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ORIGINAL PAPER

Algebraic structure through interval-valued fuzzy signature based on interval-valued fuzzy sets

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Abstract

This paper delivers three different ways to establish the initial structure of the interval-valued fuzzy signature (IVFSig). In recent years, interval-valued fuzzy set theory has proven more capable of dealing with uncertainty and vagueness than fuzzy set theory due to its increased flexibility. Therefore, the primary goal of this work is to develop an algebraic framework for an IVFSig based on the aspects of an interval-valued fuzzy set (IVFS). First, the IVFSig's are constructed with the aid of IVFSs, which may be considered the truth values of IVFSs. Second, the families of IVFSig's, as well as meet and join operators, are formulated, and then their lattice algebraic structure is verified. Third, the relation of partial ordering is established in an IVFSig family. Precisely, the addressed design is compared with recent well-known framework. Finally, the numerical illustrations provide a higher degree of representation than other existing framework.

Keywords Fuzzy sets · Interval-valued fuzzy sets · Interval-valued fuzzy signatures · Lattice · Meet and join operators

1 Introduction

Graph theory is a branch of Mathematics that studies the relationship between objects, or nodes. In a graph, the edges are referred to as the connections between two or more nodes; however, which was found by the Swiss mathematician Leonhard Euler and they solved the famous Konigsberg bridge problem. This problem raised the question of whether it was possible to walk across all seven bridges of the city of Konigsberg without crossing any bridge twice. Euler's work on this problem is considered to be the first instance of graph theory applied to a real-world problem. And also it applied to a wide range of applications, such as computer science, engineering, operations research, social sciences, medical diagnosis, and many other fields. Especially, in the medical field, which has the ability to represent complex systems and have a powerful analytical tool for identifying disease diagnosis (Chen 1997).

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On the other hand, fuzzy set (FS) theory is a mathematical concept that was developed by Zadeh in 1965, which is based on the conception of crisp set theory (Zadeh et al. 1996). Followed by Zadeh's groundbreaking idea of fuzzy sets, Goguen (1967) proposed the theory of the L-fuzzy sets. This concept was aimed to provide a more comprehensive description of a set. Thus, Goguen's idea of an L-fuzzy set has an important milestone in the evolution of fuzzy logic. Besides, IVFS was intended by Zadeh in 1975 as an extension of FS theory. They are used to represent the membership of a given element in a fuzzy set with a range of intervals instead of a single value, which allows a more precise and accurate representation of uncertainty in a system (Zadeh 1975). Moreover, the authors developed a new FS called interval-valued intuitionistic fuzzy set (Atanassov and Gargov 1989), which assigns a range of values from a lower limit and an upper limit to the elements of a set.

Following that, fuzzy modeling utilizes a mathematical construct known as a signature to model fuzzy subsets, rules, and relations. A signature is defined by a set of operators that give a description of the membership degrees of elements within the fuzzy set. It is used to model fuzzy rules, relations, systems, and control. The accuracy and effectiveness of fuzzy logic systems have been increased by the use of signatures. Also, it can be used to identify objects, estimate system behavior, locate data clusters, and so on. The fuzzy signature models allow them to make more accurate predictions and are simple to use (Pozna et al. 2012). But, it is essential to observe that the lack of proper numerical scheme analysis has been a significant obstacle whereas the pure statistical procedure has been attempted. In addition, the idea of the vector-valued fuzzy set (VVFS) was generated in the response to an industrial project that required the identification of microscopic pictures of various steel alloys based on a number of criteria, such as texture, directedness, and fineness. Subsequently, fuzzy decision-making applications adopted the VVFS concept (Koczy 1980). Later the applications prompted a hierarchically organized expansion of the VVFS formulation, resulting in the fuzzy signature idea. In detail, fuzzy signatures are mathematical functions that are used to represent the characteristics of a given data set. They are used to identify patterns in the data and make predictions about the behavior of the data. Also, they provide a way to identify patterns and relationships in data that would otherwise be difficult to determine (Koczy et al. 2021). Moreover, these are commonly used in many areas of research, including machine learning, pattern recognition (Rathnasabapathy and Palanisami 2022), and artificial intelligence and image processing (Premalatha and Dhanalakshmi 2022; Rathnasabapathy and Palanisami 2022). The authors (Koczy et al. 2021) suggested the new concept named fuzzy signature, which only consists of membership degree but not considered the intervals. Also, this study lacks a comprehensive analysis of the various properties of fuzzy signatures, as well as lacks their applications. Moreover, the fuzzy signature does not deal with uncertain situations in the statistical evaluation process.

Followed to the above study, Chen (1997) built the concept of interval-valued fuzzy hypergraphs and fuzzy partitions as a generalization of hypergraphs and fuzzy partitions, respectively. This study verifies the properties of the structures, and its applications, as well as their relationships between them. Moreover, Chen and Wang (2009) investigated the use of interval-valued fuzzy grade sheets to evaluate students' answer scripts. In (Chen 2011), the article investigated the use of similarity measures between interval-valued fuzzy numbers to analyze fuzzy risk.

Nevertheless, the authors in (Chen et al. 2012) explored a new multi-attribute decision-making method based on an interval-valued intuitionistic fuzzy weighted average operator and a fuzzy ranking method for intuitionistic fuzzy values. Then, Zeng et al. (2020) recommended an approach to solve the problem of interval-valued intuitionistic fuzzy multiple attribute decision-making in the IVIFS environment. Besides, Susniene et al. (2021) create a combined fuzzy signature model using Organizational Citizenship Behavior (OCB) and Counterproductive Work Behavior (CWB). Further, Chen and Chiou (2014), and Komarudin et al. (2021) suggested a novel approach to combining Linear Quadratic Regulator (LQR) controllers using fuzzy signature-based Particle Swarm Optimization (PSO). Furthermore, Shuichi Shuichi et al. (2021) formed an approach to cryptographic authentication systems that rely on biometrics, combining fuzzy signature and biometric authentication methods. In recent decades, Koczy et al. (2022) offered a novel similarity measure for fuzzy signatures, which was applied to employee engagement in Hungary and Lithuania. Further, the experts (Chaurasiya and Jain 2022; Banitalebi and Borzooei 2023) introduced an algorithm for solving multi-criteria healthcare waste treatment problems that have been based on the Pythagorean fuzzy entropy measure. Furthermore, based on Fuzzy Inference System (FIS) (Ejegwa et al. 2022; Lilik et al. 2022; Akram and Martino 2022), a new aggregation operator has been developed. Following this context, Ferenczi et al. (2022) suggested an approach for optimizing material handling management problems using fuzzy signatures and state-dependent weighting. Then, Singh (2022) examined the concept of bipolarity in a multi-way fuzzy context algorithm to calculate the granules of bipolar fuzzy sets. And also, the aforesaid studies do not deliver any insights into the contexts of the similarity measure. Besides, the fuzzy signature model relies too heavily on subjective measurements, which may lead to unreliable results. In this scenario, IVFSig provides an effective way to represent imprecise and uncertain information instead of a classical signature.

Based on the above thoughts, we develop a new signature called IVFSig and the contribution of this study is detailed below.

1. We present a method for constructing an algebraic framework for IVFSig's, based on the premise of IVFSs. We define a family of IVFSig's associated with a graph and a family of aggregation operators.
2. A meet and join operators are defined using the concept of a family of IVFSig's, and then their lattice algebraic structure is verified.
3. As an algebraic structure, the family of IVFSig's, the meet, and the join together form a bounded lattice and it also establishes the partial ordering relation in an IVFSig. Then, the proposed study has been compared with the existing method.
4. At last, numerical examples are given to show the performance of the addressed technique.

In light of the aforementioned studies, this paper is organized in the following manner: in Sect. 2, the study begins with the fundamental conceptions linked with graph theory.

Section 3 provides the creation of families of aggregation operators. Section 4 delivers the definitions of an IVFSig, a leaf interval-valued fuzzy sub signature, and a root interval-valued fuzzy sub signature. Based on IVFSig's the meet and join operators are formulated in Sect. 5. Then, Sect. 6 discusses the construction of partial ordering connections between IVFSig's. At last, the conclusions are presented in the final section.

2 Brief description of graph theory

In this section, we introduce several well-known notions and results which are necessary for our subsequent developments. This section provides an overview of the key concepts in graph theory that will be utilized later.

Definition 2.1 (Hallquist and Hillary 2018) A graph $G = (V, E)$ is a set of objects, where V stands for vertices or nodes and E stands for the pairwise connections between nodes.

Definition 2.2 (Koczy et al. 2021) Let the graph $G = (V, E)$, and $y, z \in V$ be the coordinates of the graph. We state that:

- If $k \geq 2$, k is an integer. A graph having node set $\{v_1, v_2, \ldots, v_k\} \subseteq V$ and edge set $\{\{v_1, v_2\},\ldots,\{v_{k-1}, v_k\}\} \subseteq E$, having $y = v_1$ and $z = v_k$, is a path yz on k nodes in G.
- A cycle is defined as a journey that starts and finishes at the same node.

Definition 2.3 (Koczy et al. 2021) Let $G = (V, E)$ be a graph. Here is what we say:

- If each pair of nodes $y, z \in V$ has a path yz, then G is connected.
- If G has no cycles and is connected, it is a tree.
- A tree G is said to be rooted if it has a node called the root and all of its edges point away from it.

Definition 2.4 (Koczy et al. 2021) $G = (V, E)$ is a rooted tree having $V = \{v_0, v_1, \ldots, v_n\}$ with v_0 as the root. Here is what we say:

- The length of any path v_i, v_j in G interconnecting the nodes v_i and v_j in V is one less than the number of nodes appearing in the path. The length of a path v_iv_j will be symbolised by the letter $l(v_i, v_j)$.
- The depth of G is represented by $d(G) = \max\{l(v_0, v_i) | v_i \in V\}$.

Definition 2.5 (Koczy et al. 2021) Consider $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ to be rooted trees. If $V_1 \subseteq V_2$ and $E_1 \subseteq E_2$, we can say that G_1 is a rooted subtree of G_2, which is indicated by $G_1 \subseteq G_2$.

Definition 2.6 (Koczy et al. 2021) Consider $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ to be rooted trees. $G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$ and $G_1 \cap G_2 = (V_1 \cap V_2, E_1 \cap E_2)$ are the union and intersection of rooted trees $G_1 \cup G_2$ and $G_1 \cap G_2$, respectively.

It is worth noting that if G_1 is a rooted subtree of G_2, then $G_1 \cup G_2 = G_2$ and $G_1 \cap G_2 = G_1$. The following example demonstrates the concepts of rooted tree union and intersection.

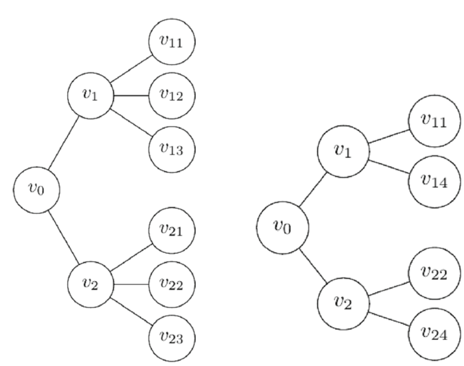
Example 2.1 Consider the rooted trees $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ demonstrated in Fig. 1. Using Definition 2.6, we can get the rooted trees $G_1 \cup G_2$ and $G_1 \cap G_2$ are illustrates in Fig. 2.

Proposition 2.1 *(Koczy et al. 2021) Let us pretend that $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$, and $G_3 = (V_3, E_3)$ are all rooted trees. The characteristics listed below are then checked:*

- $G_1 \cup G_2 = G_2 \cup G_1$
- $(G_1 \cup G_2) \cup G_3 = G_1 \cup (G_2 \cup G_3)$ and $(G_1 \cap G_2) \cap G_3 = G_1 \cap (G_2 \cap G_3)$
- $G_1 \cup (G_1 \cap G_2) = G_1$ and $G_1 \cap (G_1 \cup G_2) = G_1$.

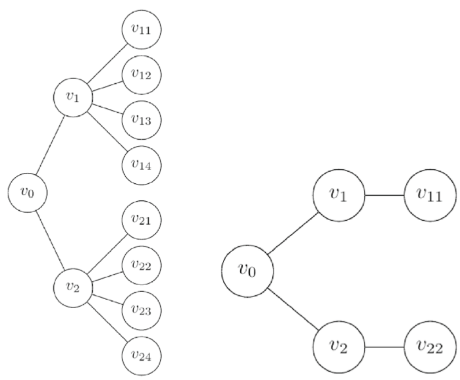
3 Organization of aggregation operators

Throughout this article, aggregation operators will also play a significant role, because they will be taken into account in the definition of IVFSig and operations between two general IVFSig's. Following that, we will provide several examples as well as the formal definition of aggregation operator.



(a) Rooted tree G_1 (b) Rooted tree G_2

Fig. 1 Rooted trees



(a) $G_1 \cup G_2$ (b) $G_1 \cap G_2$

Fig. 2 $G_1 \cup G_2$ and $G_1 \cap G_2$ of the rooted trees G_1 and G_2

Definition 3.1 (Koczy et al. 2021) Assume that $(P, \leq, \perp, \top)$ is a bounded poset. An order-preserving n-ary aggregation operator $\psi : P^n \to P$ is one that satisfies the following equalities: $\psi(\top, \ldots, \top) = \top$ and $\psi(\perp, \ldots, \perp) = \perp$.

Aggregation operators include conjunctive and disjunctive operators, as well as averaging and hybrid operators. The following illustration show some common aggregation operators.

The arithmetic mean M, geometric mean G, and harmonic mean H are aggregation operators of the averaging type specified as

$$M(c_1, c_2, \ldots, c_n) = \sum_{i=1}^{n} \frac{c_i}{n},$$

$$G(c_1, c_2, \ldots, c_n) = \left(\prod_{i=1}^{n} c_i \right)^{\frac{1}{n}},$$

and $H(c_1, c_2, \ldots, c_n) = \frac{n}{\sum_{i=1}^{n} \frac{1}{c_i}}$ on the unit interval.

Additionally, the weighted average on [0, 1] is a non-commutative aggregation operator.

$$N(c_1, c_2, \ldots, c_n) = \frac{2}{n}(n+1) \sum_{i=1}^{n} i c_i.$$

We are focused on a collection of aggregation operators $\mathcal{A}$ and an ordering relation $\leq$ that together make a complete lattice $(\mathcal{A}, \sqsubseteq)$. This pair will be referred to as a family of aggregators, and the infimum and supremum of this ordering will be indicated as $\inf_{\sqsubseteq}$ and $\sup_{\sqsubseteq}$, respectively.

One of the most fundamental and non-trivial families of aggregation operators is $\mathcal{A} = \{\psi_{\inf}, \psi_r, \psi_s, \psi_{\sup}\}$, where $\psi_{\inf} : [0,1]^2 \to [0,1]$ is described as $\psi_{\inf}(r,s) = \inf\{r,s\}$, $\psi_r, \psi_s : [0,1] \to [0,1]$ are identity mappings $\psi_r(r) = r$, $\psi_s(s) = s$, and $\psi_{\sup} : [0,1]^2 \to [0,1]$ is determined by the supremum operator $\psi_{\sup}(r,s) = \sup\{r,s\}$, for all $r,s \in [0,1]$.

The aggregation operator $\psi_{\inf}$ is linked to the $f_{\inf} : [0,1]^2 \to [0,1]$ mapping, which is given as $f_{\inf}(r,s) = \inf\{r,s\}$, ψ_r is related with the mapping $f_r : [0,1]^2 \to [0,1]$ provided by $f_r(r,s) = r$, ψ_s is linked to the $f_s : [0,1]^2 \to [0,1]$ mapping, which is defined by $f_r(r,s) = r$ and $\psi_{\sup}$ is defined as the mapping $f_{\inf} : [0,1]^2 \to [0,1]$ provided by $f_{\sup}(r,s) = \sup\{r,s\}$, for all $r,s \in [0,1]$. Because $f_{\inf}, f_r, f_s$, and $f_{\sup}$ are all binary mappings, we may say $f_{\inf} \leq f_r \leq f_{\sup}$ and $f_{\inf} \leq f_s \leq f_{\sup}$. Furthermore, the mappings of f_r and f_s are incomparable. As a result, as seen in Fig. 3, this ordering relation allows us to structure the aggregation operators hierarchically.

As with other families of four elements, it could be easily defined. For any $r,s \in [0,1]$, for instance, the aggregator operator $\psi_{\inf}$ can be replaced with any other t-norm or operator $\psi_{\perp}$ which fulfils the $\psi_{\perp}(r,s) \leq r$ and $\psi_{\perp}(r,s) \leq s$ conditions.

Similarly, for any $r,s \in [0,1]$, the aggregator $\psi_{\sup}$ can be replaced with another t-conorm or operator $\psi_{\top}$, fulfilling the inequalities $r \leq \psi_{\top}(r,s)$ and $s \leq \psi_{\top}(r,s)$.

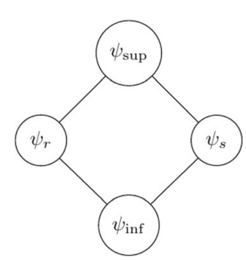
For instance, assuming a t-norm $T : [0,1]^2 \to [0,1]$ (Cretu 2001; Kesicioglu et al. 2015; Klement et al. 2002) and the aggregator operators $\psi_{\perp}, \psi_r, \psi_s, \psi_{\sup} : [0,1] \to [0,1]$ described as $\psi_{\perp}(r,s) = T(r,s)$, $\psi_r(r) = r$, $\psi_s(s) = s$, and $\psi_{\sup}(r,s) = \sup\{r,s\}$, we receive the family $\mathcal{A}_2(T) = \{\psi_{\perp}, \psi_r, \psi_s, \psi_{\sup}\}$, with the ordering shown by the Hasse diagram in Fig. 4.

The preceding family is labelled as $\mathcal{A}_2(T)$ because a similar family may be constructed for any t-norm T and an arbitrary number of variables. The similar family with 3 arguments is labelled as $\mathcal{A}_3(T)$ and is shown in Fig. 5.

In the following sections, we will propose a technique for creating families of aggregation operators from a prefixed set of aggregators. The user may require this set. For example, these operators might be useful for the user to address the application problem.

Assume $[0,1]^{[0,1] \times [0,1]}$ is the whole set of binary mappings from $[0,1] \times [0,1]$ to [0, 1], that is, $[0,1]^{[0,1] \times [0,1]} = \{f \mid f : [0,1] \times [0,1] \to [0,1]\}$. Consider the set

Fig. 3 Hasse diagram of the ordering relation on $\mathcal{A} = \{\psi_{\inf}, \psi_r, \psi_s, \psi_{\sup}\}$



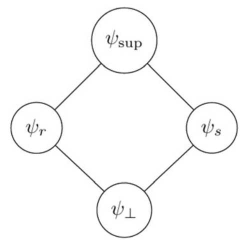


Fig. 4 Hasse diagram of the ordering relation defined on $\mathcal{A}_2(T) = \{\psi_\perp, \psi_r, \psi_s, \psi_{\sup}\}$

$\{f_\perp, f_r, f_s\} \subseteq [0,1]^{[0,1]\times[0,1]}$ in which $f_\perp(r,s) = T(r,s)$, where T is a t-norm distinct from the Godel t-norm (the minimum), $f_r(r,s) = r$, and $f_s(r,s) = s$. We shall compute the closure set of $\{f_\perp, f_r, f_s\}$ using the supremum and infimum, taking into consideration that $f_\perp \leq f_r$, $f_\perp \leq f_s$, and f_r and f_s are incomparable binary mappings. We get two new binary mappings, f_{sup} and f_{inf}, from this technique, that are defined as $f_{\text{sup}}(r,s) = \sup\{f_r(r,s), f_s(r,s)\} = \sup\{r,s\}$ and $f_{\text{inf}}(r,s) = \inf\ \{f_r(r,s),\ f_s(r,s)\} = \inf\{r,s\}$, respectively. Furthermore, because the Godel t-norm is the greatest t-norm, we can assure that the inequality $f_\perp \leq f_{\text{inf}}$ holds. As a result, by determining the supremum and infimum pointwise of every pair of mappings in the set, the set $\{f_\perp, f_r, f_s\}$ may be viewed as a generator system from which a family of binary aggregation operators $\{f_\perp, f_{\text{inf}}, f_r, f_s, f_{\text{sup}}\}$ can be produced. This family of binary aggregation operators is seen in Fig. 6.

Following the current reasoning, we may construct the family of aggregation operators with 3 arguments, the family of aggregation operators with 4 arguments, and so on.

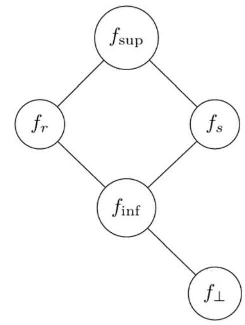


Fig. 6 Family of binary aggregation operators $\{f_\perp, f_{\inf}, f_r, f_s, f_{\sup}\}$

4 Constructing interval-valued Fuzzy signatures

Theoretically, IVFSigs have been studied, and some interesting mathematical properties have been looked into. However, there are still a lot of questions about this area of research. If the family of IVFSig's is paired with suitable meet and join operators, we want to know if it has the algebraic structure of a lattice. Following that, various definitions related to IVFSig's will be provided, along with some clarifying examples.

The notions of a rooted tree and family of aggregation operators are key to introducing the formal definition of IVFSig. Hereafter consider $G = (V, E)$ as a tree with root

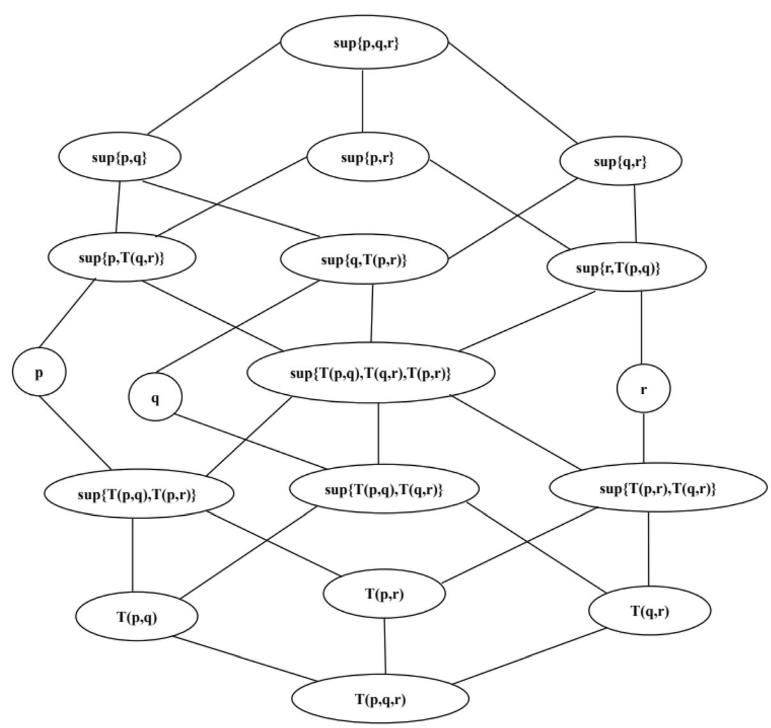


Fig. 5 Ordering relation $\mathcal{A}_3(T)$'s Hasse diagram

v_0, where the set of nodes is partitioned into 2 subsets, L and N, with a non-empty subset L containing the leaves and a subset N containing the inner nodes, respectively, fulfilling that $L \cup N = V$ and $L \cap N = \phi$, that is, $V = L \uplus N$, and that $|V| = m$, $|L| = l$. Hence, $v_0 \in N$ unless $V = \{v_0\}$, G is a single leaf.

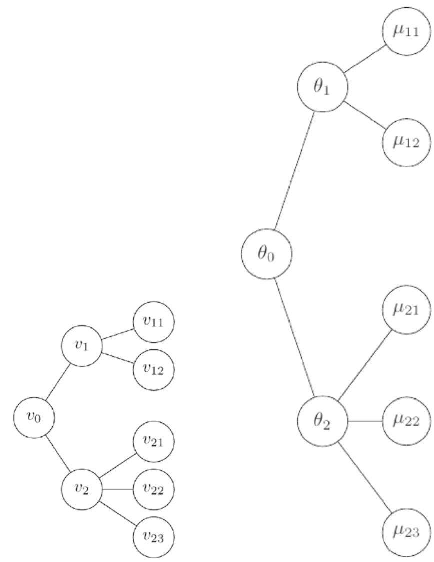
Definition 4.1 (Koczy et al. 2021) Let $\{\mathcal{A}_1, \ldots, \mathcal{A}_n\}$ denote a collection of aggregation operator families. Consider a set of fuzzy membership (M_F) degrees $\mu_j \in [0,1]$, with $j \in \{1, \ldots, l\}$, allocated to each leaf in L; and the collection of aggregation operators $\theta_i \in \mathcal{A}_i$ is applied to each inner node in N, where $i \in \{1, \ldots, n\}$. The tuple $S = \langle N, L, \{\theta_1, \theta_2, \ldots, \theta_n\}, \{\mu_1, \ldots, \mu_l\}\rangle$ is known as a fuzzy signature associated with G, the tuple $S_t = \langle N, L, \{\theta_1, \theta_2, \ldots, \theta_n\}\rangle$ is called the structure of S; $\{\mu_1, \ldots, \mu_l\}$ is the M_F degree set of S.

Depending on the length of the node to be denoted to the root, we shall give several subscripts to the nodes appearing in an IVFSig for practical purposes. As an outcome, if $v \in V$ and $l(v_0, v) = 1$, we shall write v with one subscript, if $l(v_0, v) = 2$, we will write v with two subscripts and so on. If v_0 has n_0 children, its descendants will be denoted as $v_1, v_2, \ldots, v_{n_0}$, If v_1 has n_1 children, the following will be written: $v_{1,1}, v_{1,2}, \ldots, v_{1,n_1}$. Inductively, if $v_{i_1,i_2,\ldots,i_d} \in V$ is not the root and has n_d children, the children are $v_{i_1,i_2,\ldots,i_d,1}$, $\ldots, v_{i_1,i_2,\ldots,i_d,n_d}$. When there is no chance of confusion, we shall write the subindices without commas, that is $v_{i_1 i_2,\ldots,i_d n_d}$ instead of $v_{i_1,i_2,\ldots,i_d,n_d}$.

Here, N will be segmented into $N = N_0 \cup N_1 \cup \ldots \cup N_k$ from which $k = \max\{l(v_0, v) \mid v \in V\} = d(G)$, $N_0 = v_0$ as well as we get $N_h = \{v \in V \mid l(v_0, v) = h\}$ here $h \in \{1, 2, \ldots, k\}$. It is worth noting that all nodes in N_h have precisely h subscripts. The first denoting the subscript of the ancestor node that is a direct child of the root. The first and second producing the subscript of the ancestor, that is a grandchild of the given root.

Example 4.1 Consider G be a rooted tree with $d(G) = 2$, with nodes $V = \{v_0, v_1, v_2, v_{11}, v_{21}, v_{22}, v_{23}\}$, leaves $L = \{v_{11}, v_{21}, v_{22}, v_{23}\}$, and inner nodes $N = \{v_0, v_1, v_2\}$. The root v_0 has two children, v_1 and v_2, and the node v_1 has two children, v_{11} and v_{12}. The node v_2 has three children, v_{21}, v_{22}, and v_{23}. The tree can be presented graphically using this notation. This rooted tree is shown in Fig. 7, from which we can determine the sets $N_0 = \{v_0\}$ and $N_1 = \{v_1, v_2\}$ that fulfill $N = N_0 \cup N_1$.

Assuming the fuzzy M_F degrees μ_{11},μ_{12}, μ_{21}, μ_{22}, $\mu_{23} \in [0,1]$ applied to the leaves L, the bundle of aggregation operator families $\{\mathcal{A}_2(G), \mathcal{A}_1, \mathcal{A}_3(P)\}$ is defined as follows:$\mathcal{A}_2(G)$ is the aggregation operator family constructed from the Godel t-norm with 2 variables; $\mathcal{A}_1$ is the aggregation operator family constructed from the Godel t-conorm with 2 variables; and $\mathcal{A}_3(P)$ is the family of aggregation operators is made up of the product t-norm with 3 variables. Each inner node in N is allotted with the aggregation operators $\theta_0 \in \mathcal{A}_2(G)$, $\theta_1 \in \mathcal{A}_1$, and $\theta_2 \in \mathcal{A}_3(P)$.



(a) Rooted tree G (b) Structure S_t of Example 4.1

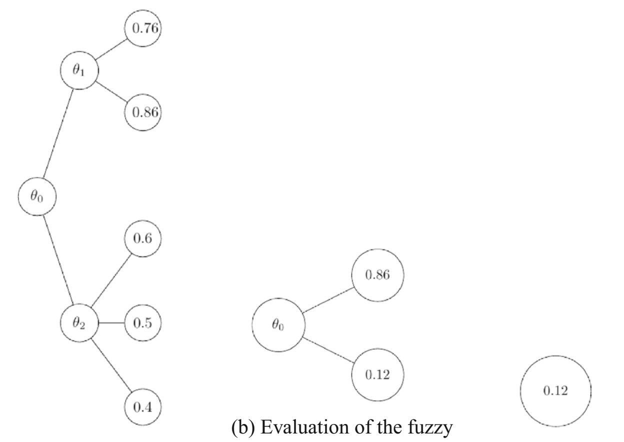
Fig. 7 Rooted tree & its fuzzy signature structure

$$\theta_0(p,q) = \min\{p,q\}$$
$$\theta_1(p,q) = \max\{p,q\}$$
$$\theta_2(p,q,r) = p * q * r.$$

The fuzzy signature, $S = \langle N, L, \{\theta_0, \theta_1, \theta_2\}, \{\mu_{11}, \mu_{12}, \mu_{21}, \mu_{22}, \mu_{23}\}\rangle$ is outlined in Fig. 7. The structure (S_t) of the fuzzy signature can also be seen in this diagram, where θ_0 is allocated to the root, and θ_1, θ_2 are assigned to the children. In Fig. 8a, a specified set of M_F degrees is evaluated, and the variables are substituted with constant values to yield the fuzzy signature, $S = \langle\ N, L, \{\theta_0, \theta_1, \theta_2\}, \{\ 0.76, 0.86, 0.6, 0.5, 0.4\}\rangle$.

Definition 4.2 (Koczy et al. 2021) The evaluation of a fuzzy signature S associated with $G = (V, E)$, denoted as $E(S)$, is the M_F degree assigned to v_0 obtained by executing all the general aggregation operators in S, starting from the M_F degrees in the leaves.

Example 4.2 Returning to Example 4.1, we can determine the evaluation of the fuzzy signature, $S = \langle\ N, L, \{\theta_0, \theta_1, \theta_2\}, \{\ 0.76, 0.86, 0.6, 0.5, 0.4\ \}\rangle$ shown in Fig. 8a. Using basic computations, we can assess that the evaluation of the fuzzy signature, $E(S) = 0.12$. This approach is presented in Fig. 8b and c.



(a) Fuzzy signature S in Example 4.1 (b) Evaluation of the fuzzy signature in Example 4.2 (c) $E(S) = 0.12$

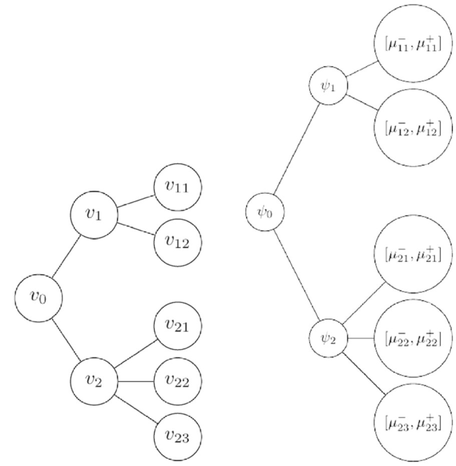
Fig. 8 Fuzzy signature and its evaluation

Definition 4.3 Let $\{\mathcal{A}_1, \ldots, \mathcal{A}_n\}$ denote a collection of aggregation operator families. Given a set of interval-valued fuzzy M_F degrees $[\mu_j^-, \mu_j^+] \in [0,1]$, with $j \in \{1, \ldots, l\}$, allocated to each leaf in L; and the collection of aggregation operators $\psi_i \in \mathcal{A}_i$ is applied to each inner node in N, where $i \in \{1, \ldots, n\}$. The tuple $I_S = \langle N, L, \{\psi_1, \ldots, \psi_n\}, \{[\mu_1^-, \mu_1^+], \ldots, [\mu_l^-, \mu_l^+]\}\rangle$ is known as an *IVFSig* associated with G, the tuple $I_{S_t} = \langle N, L, \{\psi_1, \psi_2, \ldots, \psi_n\}\rangle$ is called the structure of I_S; $\{[\mu_1^-, \mu_1^+], [\mu_2^-, \mu_2^+], \ldots, [\mu_l^-, \mu_l^+]\}$ is the M_F degree intervals set of I_S.

We will use the following example to demonstrate the concept of an IVFSig and the preceding concerns about the notation.

Example 4.3 Consider G be a rooted tree with $d(G) = 2$, with nodes $V = \{v_0, v_1, v_2, v_{11}, v_{21}, v_{22}, v_{23}\}$ leaves $L = \{v_{11}, v_{21}, v_{22}, v_{23}\}$ and inner nodes $N = \{v_0, v_1, v_2\}$. The root v_0 has two children, v_1 and v_2, and the node v_1 has two children v_{11} and v_{12}. The node v_2 has three children, v_{21}, v_{22}, and v_{23}. The tree may be presented graphically using this notation. This rooted tree is shown in Fig. 9, from which we can determine the sets $N_0 = \{v_0\}$ and $N_1 = \{v_1, v_2\}$ that fulfill $N = N_0 \cup N_1$.

Assuming the interval-valued fuzzy M_F degrees $[\mu_{11}^-, \mu_{11}^+]$, $[\mu_{12}^-, \mu_{12}^+]$, $[\mu_{21}^-, \mu_{21}^+]$, $[\mu_{22}^-, \mu_{22}^+]$, $[\mu_{23}^-, \mu_{23}^+] \in [0,1]$ applied to the leaves L, the bundle of aggregation operator families $\{\mathcal{A}_2(G), \mathcal{A}_1, \mathcal{A}_3(P)\}$, here $\mathcal{A}_2(G)$ is the aggregation operator family constructed from the Godel t-norm with 2 variables, $\mathcal{A}_1$ is the aggregation operator family constructed from the Godel t-conorm with 2 variables, and the $\mathcal{A}_3(P)$ is the family of aggregation operators is made up of the product t-norm with 3 variables. Each inner node in N is allotted with the aggregation operators $\psi_0 \in \mathcal{A}_2(G)$, $\psi_1 \in \mathcal{A}_1$, $\psi_2 \in \mathcal{A}_3(P)$, are defined as follows:



(a) Rooted tree G (b) Structure I_{S_t} of Example 4.3

Fig. 9 Rooted tree & its structure

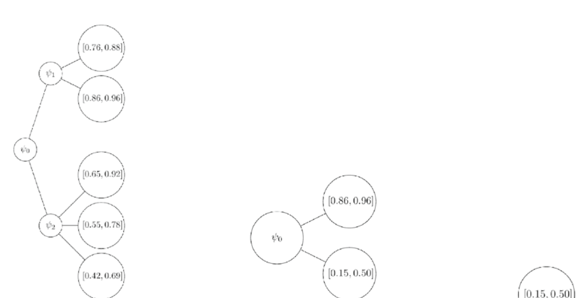
$$\psi_0(p,q) = \min\{p,q\}$$
$$\psi_1(p,q) = \max\{p,q\}$$
$$\psi_2(p,q,r) = p * q * r.$$

ψ_0 is allocated to the root, whereas ψ_1, ψ_2 are assigned to the children. Then the IVFSig, $I_S = \langle N, L, \{\psi_0, \psi_1, \psi_2\}, \{[\mu_{11}^-, \mu_{11}^+], [\mu_{21}^-, \mu_{21}^+], [\mu_{22}^-, \mu_{22}^+], [\mu_{23}^-, \mu_{23}^+]\}\rangle$ is also outlined in Fig. 9. The structure (I_{S_t}) of the IVFSig, I_S may be seen in this diagram.

In Fig. 10a, a specified set of M_F degrees is evaluated, and the variables are substituted with constant values, yielding the suitable IVFSig, $I_S = \langle N, L, \{\psi_0, \psi_1, \psi_2\}, \{$ [0.76, 0.88], [0.86, 0.96], [0.65, 0.92], [0.55, 0.78], [0.42, 0.69] $\}\rangle$.

Another intriguing concept in this paper is the assessment of an IVFSig.

Definition 4.4 An IVFSig entrusted with $G = (V, E)$ is to v_0 acquired and evaluated by performing all of the general aggregation operators in I_S, beginning with the M_F degrees



(a) Interval-valued fuzzy signature I_S in Example 4.3 (b) Evaluation $E(I_S)$ of the IVFSig, I_S in Example 4.4 (c) $E(I_S) = [0.15, 0.50]$

Fig. 10 Interval-valued fuzzy signature and its evaluation in the leaves. $E(I_S)$ represents the assessment of the IVFSig.

Example 4.4 Returning to Example 4.3, we will determine the evaluation of the IVFSig, $I_S = \langle N, L, \{\psi_0, \psi_1, \psi_2\}, \{$ [0.76, 0.88], [0.86, 0.96], [0.65, 0.92], [0.55, 0.78], [0.42, 0.69] $\}\rangle$ shown in Fig. 10a. Using basic computations, we assess that the evaluation of the IVFSig, I_S is $E(I_S) = [0.15, 0.50]$. This approach is presented in Fig. 10b and c.

The concepts and examples that follow will help us grasp the concept of IVFSig's. They also allow partial assessments of IVFSig's to be computed.

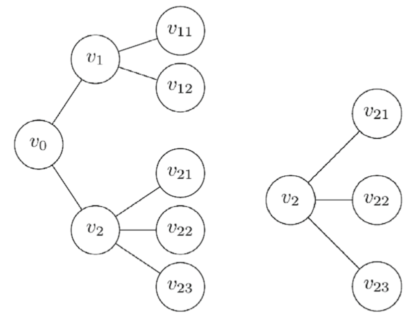
Note 1 Based on Examples 4.2 and 4.4, we have evaluated the comparison study. Figure 8 of Example 4.2 shows the evaluation of the fuzzy signatures as well as Fig. 10 of Example 4.4 provides an evaluation of the IVFSig. In this context, both examples utilized the aggregation operators accordingly. Then, the findings of both examples give the best results, but Fig. 8 does not consists the membership intervals; therefore, it obtained the result as low value. On the other hand, in Fig. 10, the outcome of the evaluation is [0.15, 0.50], which presents the intervals as well as advanced one compared to the existing one. Therefore, we conclude that the IVFSig performed better in the evaluation of fuzzy signature.

Definition 4.5 Let G be the rooted tree, with N representing the inner node collections and I_S being the IVFSig associated with G.

- The leaf subtree of G associated with $v \in N$ is the tree whose root is v and which contains the descendant nodes of v as well as the related edges. $L_G(v)$ is a leaf subtree of G related to $v \in N$.
- The leaf subtree $L_G(v)$ contains the aggregation operators and M_F degree intervals for each node in the IVFSig I_S. It is referred to as the leaf interval-valued fuzzy sub-signature of I_S associated with $v \in N$. The new signature is known as $L_{I_S}(v)$.

Because a leaf interval-valued fuzzy subsignature is a specific IVFSig, evaluating it is as simple as $E(L_{I_S}(v))$, as shown in Definition 4.4.

Example 4.5 Turning to Example 4.3, we now construct the leaf subtree of G associated with v_2, namely $L_G(v_2)$, as seen in Fig. 11. $L_{I_S}(v_2)$, the leaf interval-valued fuzzy subsignature of I_S correlated with $v_2 \in N$, is shown in Fig. 12; Furthermore, the leaf interval-valued fuzzy sub-signature of I_S associated with $v_2 \in N$, $E(L_{I_S}(v_2))$ is clearly ψ_2([0.65, 0.92], [0.55, 0.78], [0.42, 0.69]) = [0.15, 0.50].



(a) Rooted tree G (b) Leaf subtree $L_G(v_2)$ of Example 4.5

Fig. 11 Rooted tree and its one of the leaf subtrees $L_G(v_2)$

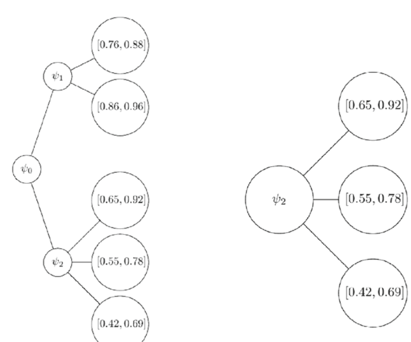
The leaf interval-valued fuzzy subsignature's dual definition is similarly valuable, and it is discussed next.

Definition 4.6 Suppose I_S be an IVFSig connected with G.

- The root subtree $R_G(v)$ of a tree G is formed by removing all descendants of v from G, and it is connected to $v \in N$.
- The root interval-valued fuzzy sub signature of an I_S associated with $v \in N$ in the root subtree $R_G(v)$ includes the evaluation of the leaf interval-valued fuzzy sub signature in the node v, where the aggregation operators and membership degrees of the original IVFSig I_S are preserved in the other nodes of the graph. This signature is called $R_{I_S}(v)$.

Unlike the leaf subtree, the evaluation of $R_{I_S}(v)$ in this instance fulfils $R_{I_S}(v) = E(I_S)$ for every $v \in N$. We will return to Example 4.3 to explain the concepts mentioned before.

Example 4.6 We would calculate the root subtree of the graph G and the root interval-valued fuzzy subsignature of I_S corresponding to $v \in N$ in the surroundings of Example 4.3, which seems to be $R_G(v_0)$ in Fig. 13 and $R_{I_S}(v_0)$ in



(a) IVFSig I_S (b) Leaf interval valued fuzzy subsignature $L_{I_S}(v_2)$ of Example 4.5

Fig. 12 IVFsig's and leaf interval valued fuzzy subsignature $L_{I_S}(v_2)$

Fig. 14. In this circumstance, $R_{I_S}(v_0) = E(I_S) = [0.15, 0.50]$.

Remark 4.1 Suppose $G = (V, E)$ indicate the rooted tree, with N designating the set of inner nodes.

- Definitions 4.5 and 4.6 show that for each $v \in N$, $R_G(v) \cup L_G(v) = G$ and $R_G(v) \cap L_G(v) = v$. In $R_G(v)$, an inner node v becomes a leaf.
- Even if the root interval-valued fuzzy sub signature of I_S linked to one of its leaves does not produce any change in I_S, Definition 4.6 may be extended to any node.

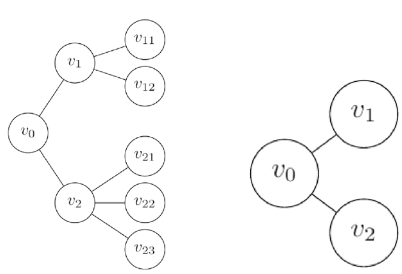
5 Join and meet operators in a family of IVFSig

IVFSig was made to describe things or events that let different structures with uncertain properties be used. The main benefit of modeling with IVFSig is that different IVFSig may appear in the same issue due to missing data on one or more of the features defined by the leaves, or even on whole feature groups defined by leaf subtrees.

A basic illustration is when an IVFSig describing a patient's body temperature is considered, as shown in the application. It may be an IVFSig's with three inner nodes, each of which has four leaves. The temperatures on the day before yesterday, yesterday, and today are represented by the three potential leaf subtrees. Each leaf in a leaf subtree represents body temperature in the morning, noon, afternoon, and evening.

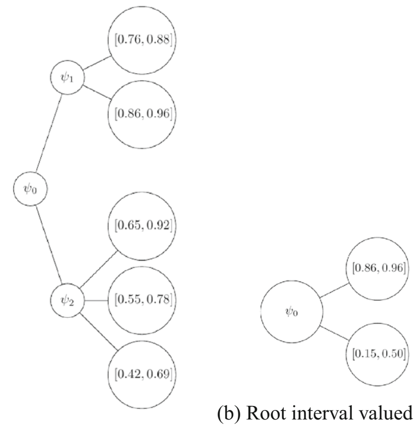
The very first leaf subtree is lacking for patients who were admitted yesterday. Today's new arrivals will miss the first and second leaf subtrees, and depending on the time, one or more leaves of the last subtree. A patient may have fewer than four temperature measurements on those days.

Even so, some individuals may only have average body temperatures throughout the whole duration. In this



(a) Rooted tree G (b) Root subtree $R_G(v_0)$ of Example 4.6

Fig. 13 Rooted tree and its root subtree $R_G(v_0)$



(a) IVFSig I_S (b) Root interval valued fuzzy subsignature $R_{I_S}(v_0)$ of Example 4.6

Fig. 14 IVFSig's and root interval valued fuzzy subsignature $R_{I_S}(v_0)$

manner, whole subtrees might be missed, or just a few leaves from some of the subtrees, or they may be replaced by single leaves assigned interval-valued fuzzy M_F degrees expressing the degree of getting "high body temperature", or simply fever, on a specific day, or in the period under inspection in general.

Considering these things, Building a database of possible IVFSig's may be very helpful that might represent information related to a certain situation. In addition, if you want to combine two or more IVFSig's, you will need to specify the meet and the join operators on them. In the following parts, we will provide a more in-depth explanation of the aforementioned ideas.

We will begin by defining a family that is constructed using a rooted tree and the sets of families of aggregation operators.

Definition 5.1 Assume that $G = (V, E)$. $N = \{v_0, v_1, \ldots, v_n\}$ is a collection of inner nodes for a tree with a root v_0, and $\mathcal{A} = \{\mathcal{A}_0, \mathcal{A}_1, \ldots, \mathcal{A}_n\}$ is a sets of families of aggregation operators. The $\mathcal{F}(G, \mathcal{A})$ family of IVFSig's, which are derived from G and $\mathcal{A}$, is defined as follows:

$$\{I_{S_k} = \langle N_k, L_k, \{\psi_{i_1}, \ldots, \psi_{i_p}\}, \{[\mu_1^-, \mu_1^+], \ldots, [\mu_q^-, \mu_q^+]\}\rangle | G_k \subseteq G \text{ and } \psi_{i_j} \in \mathcal{A}_{i_j}, \text{for all } v_{i_j} \in N_k\},$$

where $G_k = (N_k \cup L_k, E_k)$ is a subgraph of G that satisfies that $v_0 \in N_k$, and I_{S_k} is an IVFSig associated with G_k.

Definition 5.2 Let I_S be an IVFSig for the rooted tree G_{I_S} and the collection of aggregation operator families $\mathcal{A}_{I_S}$. The $\mathcal{F}(G_{I_S}, \mathcal{A}_{I_S})$ family of IVFSig's is defined as the family of IVFSig's created from I_S.

Definition 5.3 Let I_S be an IVFSig for the rooted tree G_{I_S} and the collection of aggregation operator families $\mathcal{A}_{I_S}$. Let $I_{S_1}, I_{S_2} \in \mathcal{F}(G_{I_S}, \mathcal{A}_{I_S})$ represent *IVFSig*'s associated with

$G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, respectively, where L_1, L_2 are leaf collections and N_1, N_2 are inner node collections of each tree G_1 and G_2. The IVFSig linked with the rooted tree $G_1 \cap G_2 = (V_1 \cap V2, E_1 \cap E2)$, where $N_{G_1 \cap G_2}$ represents the collection of inner nodes and $L_{G_1 \cap G_2}$ the collection of leaves, is the meet of the IVFSig's I_{S_1} and I_{S_2}, which will be designated as $I_{S_1} \wedge I_{S_2}$. The following rules will determine the M_F degrees allocated to each leaf in $L_{G_1 \cap G_2}$ and the aggregation operators applied to each inner node in $N_{G_1 \cap G_2}$:

- If $v \in N_{G_1 \cap G_2}$, the aggregation operator for v is $\psi_v = \inf_{\sqsubseteq}\{\psi_v^1, \psi_v^2\}$, where ψ_v^1 is the aggregation operator for $v \in N_1$ in I_{S_1} and ψ_v^2 is the aggregation operator for $v \in N_2$ in I_{S_2}.

  It's interesting to note that if ψ_v has fewer leaves than variables, the missing variables will have a value of zero when the IVFSig $I_{S_1} \wedge I_{S_2}$ is evaluated.
- If $v \in L_{G_1 \cap G_2}$, M_F degree given to v is computed by taking into account the following scenarios:

  (i) If $v \in L_1$ and $v \in L_2$ are present, v's M_F degree equals $[\mu_v^-, \mu_v^+] = \inf\{[\mu_v^{1-}, \mu_v^{1+}], [\mu_v^{2-}, \mu_v^{2+}]\}$, where $[\mu_v^{1-}, \mu_v^{1+}]$ represents the M_F degree allocated to $v \in L_1$ and $[\mu_v^{2-}, \mu_v^{2+}]$ represents the M_F degree allocated to $v \in L_2$.

  (ii) If $v \in L_1$ and $v \in N_2$ are present, the M_F degree given to v is $[\mu_v^-, \mu_v^+] = [\mu_v^{1-}, \mu_v^{1+}]$, where $[\mu_v^{1-}, \mu_v^{1+}]$ is $v \in L_1$'s M_F degree.

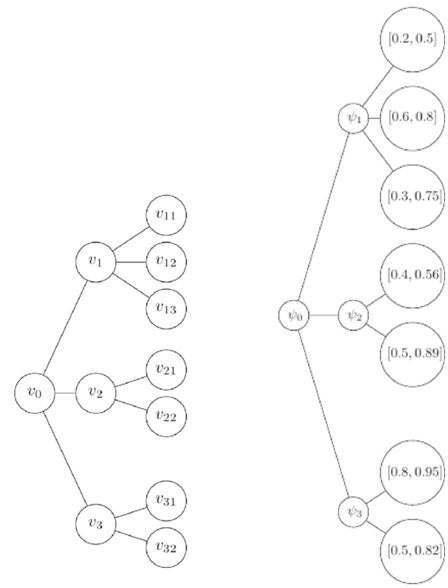
  (iii) If $v \in N_1$ and $v \in L_2$ are present, the M_F degree assigned to v is $[\mu_v^-, \mu_v^+] = [\mu_v^{2-}, \mu_v^{2+}]$, where $[\mu_v^{2-}, \mu_v^{2+}]$ is the M_F degree assigned to $v \in L_2$.

  (iv) If $v \in N_1$ and $v \in N_2$ are true, v's M_F degree is $[\mu_v^-, \mu_v^+] = 0$. When the arguments of the aggregator of v in N_1 differ from the arguments of the aggregator of v in N_2, this situation might occur.

To understand the concept of meet of two IVFSig's, we shall present an example.

Example 5.1 Assume the rooted tree $G_{I_S} = (V, E)$ in Fig. 15, where the set of nodes, the set of leaves, and the set of inner nodes are $V = \{v_0, v_1, v_2, v_3, v_{11}, v_{12}, v_{13}, v_{21}, v_{22}, v_{31}, v_{32}\}$, $L = \{v_{11}, v_{12}, v_{13}, v_{21}, v_{22}, v_{31}, v_{32}\}$, and $N = \{v_0, v_1, v_2, v_3\}$.

Consider $\mathcal{A}_{I_S} = \{\mathcal{A}_2(G), \mathcal{A}_3(G), \mathcal{A}_3(P)\}$, a collection of aggregation operator families, where $\mathcal{A}_2(G)$ is the aggregation operator family derived from the Godel t-norm and two variables, and $\mathcal{A}_3(G)$ is a family of aggregation operators described by the Godel t-norm and three variables. The $\mathcal{A}_3(P)$ operator family is defined by the product t-norm and three variables.



(a) Rooted tree G_{I_S} (b) IVFSig I_S

Fig. 15 Rooted tree G_{I_S} and IVFSig I_S of Example 5.1

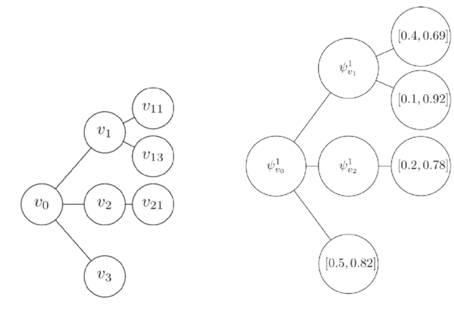
The aggregation operators $\psi_0 \in \mathcal{A}_3(G)$, $\psi_1 \in \mathcal{A}_3(P)$, $\psi_2, \psi_3 \in \mathcal{A}_2(G)$ that are provided to each inner node in N are as follows:

$$\psi_0(p, q, r) = \min\{p, q, r\}$$
$$\psi_1(p, q, r) = p * q * r$$
$$\psi_2(p, q) = \psi_3(p, q) = \min\{p, q\}, \text{ for every } p, q, r \in [0, 1].$$

Figure 15 also shows the IVFSig $I_S = \langle N, L, \{\psi_0, \psi_1, \psi_2\}\{[0.2, 0.5], [0.6, 0.8], [0.3, 0.75], [0.4, 0.56], [0.5, 0.89], [0.8, 0.95], [0.5, 0.82]\}\rangle$.

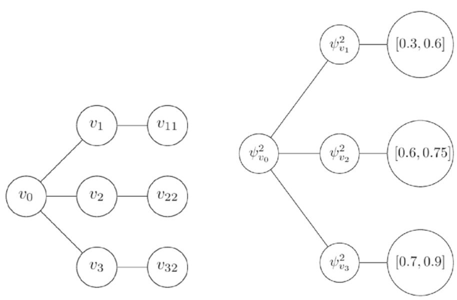
Let $I_{S_1}, I_{S_2} \in \mathcal{F}(G_{I_S}, A_{I_S})$ represent the IVFSig's for the rooted trees. $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, respectively. These are shown in Figs. 16 and 17 and are defined by the tuples $I_{S_1} = \langle N_1, L_1, \{\psi_{v_0}^1, \psi_{v_1}^1, \psi_{v_2}^1\}$ $\{[0.5, 0.82], [0.4, 0.69], [0.1, 0.92], [0.2, 0.78]\}\rangle$ and $I_{S_2} = \langle N_2, L_2, \{\psi_{v_0}^2, \psi_{v_1}^2, \psi_{v_2}^2, \psi_{v_3}^2\}$ $\{[0.3, 0.6], [0.6, 0.75], [0.7, 0.9]\}\rangle$, respectively, where $N_1 = \{v_0, v_1, v_2\}$, $L_1 = \{v_3, v_{11}, v_{13}, v_{21}\}$, $N_2 = \{v_0, v_1, v_2, v_3\}$, $L_2 = \{v_{11}, v_{22}, v_{32}\}$. For every $p, q, r \in [0, 1]$, the aggregation operators $\psi_{v_0}^1, \psi_{v_0}^2 \in \mathcal{A}_3(G)$, $\psi_{v_1}^1, \psi_{v_1}^2 \in \mathcal{A}_3(P)$, $\psi_{v_2}^1, \psi_{v_2}^2$, $\psi_{v_3}^2 \in \mathcal{A}_2(G)$ allocated to every inner node are defined as:

$$\psi_{v_0}^1(p, q, r) = \psi_{v_0}^2(p, q, r) = \max\{p, q, r\}$$
$$\psi_{v_1}^1(p, q) = p * q$$
$$\psi_{v_2}^1(p) = \psi_{v_1}^2(p) = p$$
$$\psi_{v_2}^2(q) = \psi_{v_3}^2(q) = q.$$



(a) Rooted tree G_1 (b) IVFSig I_{S_1}

Fig. 16 Rooted tree G_1 and IVFSig I_{S_1} of Example 5.1



(a) Rooted tree G_2 (b) IVFSig I_{S_2}

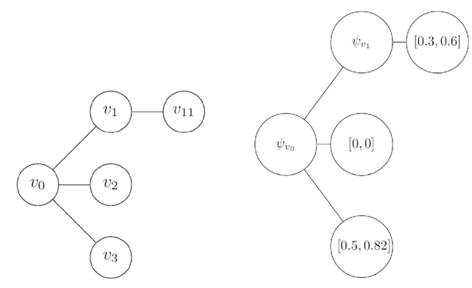
Fig. 17 Rooted tree G_2 and IVFSig I_{S_2} of Example 5.1

We will now determine the meet of the IVFSig's I_{S_1} and I_{S_2}. According to Definition 5.3, $G_1 \cap G_2$ is the rooted tree corresponding to the IVFSig $I_{S_1} \wedge I_{S_2}$, as seen in Fig. 18. The set of inner nodes is $N_{G_1\cap G_2} = \{v_0, v_1\}$, and the set of leaves is $L_{G_1\cap G_2} = \{v_2, v_3, v_{11}\}$. Using the criteria contained in Definition 5.3, the M_F degrees allotted to each leaf in $L_{G_1\cap G_2}$ and also the aggregation operators assigned to each inner node in $N_{G_1\cap G_2}$ will be calculated. With regard to the nodes in $L_{G_1\cap G_2}$, we identify the following instances:

- $v_{11} \in L_1$ and $v_{11} \in L_2$, then v_{11}'s M_F degree is $[\mu^-_{v_{11}}, \mu^+_{v_{11}}] = \inf\{[0.4, 0.69], [0.3, 0.6]\} = [0.3, 0.6]$.
- $v_2 \in N_1$ and $v_2 \in N_2$, then $\mu_{v_2} = [0, 0]$ is the M_F degree ascribed to v_2.
- $v_3 \in L_1$ and $v_3 \in N_2$, then $\mu_{v_3} = [0.5, 0.82]$ is the M_F degree ascribed to v_3.

Consider the ordering relation provided on $\mathcal{A}_3(G)$ and $\mathcal{A}_3(P)$ using the complete lattice shown in Fig. 5 for the nodes in $N_{G_1\cap G_2}$. The aggregation operators ψ_{v_0} and ψ_{v_1} given to v_0 and v_1, respectively, are as follows:

$$\psi_{v_0} = \inf_{\sqsubseteq} \{\psi^1_{v_0}, \psi^2_{v_0}\} = \psi^1_{v_0},$$



(a) $G_1 \cap G_2$ (b) $I_{S_1} \wedge I_{S_2}$

Fig. 18 Rooted tree $G_1 \cap G_2$ and IVFSig $I_{S_1} \wedge I_{S_2}$ of Example 5.1

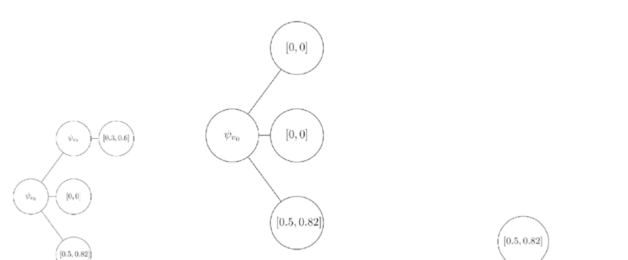
$$\psi_{v_1} = \inf_{\sqsubseteq} \{\psi^1_{v_1}, \psi^2_{v_1}\} = \psi^1_{v_1}.$$

Figure 18 depicts the IVFSig $I_{S_1} \wedge I_{S_2} = \langle N_{G_1\cap G_2}, L_{G_1\cap G_2}, \{\psi_{v_0}, \psi_{v_1}\}$ {[0, 0], [0.5, 0.82], [0.3, 0.6]}$\rangle$.

Look at the fact that ψ_{v_1} has fewer leaves than variables. As a result, in the assessment of the IVFSig $I_{S_1} \wedge I_{S_2}$, the value of the missing variables will be 0. To acquire the assessment of the IVFSig $I_{S_1} \wedge I_{S_2}$, we will do the necessary computations $\psi_{v_1} = ([0.3, 0.4] * [0, 0]) = [0, 0]$ and $\psi_{v_0} =$ max ([0,0], [0, 0], [0.5, 0.82]) = [0.5, 0.82]. As a result, we arrive at the conclusion that $E(I_{S_1} \wedge I_{S_2}) = [0.5, 0.82]$. Figure 19 shows the step by step evaluation of $I_{S_1} \wedge I_{S_2}$.

In addition, the join of two IVFSig's $I_{S_1}, I_{S_2} \in \mathcal{F}(G_{I_S}, \mathcal{A}_{I_S})$ will be computed by combining the rooted trees associated with I_{S_1}, I_{S_2} and assigning suitable labels to the nodes in the resulting tree. The labels of the nodes will be generated using the supremum operator in this example.

Definition 5.4 Let I_S be an IVFSig linked to the rooted tree G_{I_S} and, the collection of aggregation operator families $\mathcal{A}_{I_S}$. Let $I_{S_1}, I_{S_2} \in \mathcal{F}(G_{I_S}, \mathcal{A}_{I_S})$ be IVFSig's for $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, respectively, where L_1, L_2 are leaf sets and N_1, N_2 are inner node sets of each tree G_1 and G_2. The join of the IVFSig's I_{S_1} and I_{S_2}, denoted as $I_{S_1} \vee I_{S_2}$, is the IVFSig associated with the rooted tree $G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$, where $N_{G_1\cup G_2}$ represents the set of inner nodes and $L_{G_1\cup G_2}$ the set of leaves. The



(a) $I_{S_1} \wedge I_{S_2}$ (b) Evaluation $E(I_{S_1} \wedge I_{S_2})$ of the IVFSig, $I_{S_1} \wedge I_{S_2}$ in Example 5.1 (c) $E(I_{S_1} \wedge I_{S_2}) = [0.5, 0.82]$

Fig. 19 Evaluation $E(I_{S_1} \wedge I_{S_2})$ of the IVFSig $I_{S_1} \wedge I_{S_2}$ in Example 5.1

following rules will determine the M_F intervals allocated to each leaf in $L_{G_1 \cup G_2}$ and the aggregation operators applied to each inner node in $N_{G_1 \cup G_2}$:

- If $N_{G_1 \cup G_2}$, the aggregation operator allocated to v is determined by taking into account the following cases:
  - (i) If $v \in N_1$ and $v \in N_2$ are both true, then the aggregation operator assigned to v is: $\psi_v = \sup_{\sqsubseteq} \{\psi_v^1, \psi_v^2\}$, where ψ_v^1 is the aggregation operator allotted to $v \in N_1$ in I_{S_1} and ψ_v^2 is the aggregation operator given to $v \in N_2$ in I_{S_2}.
  - (ii) If $v \in N_1$ and $v \in L_2$ are true, the aggregation operator for v is $\psi_v = \psi_v^1$, where ψ_v^1 is the aggregation operator for $v \in N_1$.
  - (iii) Symmetrically, if $v \in L_1$ and $v \in N_2$ are true, the aggregation operator for v is $\psi_v = \psi_v^2$, where ψ_v^2 is the aggregation operator for $v \in N_2$.
  - (iv) If $v \in N_1$ and $v \notin V_2$, the aggregation operator for v is $\psi_v = \psi_v^1$, where ψ_v^1 is the aggregation operator allocated to $v \in N_1$.
  - (v) If $v \notin N_1$ and $v \in V_2$, the aggregation operator for v is $\psi_v = \psi_v^2$, where ψ_v^2 is the aggregation operator allocated to $v \in N_2$.
- If $v \in L_{G_1 \cup G_2}$, the M_F degree given to v is determined taking into account the following scenarios:
  - (i) If $v \in L_1$ and $v \in L_2$ are used, the M_F degree assigned to v is $[\mu_v^-, \mu_v^+] = \sup\{[\mu_v^{1-}, \mu_v^{1+}], [\mu_v^{2-}, \mu_v^{2+}]\}$, where $[\mu_v^{1-}, \mu_v^{1+}]$ is the M_F degree $v \in L_1$ and $[\mu_v^{2-}, \mu_v^{2+}]$ is the M_F degree $v \in L_2$.
  - (ii) If $v \in L_1$ and $v \notin V_2$, $[\mu_v^-, \mu_v^+] = [\mu_v^{1-}, \mu_v^{1+}]$, is the M_F degree granted to v, where μ_v^1 is the M_F degree granted to $v \in L_1$.
  - (iii) If $v \notin V_1$ and $v \in L_2$, $[\mu_v^-, \mu_v^+] = [\mu_v^{2-}, \mu_v^{2+}]$, is the M_F degree granted to v, where $[\mu_v^{2-}, \mu_v^{2+}]$ is the M_F degree granted to $v \in L_2$.

Example 5.2 In the setting of Example 5.1, we will calculate the join of the *IVFSig*'s I_{S_1} and I_{S_2}, which are associated with the rooted trees G_1 and G_2 shown in Figs. 16 and 17, respectively. By applying Definition 5.4, we can see that the union of G_1 and G_2 is the rooted tree corresponding to the *IVFSig*'s $I_{S_1} \vee I_{S_2}$, as shown in Fig. 20. We can determine the set of inner nodes $N_{G_1 \cup G_2} = \{v_0, v_1, v_2, v_3\}$ and the set of leaves $L_{G_1 \cup G_2} = \{v_{11}, v_{13}, v_{21}, v_{22}, v_{32}\}$. The membership degrees assigned to each leaf and the aggregation operators assigned to each inner node will be determined by using the rules in Definition 5.4. With respect to the vertices in $L_{G_1 \cup G_2}$, we can distinguish the following cases:

- $v_{11} \in L_1$ and $v_{11} \in L_2$, then v_{11}'s M_F degree is $[\mu_{v_{11}}^-, \mu_{v_{11}}^+] = \sup\{[0.4, 0.69], [0.3, 0.6]\} = [0.4, 0.69]$.
- $v_{13} \in L_1$ and $v_{13} \notin v_2$, then v_{13}'s M_F degree is $[\mu_{v_{13}}^-, \mu_{v_{13}}^+] = [0.1, 0.92]$.
- $v_{21} \in L_1$ and $v_{21} \notin v_2$, then v_{21}'s M_F degree is $[\mu_{v_{21}}^-, \mu_{v_{21}}^+] = [0.2, 0.78]$.
- $v_{22} \notin v_1$ and $v_{22} \in L_2$, then v_{22}'s M_F degree is $[\mu_{v_{22}}^-, \mu_{v_{22}}^+] = [0.6, 0.75]$.
- $v_{32} \notin v_1$ and $v_{32} \in L_2$, then v_{32}'s M_F degree is $[\mu_{v_{32}}^-, \mu_{v_{32}}^+] = [0.7, 0.9]$.

Considering the ordering relation defined on $\mathcal{A}_3(G)$ and $\mathcal{A}_3(P)$ by means of the complete lattice shown in Fig. 5, we can assign aggregation operators to the vertices in $N_{G_1 \cap G_2}$:

- $v_0 \in N_1$ and $v_0 \in N_2$, then v_0's aggregation operator is $\psi_{v_0} = \sup_{\sqsubseteq}\{\psi_{v_0}^1, \psi_{v_0}^2\} = \psi_{v_0}^1 = \psi_{v_0}^2$.
- $v_1 \in N_1$ and $v_1 \in N_2$, then v_1's aggregation operator is $\psi_{v_1} = \sup_{\sqsubseteq}\{\psi_{v_1}^1, \psi_{v_1}^2\} = \psi_{v_1}^2$.
- $v_2 \in N_1$ and $v_2 \in N_2$, then v_2's aggregation operator is $\psi_{v_2} = \sup_{\sqsubseteq}\{\psi_{v_2}^1, \psi_{v_2}^2\} = \psi_{\sup}$, where $\psi_{\sup} = \sup\{a, b\}$, for all $a, b \in [0, 1]$.
- $v_3 \in L_1$ and $v_3 \in N_2$, then v_3's aggregation operator is $\psi_{v_3} = \psi_{v_3}^2$.

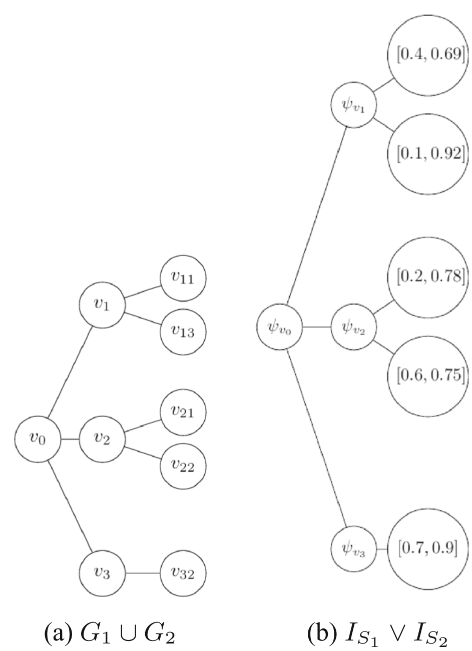


Fig. 20 Rooted tree $G_1 \cup G_2$ and IVFSig $I_{S_1} \vee I_{S_2}$ of Example 5.2

Fig. 20 shows the *IVFSig*'s $I_{S_1} \vee I_{S_2}$, which is represented by the tuple $I_{S_1} \vee I_{S_2} = \langle N_{G_1 \cup G_2}, L_{G_1 \cup G_2}, \{\psi_{v_0}, \psi_{v_1}, \psi_{v_2}, \psi_{v_3}\}$, {[0.4, 69], [0.1, 0.92], [0.2, 0.78], [0.6, 0.75], [0.7, 0.9]}⟩.

The evaluation of this *IVFSig*'s is $E(I_{S_1} \vee I_{S_2})$= $\max\{\psi^2_{v_1}([0.4, 0.69], [0.1, 0.92]), \psi_{\sup}([0.2, 0.78], [0.6, 0.75]), \psi^2_{v_3}([0.7, 0.9])\}$.

$$E(I_{S_1} \vee I_{S_2}) = \max\{[0.4, 0.92], [0.6, 0.78], [0.7, 0.9]\} \\ = [0.7, 0.92].$$

The results demonstrate that the meet and join operators of the family $\mathcal{F}(G_{I_S}, \mathcal{A}_{I_S})$ give rise to a lattice structure as an abstract algebra (Birkhoff 1940).

Theorem 5.1 *Let I_S be an IVFSig for the rooted tree G_{I_S} and the collection of aggregation operator families $\mathcal{A}_{I_S}$. A lattice is formed by the family $\mathcal{F}(G_{I_S}, \mathcal{A}_{I_S})$, the meet operator $\wedge$, and the join operator $\vee$.*

Proposition 5.1 *Let $I_S = \langle N, L, \{\psi_0, \psi_1, \ldots, \psi_n\}, \{[\mu^-_1, \mu^+_1], \ldots, [\mu^-_l, \mu^+_l]\}\rangle$ be an IVFSig, linked to the rooted tree G_{I_S} and the collection of aggregation operator families $\mathcal{A}_{I_S} = \{\mathcal{A}_0, \mathcal{A}_1, \ldots, \mathcal{A}_n\}$. Then, for each $i \in \{1, 2, \ldots, n\}$, the tuple $(\mathcal{F}(G_{I_S}, \mathcal{A}_{I_S}), \wedge, \vee)$ is a bounded lattice, with the least element being the IVFSig $I_{S_\perp} = \langle \phi, \{v_0\}, \phi, \{[0, 0]\}\rangle$ and the biggest element being the IVFSig $I_{S_\top} = \langle N, L, \{\psi^\top_0, \psi^\top_1, \ldots, \psi^\top_n\}, \{[1, 1], \ldots, [1, 1]\}\rangle$, with $\psi^\top_i$ being the greatest element in $\mathcal{A}_i$.*

6 Relations of partial ordering in an IVFSig family

We concentrate on ordering two IVFSigs from the same family after introducing the join and the meet operators. We will now analyze an IVFSig I_S associated with a rooted tree G_{I_S} and a collection of aggregation operator families $\mathcal{A}_{I_S}$. Let $I_{S_1}, I_{S_2} \in \mathcal{F}(G_{I_S}, \mathcal{A}_{I_S})$ be IVFSig's for $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, respectively, where L_1, L_2 are leaf sets and N_1, N_2 are inner node sets of each tree G_1 and G_2. Perhaps it is far natural to assume that the most direct manner to outline the ordering relation, $\preceq$ among I_{S_1} and I_{S_2} is to compare any two IVFSig's only by their evaluations, that is by equivalence: $I_{S_1} \preceq I_{S_2}$ if and only if $E(I_{S_1}) \leq E(I_{S_2})$. In any case, the assessment is not the foremost important feature of each IVFSig. This is often why we attempt to function with the IVFSig's from their graphs. Hence, it appears that a conceivable request based on the evaluations does not make sense when we need to operate with the proper IVFSig's. Without a doubt, this definition isn't compatible with the supremum and infimum operators, as we are going to show next.

Example 6.1 Let $G_{I_S} = (V, E)$ be the rooted tree represented in Fig. 21, where the nodes $V = \{v_0, v_1, v_2, v_{11}, v_{12}, v_{21}, v_{22}, v_{23}\}$, the leaves $L = \{v_{11}, v_{12}, v_{21}, v_{22}, v_{23}\}$, and the inner nodes $N = \{v_0, v_1, v_2\}$ may be identified. Let $\mathcal{A}_{I_S} = \{\mathcal{A}_2(G), \mathcal{A}_3(P)\}$ be a set of aggregation operator families, where $\mathcal{A}_2(G)$ is the aggregation operator family formed from the Godel t-norm and 2 variables, and $\mathcal{A}_3(P)$ is the aggregation operator family formed from the product t-norm and 3 variables. Every inner node in N is assigned the aggregation operators $\psi_0, \psi_1 \in \mathcal{A}_2(G)$ and $\psi_2 \in \mathcal{A}_3(P)$, which are specified as

$$\psi_0(p, q) = \psi_1(p, q) = \min\{p, q\}$$
$$\psi_2(p, q, r) = p * q * r.$$

Figure 21 also shows the IVFSig, $I_S = \langle N, L, \{\psi_0, \psi_1, \psi_2\}, \{[0.2, 0.5], [0.5, 0.7], [0.7, 0.82], [0.5, 0.65], [0.3, 0.78]\}\rangle$.

Let $I_{S_1}, I_{S_2} \in \mathcal{F}(G_{I_S}, \mathcal{A}_{I_S})$ represent the IVFSig's shown in Figs. 22 and 23, which are connected with $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, respectively. These IVFSig's are $I_{S_1} = \langle N_1, L_1, \{\psi^1_{v_0}\}\{[0.61, 0.88], [0.62, 0.98]\}\rangle$ and $I_{S_2} = \langle N_2, L_2, \{\psi^2_{v_0}, \psi^2_{v_1}, \psi^2_{v_2}\}\{[0.8, 0.9], [0.6, 0.8], [0.7, 0.8], [0.5, 0.7]\}\rangle$, where $N_1 = \{v_0\}$, $L_1 = \{v_1, v_2\}$, $N_2 = \{v_0, v_1, v_2\}$, $L_2 = \{v_{11}, v_{12}, v_{21}, v_{22}\}$, and the aggregation operators $\psi^1_{v_0}, \psi^2_{v_0}, \psi^2_{v_1} \in \mathcal{A}_2(G)$, and $\psi^2_{v_2} \in \mathcal{A}_3(P)$, given to each inner node are defined as:

$$\psi^1_{v_0}(p, q) = \psi^2_{v_0}(p, q) = \psi^2_{v_1}(p, q) = \min\{p, q\}$$
$$\psi^2_{v_2}(p, q) = p * q.$$

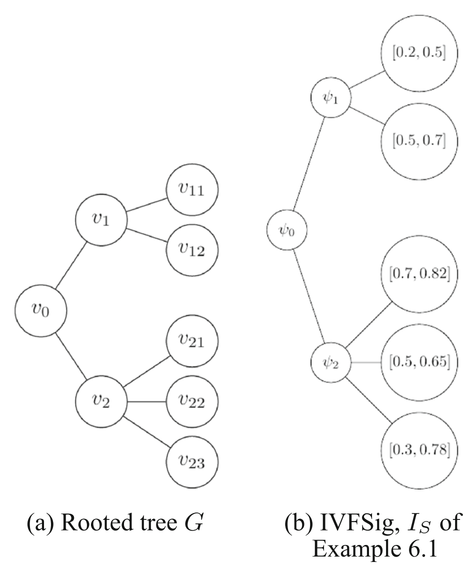


Fig. 21 Rooted tree and its IVFSig

From Definition 5.3, we are able to compute the meet of the IVFSig's I_{S_1} and I_{S_2}, acquiring the IVFSig $I_{S_1} \wedge I_{S_2}$ proven in Fig. 24. Note that $I_{S_1} \wedge I_{S_2} = \langle N_{G_1 \cap G_2}, L_{G_1 \cap G_2}, \{\psi_{v_0}\}\{[0.61, 0.88], [0.62, 0.98]\}\rangle$, wherein the inner node set is $N_{G_1 \cap G_2} = \{v_0\}$, the leaf set $L_{G_1 \cap G_2} = \{v_1, v_2\}$ and the aggregation operator is $\psi_{v_0}(x, y) = \min\{x, y\}$.

With a few basic calculations, we can find that $E(I_{S_1}) = [0.61, 0.88]$, $E(I_{S_2}) = [0.35, 0.56]$, and $E(L_{I_{S_1} \wedge I_{S_2}}(v_0)) = [0.61, 0.88]$. We can confirm that $I_{S_2} \preceq I_{S_1}$ using the ordering relation between IVFSig's given just above this example, because $E(I_{S_2}) = [0.35, 0.56] \;\; \leq E(I_{S_1}) = [0.61, 0.88]$. Therefore, as seen below, the equality $E(I_{S_1} \wedge I_{S_2}) = \inf\{E(I_{S_1}), E(I_{S_2})\}$ is not derived:

$$\begin{aligned} E(I_{S_1} \wedge I_{S_2}) &= [0.61, 0.88] > [0.35, 0.56] \\ &= \inf\{[0.61, 0.88], [0.35, 0.56]\} \\ &= \inf\{E(I_{S_1}), E(I_{S_2})\}. \end{aligned}$$

Because we are interested in ordering $\preceq$ so that $(\mathcal{F}(G_{I_S}, \mathcal{A}_{I_S}), \preceq)$ is a lattice, the natural ordering from the supremum and infimum operators, as described in lattice theory, will be addressed in the following.

Definition 6.1 Consider $I_{S_1}, I_{S_2} \in \mathcal{F}(G_{I_S}, \mathcal{A}_{I_S})$ to be an IVFSig's. From the meet operator, the ordering relation $\preceq$ between I_{S_1} and I_{S_2} is defined as:

$I_{S_1} \preceq I_{S_2}$ if and only if $I_{S_1} \wedge I_{S_2} = I_{S_1}$.

From the join operator, the ordering relation $\preceq$ between I_{S_1} and I_{S_2} is defined as:

$I_{S_1} \preceq I_{S_2}$

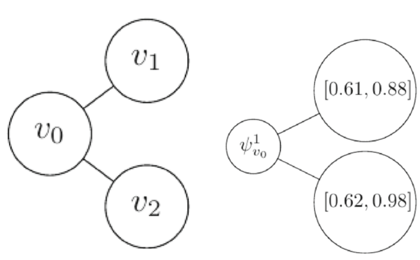
if and only if $I_{S_1} \vee I_{S_2} = I_{S_2}$.

As a result, this is the ordering that permits the lattice of the $\mathcal{F}(G_{I_S}, \mathcal{A}_{I_S})$ family to be seen as an ordered set in a clear manner. The following proposition gives a description of this ordering relation.

Proposition 6.1 *Consider $I_{S_1}, I_{S_2} \in \mathcal{F}(G_{I_S}, \mathcal{A}_{I_S})$ to be IVFSig's connected with the rooted trees, $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, respectively, where L_1 and L_2 are the leaf sets and N_1, N_2 are the inner node sets of each tree, G_1 and G_2. Therefore, if $I_{S_1} \preceq I_{S_2}$, if and only if the following assertions are true:*

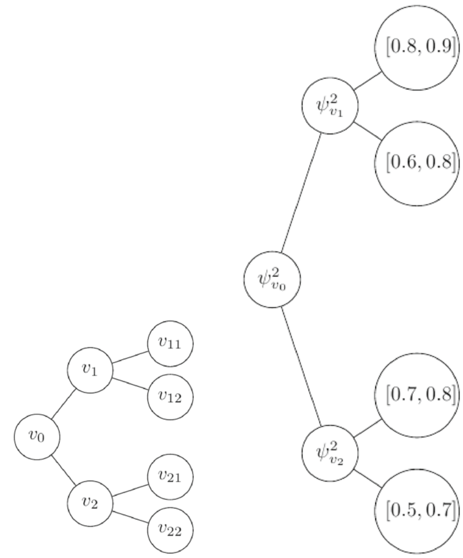
(a) $G_1 \subseteq G_2$.
(b) If $v \in N_1$ is present, then $\psi_v^1 \sqsubseteq \psi_v^2$, where ψ_v^1 represents the aggregation operator allocated to $v \in N_1$ in I_{S_1} and ψ_v^2 represents the aggregation operator allocated to $v \in N_2$ in I_{S_2}.
(c) If $v \in L_1 \cap L_2$, then $[\mu_v^{1-}, \mu_v^{1+}] \leq [\mu_v^{2-}, \mu_v^{2+}]$, where $[\mu_v^{1-}, \mu_v^{1+}]$ represents the M_F degree allotted to $v \in L_1$ in I_{S_1} and $[\mu_v^{2-}, \mu_v^{2+}]$ represents the M_F degree allotted to $v \in L_2$ in I_{S_2}.

Proof To begin, we will assume that $I_{S_1} \preceq I_{S_2}$ and demonstrate that the propositions (a), (b), and (c) are satisfied:



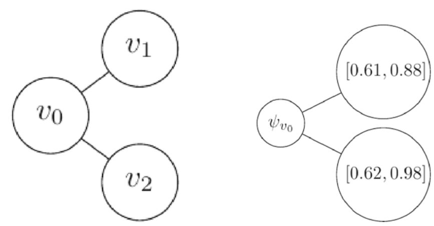
(a) Rooted tree G_1 (b) IVFSig, I_{S_1} of Example 6.1

Fig. 22 Rooted tree and its IVFSig I_{S_1}



(a) Rooted tree G_2 (b) IVFSig, I_{S_2} of Example 6.1

Fig. 23 Rooted tree and its IVFSig I_{S_2}



(a) Rooted tree $G_1 \cap G_2$ (b) $I_{S_1} \wedge I_{S_2}$

Fig. 24 Rooted tree $G_1 \cap G_2$ and IVFSig $I_{S_1} \wedge I_{S_2}$ of Example 6.1

(a) As for $I_{S_1} \preceq I_{S_2}$, we have $I_{S_1} \wedge I_{S_2} = I_{S_1}$. As a result, the rooted tree $G_1 \cap G_2$ relating to the IVFSig $I_{S_1} \wedge I_{S_2}$ is equivalent to the rooted tree G_1 relating to the IVFSig I_{S_1}, that is, $G_1 \cap G_2 = G_1$. This brings us to $G_1 \subseteq G_2$.

(b) Because $I_{S_1} \wedge I_{S_2} = I_{S_1}$, if $v \in N_1$ is true, then $v \in N_2$ and $v \in N_{G_1 \cap G_2}$ are true. According to Definition 5.3, the aggregation operator ψ_v allocated to $v \in N_{G_1 \cap G_2}$ in $I_{S_1} \wedge I_{S_2}$ is $\psi_v = \inf\{\psi_v^1, \psi_v^2\}$, ψ_v^1 is the aggregation operator allocated to $v \in N_1$ in I_{S_1} and ψ_v^2 is the aggregation operator given to $v \in N_2$ in I_{S_2}.

Taking into consideration that $I_{S_1} \wedge I_{S_2} = I_{S_1}$, we find that $\psi_v = \inf_{\sqsubseteq}\{\psi_v^1, \psi_v^2\} = \psi_v^1$, and so, because the family $(\mathcal{A}_v, \sqsubseteq)$ associated with v is a lattice, the inequality $\psi_v^1 \sqsubseteq \psi_v^2$ holds.

(c) If $v \in L_1 \cap L_2$, then $v \in L_{G_1 \cap G_2}$ follows. According to Definition 5.3, the M_F degree $[\mu_v^-, \mu_v^+]$ allocated to $v \in L_{G_1 \cap G_2}$ in $I_{S_1} \wedge I_{S_2}$ is $[\mu_v^-, \mu_v^+] = \inf\ \{[\mu_v^{1-}, \mu_v^{1+}], [\mu_v^{2-}, \mu_v^{2+}]\}$, $[\mu_v^{1-}, \mu_v^{1+}]$ is the M_F degree allocated to $v \in L_1$ in I_{S_1} and $[\mu_v^{2-}, \mu_v^{2+}]$ is the M_F degree allocated to $v \in L_2$ in I_{S_2}. Taking into mind that $I_{S_1} \wedge I_{S_2} = I_{S_1}$, we have $[\mu_v^-, \mu_v^+] = \inf\{[\mu_v^{1-}, \mu_v^{1+}], [\mu_v^{2-}, \mu_v^{2+}]\} = [\mu_v^{1-}, \mu_v^{1+}]$ and, as a result, the inequality $[\mu_v^{1-}, \mu_v^{1+}] \leq [\mu_v^{2-}, \mu_v^{2+}]$ holds.

We shall now describe the counterpart. Assuming that claims (a), (b), and (c) are accurate, we will demonstrate that $I_{S_1} \preceq I_{S_2}$ is true.

We know from assertion (a) that $G_1 \subseteq G_2$, thus we may be certain that $G_1 \cap G_2 = G_1$. If $v \in N_1$ is true, then $v \in N_{G_1 \cap G_2}$ is true, so $G_1 \cap G_2 = G_1$. Using assertion (b) and Definition 5.3, we can calculate that $\psi_v = \inf_{\sqsubseteq}\{\psi_v^1, \psi_v^2\} = \psi_v^1$. As a result, the aggregation operator ψ_v given to $v \in N_{G_1 \cap G_2}$ in $I_{S_1} \wedge I_{S_2}$ is equivalent to the aggregation operator allocated to $v \in N_1$ in I_{S_1}. If $v \in L_1$ is present, then $v \in N_2$ is present, and finally $v \in L_{G_1 \cap G_2}$ is present. Because $[\mu_v^-, \mu_v^+] = [\mu_v^{1-}, \mu_v^{1+}]$ according to Definition 5.3, then the M_F degree $[\mu_v^-, \mu_v^+]$ allocated to $v \in L_{G_1 \cap G_2}$ in $I_{S_1} \wedge I_{S_2}$ corresponds to the M_F degree allocated to $v \in L_1$ in I_{S_1}.

Both values also correspond in the last instance, that is, if $v \in L_1 \cap L_2$ then $v \in L_{G_1 \cap G_2}$, since $G_1 \cap G_2 = G_1$, and we have $[\mu_v^-, \mu_v^+] = \inf\{[\mu_v^{1-}, \mu_v^{1+}], [\mu_v^{2-}, \mu_v^{2+}]\} = [\mu_v^{1-}, \mu_v^{1+}]$ using statement (c) and Definition 5.3. Taking these factors into account, we can be certain that $I_{S_1} \wedge I_{S_2} = I_{S_1}$ and, as a result, $I_{S_1} \preceq I_{S_2}$. □

The above description is quite obvious. Two IVFSig's, $I_{S_1}, I_{S_2} \in \mathcal{F}(G_{I_S}, \mathcal{A}_{I_S})$, can be evaluated if G_1 is a subtree of the tree G_2. Furthermore, It is possible to verify that I_{S_1} is lesser than or equivalent to I_{S_2}. If the M_F degree allotted to I_{S_1} is lesser than or equivalent to the one designated to the corresponding leaf in I_{S_2} in all coinciding leaves, If the aggregation operators in the inner nodes of I_{S_1} are lesser than or equivalent to the ones designated for the corresponding inner nodes of I_{S_2}. Returning to Example 6.1, it's clear that the IVFSig's in Figs 22 and 23 are comparable. We have $I_{S_1} \leq I_{S_2}$ because Definition 6.1 holds.

7 Conclusion

In this paper, we designed a new framework like IVFSig rely on the IVFS environment. Firstly, we explored some definitions, including leaf and root interval-valued fuzzy sub-signatures. Further, we created a family of IVFSig's associated with a graph and a family of aggregation operators; furthermore, join-and-meet operators, a partial ordering, an ordered set, and a lattice structure were employed to define the family of IVFSig's. Moreover, some of the numerical examples are supplied to compare the proposed method with the existing one. Lastly, these illustrations prove the efficacy of the suggested method.

The authors declare that the join and meet operators as well as partial ordering are enough for the lattice structure verification. In the future, we will explore some other operations at the next level. Likewise, the authors plan to investigate the interval-valued intuitionistic fuzzy signature and its applications.

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Data availability There are no data associated with this study.

Declarations

Conflict of interest There are no conflicts of interest to declare.

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