0.2\mid 0.6,0.3\mid 0.4\}\rangle$	$\langle\{0.4\mid 0.2,0.7\mid 0.8\},\{0.1\mid 0.5,0.3\mid 0.5\}\rangle$
x_2	$\langle\{0.4\mid 0.5,0.5\mid 0.5\},\{0.2\mid 0.65,0.3\mid 0.35\}\rangle$	$\langle\{0.2\mid 0.5,0.3\mid 0.5\},\{0.1\mid 0.5,0.2\mid 0.5\}\rangle$	$\langle\{0.3\mid 0.35,0.4\mid 0.65\},\{0.3\mid 0.7,0.4\mid 0.3\}\rangle$	$\langle\{0.3\mid 0.5,0.6\mid 0.5\},\{0.2\mid 0.35,0.3\mid 0.65\}\rangle$
x_3	$\langle\{0.4\mid 0.4,0.5\mid 0.6\},\{0.2\mid 0.6,0.3\mid 0.4\}\rangle$	$\langle\{0.25\mid 0.6,0.4\mid 0.4\},\{0.1\mid 0.9,0.2\mid 0.1\}\rangle$	$\langle\{0.4\mid 0.5,0.5\mid 0.5\},\{0.2\mid 0.5,0.3\mid 0.5\}\rangle$	$\langle\{0.4\mid 0.35,0.6\mid 0.65\},\{0.2\mid 0.5,0.3\mid 0.5\}\rangle$
x_4	$\langle\{0.5\mid 0.6,0.6\mid 0.4\},\{0.1\mid 0.4,0.2\mid 0.6\}\rangle$	$\langle\{0.25\mid 0.2,0.35\mid 0.2\},\{0.1\mid 1\}\rangle$	$\langle\{0.1\mid 0.6,0.2\mid 0.4\},\{0.5\mid 0.6,0.6\mid 0.4\}\rangle$	$\langle\{0.1\mid 0.2,0.2\mid 0.8\},\{0.3\mid 0.4,0.8\mid 0.6\}\rangle$

The distance formula proposed in this paper not only considers the distance of the product between the MDs and the corresponding probabilities and the distance of the product between the NMDs and the corresponding probabilities, but also considers the specific distribution of MDs and NMDs to improve the distance measure between PDHFEs. The determination of attribute weights in this study is obtained based on entropy and cross-entropy, considering the individual effects of PDHFS and their interactions. The combination of entropy and cross-entropy can consider both the degree of fuzziness and the degree of deviation, which is important for multi-attribute decision-making.

7 Conclusions

In this study, we systematically reviewed related studies on PDHFS, which can more comprehensively represent fuzzy information and uncertainty in multi-attribute decision-making. We found that the distance and entropy measures of the PDHFS have some shortcomings, and the definition of cross-entropy has not been given. Therefore, in this study, we improve the distance measure so that the distance formula can effectively compare the distances between different PDHFSs to compensate for the shortcomings of existing distance measures. The entropy measure is improved, and the new entropy measure considers the uncertainty caused by the difference between different MDs (different NMDs) and the uncertainty caused by the proximity between the probabilities of the respective MDs (or NMDs). Cross-entropy was proposed to fill the research gap in the cross-entropy of PDHFS. A new multi-attribute decision method in the PDHFS environment was proposed and applied to protective decision-making for the release of radioactive substances. The new multi-attribute decision method provides rich and comprehensive decision information and objective attribute weights to the protective decision-making for the release of radioactive substances. The applicability of the new multi-attribute decision method is explained, and its effectiveness is confirmed by comparative analysis.

Considering that there is little research on group contingency decision-making methods for PDHFSs, solving the consensus problem in group decision-making requires further research.

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Declarations

Conflict of interest The authors declare no conflict of interest.

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