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ORIGINAL PAPER

Multi-attribute group decision-making based on probabilistic dual hesitant fuzzy Maclaurin symmetric mean operators

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Abstract
In this study, we put forward the dual Maclaurin symmetric mean (DMSM) and the Maclaurin symmetric mean (MSM) operators with the context of the probabilistic dual hesitant fuzzy set (PDHFS), which can address the issues in previous probabilistic dual hesitant fuzzy aggregation operators. Some novel operators based on MSM and DMSM for aggregating PDHF information are prepared, followed by several properties and special cases. Namely, the PDHFMSM, weighted PDHFMSM (WPDHFMSM), PDHFDMSM, weighted PDHFDMSM (WPDHFDMSM) operators. Furthermore, some necessary characteristics and exceptional cases concerning different parametric values of these operators are discussed. Additionally, two new methods based on the WPDHFMSM and WPDHFDMSM operators have been developed with the help of COPRAS technique to deal with multi-attribute group decision-making problems. Lastly, the validity and effectiveness of the intended methods are demonstrated through a case study on selecting the best photovoltaic cells.

Keywords Maclaurin symmetric mean · Dual Maclaurin symmetric mean · Probabilistic hesitant fuzzy set · Multi-criteria group decision-making

1 Introduction

Multi-attribute group decision-making (MAGDM) evolved into a powerful course that assists many decision-makers (DMs) in finding the best possible outcomes. Because of the complexity and haziness of human perception, it is challenging to use accurate values to represent alternative attribute values in MAGDM problems. Many ideas have been developed to solve the issue described above; Zadeh (1965) addresses the uncertain information using fuzzy sets (FSs), which involves the degree of membership to evaluate alternatives and has been studied widely. However, FSs only have a degree of membership, and disclosing more complex and vague information is usually difficult. Many extensions have been made based on these FSs to deal with this situation. For example, Torra & Narukawa (2009) presented the notion of hesitant FSs (HFSs), in which the degree of membership has a collection of several possible values. Zhu et al (2012) extended HFSs to novel Dual HFSs (DHFSs). It allows DMs to deliver multiple-preference values for both the membership and non-membership functions. So DHFSs can help the DMs to capture the original information as much as possible. However, the importance of membership and non-membership degree values, which signify the preference information of DMs, are not considered in above-mentioned FSs. This ignorance will lead to loss of information, and the decisions' consequences will be affected. Thus, it is necessary to describe the importance degree of each value. To address the issue, Hao et al (2017) proposed PDHFSs as a powerful tool to represent incomplete information by embedding the characteristics of DHFSs and their occurrence probability.

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In every decision-making problem, the critical part is the fusion of data from the experts/group. The problem is solved by merging all the input values into a collective value using aggregation operators (Grabisch et al 2009 and Komornikova & Mesiar 2011). For different fuzzy environments, separate aggregation operators were proposed, such as geometric mean, arithmetic mean, Bonferroni mean (2017), power aggregation, MSM, and DMSM (2015) operators, and so forth. Most of them presume no correlation between the multi-input values. Remarkably, the MSM and DMSM operators have a desirable characteristic to seize the interdependence between multiple-preference values (2020). In contrast, the power-mean operators, geometric mean operators, and arithmetic mean operators do not capture the interrelationship between input arguments. Bonferroni mean operators can only reflect the interdependence among two arguments. Thus, many extensions regarding MSM and DMSM operators have been made. For example, Li et al (2016) put forward the MSM operator to handle the HF information, proposed two novel operators, HFMSM and weighted HFMSM operators, and examined its desirable properties. Qin & Liu (2015) designed the intuitionistic fuzzy MSM operators and discussed their monotonicity, commutativity, idempotency, and boundedness. For the linguistic information, Ju et al (2016) put forward the MSM operators for the linguistic environment to fuse information. Zhang (2020) investigates the MSM within the context of dual hesitant fuzzy sets and develops the dual hesitant fuzzy Maclaurin symmetric mean. Recently Darko & Liang (2020) extended the MSM and DMSM operators for the DHF environment in aggregating the fuzzy information and put forward several new operators, namely DHFMSM, weighted DHFMSM (WDHFMSM), DHFDMSM, and weighted DHFDMSM (WDHFDMSM). By utilizing the MSM and DMSM operators, Darko & Liang (2020) also extend the COPRAS method. The analysis clearly shows that the MSM and DMSM are valuable tools for analyzing DMs' risk attitudes and accurately depicting the interrelationship among multi-input arguments.

However, some drawbacks in aggregation methods of PDHFSs need to be addressed.

(a) The existing methods (operators) do not examine the association of the multiple-preference values throughout the process of aggregation. PDHF weighted averaging (PDHFWA) operator presented by Hao et al (2017) is based upon the independent postulate that the input values are not associated. Moreover, the PDHF weighted Einstein averaging (PDHFWEA), and the PDHF weighted Einstein geometric (PDHFWEG) operators given by Garg & Kaur (2018) consider only the correlation among the two input arguments. There are situations where multiple-preference values are interrelated in the real world, not just one or two. Consequently, the existing operators in different fuzzy environments cannot handle the issue.

(b) Furthermore, operators designed for PDHFSs do not have an internal mechanism for modelling DMs' risk behaviour. It is vital because the DMs' risky attitude greatly affects the decision's outcome.

The analysis, as mentioned earlier, clearly depicted that many studies on PDHFSs cannot capture the interrelationship of multiple-preference values throughout the aggregation process. Hence, there is a dire need to find solutions to the problems given below.

1. How can DMs model the interdependence between multiple-preference values throughout decision-making under a PDHF environment?
2. How can the risk attitude of DMs be considered while aggregating PDHF information?

In the light of the MSM and DMSM characteristics, we put these operators into the PDHF environment and originated some novel operators, namely PDHFMSM, PDHFDMSM, weighted PDHFMSM (WPDHFMSM), and weighted PDHFDMSM (WPDHFDMSM).

Many fuzzy decision-making techniques like ELECTRE (2019), TOPSIS (2013), WASPAS (2019) and VIKOR (2020), MABAC, VIKOR (2022), complex proportional assessment COPRAS (1994) and analytical hierarchy process (AHP) have been established to address MAGDM under different fuzzy environments. The purpose of these methods is to provide DMs with an appropriate way to rank the desired alternative. COPRAS has recently attracted a great deal of interest among these methods. This method permits DMs to assess the relative significance and utility degree among alternatives involving their weights, multiple criteria, and performance values of the alternatives concerning all the attributes. Chatterjee et al (2011) conducted a comparative analysis among different techniques. They revealed that the COPRAS-based approach is much better than AHP, TOPSIS, and VIKOR because it takes less estimation time and is straightforward. There are many applications in the literature based on the COPRAS technique. For instance, Goswami & Behera (2021) proposed a hybrid method, COPRAS-ARAS, by integrating the COPRAS and additive ratio assessment (ARAS). With the help of integrated COPRAS and ARAS, they solve an industry's material-handling equipment selection problem. Kumari & Mishra (2020) use the COPRAS-based method to select green suppliers based on intuitionistic fuzzy information. Balali et al (2021) proposed a novel mechanism to solve practical risks on human resources threats in natural gas projects using the COPRAS method. Using fuzzy COPRAS, Garg et al (2018) developed MADM based parametric technique for selecting and ranking e-learning websites. Literature analysis reveals that no

COPRAS method exists to resolve the decision-making problems under a PDHF framework. Moreover, the existing methods fail to model the interrelationship of the multiple-input arguments under the PDHF environment. Thus, addressing the above-mentioned issue in the decision analysis requires an operator to capture the interdependence of multiple inputs is necessary.

In the light of this, we intensify the COPRAS approach to impact it to modify the PDHF environment. The main advantages of this approach are as follows.

(a) Due to the balanced assessment, it makes decisions based on two aspects of attributes: cost and benefit.

(b) Its feature to find the utility degree of the alternatives makes it easy to ascertain the distance between each alternative and select the optimal one. By using the capacity of COPRAS, MSM, and DMSM, this article aims to resolve the shortcomings of the DHFS by presenting two new MAGDM approaches. We proposed the COPRAS approaches by employing the WPDHFMSM and WPDHFDMSM operators to integrate the multiple-preferences values. The presented approach cannot only distinguish optimal alternatives but also solve the interrelation problem between multiple-preferences values in real situations. The main contributions of this article are summarized below.

1. To model the interdependence of the PDHF information, we develop novel operators with the aid of MSM and DMSM.
2. This article reflects the risk attitude of DMs based on the monotonicity regarding parameters in the MSM and DMSM operators.
3. To solve MAGDM, this article structures two novel methods utilizing the WPDHFMSM, WPDHFDMSM, and COPRAS.
4. The strengths and weaknesses of the proposed MAGDM framework are realized in the theoretical and numeric sense by comparison with other methods.

Accordingly, the remaining part of the article is arranged as follows. In Sect. 2, fundamental definitions connected to PDHFSs, MSM, and DMSM operators are elaborated. The following section shows the details of the PDHFMSM operator and its weighted structure WPDHFMSM. Section 4 examines PDHFDMSM aggregation operators with their weighted form, WPDHFDMSM, and desirable features. In Sect. 5, stages of the COPRAS method are given in detail to resolve the MAGDM problems with the help of WPDHFMSM and WPDHFDMSM operators. Section 6 applies the developed methods to rank the best photovoltaic cells, including case background, comparison, implementation, and discussion. The last section sketches the ultimate results and possible directions for further study.

2 Preliminaries

We begin this section by reviewing some existing concepts, including DHFSs, PDHFSs, MSM, and DMSM.

Definition 1 (Zhu et al. 2012) For any set X, DHFS D on X is described as

$$D = \{\langle x, h(x), g(x)\rangle \mid x \in X\},$$

where $h(x)$ and $g(x)$ are the degree of membership and non-membership to the set D respectively, having finite set of values in [0, 1]. Also, there is

$$0 \le \gamma, \eta \le 1, 0 \le \gamma^+ + \eta^+ \le 1,$$

where $\gamma \in h(x)$, $\eta \in g(x)$, $\gamma^+ = \max\{\gamma \mid \gamma \in h(x)\}$ and $\eta^+ = \max \eta \mid \eta \in g(x)$ for all $x \in X$. For ease of use, the pair $d(x) = (h(x), g(x))$ is called the DHFE and denoted by $d = (h, g)$ along with the conditions: $\gamma \in h$, $\eta \in g$ and $\gamma, \eta \in [0, 1]$. Apparently, if $g = \phi$ and $h \ne \phi$,then the DHFS turn down to HFS; if h and g have only single element, then DHFS turn down to the IFS.

Definition 2 Consider $d_1 = (h_1, g_1)$ and $d_2 = (h_2, g_2)$ are two DHFEs. The elementary operations of the DHFEs are described as

1. The **complement** of the DHFS:
$$d_1^c = \left\{\cup_{\gamma \in h, \eta \in g}(\eta, \gamma)\right\}$$
2. The $\oplus-$**union** of the DHFS:
$$d_1 \oplus d_2 = \left\{\cup_{\gamma_1 \in h_1, \gamma_2 \in h_2}(\gamma_1 + \gamma_2 - \gamma_1\gamma_2), \cup_{\eta_1 \in g_1, \eta_2 \in g_2}(\eta_1\eta_2)\right\}$$
3. The $\otimes-$**intersection** of the DHFS:
$$d_1 \otimes d_2 = \left\{\cup_{\gamma_1 \in h_1, \gamma_2 \in h_2}(\gamma_1\gamma_2), \cup_{\eta_1 \in g_1, \eta_2 \in g_2}(\eta_1 + \eta_2 - \eta_1\eta_2)\right\}$$
4. $\lambda d_1 = \left\{\cup_{\gamma_1 \in h_1}\left(1 - (1 - \gamma_1)^\lambda\right), \cup_{\eta_1 \in g_1}\left(\eta_1^\lambda\right)\right\}, \lambda \ge 0$
5. $d_1^\lambda = \left\{\cup_{\gamma_1 \in h_1}(\gamma_1)^\lambda, \cup_{\eta_1 \in g_1}\left(1 - (1 - \eta_1)^\lambda\right)\right\}, \lambda \ge 0.$

According to many DMs, preferences usually based on DHFS are not homogeneous which causes problems in decision-making results. Thus, to strengthen the preferences issues in decision-making based on DHFS, Hao et al (2017) introduced the concept of PDHFS described as:

Definition 3 (Hao et al 2017) Suppose X be any set, PDHFS on X can be expressed by an expression

$$D_p = \{\langle x, h(x) \mid p(x), g(x) \mid q(x) : x \in X\rangle\},$$

where $h(x)$ and $g(x)$ are two sets of finite values in [0, 1], symbolize the membership and the non-membership degrees of $x \in X$ to the set D, respectively. $p(x)$ and $q(x)$

are the corresponding probabilities for these two types of degrees and satisfies the following requirements:

1. $\gamma, \eta \in [0,1]$ and $0 \leq \gamma^+ + \eta^+ \leq 1$;
2. $p_i \in [0,1], q_j \in [0,1]$ and $\sum_{i=1}^{\#h} p_i = 1, \sum_{j=1}^{\#g} q_j = 1$,

where $\gamma \in h(x)$, $\eta \in g(x)$, $\gamma^+ = \max\{\gamma \mid \gamma \in h(x)\}$, $\eta^+ = \max \eta \mid \eta \in g(x), p_i \in p(x)$ and $q_i \in q(x)$ for all $x \in X$. $\#h$ and $\#g$ represents the number of elements in $h(x) \mid p(x)$ and $g(x) \mid q(x)$, respectively. For convenience, we call the pair $d_p(x) = (h(x) \mid p(x), g(x) \mid q(x))$ as the PDHFE, represented by $P = (h \mid p, g \mid q)$.

To compare two PDHFEs, Hao et al (2017) proposed the definitions of score and deviation function for PDHFEs.

Definition 4 The score function for the given PDHFE $P = (h|p, g|q)$, is calculated as

$$s(P) = \sum_{\gamma_i \in h}^{\#h} \gamma_i p_i - \sum_{\eta_j \in g}^{\#g} \eta_j q_j.$$

Definition 5 For the given a PDHFE $P = (h|p, g|q)$, deviation function is calculated as

$$\sigma(P) = \left(\sum_{\gamma_i \in h}^{\#h} (\gamma_i - s)^2 p_i + \sum_{\eta_j \in g}^{\#g} (\eta_j - s)^2 q_j \right)^{\frac{1}{2}}.$$

According to the score and deviation function, two PDHFEs, P_1 and P_2, are compared as follows:

1. If $s(P_1) > s(P_2)$, then $P_1 > P_2$ and vice versa.
2. If $s(P_1) = s(P_2)$, and $\sigma(P_1) > \sigma(P_2)$, then $P_1 < P_2$ and vice versa.
3. If $s(P_1) = s(P_2)$, and $\sigma(P_1) = \sigma(P_2)$, then $P_1 = P_2$.

Definition 6 Consider $P = (h|p_h, g|q_g)$, $P_1 = (h_1|p_{h_1}, g_1|q_{g_1})$ and $P_2 = (h_2|p_{h_2}, g_2|q_{g_2})$ are three PDHFEs. The basic operations between the PDHFEs are outlined as below:

1. The **complement** of the DHFS:
$$P^c = \left\{ \cup_{\gamma \in h, \eta \in g} (\eta \mid q_\eta, \gamma \mid p_\gamma) \right\}$$
2. The $\oplus-$ **union** of the DHFS:
$$P_1 \oplus P_2 = \left\{ \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2} (\gamma_1 + \gamma_2 - \gamma_1 \gamma_2) \mid p_{\gamma_1} p_{\gamma_2}, \cup_{\eta_1 \in g_1, \eta_2 \in g_2} (\eta_1 \eta_2) \mid q_{\eta_1} q_{\eta_2} \right\}$$
3. The $\otimes-$ **intersection** of the DHFS:
$$P_1 \otimes P_2 = \left\{ \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2} (\gamma_1 \gamma_2) \mid p_{\gamma_1} p_{\gamma_2}, \cup_{\eta_1 \in g_1, \eta_2 \in g_2} (\eta_1 + \eta_2 - \eta_1 \eta_2) \mid q_{\eta_1} q_{\eta_2} \right\}$$
4. $\lambda P = \left\{ \cup_{\gamma \in h} \left(1 - (1-\gamma)^\lambda\right) \mid p_\gamma, \cup_{\eta \in g} \left(\eta^\lambda\right) \mid q_\eta \right\}, \lambda \geq 0$
5. $P^\lambda = \left\{ \cup_{\gamma \in h} (\gamma)^\lambda \mid p_\gamma, \cup_{\eta \in g} \left(1 - (1-\eta)^\lambda\right) \mid q_\eta \right\}, \lambda \geq 0$

2.1 Maclaurin symmetric mean operator

The MSM operator was initially presented by Maclaurin Maclaurin (1729), a traditional mean type operator utilized in the theory of information fusion and appropriate to integrate numeric values. It can reflect the interdependence of multiple-input arguments, as described below:

Definition 7 The MSM operator is the relation for a set of numbers $a_j (j = 1, 2, 3, ..., n)$ defined as

$$MSM^{(m)}(a_1, a_2, a_3, \ldots, a_n) = \left(\frac{\sum_{1 \leq i_1 \leq i_2 \leq \ldots \leq i_m \leq n} \prod_{j=1}^m a_{i_j}}{\binom{n}{m}} \right)^{\frac{1}{m}},$$

where $(i_1, i_2, ..., i_m)$ traverses all the k-tuple combination of $(1, 2, 3, ..., n), m = 1, 2, 3, ...n$ and $\binom{n}{m}$ denotes the binomial coefficient (BC). The useful features of the MSM operator are given below:

1. $MSM^{(m)}(0, 0, 0, ..., 0) = 0$;
2. $MSM^{(m)}(a, a, a, ..., a) = a$;
3. If $a_j \leq b_j$, then $MSM^{(m)}(a_1, a_2, a_3, ..., a_n) \leq MSM^{(m)}(b_1, b_2, b_3, ..., b_n)$ for all j;
4. $\min_j\{a_j\} \leq MSM^{(m)}(a_1, a_2, a_3, ..., a_n) \leq \max_j\{a_j\}$.

2.2 Dual Maclaurin symmetric mean operator

Qin & Liu (2015) introduced the DMSM operator, which is the result of the combination of geometric mean and the MSM, defined as follows:

Definition 8 (2015) Consider a set of positive numbers $a_j (j = 1, 2, 3, ..., n)$. The DMSM operator is defined by the relation

$$DMSM^{(m)}(a_1, a_2, a_3, ..., a_n) = \frac{1}{m} \left(\left(\prod_{1 \leq i_1 \leq i_2 \leq \ldots \leq i_m \leq n} \sum_{j=1}^k a_{i_j} \right)^{\frac{1}{\binom{n}{m}}} \right),$$

where $(i_1, i_2, \ldots, i_m)$ traverses all the k-tuple combination of $(1,2,3,\ldots,n), m = 1,2,3,\ldots n$ and $\binom{n}{m}$ denotes the BC. The following are the obvious properties of the MSM operator:

1. $DMSM^{(m)}(0,0,0,\ldots,0) = 0$;
2. $DMSM^{(m)}(a,a,a,\ldots,a) = a$;
3. If $a_j \le b_j$, then

   $$DMSM^{(m)}(a_1,a_2,a_3,\ldots,a_n) \le DMSM^{(m)}(b_1,b_2,b_3,\ldots,b_n)$$

   for all j;
4. $\min_j\{a_j\} \le DMSM^{(m)}(a_1,a_2,a_3,\ldots,a_n) \le \max_j\{a_j\}$.

Due to the ability to reflect the interrelationship among multiple-preference values, MSM and DMSM operators have received more attention from researchers in many fields. The emergence of different extended forms of MSM and DMSM operators enables DMs to express their evaluation information about alternatives more comprehensively. For instance, by utilizing these operators, Darko & Liang (2020) extended the COPRAS method and proposed two novel ways, DHFMSM-COPARS and DHFDMSM-COPRAS, to rank the alternatives. Wei & Lu (2018) put forward the MSM operator for the Pythagorean fuzzy environment. Moreover, Feng et al (2019) studied 2-tuple linguistic FSs in view of MSM operators. Besides, Liu & Li (2019) studied a MADM method based on a generalized MSM operator for probabilistic linguistic information.

3 Probabilistic dual hesitant fuzzy Maclaurin symmetric mean

There exist many circumstances where the preferences are presented by PDHF information. From now on, we incorporate the MSM operator to accommodate the PDHF information and accordingly propose new operators, investigating its properties and operational laws in this section. The details and specific contents are described below:

3.1 PDHFMSM

Definition 9 Given $P_j (j = 1,2,3,\ldots,n)$ be a set of PDHFEs, then the PDHFMSM operator is defined as follows:

$$PDHFMSM^{(m)}(P_1,P_2,P_3,\ldots,P_n) = \left(\frac{\sum_{1\le i_1\le i_2\le\ldots\le i_m\le n}\prod_{j=1}^{m} P_{i_j}}{\binom{n}{m}}\right)^{\frac{1}{m}},$$

where $(i_1, i_2, \ldots, i_m)$ traverses all the k-tuple combination of $(1,2,3,\ldots,n), m = 1,2,3,\ldots n$ and $\binom{n}{m}$ denotes the BC.

Based on the operation laws of the PDHFSs given in Definition 6, we develop the following results from the Definition 9.

Proposition 1 *Let* $P_j = \left(h_j|p_{h_j}, g_j|q_{g_j}\right)(j = 1,2,3,\ldots,n)$ *are the group of PDHFEs, where* $m = 1,2,3,\ldots,n$*. On the basis of MSM, we have:*

$$\bigotimes_{j=1}^{m} P_{i_j} = \left\{\cup_{\gamma_{i_j}\in h_j}\prod_{j=1}^{m}\left(\gamma_{i_j}\right) \mid \prod_{j=1}^{m} p_{\gamma_{i_j}}, \cup_{\eta_{i_j}\in g_j}\left(1-\prod_{j=1}^{m}\left(1-\eta_{i_j}\right)\right) \mid \prod_{j=1}^{m} q_{\eta_{i_j}}\right\}.$$

Proof Consider any two PDHFEs $P_j = \left(h_j|p_{h_j}, g_j|q_{g_j}\right)$ and $P_k = \left(h_k|p_{h_k}, g_k|q_{g_k}\right)$.

$$P_{i_j}\otimes P_{i_k} = \left\{\cup_{\gamma_{i_j}\in h_j,\gamma_{i_k}\in h_k}\left(\gamma_{i_j}\gamma_{i_k}\right) \mid p_{\gamma_{i_j}}p_{\gamma_{i_k}}, \cup_{\eta_{i_j}\in g_j,\eta_{i_k}\in g_k}\left(\eta_{i_j}+\eta_{i_k}-\eta_{i_j}\eta_{i_k}\right) \mid q_{\eta_{i_j}}q_{\eta_{i_k}}\right\}$$
$$= \left\{\cup_{\gamma_{i_j}\in h_j,\gamma_{i_k}\in h_k}\left(\gamma_{i_j}\gamma_{i_k}\right) \mid p_{\gamma_{i_j}}p_{\gamma_{i_k}}, \cup_{\eta_{i_j}\in g_j,\eta_{i_k}\in g_k}\left(1-\left(1-\eta_{i_j}\right)\left(1-\eta_{i_k}\right)\right) \mid q_{\eta_{i_j}}q_{\eta_{i_k}}\right\}$$
$$= \left\{\cup_{\gamma_{i_j}\in h_j}\prod_{j=1}^{m}\left(\gamma_{i_j}\right) \mid \prod_{j=1}^{m} p_{\gamma_{i_j}}, \cup_{\eta_{i_j}\in g_j}\left(1-\prod_{j=1}^{m}\left(1-\eta_{i_j}\right)\right) \mid \prod_{j=1}^{m} q_{\eta_{i_j}}\right\}.$$

□

Proposition 2 *Let* $P_j = \left(h_j|p_{h_j}, g_j|q_{g_j}\right)(j = 1,2,3,\ldots,n)$ *are the group of PDHFEs, where* $m = 1,2,3,\ldots,n$*. On the basis of Definition 7, we have*

$$\oplus_{1\le i_1\le i_2\le\ldots\le i_m\le n}\left(\otimes_{j=1}^{m}P_{i_j}\right)=\left\{\begin{array}{l}\cup_{\gamma_{i_j}\in h_j}\left(1-\prod_{1\le i_1\le i_2\le\ldots\le i_m\le n}\left(1-\prod_{j=1}^{m}\left(\gamma_{i_j}\right)\right)\right)\mid\prod_{j=1}^{m}p_{\gamma_{i_j}},\\ \cup_{\eta_{i_j}\in g_j}\left(\prod_{1\le i_1\le i_2\le\ldots\le i_m\le n}\left(1-\prod_{j=1}^{m}\left(1-\eta_{i_j}\right)\right)\right)\mid\prod_{j=1}^{m}q_{\eta_{i_j}}.\end{array}\right\}$$

Proof Consider the two parameters (1) j; and (2) t. By utilizing the above Proposition 1, we have:

$$\otimes_{j=1}^{m}P_{i_j}=\left\{\cup_{\gamma_{i_j}\in h_j}\prod_{j=1}^{m}\left(\gamma_{i_j}\right)\mid\prod_{j=1}^{m}p_{\gamma_{i_j}},\ \cup_{\eta_{i_j}\in g_j}\left(1-\prod_{j=1}^{m}\left(1-\eta_{i_j}\right)\right)\mid\prod_{j=1}^{m}q_{\eta_{i_j}}\right\},$$

$$\otimes_{t=1}^{m}P_{i_t}=\left\{\cup_{\gamma_{i_t}\in h_t}\prod_{t=1}^{m}\left(\gamma_{i_t}\right)\mid\prod_{t=1}^{m}p_{\gamma_{i_t}},\ \cup_{\eta_{i_t}\in g_t}\left(1-\prod_{t=1}^{m}\left(1-\eta_{i_t}\right)\right)\mid\prod_{t=1}^{m}q_{\eta_{i_t}}\right\}.$$

Based on the operation law of PHFEs given in Definition 6, the sum is computed as follows:

$$\begin{aligned}&\left(\otimes_{j=1}^{m}P_{i_j}\right)\oplus\left(\otimes_{t=1}^{m}P_{i_t}\right)\\&=\left(\begin{array}{l}\left\{\cup_{\gamma_{i_j}\in h_j}\prod_{j=1}^{m}\left(\gamma_{i_j}\right)\mid\prod_{j=1}^{m}p_{\gamma_{i_j}},\cup_{\eta_{i_j}\in g_j}\left(1-\prod_{j=1}^{m}\left(1-\eta_{i_j}\right)\right)\mid\prod_{j=1}^{m}q_{\eta_{i_j}}\right\}\\ \oplus\left\{\cup_{\gamma_{i_t}\in h_t}\prod_{t=1}^{m}\left(\gamma_{i_t}\right)\mid\prod_{t=1}^{m}p_{\gamma_{i_t}},\cup_{\eta_{i_t}\in g_t}\left(1-\prod_{t=1}^{m}\left(1-\eta_{i_t}\right)\right)\mid\prod_{t=1}^{m}q_{\eta_{i_t}}\right\}\end{array}\right)\\&=\left(\begin{array}{l}\left\{\cup_{\gamma_{i_j}\in h_j,\gamma_{i_t}\in h_t}\left(\prod_{j=1}^{m}\left(\gamma_{i_j}\right)+\prod_{t=1}^{m}\left(\gamma_{i_t}\right)-\prod_{j=1}^{m}\left(\gamma_{i_j}\right)\prod_{t=1}^{m}\left(\gamma_{i_t}\right)\right)\mid p_{\gamma_{i_j}}p_{\gamma_{i_t}}\right\},\\ \left\{\cup_{\eta_{i_j}\in g_j,\eta_{i_t}\in g_t}\left(1-\prod_{j=1}^{m}\left(1-\eta_{i_j}\right)\right)\left(1-\prod_{t=1}^{m}\left(1-\eta_{i_t}\right)\right)\mid q_{\eta_{i_j}}q_{\eta_{i_t}}\right\}\end{array}\right)\\&=\left(\begin{array}{l}\left\{\cup_{\gamma_{i_j}\in h_j,\gamma_{i_t}\in h_t}\left(1-\left(1-\prod_{j=1}^{m}\left(\gamma_{i_j}\right)\right)\left(1-\prod_{t=1}^{m}\left(\gamma_{i_t}\right)\right)\right)\mid p_{\gamma_{i_j}}p_{\gamma_{i_t}}\right\},\\ \left\{\cup_{\eta_{i_j}\in g_j,\eta_{i_t}\in g_t}\left(1-\prod_{j=1}^{m}\left(1-\eta_{i_j}\right)\right)\left(1-\prod_{t=1}^{m}\left(1-\eta_{i_t}\right)\right)\mid q_{\eta_{i_j}}q_{\eta_{i_t}}\right\}\end{array}\right)\\&=\left(\begin{array}{l}\left\{\cup_{\gamma_{i_j}\in h_j}\left(1-\prod_{1\le i_1\le i_2\le\ldots\le i_m\le n}\left(1-\prod_{j=1}^{m}\left(\gamma_{i_j}\right)\right)\right)\mid\prod_{j=1}^{m}p_{\gamma_{i_j}}\right\},\\ \left\{\cup_{\eta_{i_j}\in g_j}\left(\prod_{1\le i_1\le i_2\le\ldots\le i_m\le n}\left(1-\prod_{j=1}^{m}\left(1-\eta_{i_j}\right)\right)\right)\mid\prod_{j=1}^{m}q_{\eta_{i_j}}\right\}\end{array}\right)\end{aligned}$$

which completes the proof of Proposition. □

Proposition 3 *Let* $P_j=\left(h_j|p_{h_j},g_j|q_{g_j}\right)(j=1,2,3,\ldots,n)$ *are the group of PDHFEs, where* $m=1,2,3,\ldots,n$. *On the basis of Definition 7, we have*

$$\frac{1}{\binom{m}{n}}\left(\oplus_{1\le i_1\le i_2\le\ldots\le i_m\le n}\left(\otimes_{j=1}^{m}P_{i_j}\right)\right)=\left\{\begin{array}{l}\cup_{\gamma_{i_j}\in h_j}\left(1-\prod_{1\le i_1\le i_2\le\ldots\le i_m\le n}\left(1-\prod_{j=1}^{m}\left(\gamma_{i_j}\right)\right)^{\frac{1}{\binom{n}{m}}}\right)\mid\prod_{j=1}^{m}p_{\gamma_{i_j}},\\ \cup_{\eta_{i_j}\in g_j}\left(\prod_{1\le i_1\le i_2\le\ldots\le i_m\le n}\left(1-\prod_{j=1}^{m}\left(1-\eta_{i_j}\right)\right)^{\frac{1}{\binom{n}{m}}}\right)\mid\prod_{j=1}^{m}q_{\eta_{i_j}}\end{array}\right\}.$$

Proof By utilizing the above Proposition 2 and operation of PDHFEs given in Definition 6, we have

$$\begin{aligned}&\frac{1}{\binom{m}{n}}\left(\oplus_{1\le i_1\le i_2\le\ldots\le i_m\le n}\left(\otimes_{j=1}^{m}P_{i_j}\right)\right)\\&=\frac{1}{\binom{m}{n}}\left\{\begin{array}{l}\cup_{\gamma_{i_j}\in h_j}\left(1-\prod_{1\le i_1\le i_2\le\ldots\le i_m\le n}\left(1-\prod_{j=1}^{m}\left(\gamma_{i_j}\right)\right)\right)\mid\prod_{j=1}^{m}p_{\gamma_{i_j}},\\ \cup_{\eta_{i_j}\in g_j}\left(\prod_{1\le i_1\le i_2\le\ldots\le i_m\le n}\left(1-\prod_{j=1}^{m}\left(1-\eta_{i_j}\right)\right)\right)\mid\prod_{j=1}^{m}q_{\eta_{i_j}}\end{array}\right\}.\\&=\left\{\begin{array}{l}\cup_{\gamma_{i_j}\in h_j}\left(1-\prod_{1\le i_1\le i_2\le\ldots\le i_m\le n}\left(1-\prod_{j=1}^{m}\left(\gamma_{i_j}\right)\right)^{\frac{1}{\binom{n}{m}}}\right)\mid\prod_{j=1}^{m}p_{\gamma_{i_j}},\\ \cup_{\eta_{i_j}\in g_j}\left(\prod_{1\le i_1\le i_2\le\ldots\le i_m\le n}\left(1-\prod_{j=1}^{m}\left(1-\eta_{i_j}\right)\right)^{\frac{1}{\binom{n}{m}}}\right)\mid\prod_{j=1}^{m}q_{\eta_{i_j}}.\end{array}\right.\end{aligned}$$

Thus, proof of Proposition is completed. □

Theorem 1 *Let* $P_j=\left(h_j|p_{h_j},g_j|q_{g_j}\right)(j=1,2,3,\ldots,n)$ *are the group of PDHFEs, where* $m=1,2,3,\ldots,n$, *Then the accumulated result using the PDHFMSM operator is also PDHFE, described as*

$$PDHFMSM^{(m)}(P_1, P_2, P_3, ..., P_n) = \left\{ \begin{array}{c} \cup_{\gamma_{i_j} \in h_j} \left(1 - \prod_{1 \le i_1 \le i_2 \le ... \le i_m \le n} \left(1 - \prod_{j=1}^{m} \left(\gamma_{i_j} \right) \right)^{\frac{1}{\binom{n}{m}}} \right)^{\frac{1}{m}} \mid \prod_{j=1}^{m} p_{\gamma_{i_j}}, \\ \cup_{\eta_{i_j} \in g_j} \left(1 - \left(1 - \prod_{1 \le i_1 \le i_2 \le ... \le i_m \le n} \left(1 - \prod_{j=1}^{m} \left(1 - \eta_{i_j} \right) \right)^{\frac{1}{\binom{n}{m}}} \right)^{\frac{1}{m}} \right) \mid \prod_{j=1}^{m} q_{\eta_{i_j}} \end{array} \right\}.$$

Proof Form the Proposition 3 and operational law of PDHFEs given in Definition 6,

$$\left(\frac{1}{\binom{m}{n}} \left(\bigoplus_{1 \le i_1 \le i_2 \le ... \le i_m \le n} \left(\bigotimes_{j=1}^{m} P_{i_j} \right) \right) \right)^{\frac{1}{m}}$$

$$= \left\{ \begin{array}{c} \cup_{\gamma_{i_j} \in h_j} \left(1 - \prod_{1 \le i_1 \le i_2 \le ... \le i_m \le n} \left(1 - \prod_{j=1}^{m} \left(\gamma_{i_j} \right) \right)^{\frac{1}{\binom{n}{m}}} \right) \mid \prod_{j=1}^{m} p_{\gamma_{i_j}}, \\ \cup_{\eta_{i_j} \in g_j} \left(\prod_{1 \le i_1 \le i_2 \le ... \le i_m \le n} \left(1 - \prod_{j=1}^{m} \left(1 - \eta_{i_j} \right) \right)^{\frac{1}{\binom{n}{m}}} \right) \mid \prod_{j=1}^{m} q_{\eta_{i_j}} \end{array} \right\}^{\frac{1}{m}}$$

$$= \left\{ \begin{array}{c} \cup_{\gamma_{i_j} \in h_j} \left(1 - \prod_{1 \le i_1 \le i_2 \le ... \le i_m \le n} \left(1 - \prod_{j=1}^{m} \left(\gamma_{i_j} \right) \right)^{\frac{1}{\binom{n}{m}}} \right)^{\frac{1}{m}} \mid \prod_{j=1}^{m} p_{\gamma_{i_j}}, \\ \cup_{\eta_{i_j} \in g_j} \left(1 - \left(1 - \prod_{1 \le i_1 \le i_2 \le ... \le i_m \le n} \left(1 - \prod_{j=1}^{m} \left(1 - \eta_{i_j} \right) \right)^{\frac{1}{\binom{n}{m}}} \right)^{\frac{1}{m}} \right) \mid \prod_{j=1}^{m} q_{\eta_{i_j}}. \end{array} \right\}$$

□

In view of Theorem 1, few particular remarks with respect to the parameter m are explained below.

Remark 1 When $m = 1$, the PDHFMSM operator turn down to PDHF average operator as follows:

$$PDHFMSM^{(1)}(P_1, P_2, P_3, \ldots, P_n)$$

$$= \left\{ \begin{array}{c} \cup_{\gamma_{i_j} \in h_j} \left(1 - \prod_{1 \le i_1 \le n} \left(1 - \prod_{j=1}^{1} \left(\gamma_{i_j} \right) \right)^{\frac{1}{\binom{n}{1}}} \right)^{\frac{1}{1}} \mid \prod_{j=1}^{1} p_{\gamma_{i_j}}, \\ \cup_{\eta_{i_j} \in g_j} \left(1 - \left(1 - \prod_{1 \le i_1 \le n} \left(1 - \prod_{j=1}^{1} \left(1 - \eta_{i_j} \right) \right)^{\frac{1}{\binom{n}{1}}} \right)^{\frac{1}{1}} \right) \mid \prod_{j=1}^{1} q_{\eta_{i_j}} \end{array} \right\}$$

$$= \left\{ \cup_{\gamma_{i_1} \in h_1} \left(1 - \prod_{1 \le i_1 \le n} \left(1 - \left(\gamma_{i_1} \right) \right)^{\frac{1}{n}} \right) \mid p_{\gamma_{i_1}}, \cup_{\eta_{i_1} \in g_1} \left(\prod_{1 \le i_1 \le n} \left(\left(1 - \left(1 - \eta_{i_1} \right) \right)^{\frac{1}{n}} \right) \right) \mid q_{\eta_{i_1}} \right\}$$

$$= \left\{ \cup_{\gamma_{i_1} \in h_1} \left(1 - \prod_{1 \le i_1 \le n} \left(1 - \left(\gamma_{i_1} \right) \right)^{\frac{1}{n}} \right) \mid p_{\gamma_{i_1}}, \cup_{\eta_{i_1} \in g_1} \left(\prod_{1 \le i_1 \le n} \left(\eta_{i_1} \right)^{\frac{1}{n}} \right) \mid q_{\eta_{i_1}} \right\}.$$

Let $i_1 = i$, Then

$$= \left\{ \cup_{\gamma_i \in h_i} \left(1 - \prod_{i=1}^{n} \left(1 - \left(\gamma_i \right) \right)^{\frac{1}{n}} \right) \mid \prod_{i=1}^{n} p_{\gamma_i}, \cup_{\eta_i \in g_i} \left(\prod_{i=1}^{n} \left(\eta_i \right)^{\frac{1}{n}} \right) \mid \prod_{i=1}^{n} q_{\eta_i} \right\}.$$

Remark 2 When $m = 2$, the PDHFMSM operator turn into the PDHF Bonferroni mean (PDHFBM) operator as follows:

$$PDHFMSM^{(2)}(P_1, P_2, P_3, \ldots, P_n)$$

$$= \left\{ \begin{array}{c} \cup_{\gamma_{i_j} \in h_j} \left(1 - \prod_{1 \le i_1 \le i_2 \le n} \left(1 - \prod_{j=1}^{2} \left(\gamma_{i_j} \right) \right)^{\frac{1}{\binom{n}{2}}} \right)^{\frac{1}{2}} \mid \prod_{j=1}^{2} p_{\gamma_{i_j}}, \\ \cup_{\eta_{i_j} \in g_j} \left(1 - \left(1 - \prod_{1 \le i_1 \le i_2 \le n} \left(1 - \prod_{j=1}^{2} \left(1 - \eta_{i_j} \right) \right)^{\frac{1}{\binom{n}{2}}} \right)^{\frac{1}{2}} \right) \mid \prod_{j=1}^{2} q_{\eta_{i_j}} \end{array} \right\}$$

$$= \left\{ \begin{array}{c} \cup_{\gamma_{i_1} \in h_1, \gamma_{i_2} \in h_2} \left(1 - \prod_{1 \le i_1 \le i_2 \le n} \left(1 - \left(\gamma_{i_1} \gamma_{i_2} \right) \right)^{\frac{2}{n(n-1)}} \right)^{\frac{1}{2}} \mid p_{\gamma_{i_1}} p_{\gamma_{i_2}}, \\ \cup_{\eta_{i_1} \in g_1, \eta_{i_2} \in g_2} \left(1 - \left(1 - \prod_{1 \le i_1 \le i_2 \le n} \left(1 - \left(1 - \eta_{i_1} \right) \left(1 - \eta_{i_2} \right) \right)^{\frac{2}{n(n-1)}} \right)^{\frac{1}{2}} \right) \mid q_{\eta_{i_1}} q_{\eta_{i_2}} \end{array} \right\}$$

$$= \left\{ \begin{array}{c} \cup_{\gamma_{i_1} \in h_1, \gamma_{i_2} \in h_2} \left(1 - \prod_{i_1, i_2 = 1; i_1 \ne i_2}^{n} \left(1 - \left(\gamma_{i_1} \gamma_{i_2} \right) \right)^{\frac{1}{2} \cdot \frac{2}{n(n-1)}} \right)^{\frac{1}{2}} \mid p_{\gamma_{i_1}} p_{\gamma_{i_2}}, \\ \cup_{\eta_{i_1} \in g_1, \eta_{i_2} \in g_2} \left(1 - \left(1 - \prod_{i_1, i_2 = 1; i_1 \ne i_2}^{n} \left(1 - \left(1 - \eta_{i_1} \right) \left(1 - \eta_{i_2} \right) \right)^{\frac{1}{2} \cdot \frac{2}{n(n-1)}} \right)^{\frac{1}{2}} \right) \mid q_{\eta_{i_1}} q_{\eta_{i_2}} \end{array} \right\}$$

$$= \left\{ \begin{array}{c} \cup_{\gamma_{i_1} \in h_1, \gamma_{i_2} \in h_2} \left(1 - \prod_{i_1, i_2 = 1; i_1 \ne i_2}^{n} \left(1 - \left(\gamma_{i_1} \gamma_{i_2} \right) \right)^{\frac{1}{n(n-1)}} \right)^{\frac{1}{2}} \mid p_{\gamma_{i_1}} p_{\gamma_{i_2}}, \\ \cup_{\eta_{i_1} \in g_1, \eta_{i_2} \in g_2} \left(1 - \left(1 - \prod_{i_1, i_2 = 1; i_1 \ne i_2}^{n} \left(1 - \left(1 - \eta_{i_1} \right) \left(1 - \eta_{i_2} \right) \right)^{\frac{1}{n(n-1)}} \right)^{\frac{1}{2}} \right) \mid q_{\eta_{i_1}} q_{\eta_{i_2}} \end{array} \right\}$$

$$= PDHFBM^{1,1}(P_1, P_2, P_3, \ldots, P_n).$$

Remark 3 When $m = n$, the PDHFMSM operator take shape of the PDHF geometric mean (PDHFGM) operator as follows:

$$PDHFMSM^{(n)}(P_1, P_2, P_3, \ldots, P_n)$$

$$= \left\{ \begin{array}{c} \cup_{\gamma_{i_j} \in h_j} \left(1 - \prod_{1 \le i_1 \le i_2 \le \ldots \le n} \left(1 - \prod_{j=1}^{n} \left(\gamma_{i_j} \right) \right)^{\frac{1}{\binom{n}{n}}} \right)^{\frac{1}{n}} \mid \prod_{j=1}^{n} p_{\gamma_{i_j}}, \\ \cup_{\eta_{i_j} \in g_j} \left(1 - \left(1 - \prod_{1 \le i_1 \le i_2 \le \ldots \le n} \left(1 - \prod_{j=1}^{n} \left(1 - \eta_{i_j} \right) \right)^{\frac{1}{\binom{n}{n}}} \right)^{\frac{1}{n}} \right) \mid \prod_{j=1}^{n} q_{\eta_{i_j}} \end{array} \right\}$$

$$= \left\{ \begin{array}{c} \cup_{\gamma_{i_j} \in h_j} \left(1 - \left(1 - \prod_{j=1}^{n} \left(\gamma_{i_j} \right) \right)^{\frac{1}{\binom{n}{n}}} \right)^{\frac{1}{n}} \mid \prod_{j=1}^{n} p_{\gamma_{i_j}}, \\ \cup_{\eta_{i_j} \in g_j} \left(1 - \left(1 - \left(1 - \prod_{j=1}^{n} \left(1 - \eta_{i_j} \right) \right)^{\frac{1}{\binom{n}{n}}} \right)^{\frac{1}{n}} \right) \mid \prod_{j=1}^{n} q_{\eta_{i_j}}. \end{array} \right\}$$

Let $i_j = i$; then

$$= \left\{ \begin{array}{c} \cup_{\gamma_i \in h_i} \left(1 - \left(1 - \prod_{i=1}^{n} (\gamma_i) \right) \right)^{\frac{1}{n}} \mid \prod_{i=1}^{n} p_{\gamma_i}, \\ \cup_{\eta_i \in g_i} \left(1 - \left(1 - \left(1 - \prod_{i=1}^{n} (1 - \eta_i) \right) \right)^{\frac{1}{n}} \right) \mid \prod_{i=1}^{n} q_{\eta_i} \end{array} \right\}$$

$$= \left\{ \cup_{\gamma_i \in h_i} \prod_{i=1}^{n} (\gamma_i)^{\frac{1}{n}} \mid \prod_{i=1}^{n} p_{\gamma_i},\ \cup_{\eta_i \in g_i} \left(1 - \prod_{i=1}^{n} (1 - \eta_i)^{\frac{1}{n}} \right) \mid \prod_{i=1}^{n} q_{\eta_i} \right\}.$$

In the following, we use an example to show the characteristic using the PDHFMSM operator.

Example 1 Let $P_1 = \{(0.4 \mid 1), (0.1 \mid 0.6, 0.3 \mid 0.4)\}, P_2 = \{(0.1 \mid 0.4, 0.2 \mid 0.6), 0.6 \mid \ \ 0.8, 0.7 \mid 0.2\}$, and $P_3 = \{(0.6 \mid 0.8, 0.7 \mid 0.2), (0.4 \mid 1)\}$ are the three PDHFEs. For $m = 2$, the accumulated PDHFEs utilizing the PDHFMSM operator given in the above Theorem 1, is calculated as

$$PDHFMSM^{(2)}(P_1, P_2, P_3)$$

$$= \left\{ \begin{array}{c} \cup_{\gamma_{i_j} \in h_j} \left(1 - \prod_{1 \le i_2 \le 3} \left(1 - \prod_{j=1}^{2} \left(\gamma_{i_j} \right) \right)^{\frac{1}{\binom{3}{2}}} \right)^{\frac{1}{2}} \mid \prod_{j=1}^{2} p_{\gamma_{i_j}}, \\ \cup_{\eta_{i_j} \in g_j} \left(1 - \left(1 - \prod_{1 \le i_2 \le 3} \left(1 - \prod_{j=1}^{2} \left(1 - \eta_{i_j} \right) \right)^{\frac{1}{\binom{3}{2}}} \right)^{\frac{1}{2}} \right) \mid \prod_{j=1}^{2} q_{\eta_{i_j}} \end{array} \right\}$$

$$= \left\{ \begin{array}{c} \left(\begin{array}{l} \left(1-(1-(0.4\times0.1))^{\frac{1}{3}}\times(1-(0.4\times0.6))^{\frac{1}{3}}\times(1-(0.1\times0.6))^{\frac{1}{3}}\right)^{\frac{1}{2}} | 1\times0.4\times0.8, \\ \left(1-(1-(0.4\times0.1))^{\frac{1}{3}}\times(1-(0.4\times0.7))^{\frac{1}{3}}\times(1-(0.1\times0.7))^{\frac{1}{3}}\right)^{\frac{1}{2}} | 1\times0.4\times0.2, \\ \left(1-(1-(0.4\times0.2))^{\frac{1}{3}}\times(1-(0.4\times0.6))^{\frac{1}{3}}\times(1-(0.2\times0.6))^{\frac{1}{3}}\right)^{\frac{1}{2}} | 1\times0.6\times0.8, \\ \left(1-(1-(0.4\times0.2))^{\frac{1}{3}}\times(1-(0.4\times0.7))^{\frac{1}{3}}\times(1-(0.2\times0.7))^{\frac{1}{3}}\right)^{\frac{1}{2}} | 1\times0.6\times0.2, \end{array} \right), \\ \left(\begin{array}{l} 1-\left(1-(1-(1-0.1)(1-0.6))^{\frac{1}{3}}\times(1-(1-0.1)(1-0.4))^{\frac{1}{3}}\times(1-(1-0.6)(1-0.4))^{\frac{1}{3}}\right)^{\frac{1}{2}} | 0.6\times0.8\times1, \\ 1-\left(1-(1-(1-0.1)(1-0.7))^{\frac{1}{3}}\times(1-(1-0.1)(1-0.4))^{\frac{1}{3}}\times(1-(1-0.7)(1-0.4))^{\frac{1}{3}}\right)^{\frac{1}{2}} | 0.6\times0.2\times1, \\ 1-\left(1-(1-(1-0.3)(1-0.6))^{\frac{1}{3}}\times(1-(1-0.3)(1-0.4))^{\frac{1}{3}}\times(1-(1-0.6)(1-0.4))^{\frac{1}{3}}\right)^{\frac{1}{2}} | 0.4\times0.8\times1, \\ 1-\left(1-(1-(1-0.3)(1-0.7))^{\frac{1}{3}}\times(1-(1-0.3)(1-0.4))^{\frac{1}{3}}\times(1-(1-0.7)(1-0.4))^{\frac{1}{3}}\right)^{\frac{1}{2}} | 0.4\times0.2\times1 \end{array} \right) \end{array} \right\}$$

$$= \{(0.3437 \mid 0.32, 0.3701 \mid 0.08, 0.3866 \mid 0.48, 0.4136 \mid 0.12), (0.3732 \mid 0.48, 0.4089 \mid 0.12, 0.4362 \mid 0.32, 0.4724 \mid 0.08)\}.$$

In subsequent, we scrutinize some useful properties of PDHFMSM operator.

Property 1. [**Idem − potency**] Let $P_j = \left(h_j|p_{h_j}, g_j|q_{g_j}\right)(j = 1, 2, 3, ..., n)$ be a collection of PDHFEs. If all P_j are equal, i.e. $P_j = P = \left(h|p_h, g|q_g\right)$ for all j, then

$$PDHFMSM^{(m)}(P_1, P_2, P_3, ..., P_n) = P = \left(h|p_h, g|q_g\right)$$

Property 2. [**Monotonic**] Let $P_j = \left(h_j|p_{h_j}, g_j|q_{g_j}\right)$ and $P_j^{'} = \left(h_j^{'} \mid p_{h_j^{'}}, g_j^{'} \mid q_{g_j^{'}}\right)(j = 1, 2, 3, ..., n)$ be two group of PHFSs. For each element in the P_j and $P_j^{'}$, there are $\gamma_{h_j} \le \gamma_{h_j^{'}}$ and $\eta_{g_j} \le \eta_{g_j^{'}}$ while the probabilities are same, i.e. $p_{h_j} = p_{h_j^{'}}$ and $q_{g_j} = q_{g_j^{'}}$. Then take advantage of PDHFMSM operator,

$$PDHFMSM^{(m)}(P_1, P_2, P_3, ..., P_n) \le PDHFMSM^{(m)}\left(P_1^{'}, P_2^{'}, P_3^{'}, ..., P_n^{'}\right)$$

Property 3. [**Commutative**] Suppose $P_j = \left(h_j \mid p_{h_j}, g_j \mid q_{g_j}\right)(j = 1, 2, 3, ..., n)$ be a set of PDHFEs, and $P_j^{'}$ is any permutation of $P_j(j = 1, 2, 3, ..., n)$, then

$$PDHFMSM^{(m)}(P_1, P_2, P_3, ..., P_n) = PDHFMSM^{(m)}\left(P_1^{'}, P_2^{'}, P_3^{'}, ..., P_n^{'}\right)$$

3.2 Weighted PDHFMSM

From Definition 9, it can easily be observed that the PDHFMSM operator does not consider the weights of the multiple-input arguments. Nevertheless, in many decision-making problems, particularly in MADM, the importance of arguments contribute significantly to the process of aggregation. Next, we address the shortcomings of the PDHFMSM operator and propose the weighted PDHFMSM (WPDHFMSM) operator as follows:

Definition 10 Given $P_j(j=1,2,3,\ldots,n)$ be the set of PDHFEs, then there exist a weight vector $w_j=(w_1,w_2,w_3,\ldots,w_n)^T$ satisfying $w_j>0$ and $\sum_{j=1}^{n} w_j=1$, where w_j represents the importance degree of P_j. Then the WPDHFMSM operator is defined as follows:

$$WPDHFMSM^{(m)}(P_1,P_2,P_3,\ldots,P_n)=\left(\frac{\sum_{1\le i_1\le i_2\le\ldots\le i_m\le n}\prod_{j=1}^{m}\left(w_{i_j}.P_{i_j}\right)}{\binom{n}{m}}\right)^{\frac{1}{m}},$$

where $(i_1,i_2,\ldots,i_m)$ traverses all the k-tuple combination of $(1,2,3,\ldots,n), m=1,2,3,\ldots n$ and $\binom{n}{m}$ denotes the BC.

According to the Definition 10, 6, and Theorem 1, we can set up the following theorem:

Theorem 2 *Given $P_j(j=1,2,3,\ldots,n)$ be the collection of PDHFEs, then there exist a weight vector $w_j=(w_1,w_2,w_3,\ldots,w_n)^T$ satisfying $w_j>0$ and $\sum_{j=1}^{n} w_j=1$, where w_j represents the importance degree of P_j, then the accumulated results by employing the WPDHFMSM is also a PDHFE,and we have*

$$WPDHFMSM^{(m)}(P_1,P_2,P_3,\ldots,P_n)=\left\{\begin{array}{l}\cup_{\gamma_{i_j}\in h_j}\left(1-\prod_{1\le i_1\le i_2\le\ldots\le i_m\le n}\left(1-\prod_{j=1}^{m}\left(1-\left(1-\gamma_{i_j}\right)^{w_{i_j}}\right)\right)^{\frac{1}{\binom{n}{m}}}\right)^{\frac{1}{m}} \mid \prod_{j=1}^{m} p_{\gamma_{i_j}}, \\ \cup_{\eta_{i_j}\in g_j}\left(1-\left(1-\prod_{1\le i_1\le i_2\le\ldots\le i_m\le n}\left(1-\prod_{j=1}^{m}\left(1-\eta_{i_j}^{w_{i_j}}\right)\right)^{\frac{1}{\binom{n}{m}}}\right)^{\frac{1}{m}}\right) \mid \prod_{j=1}^{m} q_{\eta_{i_j}}\end{array}\right\}$$

Proof By utilizing the laws given in the Definition 6, we have

$$w_{i_j}P_{i_j}=w_{i_j}\left\{\cup_{\gamma_{i_j}\in h_j}\left(\gamma_{i_j}\mid p_{\gamma_{i_j}}\right),\cup_{\eta_{i_j}\in g_j}\left(\eta_{i_j}\mid q_{\eta_{i_j}}\right)\right\}=\left\{\cup_{\gamma_{i_j}\in h_j}\left(1-\left(1-\gamma_{i_j}\right)^{w_{i_j}}\right)\mid p_{\gamma_{i_j}},\cup_{\eta_{i_j}\in g_j}\left(\eta_{i_j}^{w_{i_j}}\mid q_{\eta_{i_j}}\right)\right\}$$

Now from the sequel of Definition 9 and Theorem 1, we obtain:

$$WPDHFMSM^{(m)}(P_1, P_2, P_3, ..., P_n) = \left(\frac{\sum_{1 \le i_1 \le i_2 \le ... \le i_m \le n} \left(\prod_{j=1}^{m} \left(w_{i_j} . P_{i_j} \right) \right)}{\binom{n}{m}} \right)^{\frac{1}{m}}$$

$$= \left(\frac{\sum_{1 \le i_1 \le i_2 \le ... \le i_m \le n} \prod_{j=1}^{m} \left\{ \cup_{\gamma_{i_j} \in h_j} \left(1 - \left(1 - \gamma_{i_j} \right)^{w_{i_j}} \right) \mid p_{\gamma_{i_j}}, \cup_{\eta_{i_j} \in g_j} \left(\eta_{i_j}^{w_{i_j}} \mid q_{\eta_{i_j}} \right) \right\}}{\binom{n}{m}} \right)^{\frac{1}{m}}$$

$$= \left\{ \begin{array}{l} \cup_{\gamma_{i_j} \in h_j} \left(\left(1 - \prod_{1 \le i_1 \le i_2 \le ... \le i_m \le n} \left(1 - \prod_{j=1}^{m} \left(1 - \left(1 - \gamma_{i_j} \right)^{w_{i_j}} \right) \right) \right)^{\frac{1}{\binom{n}{m}}} \right)^{\frac{1}{m}} \mid \prod_{j=1}^{m} p_{\gamma_{i_j}}, \\ \cup_{\eta_{i_j} \in g_j} \left(1 - \left(1 - \prod_{1 \le i_1 \le i_2 \le ... \le i_m \le n} \left(1 - \prod_{j=1}^{m} \left(1 - \eta_{i_j}^{w_{i_j}} \right) \right)^{\frac{1}{\binom{n}{m}}} \right)^{\frac{1}{m}} \right) \mid \prod_{j=1}^{m} q_{\eta_{i_j}} \end{array} \right\}$$

Hence, proof is completed. □

According to the different values of parameter m, we can figure out the following remarks from Theorem 2:

Remark 4 When $m = 1$, the WPDHFMSM operator come into being the weighted PDHF average operator as follows:

$$WPDHFMSM^{(1)}(P_1, P_2, P_3, ..., P_n)$$

$$= \left\{ \begin{array}{l} \cup_{\gamma_{i_j} \in h_j} \left(1 - \prod_{1 \le i_1 \le n} \left(1 - \prod_{j=1}^{1} \left(1 - \left(1 - \gamma_{i_j} \right)^{w_{i_j}} \right) \right)^{\frac{1}{\binom{n}{1}}} \right)^{\frac{1}{1}} \mid \prod_{j=1}^{1} p_{\gamma_{i_j}}, \\ \cup_{\eta_{i_j} \in g_j} \left(1 - \left(1 - \prod_{1 \le i_1 \le n} \left(1 - \prod_{j=1}^{1} \left(1 - \eta_{i_j}^{w_{i_j}} \right) \right)^{\frac{1}{\binom{n}{1}}} \right)^{\frac{1}{1}} \right) \mid \prod_{j=1}^{1} q_{\eta_{i_j}} \end{array} \right\}$$

$$= \left\{ \cup_{\gamma_{i_j} \in h_j} \left(1 - \prod_{i=1}^{n} \left(1 - \gamma_{i_j} \right)^{w_{i_j}} \right)^{\frac{1}{n}} \mid \prod_{i=1}^{n} p_{\gamma_{i_j}}, \cup_{\eta_{i_j} \in g_j} \prod_{i=1}^{n} \left(\eta_{i_j}^{w_{i_j}} \right)^{\frac{1}{n}} \mid \prod_{i=1}^{n} q_{\eta_{i_j}} \right\}$$

Remark 5 When $m = 2$, the WPDHFMSM operator come into being the weighted PDHF Bonferoni mean (WPDHFBM) operator as follows:

$$WPDHFMSM^{(2)}(P_1,P_2,P_3,\ldots,P_n)$$
$$=\left\{\begin{array}{l}\cup_{\gamma_{i_j}\in h_j}\left(1-\prod_{1\le i_1\le i_2\le n}\left(1-\prod_{j=1}^{2}\left(1-\left(1-\gamma_{i_j}\right)^{w_{i_j}}\right)\right)^{\frac{1}{\binom{n}{2}}}\right)^{\frac{1}{2}} \mid \prod_{j=1}^{2}p_{\gamma_{i_j}},\\ \cup_{\eta_{i_j}\in g_j}\left(1-\left(1-\prod_{1\le i_1\le i_2\le n}\left(1-\prod_{j=1}^{2}\left(1-\eta_{i_j}^{w_{i_j}}\right)\right)^{\frac{1}{\binom{n}{2}}}\right)^{\frac{1}{2}}\right) \mid \prod_{j=1}^{2}q_{\eta_{i_j}}\end{array}\right\}$$
$$=\left\{\begin{array}{l}\cup_{\gamma_{i_1}\in h_1,\gamma_{i_2}\in h_2}\left(1-\prod_{i_1,i_2=1;i_1\ne i_2}^{n}\left(1-\left(1-\left(1-\gamma_{i_1}\right)^{w_{i_1}}\right)\left(1-\left(1-\gamma_{i_2}\right)^{w_{i_2}}\right)\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{2}} \mid p_{\gamma_{i_1}}p_{\gamma_{i_2}},\\ \cup_{\eta_{i_1}\in g_1,\eta_{i_2}\in g_2}\left(1-\left(1-\prod_{i_1,i_2=1;i_1\ne i_2}^{n}\left(1-\left(1-\eta_{i_1}^{w_{i_1}}\right)\left(1-\eta_{i_2}^{w_{i_2}}\right)\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{2}}\right) \mid q_{\eta_{i_1}}q_{\eta_{i_2}}\end{array}\right\}$$

Remark 6 When $m=n$, the WPDHFMSM operator come into being the weighted PDHF geometric mean (WPDHFGM) operator as follows:

$$WPDHFMSM^{(n)}(P_1,P_2,P_3,\ldots,P_n)$$
$$=\left\{\begin{array}{l}\cup_{\gamma_{i_j}\in h_j}\left(1-\prod_{1\le i_1\le i_2\le\ldots\le i_n\le n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\gamma_{i_j}\right)^{w_{i_j}}\right)\right)^{\frac{1}{\binom{n}{n}}}\right)^{\frac{1}{n}} \mid \prod_{j=1}^{n}p_{\gamma_{i_j}},\\ \cup_{\eta_{i_j}\in g_j}\left(1-\left(1-\prod_{1\le i_1\le i_2\le\ldots\le i_n\le n}\left(1-\prod_{j=1}^{n}\left(1-\eta_{i_j}^{w_{i_j}}\right)\right)^{\frac{1}{\binom{n}{n}}}\right)^{\frac{1}{n}}\right) \mid \prod_{j=1}^{n}q_{\eta_{i_j}}\end{array}\right\}$$
$$=\left\{\cup_{\gamma_i\in h_i}\left(\prod_{i=1}^{n}\gamma_i^{w_i}\right) \mid \prod_{i=1}^{n}p_{\gamma_i},\cup_{\eta_i\in gi}\left(1-\prod_{i=1}^{n}(1-\eta_i^{w_i})\right) \mid \prod_{i=1}^{n}q_{\eta_i}\right\}$$

In the following, we use an example to show the characteristic using the WPDHFMSM operator.

Example 2 Let $P_1=\{(0.4\mid 1),(0.1\mid 0.6,0.3\mid 0.4)\},P_2=\{(0.1\mid 0.4,0.2\mid 0.6),0.6\mid 0.8,0.7\mid 0.2\}$ and $P_3=\{(0.6\mid 0.8,0.7\mid 0.2),(0.4\mid 1)\}$ are the three PDHFEs and weight vector for these elements is $w=(0.5,0.2,0.3)^T$. For $m=2$, the accumulated PDHFEs utilizing the WPDHFMSM operator given in Theorem 2, is calculated as

$$
\begin{aligned}
&WPDHFMSM^{(2)}(P_1,P_2,P_3)\\
&=\left\{\begin{array}{l}
\cup_{\gamma_{i_j}\in h_j}\left(\left(1-\prod_{1\le i_2\le 3}\left(1-\prod_{j=1}^{2}\left(1-\left(1-\gamma_{i_j}\right)^{w_{i_j}}\right)\right)^{\frac{1}{\binom{3}{2}}}\right)^{\frac{1}{2}} \mid \prod_{j=1}^{2}p_{\gamma_{i_j}},\right.\\
\left.\cup_{\eta_{i_j}\in g_j}\left(1-\left(1-\prod_{1\le i_2\le 3}\left(1-\prod_{j=1}^{2}\left(1-\eta_{i_j}^{w_{i_j}}\right)\right)^{\frac{1}{\binom{3}{2}}}\right)^{\frac{1}{2}}\right) \mid \prod_{j=1}^{2}q_{\eta_{i_j}}\right)
\end{array}\right\}\\
&=\left\{\begin{array}{c}
\left(\begin{array}{l}
\left(1-\left(1-\left(1-(1-0.4)^{0.5}\right)\times\left(1-(1-0.1)^{0.2}\right)\right)^{\frac{1}{3}}\times\left(1-\left(1-(1-0.4)^{0.5}\right)\times\left(1-(1-0.6)^{0.3}\right)\right)^{\frac{1}{3}}\times\left(1-\left(1-(1-0.1)^{0.2}\right)\times\left(1-(1-0.6)^{0.3}\right)\right)^{\frac{1}{3}}\right)^{\frac{1}{2}} \mid 1.0\times 0.4\times 0.8,\\
\left(1-\left(1-\left(1-(1-0.4)^{0.5}\right)\times\left(1-(1-0.1)^{0.2}\right)\right)^{\frac{1}{3}}\times\left(1-\left(1-(1-0.4)^{0.5}\right)\times\left(1-(1-0.7)^{0.3}\right)\right)^{\frac{1}{3}}\times\left(1-\left(1-(1-0.1)^{0.2}\right)\times\left(1-(1-0.7)^{0.3}\right)\right)^{\frac{1}{3}}\right)^{\frac{1}{2}} \mid 1.0\times 0.4\times 0.2,\\
\left(1-\left(1-\left(1-(1-0.4)^{0.5}\right)\times\left(1-(1-0.2)^{0.2}\right)\right)^{\frac{1}{3}}\times\left(1-\left(1-(1-0.4)^{0.5}\right)\times\left(1-(1-0.6)^{0.3}\right)\right)^{\frac{1}{3}}\times\left(1-\left(1-(1-0.2)^{0.2}\right)\times\left(1-(1-0.6)^{0.3}\right)\right)^{\frac{1}{3}}\right)^{\frac{1}{2}} \mid 1.0\times 0.6\times 0.8,\\
\left(1-\left(1-\left(1-(1-0.4)^{0.5}\right)\times\left(1-(1-0.2)^{0.2}\right)\right)^{\frac{1}{3}}\times\left(1-\left(1-(1-0.4)^{0.5}\right)\times\left(1-(1-0.7)^{0.3}\right)\right)^{\frac{1}{3}}\times\left(1-\left(1-(1-0.2)^{0.2}\right)\times\left(1-(1-0.7)^{0.3}\right)\right)^{\frac{1}{3}}\right)^{\frac{1}{2}} \mid 1.0\times 0.6\times 0.2,
\end{array}\right),\\
\left(\begin{array}{l}
1-\left(1-(1-(1-0.1^{0.5})(1-0.6^{0.2}))^{\frac{1}{3}}\times(1-(1-0.1^{0.5})(1-0.4^{0.3}))^{\frac{1}{3}}\times(1-(1-0.6^{0.2})(1-0.4^{0.3}))^{\frac{1}{3}}\right)^{\frac{1}{2}} \mid 0.6\times 0.8\times 1.0,\\
1-\left(1-(1-(1-0.1^{0.5})(1-0.7^{0.2}))^{\frac{1}{3}}\times(1-(1-0.1^{0.5})(1-0.4^{0.3}))^{\frac{1}{3}}\times(1-(1-0.7^{0.2})(1-0.4^{0.3}))^{\frac{1}{3}}\right)^{\frac{1}{2}} \mid 0.6\times 0.2\times 1.0,\\
1-\left(1-(1-(1-0.3^{0.5})(1-0.6^{0.2}))^{\frac{1}{3}}\times(1-(1-0.3^{0.5})(1-0.4^{0.3}))^{\frac{1}{3}}\times(1-(1-0.6^{0.2})(1-0.4^{0.3}))^{\frac{1}{3}}\right)^{\frac{1}{2}} \mid 0.4\times 0.8\times 1.0,\\
1-\left(1-(1-(1-0.3^{0.5})(1-0.7^{0.2}))^{\frac{1}{3}}\times(1-(1-0.3^{0.5})(1-0.4^{0.3}))^{\frac{1}{3}}\times(1-(1-0.7^{0.2})(1-0.4^{0.3}))^{\frac{1}{3}}\right)^{\frac{1}{2}} \mid 0.4\times 0.2\times 1.0
\end{array}\right)
\end{array}\right\}\\
&=\{(0.1469\mid 0.32,0.1640\mid 0.08,0.1583\mid 0.48,0.1756\mid 0.12),(0.7057\mid 0.48,0.7203\mid 0.12,0.7563\mid 0.32,0.7698\mid 0.08)\}
\end{aligned}
$$

4 Probabilistic dual hesitant fuzzy dual Maclaurin symmetric mean

To manage the situation where the arguments are represented by PDHF information. Hereinafter referred, we incorporate the DMSM, to accommodate the PDHF information and accordingly propose novel aggregation operators, investigating its properties and operation laws in this section. Therefore, we propose the PDHFDMSM and the WPDHFDMSM aggregation operators. The details and specific content are defined as follows:

4.1 PDHFDMSM

Based on Definition 3 and 8, we can develop the PDHFDMSM operator as follows:

Definition 11 Suppose $P_j(j=1,2,3,\ldots,n)$ be a set of PDHFEs, and $m=(1,2,3,\ldots,n)$. If

$$
PDHFDMSM^{(m)}(P_1,P_2,P_3,\ldots,P_n)=\frac{1}{m}\left(\prod_{1\le i_1\le i_2\le\ldots\le i_m\le n}\left(\sum_{j=1}^{m}P_{i_j}\right)^{\frac{1}{\binom{n}{m}}}\right),
$$

where $(i_1,i_2,\ldots,i_m)$ traverses all the k-tuple combination of $(1,2,3,\ldots,n)$, and $\binom{n}{m}$ denotes the BC.

In accordance with operational laws of the PDHFS given in Definition 6, we develop the following results from the Definition 11.

Proposition 4 *Suppose* $P_j=\left(h_j\mid p_{h_j},g_j\mid q_{g_j}\right)(j=1,2,3,\ldots,n)$ *are the group of PDHFEs, where* $m=1,2,3,\ldots,n$. *On the basis of Definition , we have*

$$\bigoplus_{j=1}^{n} P_{i_j} = \left\{ \cup_{\gamma_{i_j} \in h_j} \left(1 - \prod_{j=1}^{m} \left(1 - \gamma_{i_j} \right) \right) \mid \prod_{j=1}^{m} p_{\gamma_{i_j}}, \cup_{\eta_{i_j} \in g_j} \left(\prod_{j=1}^{m} \left(\eta_{i_j} \right) \right) \mid \prod_{j=1}^{m} q_{\eta_{i_j}} \right\}$$

Proof Consider any two PDHFEs $P_j = \left(h_j \mid p_{h_j}, g_j \mid q_{g_j} \right)$ and $P_k = \left(h_k \mid p_{h_k}, g_k \mid q_{g_k} \right)$.

$$\begin{aligned} P_{i_j} \oplus P_{i_k} &= \left\{ \cup_{\gamma_{i_j} \in h_j, \gamma_{i_k} \in h_k} \left(\gamma_{i_j} + \gamma_{i_k} - \gamma_{i_j}\gamma_{i_k} \right) \mid p_{\gamma_{i_j}} p_{\gamma_{i_k}}, \cup_{\eta_{i_j} \in g_j, \eta_{i_k} \in g_k} \left(\eta_{i_j}\eta_{i_k} \right) \mid q_{\eta_{i_j}} q_{\eta_{i_k}} \right\} \\ &= \left\{ \cup_{\gamma_{i_j} \in h_j, \gamma_{i_k} \in h_k} \left(1 - \left(1 - \gamma_{i_j} \right)\left(1 - \gamma_{i_k} \right) \right) \mid p_{\gamma_{i_j}} p_{\gamma_{i_k}}, \cup_{\eta_{i_j} \in g_j, \eta_{i_k} \in g_k} \left(\eta_{i_j}\eta_{i_k} \right) \mid q_{\eta_{i_j}} q_{\eta_{i_k}} \right\} \\ &= \left\{ \cup_{\gamma_{i_j} \in h_j} \prod_{j=1}^{m} \left(1 - \gamma_{i_j} \right) \mid \prod_{j=1}^{m} p_{\gamma_{i_j}}, \cup_{\eta_{i_j} \in g_j} \prod_{j=1}^{m} \left(\eta_{i_j} \right) \mid \prod_{j=1}^{m} q_{\eta_{i_j}} \right\} \end{aligned}$$

Hence, proof of proposition is completed. □

Proposition 5 *Suppose* $P_j = \left(h_j \mid p_{h_j}, g_j \mid q_{g_j} \right)(j = 1, 2, 3, \ldots, n)$ *are the group of PDHFEs, where* $m = 1, 2, 3, \ldots, n$. *On the basis of Definition 11, we have*

$$\left(\bigoplus_{j=1}^{n} P_{i_j} \right)^{\frac{1}{\binom{n}{m}}} = \left\{ \cup_{\gamma_{i_j} \in h_j} \left(1 - \prod_{j=1}^{m} \left(1 - \gamma_{i_j} \right) \right)^{\frac{1}{\binom{n}{m}}} \mid \prod_{j=1}^{m} p_{\gamma_{i_j}}, \cup_{\eta_{i_j} \in g_j} \left(1 - \left(1 - \prod_{j=1}^{m} \left(\eta_{i_j} \right) \right)^{\frac{1}{\binom{n}{m}}} \right) \mid \prod_{j=1}^{m} q_{\eta_{i_j}} \right\}.$$

Proof In consideration of Proposition 4 and the operational laws given in the Definition 6, we get

$$\begin{aligned} &\left(\bigoplus_{j=1}^{n} P_{i_j} \right)^{\frac{1}{\binom{n}{m}}} = \left\{ \cup_{\gamma_{i_j} \in h_j} \left(1 - \prod_{j=1}^{m} \left(1 - \gamma_{i_j} \right) \right) \mid \prod_{j=1}^{m} p_{\gamma_{i_j}}, \cup_{\eta_{i_j} \in g_j} \left(\prod_{j=1}^{m} \left(\eta_{i_j} \right) \right) \mid \prod_{j=1}^{m} q_{\eta_{i_j}} \right\}^{\frac{1}{\binom{n}{m}}} \\ &= \left\{ \cup_{\gamma_{i_j} \in h_j} \left(1 - \prod_{j=1}^{m} \left(1 - \gamma_{i_j} \right) \right)^{\frac{1}{\binom{n}{m}}} \mid \prod_{j=1}^{m} p_{\gamma_{i_j}}, \cup_{\eta_{i_j} \in g_j} \left(1 - \left(1 - \prod_{j=1}^{m} \left(\eta_{i_j} \right) \right)^{\frac{1}{\binom{n}{m}}} \right) \mid \prod_{j=1}^{m} q_{\eta_{i_j}} \right\} \end{aligned}$$

Hence, completed. □

Proposition 6 *Suppose* $P_j = \left(h_j \mid p_{h_j}, g_j \mid q_{g_j} \right)(j = 1, 2, 3, \ldots, n)$ *are the group of PDHFEs, where* $m = 1, 2, 3, \ldots, n$. *On the basis of Definition 11, we have*

$$\bigotimes_{1 \le i_1 \le i_2 \le \ldots \le i_m \le n} \left(\bigoplus_{j=1}^{n} P_{i_j} \right)^{\frac{1}{\binom{n}{m}}} = \left\{ \begin{array}{c} \cup_{\gamma_{i_j} \in h_j} \prod_{1 \le i_1 \le i_2 \le \ldots \le i_m \le n} \left(1 - \prod_{j=1}^{m} \left(1 - \gamma_{i_j} \right) \right)^{\frac{1}{\binom{n}{m}}} \mid \prod_{j=1}^{m} p_{\gamma_{i_j}}, \\ \cup_{\eta_{i_j} \in g_j} \left(1 - \prod_{1 \le i_1 \le i_2 \le \ldots \le i_m \le n} \left(1 - \prod_{j=1}^{m} \left(\eta_{i_j} \right) \right)^{\frac{1}{\binom{n}{m}}} \right) \mid \prod_{j=1}^{m} q_{\eta_{i_j}}. \end{array} \right\}$$

Proof Consider the two parameters (1) j; and (2) t, make use of the Proposition 5, we get

$$\left(\bigoplus_{j=1}^{n} P_{i_j}\right)^{\frac{1}{\binom{n}{m}}} = \left\{ \cup_{\gamma_{i_j} \in h_j} \left(1 - \prod_{j=1}^{m}\left(1-\gamma_{i_j}\right)\right)^{\frac{1}{\binom{n}{m}}} \mid \prod_{j=1}^{m} p_{\gamma_{i_j}}, \cup_{\eta_{i_j} \in g_j} \left(1 - \left(1 - \prod_{j=1}^{m}\left(\eta_{i_j}\right)\right)^{\frac{1}{\binom{n}{m}}}\right) \mid \prod_{j=1}^{m} q_{\eta_{i_j}} \right\}$$

$$\left(\bigoplus_{t=1}^{n} P_{i_t}\right)^{\frac{1}{\binom{n}{m}}} = \left\{ \cup_{\gamma_{i_t} \in h_t} \left(1 - \prod_{t=1}^{m}\left(1-\gamma_{i_t}\right)\right)^{\frac{1}{\binom{n}{m}}} \mid \prod_{t=1}^{m} p_{\gamma_{i_t}}, \cup_{\eta_{i_t} \in g_t} \left(1 - \left(1 - \prod_{t=1}^{m}\left(\eta_{i_t}\right)\right)^{\frac{1}{\binom{n}{m}}}\right) \mid \prod_{t=1}^{m} q_{\eta_{i_t}} \right\}$$

Based on the operation law of PHFEs given in Definition 6, the product is computed as follows:

$$\left(\bigoplus_{j=1}^{n} P_{i_j}\right)^{\frac{1}{\binom{n}{m}}} \otimes \left(\bigoplus_{t=1}^{n} P_{i_t}\right)^{\frac{1}{\binom{n}{m}}}$$

$$= \left(\begin{array}{c} \left\{ \cup_{\gamma_{i_j} \in h_j} \left(1 - \prod_{j=1}^{m}\left(1-\gamma_{i_j}\right)\right)^{\frac{1}{\binom{n}{m}}} \mid \prod_{j=1}^{m} p_{\gamma_{i_j}}, \cup_{\eta_{i_j} \in g_j} \left(1 - \left(1 - \prod_{j=1}^{m}\left(\eta_{i_j}\right)\right)^{\frac{1}{\binom{n}{m}}}\right) \mid \prod_{j=1}^{m} q_{\eta_{i_j}} \right\} \otimes \\ \left\{ \cup_{\gamma_{i_t} \in h_t} \left(1 - \prod_{t=1}^{m}\left(1-\gamma_{i_t}\right)\right)^{\frac{1}{\binom{n}{m}}} \mid \prod_{t=1}^{m} p_{\gamma_{i_t}}, \cup_{\eta_{i_t} \in g_t} \left(1 - \left(1 - \prod_{t=1}^{m}\left(\eta_{i_t}\right)\right)^{\frac{1}{\binom{n}{m}}}\right) \mid \prod_{t=1}^{m} q_{\eta_{i_t}} \right\} \end{array} \right)$$

$$= \left\{ \begin{array}{l} \cup_{\gamma_{i_j} \in h_j, \gamma_{i_t} \in h_t} \left(\left(1 - \prod_{j=1}^{m}\left(1-\gamma_{i_j}\right)\right)^{\frac{1}{\binom{n}{m}}} \left(1 - \prod_{t=1}^{m}\left(1-\gamma_{i_t}\right)\right)^{\frac{1}{\binom{n}{m}}} \right) \mid p_{\gamma_{i_j}} p_{\gamma_{i_t}}, \\ \cup_{\eta_{i_j} \in g_j, \eta_{i_t} \in g_t} \left(1 - \left(1 - \prod_{j=1}^{m}\left(\eta_{i_j}\right)\right)^{\frac{1}{\binom{n}{m}}} \left(1 - \prod_{t=1}^{m}\left(\eta_{i_t}\right)\right)^{\frac{1}{\binom{n}{m}}} \right) \mid q_{\eta_{i_j}} q_{\eta_{i_t}} \end{array} \right\}$$

$$= \left\{ \begin{array}{l} \cup_{\gamma_{i_j} \in h_j} \prod_{1 \le i_1 \le i_2 \le \ldots \le i_m \le n} \left(1 - \prod_{j=1}^{m}\left(1-\gamma_{i_j}\right)\right)^{\frac{1}{\binom{n}{m}}} \mid \prod_{j=1}^{m} p_{\gamma_{i_j}}, \\ \cup_{\eta_{i_j} \in g_j} \left(1 - \prod_{1 \le i_1 \le i_2 \le \ldots \le i_m \le n} \left(1 - \prod_{j=1}^{m}\left(\eta_{i_j}\right)\right)^{\frac{1}{\binom{n}{m}}} \right) \mid \prod_{j=1}^{m} q_{\eta_{i_j}} \end{array} \right\}$$

Hence, ended. □

As stated in operations of PDHFEs given in Definition 6, we can extract a theorem below:

Theorem 3 *Suppose* $P_j = \left(h_j \mid p_{h_j}, g_j \mid q_{g_j}\right)(j = 1, 2, 3, \ldots, n)$ *are the group of PDHFEs, where* $m = 1, 2, 3, \ldots, n$, *Then the accumulated result by using the PDHFDMSM operator is also PDHFE, described as*

$$PDHFDMSM^{(m)}(P_1,P_2,P_3,\ldots,P_n) = \left\{ \begin{array}{c} \cup_{\gamma_{i_j}\in h_j}\left(1-\left(1-\prod_{1\le i_1\le i_2\le\ldots\le i_m\le n}\left(1-\prod_{j=1}^{m}\left(1-\gamma_{i_j}\right)\right)^{\frac{1}{\binom{n}{m}}}\right)^{\frac{1}{m}}\right) \mid \prod_{j=1}^{m} p_{\gamma_{i_j}}, \\ \cup_{\eta_{i_j}\in g_j}\left(1-\prod_{1\le i_1\le i_2\le\ldots\le i_m\le n}\left(1-\prod_{j=1}^{m}\left(\eta_{i_j}\right)\right)^{\frac{1}{\binom{n}{m}}}\right)^{\frac{1}{m}} \mid \prod_{j=1}^{m} q_{\eta_{i_j}} \end{array} \right\}$$

Proof In the light of Proposition 6 and the Definition 6, we get

$$\frac{1}{m}\left(\bigotimes_{1\le i_1\le i_2\le\ldots\le i_m\le n}\left(\bigoplus_{j=1}^{n} P_{i_j}\right)^{\frac{1}{\binom{n}{m}}}\right)$$

$$= \frac{1}{m}\left\{ \begin{array}{c} \cup_{\gamma_{i_j}\in h_j}\prod_{1\le i_1\le i_2\le\ldots\le i_m\le n}\left(1-\prod_{j=1}^{m}\left(1-\gamma_{i_j}\right)\right)^{\frac{1}{\binom{n}{m}}} \mid \prod_{j=1}^{m} p_{\gamma_{i_j}}, \\ \cup_{\eta_{i_j}\in g_j}\left(1-\prod_{1\le i_1\le i_2\le\ldots\le i_m\le n}\left(1-\prod_{j=1}^{m}\left(\eta_{i_j}\right)\right)^{\frac{1}{\binom{n}{m}}}\right) \mid \prod_{j=1}^{m} q_{\eta_{i_j}} \end{array} \right\}$$

$$= \left\{ \begin{array}{c} \cup_{\gamma_{i_j}\in h_j}\left(1-\left(1-\prod_{1\le i_1\le i_2\le\ldots\le i_m\le n}\left(1-\prod_{j=1}^{m}\left(1-\gamma_{i_j}\right)\right)^{\frac{1}{\binom{n}{m}}}\right)^{\frac{1}{m}}\right) \mid \prod_{j=1}^{m} p_{\gamma_{i_j}}, \\ \cup_{\eta_{i_j}\in g_j}\left(1-\prod_{1\le i_1\le i_2\le\ldots\le i_m\le n}\left(1-\prod_{j=1}^{m}\left(\eta_{i_j}\right)\right)^{\frac{1}{\binom{n}{m}}}\right)^{\frac{1}{m}} \mid \prod_{j=1}^{m} q_{\eta_{i_j}} \end{array} \right\}$$

Hence, proved. □

In the light of the Theorem 3, we can find some exceptional remarks of the PDHFDMSM operator centered on different values of the parameter m.

Remark 7 When $m = 1$, the PDHFDMSM operator come into being the PDHF geometric mean operator as follows:

$PDHFDMSM^{(1)}(P_1,P_2,P_3,\ldots,P_n)$

$$=\left\{\begin{array}{c}\cup_{\gamma_{i_j}\in h_j}\left(1-\left(1-\prod_{1\le i_1\le n}\left(1-\prod_{j=1}^{1}\left(1-\gamma_{i_j}\right)\right)^{\frac{1}{\binom{n}{1}}}\right)^{\frac{1}{1}}\right)\mid\prod_{j=1}^{1}p_{\gamma_{i_j}},\\ \cup_{\eta_{i_j}\in g_j}\left(1-\prod_{1\le i_1\le n}\left(1-\prod_{j=1}^{1}\left(\eta_{i_j}\right)\right)^{\frac{1}{\binom{n}{1}}}\right)^{\frac{1}{1}}\mid\prod_{j=1}^{1}q_{\eta_{i_j}}\end{array}\right\}$$

$$=\left\{\cup_{\gamma_{i_1}\in h_1}\left(\prod_{1\le i_1\le n}\left(1-\left(1-\gamma_{i_1}\right)\right)^{\frac{1}{n}}\right)\mid p_{\gamma_{i_1}},\ \cup_{\eta_{i_1}\in g_1}\left(1-\prod_{1\le i_1\le n}\left(1-\left(\eta_{i_1}\right)\right)^{\frac{1}{n}}\right)\mid q_{\eta_{i_1}}\right\}$$

$$=\left\{\cup_{\gamma_{i_1}\in h_1}\left(\prod_{1\le i_1\le n}\left(\gamma_{i_1}\right)^{\frac{1}{n}}\right)\mid p_{\gamma_{i_1}},\ \cup_{\eta_{i_1}\in g_1}\left(1-\prod_{1\le i_1\le n}\left(1-\left(\eta_{i_1}\right)\right)^{\frac{1}{n}}\right)\mid q_{\eta_{i_1}}\right\}$$

Let $i_1=i$, Then

$$=\left\{\cup_{\gamma_i\in h_i}\left(\prod_{i=1}^{n}\left(\gamma_i\right)^{\frac{1}{n}}\right)\mid\prod_{i=1}^{n}p_{\gamma_i},\ \cup_{\eta_i\in g_j}\left(1-\prod_{i=1}^{n}\left(1-\left(\eta_i\right)\right)^{\frac{1}{n}}\right)\mid\prod_{i=1}^{n}q_{\eta_i}\right\}$$

Remark 8 When $m=2$, the PDHFDMSM operator come into being the PDHF geometric Bonferroni mean (PDHFGBM) operator as follows:

$PDHFDMSM^{(2)}(P_1,P_2,P_3,\ldots,P_n)$

$$=\left\{\begin{array}{c}\cup_{\gamma_{i_j}\in h_j}\left(1-\left(1-\prod_{1\le i_1\le i_2\le n}\left(1-\prod_{j=1}^{2}\left(1-\gamma_{i_j}\right)\right)^{\frac{1}{\binom{n}{2}}}\right)^{\frac{1}{2}}\right)\mid\prod_{j=1}^{2}p_{\gamma_{i_j}},\\ \cup_{\eta_{i_j}\in g_j}\left(1-\prod_{1\le i_1\le i_2\le n}\left(1-\prod_{j=1}^{2}\left(\eta_{i_j}\right)\right)^{\frac{1}{\binom{n}{2}}}\right)^{\frac{1}{2}}\mid\prod_{j=1}^{2}q_{\eta_{i_j}}\end{array}\right\}$$

$$=\left\{\begin{array}{c}\cup_{\gamma_{i_1}\in h_1,\gamma_{i_2}\in h_2}\left(1-\left(1-\prod_{1\le i_1\le i_2\le n}\left(1-\left(1-\gamma_{i_1}\right)\left(1-\gamma_{i_2}\right)\right)^{\frac{2}{n(n-1)}}\right)^{\frac{1}{2}}\right)\mid p_{\gamma_{i_1}}p_{\gamma_{i_2}},\\ \cup_{\eta_{i_1}\in g_1,\eta_{i_2}\in g_2}\left(1-\prod_{1\le i_1\le i_2\le n}\left(1-\left(\eta_{i_1}\right)\left(\eta_{i_2}\right)\right)^{\frac{2}{n(n-1)}}\right)^{\frac{1}{2}}\mid q_{\eta_{i_1}}q_{\eta_{i_2}}\end{array}\right\}$$

$$=\left\{\begin{array}{c}\cup_{\gamma_{i_1}\in h_1,\gamma_{i_2}\in h_2}\left(1-\left(1-\left(\prod_{i_1,i_2=1;i_1\ne i_2}\left(1-\left(1-\gamma_{i_1}\right)\left(1-\gamma_{i_2}\right)\right)\right)^{\frac{1}{2}\frac{2}{2n(n-1)}}\right)^{\frac{1}{2}}\right)\mid p_{\gamma_{i_1}}p_{\gamma_{i_2}},\\ \cup_{\eta_{i_1}\in g_1,\eta_{i_2}\in g_2}\left(1-\left(\prod_{i_1,i_2=1;i_1\ne i_2}\left(1-\left(\eta_{i_1}\right)\left(\eta_{i_2}\right)\right)\right)^{\frac{1}{2}\frac{2}{2n(n-1)}}\right)^{\frac{1}{2}}\mid q_{\eta_{i_1}}q_{\eta_{i_2}}\end{array}\right\}$$

$$=\left\{\begin{array}{c}\cup_{\gamma_{i_1}\in h_1,\gamma_{i_2}\in h_2}\left(1-\left(1-\left(\prod_{i_1,i_2=1;i_1\ne i_2}\left(1-\left(1-\gamma_{i_1}\right)\left(1-\gamma_{i_2}\right)\right)\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{2}}\right)\mid p_{\gamma_{i_1}}p_{\gamma_{i_2}},\\ \cup_{\eta_{i_1}\in g_1,\eta_{i_2}\in g_2}\left(1-\left(\prod_{i_1,i_2=1;i_1\ne i_2}\left(1-\left(\eta_{i_1}\right)\left(\eta_{i_2}\right)\right)\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{2}}\mid q_{\eta_{i_1}}q_{\eta_{i_2}}\end{array}\right\}$$

$$=PDHFGBM^{(1,1)}(P_1,P_2,P_3,\ldots,P_n)$$

Remark 9 When $m = n$, the PDHFDMSM operator takes the form of the PDHF mean operator as described below:

$$PDHFDMSM^{(n)}(P_1, P_2, P_3, \ldots, P_n)$$

$$= \left\{ \begin{array}{l} \cup_{\gamma_{i_j} \in h_j} \left(1 - \left(1 - \prod_{1 \le i_1 \le i_2 \le \ldots \le i_n \le n} \left(1 - \prod_{j=1}^{n} \left(1 - \gamma_{i_j} \right) \right)^{\frac{1}{\binom{n}{n}}} \right)^{\frac{1}{n}} \right) \mid \prod_{j=1}^{n} p_{\gamma_{i_j}}, \\ \cup_{\eta_{i_j} \in g_j} \left(1 - \prod_{1 \le i_1 \le i_2 \le \ldots \le i_n \le n} \left(1 - \prod_{j=1}^{n} \left(\eta_{i_j} \right) \right)^{\frac{1}{\binom{n}{n}}} \right)^{\frac{1}{n}} \mid \prod_{j=1}^{n} q_{\eta_{i_j}} \end{array} \right\}$$

$$= \left\{ \begin{array}{l} \cup_{\gamma_{i_j} \in h_j} \left(1 - \left(1 - \left(1 - \prod_{j=1}^{n} \left(1 - \gamma_{i_j} \right) \right)^{\frac{1}{1}} \right)^{\frac{1}{n}} \right) \mid \prod_{j=1}^{n} p_{\gamma_{i_j}}, \\ \cup_{\eta_{i_j} \in g_j} \left(1 - \left(1 - \prod_{j=1}^{n} \left(\eta_{i_j} \right) \right)^{\frac{1}{1}} \right)^{\frac{1}{n}} \mid \prod_{j=1}^{n} q_{\eta_{i_j}} \end{array} \right\}$$

$$= \left\{ \cup_{\gamma_{i_j} \in h_j} \left(1 - \prod_{j=1}^{n} \left(1 - \gamma_{i_j} \right)^{\frac{1}{n}} \right) \mid \prod_{j=1}^{n} p_{\gamma_{i_j}},\ \cup_{\eta_{i_j} \in g_j} \prod_{j=1}^{n} \left(\eta_{i_j} \right)^{\frac{1}{n}} \mid \prod_{j=1}^{n} q_{\eta_{i_j}} \right\}$$

For a better understanding of PDHFDMSM, we provide an example to show the calculation of the proposed operator.

Example 3 Let $P_1 = \{(0.4 \mid 1), (0.1 \mid 0.6, 0.3 \mid 0.4)\}$, $P_2 = \{(0.1 \mid 0.4, 0.2 \mid 0.6), 0.6 \mid 0.8, 0.7 \mid 0.2\}$, and $P_3 = \{(0.6 \mid 0.8, 0.7 \mid 0.2), (0.4 \mid 1)\}$ are the three PDHFEs. For $m = 2$, the accumulated PDHFEs utilizing the PDHFDMSM operator given in the above Theorem 3 is calculated as

$$PDHFDMSM^{(2)}(P_1, P_2, P_3)$$

$$= \left\{ \begin{array}{l} \cup_{\gamma_{i_j} \in h_j} \left(1 - \left(1 - \prod_{1 \le i_1 \le i_2 \le 3} \left(1 - \prod_{j=1}^{2} \left(1 - \gamma_{i_j} \right) \right)^{\frac{1}{\binom{3}{2}}} \right)^{\frac{1}{2}} \right) \mid \prod_{j=1}^{2} p_{\gamma_{i_j}}, \\ \cup_{\eta_{i_j} \in g_j} \left(1 - \prod_{1 \le i_1 \le i_2 \le 3} \left(1 - \prod_{j=1}^{2} \left(\eta_{i_j} \right) \right)^{\frac{1}{\binom{3}{2}}} \right)^{\frac{1}{2}} \mid \prod_{j=1}^{2} q_{\eta_{i_j}} \end{array} \right\}$$

$$= \left\{ \begin{array}{l} \left(\begin{array}{l} 1 - \left(1 - (1 - (1-0.4)(1-0.1))^{\frac{1}{3}} \times (1 - (1-0.4)(1-0.6))^{\frac{1}{3}} \times (1 - (1-0.1)(1-0.6))^{\frac{1}{3}} \right)^{\frac{1}{2}} \mid 1 \times 0.4 \times 0.8, \\ 1 - \left(1 - (1 - (1-0.4)(1-0.1))^{\frac{1}{3}} \times (1 - (1-0.4)(1-0.7))^{\frac{1}{3}} \times (1 - (1-0.1)(1-0.7))^{\frac{1}{3}} \right)^{\frac{1}{2}} \mid 1 \times 0.4 \times 0.2, \\ 1 - \left(1 - (1 - (1-0.4)(1-0.2))^{\frac{1}{3}} \times (1 - (1-0.4)(1-0.6))^{\frac{1}{3}} \times (1 - (1-0.2)(1-0.6))^{\frac{1}{3}} \right)^{\frac{1}{2}} \mid 1 \times 0.6 \times 0.8, \\ 1 - \left(1 - (1 - (1-0.4)(1-0.2))^{\frac{1}{3}} \times (1 - (1-0.4)(1-0.7))^{\frac{1}{3}} \times (1 - (1-0.2)(1-0.7))^{\frac{1}{3}} \right)^{\frac{1}{2}} \mid 1 \times 0.6 \times 0.2 \end{array} \right), \\ \left(\begin{array}{l} \left(1 - (1 - (0.1 \times 0.6))^{\frac{1}{3}} \times (1 - (0.1 \times 0.4))^{\frac{1}{3}} \times (1 - (0.6 \times 0.4))^{\frac{1}{3}} \right)^{\frac{1}{2}} \mid 0.6 \times 0.8 \times 1, \\ \left(1 - (1 - (0.1 \times 0.7))^{\frac{1}{3}} \times (1 - (0.1 \times 0.4))^{\frac{1}{3}} \times (1 - (0.7 \times 0.4))^{\frac{1}{3}} \right)^{\frac{1}{2}} \mid 0.6 \times 0.2 \times 1, \\ \left(1 - (1 - (0.3 \times 0.6))^{\frac{1}{3}} \times (1 - (0.3 \times 0.4))^{\frac{1}{3}} \times (1 - (0.6 \times 0.4))^{\frac{1}{3}} \right)^{\frac{1}{2}} \mid 0.4 \times 0.8 \times 1, \\ \left(1 - (1 - (0.3 \times 0.7))^{\frac{1}{3}} \times (1 - (0.3 \times 0.4))^{\frac{1}{3}} \times (1 - (0.7 \times 0.4))^{\frac{1}{3}} \right)^{\frac{1}{2}} \mid 0.4 \times 0.2 \times 1, \end{array} \right) \end{array} \right\}$$

$$= \{(0.3732 \mid 0.32, 0.4089 \mid 0.08, 0.4045 \mid 0.48, 0.4404 \mid 0.12),\ (0.3437 \mid 0.48, 0.3701 \mid 0.12, 0.4260 \mid 0.32, 0.4539 \mid 0.08)\}$$

In the following, we scrutinize some useful characteristics of the PDHFDMSM operator.

Property 5. [**Idempotent**] Let $P_j = \left(h_j \mid p_{h_j}, g_j \mid q_{g_j}\right)(j = 1,2,3,...,n)$ be a collection of PDHFEs. If all P_j are equal, i.e. $P_j = P = \left(h \mid p_h, g \mid q_g\right)$ for all j, then

$$PDHFDMSM^{(m)}(P_1, P_2, P_3, ..., P_n) = P = \left(h \mid p_h, g \mid q_g\right)$$

Property 6. [**Monotonic**] Let $P_j = \left(h_j \mid p_{h_j}, g_j \mid q_{g_j}\right)$ and $P_j^{'} = \left(h_j^{'} \mid p_{h_j^{'}}, g_j^{'} \mid q_{g_j^{'}}\right)(j = 1,2,3,...,n)$ be two group of PHFSs. For each element in the P_j and $P_j^{'}$, there are $\gamma_{h_j} \le \gamma_{h_j^{'}}$ and $\eta_{g_j} \le \eta_{g_j^{'}}$ while the probabilities are same, i.e. $p_{h_j} = p_{h_j^{'}}$ and $q_{g_j} = q_{g_j^{'}}$. Then take advantage of PDHFMSM operator,

$$PDHFDMSM^{(m)}(P_1, P_2, P_3, ..., P_n) \le PDHFDMSM^{(m)}\left(P_1^{'}, P_2^{'}, P_3^{'}, ..., P_n^{'}\right)$$

Property 7. [**Commutative**] Let $P_j = \left(h_j \mid p_{h_j}, g_j \mid q_{g_j}\right)(j = 1,2,3,...,n)$ be a collection of PDHFEs, and $P_j^{'}$ is any permutation of $P_j(j = 1,2,3,...,n)$, then

$$PDHFDMSM^{(m)}(P_1, P_2, P_3, ..., P_n) = PDHFDMSM^{(m)}\left(P_1^{'}, P_2^{'}, P_3^{'}, ..., P_n^{'}\right)$$

4.2 Weighted PDHFDMSM

From the Definition 11, it can easily be observed that the PDHFMSM operator does not account for the weights of the multiple-input values. Nevertheless, in many decision-making problems, particularly in MADM, the importance of arguments contribute a significant role in the aggregation process. Therefore, we propose the weighted PDHFDMSM (WPDHFDMSM) operator as described below:

Definition 12 Given $P_j(j = 1,2,3,...,n)$ be the collection of PDHFEs, then there exist a weight vector $w_j = (w_1, w_2, w_3, ..., w_n)^T$ satisfying $w_j > 0$ and $\sum_{j=1}^{n} w_j = 1$, where w_j represents the importance degree of P_j. Then the WPDHFDMSM operator is defined as follows:

$$WPDHFDMSM^{(m)}(P_1, P_2, P_3, ..., P_n) = \frac{1}{m}\left(\prod\nolimits_{1 \le i_1 \le i_2 \le ... \le i_m \le n}\left(\sum_{j=1}^{m}\left(P_{i_j}^{w_{i_j}}\right)\right)^{\binom{n}{m}}\right),$$

where $(i_1, i_2, ..., i_m)$ traverses all the k-tuple combination of $(1,2,3,...,n), m = 1,2,3,...n$ and $\binom{n}{m}$ denotes the BC.

According to the consequence of Definition 12, the operational laws given in Definition 6 and Theorem 3, we can set up the following theorem:**Theorem 4** *Given $P_j(j = 1,2,3,...,n)$ be the collection of PDHFEs, then there exist a weight vector $w_j = (w_1, w_2, w_3, ..., w_n)^T$ satisfying $w_j > 0$ and $\sum_{j=1}^{n} w_j = 1$, where w_j represents the importance degree of P_j, then the accumulated results by employing the WPDHFDMSM is also a PDHFE, and we have*

$$WPDHFDMSM^{(m)}(P_1, P_2, P_3, \ldots, P_n) = \left\{ \begin{array}{l} \cup_{\gamma_{i_j} \in h_j} \left(1 - \left(1 - \prod_{1 \le i_1 \le i_2 \le \ldots \le i_m \le n} \left(1 - \prod_{j=1}^{m} \left(1 - \gamma_{i_j}^{w_{i_j}} \right) \right)^{\frac{1}{\binom{n}{m}}} \right)^{\frac{1}{m}} \right) \mid \prod_{j=1}^{m} p_{\gamma_{i_j}}, \\ \cup_{\eta_{i_j} \in g_j} \left(1 - \prod_{1 \le i_1 \le i_2 \le \ldots \le i_m \le n} \left(1 - \prod_{j=1}^{m} \left(1 - \left(1 - \eta_{i_j} \right)^{w_{i_j}} \right) \right)^{\frac{1}{\binom{n}{m}}} \right)^{\frac{1}{m}} \mid \prod_{j=1}^{m} q_{\eta_{i_j}} \end{array} \right\}$$

Proof By utilizing the laws given in the Definition 6, we obtain:

$$\left(P_{i_j}\right)^{w_{i_j}} = \left\{ \cup_{\gamma_{i_j} \in h_j} \gamma_{i_j} \mid p_{\gamma_{i_j}},\ \cup_{\eta_{i_j} \in g_j} \left(\eta_{i_j} \mid q_{\eta_{i_j}} \right) \right\}^{w_{i_j}} = \left\{ \cup_{\gamma_{i_j} \in h_j} \left(\gamma_{i_j}^{w_{i_j}} \right) \mid p_{\gamma_{i_j}},\ \cup_{\eta_{i_j} \in g_j} \left(1 - \left(1 - \eta_{i_j} \right)^{w_{i_j}} \right) \mid q_{\eta_{i_j}} \right\}$$

Now from the sequel of Definition 11 and Theorem 3, we can obtain:

$$WPDHFDMSM^{(m)}(P_1, P_2, P_3, \ldots, P_n) = \frac{1}{m} \left(\prod_{1 \le i_1 \le i_2 \le \ldots \le i_m \le n} \left(\sum_{j=1}^{m} \left(P_{i_j}^{w_{i_j}} \right) \right)^{\binom{n}{m}} \right)$$

$$= \frac{1}{m} \left(\prod_{1 \le i_1 \le i_2 \le \ldots \le i_m \le n} \left(\sum_{j=1}^{m} \left\{ \cup_{\gamma_{i_j} \in h_j} \left(\gamma_{i_j}^{w_{i_j}} \right) \mid p_{\gamma_{i_j}}, \cup_{\eta_{i_j} \in g_j} \left(1 - \left(1 - \eta_{i_j} \right)^{w_{i_j}} \mid q_{\eta_{i_j}} \right) \right\} \right)^{\binom{n}{m}} \right)$$

$$= \left\{ \begin{array}{l} \cup_{\gamma_{i_j} \in h_j} \left(1 - \left(1 - \prod_{1 \le i_1 \le i_2 \le \ldots \le i_m \le n} \left(1 - \prod_{j=1}^{m} \left(1 - \gamma_{i_j}^{w_{i_j}} \right) \right)^{\frac{1}{\binom{n}{m}}} \right)^{\frac{1}{m}} \right) \mid \prod_{j=1}^{m} p_{\gamma_{i_j}}, \\ \cup_{\eta_{i_j} \in g_j} \left(1 - \prod_{1 \le i_1 \le i_2 \le \ldots \le i_m \le n} \left(1 - \prod_{j=1}^{m} \left(1 - \left(1 - \eta_{i_j} \right)^{w_{i_j}} \right) \right)^{\frac{1}{\binom{n}{m}}} \right)^{\frac{1}{m}} \mid \prod_{j=1}^{m} q_{\eta_{i_j}} \end{array} \right\}$$

Hence, completed. □

For different values of parameter m, we can figure out the following remarks from Theorem 4:***Remark 10*** When $m = 1$, the WPDHFMSM operator come into being the WPDHF geometric mean operator as given below:

$$WPDHFDMSM^{(1)}(P_1, P_2, P_3, \ldots, P_n) = \left\{ \begin{array}{l} \cup_{\gamma_{i_j} \in h_j} \left(1 - \left(1 - \prod_{1 \le i_1 \le n} \left(1 - \prod_{j=1}^{1} \left(1 - \gamma_{i_j}^{w_{i_j}} \right) \right)^{\frac{1}{\binom{n}{1}}} \right)^{\frac{1}{1}} \right) \mid \prod_{j=1}^{1} p_{\gamma_{i_j}}, \\ \cup_{\eta_{i_j} \in g_j} \left(1 - \prod_{1 \le i_1 \le n} \left(1 - \prod_{j=1}^{1} \left(1 - \left(1 - \eta_{i_j} \right)^{w_{i_j}} \right) \right)^{\frac{1}{\binom{n}{1}}} \right)^{\frac{1}{1}} \mid \prod_{j=1}^{1} q_{\eta_{i_j}} \end{array} \right\}$$

$$= \left\{ \cup_{\gamma_i \in h_i} \prod_{i=1}^{n} (\gamma_i^{w_i}) \mid \prod_{i=1}^{n} p_{\gamma_i},\ \cup_{\eta_i \in g_i} \left(1 - \prod_{i=1}^{n} (1 - \eta_i)^{w_i} \right) \mid \prod_{i=1}^{n} q_{\eta_i} \right\}.$$

Remark 11 When $m=2$, the WPDHFDMSM operator come into being the WPDHF geometric Bonferroni mean (WPDHFGBM) operator as follows:

$$WPDHFDMSM^{(2)}(P_1,P_2,P_3,\ldots,P_n)=\left\{\begin{array}{l}\cup_{\gamma_{i_j}\in h_j}\left(1-\left(1-\prod_{1\le i_1\le i_2\le n}\left(1-\prod_{j=1}^{2}\left(1-\gamma_{i_j}^{w_{i_j}}\right)\right)^{\frac{1}{\binom{n}{2}}}\right)^{\frac{1}{2}}\right)\mid \prod_{j=1}^{2}p_{\gamma_{i_j}},\\ \cup_{\eta_{i_j}\in g_j}\left(1-\prod_{1\le i_1\le i_2\le n}\left(1-\prod_{j=1}^{2}\left(1-\left(1-\eta_{i_j}\right)^{w_{i_j}}\right)\right)^{\frac{1}{\binom{n}{2}}}\right)^{\frac{1}{2}}\mid \prod_{j=1}^{2}q_{\eta_{i_j}}\end{array}\right\}$$

$$=\left\{\begin{array}{l}\cup_{\gamma_{i_1}\in h_1,\gamma_{i_2}\in h_2}\left(1-\left(1-\prod_{i_1,i_2=1;i_1\ne i_2}\left(1-\left(1-\gamma_{i_1}^{w_{i_1}}\right)\left(1-\gamma_{i_2}^{w_{i_2}}\right)\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{2}}\right)\mid p_{\gamma_{i_1}}p_{\gamma_{i_2}},\\ \cup_{\eta_{i_1}\in g_1,\eta_{i_2}\in g_2}\left(1-\prod_{i_1,i_2=1;i_1\ne i_2}\left(1-\left(1-\left(1-\eta_{i_1}\right)^{w_{i_1}}\right)\left(1-\left(1-\eta_{i_2}\right)^{w_{i_2}}\right)\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{2}}\mid q_{\eta_{i_1}}q_{\eta_{i_2}}\end{array}\right\}$$

Remark 12 When $m=n$, the WPDHFDMSM operator come into being the WPDHF geometric mean operator as follows:

$$WPDHFDMSM^{(n)}(P_1,P_2,P_3,\ldots,P_n)=\left\{\begin{array}{l}\cup_{\gamma_{i_j}\in h_j}\left(1-\left(1-\prod_{1\le i_1\le i_2\le\ldots\le i_n\le n}\left(1-\prod_{j=1}^{n}\left(1-\gamma_{i_j}^{w_{i_j}}\right)\right)^{\frac{1}{\binom{n}{n}}}\right)^{\frac{1}{n}}\right)\mid \prod_{j=1}^{n}p_{\gamma_{i_j}},\\ \cup_{\eta_{i_j}\in g_j}\left(1-\prod_{1\le i_1\le i_2\le\ldots\le i_n\le n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\eta_{i_j}\right)^{w_{i_j}}\right)\right)^{\frac{1}{\binom{n}{n}}}\right)^{\frac{1}{n}}\mid \prod_{j=1}^{n}q_{\eta_{i_j}}\end{array}\right\}$$

$$=\left\{\cup_{\gamma_i\in h_i}\left(1-\prod_{i=1}^{n}(1-\gamma_i)^{w_i}\right)\mid\prod_{i=1}^{n}p_{\gamma_i},\ \cup_{\eta_i\in g_i}\prod_{i=1}^{n}(\eta_i)^{w_i}\mid\prod_{i=1}^{n}q_{\eta_{i_j}}\right\}$$

Now an example is given below to better understand the action of the WPDHFDMSM operator.

Example 4 Let $P_1=\{(0.4\mid 1),(0.1\mid 0.6,0.3\mid 0.4)\}$, $P_2=\{(0.1\mid 0.4,0.2\mid 0.6),0.6\mid\ 0.8,0.7\mid 0.2\}$, and $P_3=\{(0.6\mid 0.8,0.7\mid 0.2),(0.4\mid 1)\}$ are the three PDHFEs and weight vector for these elements is $w=(0.5,0.2,0.3)^T$. For $m=2$, the accumulated PDHFEs utilizing the WPDHFDMSM operator given in Theorem 4, is calculated as

$$
WPDHFDMSM^{(2)}(P_1,P_2,P_3)=\left\{\begin{array}{l}
\cup_{\gamma_{i_j}\in h_j}\left(1-\left(1-\prod_{1\le i_2\le 3}\left(1-\prod_{j=1}^{2}\left(1-\gamma_{i_j}^{w_{i_j}}\right)\right)^{\frac{1}{\binom{3}{2}}}\right)^{\frac{1}{2}}\right)|\prod_{j=1}^{2}p_{\gamma_{i_j}},\\
\cup_{\eta_{i_j}\in g_j}\left(1-\prod_{1\le i_2\le 3}\left(1-\prod_{j=1}^{2}\left(1-\left(1-\eta_{i_j}\right)^{w_{i_j}}\right)\right)^{\frac{1}{\binom{3}{2}}}\right)^{\frac{1}{2}}|\prod_{j=1}^{2}q_{\eta_{i_j}}
\end{array}\right\}
$$

$$
=\left\{\begin{array}{c}
\left(\begin{array}{l}
1-\left(1-(1-(1-0.4^{0.5})(1-0.1^{0.2}))^{\frac{1}{3}}\times(1-(1-0.4^{0.5})(1-0.6^{0.3}))^{\frac{1}{3}}\times(1-(1-0.1^{0.2})(1-0.6^{0.3}))^{\frac{1}{3}}\right)^{\frac{1}{2}}|1\times0.4\times0.8,\\
1-\left(1-(1-(1-0.4^{0.5})(1-0.1^{0.2}))^{\frac{1}{3}}\times(1-(1-0.4^{0.5})(1-0.7^{0.3}))^{\frac{1}{3}}\times(1-(1-0.1^{0.2})(1-0.7^{0.3}))^{\frac{1}{3}}\right)^{\frac{1}{2}}|1\times0.4\times0.2,\\
1-\left(1-(1-(1-0.4^{0.5})(1-0.2^{0.2}))^{\frac{1}{3}}\times(1-(1-0.4^{0.5})(1-0.6^{0.3}))^{\frac{1}{3}}\times(1-(1-0.2^{0.2})(1-0.6^{0.3}))^{\frac{1}{3}}\right)^{\frac{1}{2}}|1\times0.6\times0.8,\\
1-\left(1-(1-(1-0.4^{0.5})(1-0.2^{0.2}))^{\frac{1}{3}}\times(1-(1-0.4^{0.5})(1-0.7^{0.3}))^{\frac{1}{3}}\times(1-(1-0.2^{0.2})(1-0.7^{0.3}))^{\frac{1}{3}}\right)^{\frac{1}{2}}|1\times0.6\times0.2
\end{array}\right),\\
\left(\begin{array}{l}
\left(1-(1-(0.1^{0.5}\times0.6^{0.2}))^{\frac{1}{3}}\times(1-(0.1^{0.5}\times0.4^{0.3}))^{\frac{1}{3}}\times(1-(0.6^{0.2}\times0.4^{0.3}))^{\frac{1}{3}}\right)^{\frac{1}{2}}|0.6\times0.8\times1,\\
\left(1-(1-(0.1^{0.5}\times0.7^{0.2}))^{\frac{1}{3}}\times(1-(0.1^{0.5}\times0.4^{0.3}))^{\frac{1}{3}}\times(1-(0.7^{0.2}\times0.4^{0.3}))^{\frac{1}{3}}\right)^{\frac{1}{2}}|0.6\times0.2\times1,\\
\left(1-(1-(0.3^{0.5}\times0.6^{0.2}))^{\frac{1}{3}}\times(1-(0.3^{0.5}\times0.4^{0.3}))^{\frac{1}{3}}\times(1-(0.6^{0.2}\times0.4^{0.3}))^{\frac{1}{3}}\right)^{\frac{1}{2}}|0.4\times0.8\times1,\\
\left(1-(1-(0.3^{0.5}\times0.7^{0.2}))^{\frac{1}{3}}\times(1-(0.3^{0.5}\times0.4^{0.3}))^{\frac{1}{3}}\times(1-(0.7^{0.2}\times0.4^{0.3}))^{\frac{1}{3}}\right)^{\frac{1}{2}}|0.4\times0.2\times1,
\end{array}\right)
\end{array}\right\}
$$

$$
=\{(0.7155|0.32,0.7330|0.08,0.7459|0.48,0.7633|0.12),(0.1151|0.48,0.1276|0.12,0.1574|0.32,0.1719|0.08)\}
$$

5 Extended COPRAS method based on the PDHFMSM and PDHFDMSM

The COPRAS approach is a MAGDM technique first introduced by Zavadskas et al (1994) has recently attracted much investigation. In practical application, COPRAS is used to maximize and minimize the values of the index to consider the effect of maximizing and minimizing the attribute index on separate evaluations of results. Because of the excellent characteristic of the COPRAS method, we modify this method by employing the WPDHFMSM and WPDHFDMSM operators to integrate the evaluation values, and the ranking procedure for the MAGDM problem is defined as follows:

5.1 The description of the MAGDM problems under PDHF environment

Assume that experts $e^l(l=1,2,3,...n)$ give the evaluations values $P_{ij}=\left(h_{ij}|p_{ij},g_{ij}|q_{ij}\right)$ of each alternative $A_i(i=1,2,3,...,s)$ relative to each attribute $\beta_j(j=1,2,3,...,t)$ in the form of PDHFEs. $w=(w_1,w_2,w_3,...,w_t)^T$ represents the weights attribute with conditions $w_j>0$ and $\sum_{j=1}^{t}w_j=1$ and $\lambda=(\lambda_1,\lambda_2,\lambda_3,...,\lambda_n)^T$ are the weights information of the experts with the requirements $w_l>0$ and $\sum_{l=1}^{m}\lambda_l=1$. The evaluation information is summarize in the decision matrix $M^l=\left[P_{ij}^l\right]_{s\times t}$ as follows:

$$
M^l=\begin{array}{c}A_1\\ \\ A_2\\ \\ \vdots\\ \\ A_s\end{array}\left[\begin{array}{cccc}
P_{11}^l=\left(h_{11}^l\mid p_{11},g_{11}^l\mid q_{11}\right) & P_{12}^l=\left(h_{12}^l\mid p_{12},g_{12}^l\mid q_{12}\right) & \cdots & P_{1t}^l=\left(h_{1t}^l\mid p_{1t},g_{1t}^l\mid q_{1t}\right)\\
\\
P_{21}^l=\left(h_{21}^l\mid p_{21},g_{21}^l\mid q_{21}\right) & P_{22}^l=\left(h_{22}^l\mid p_{22},g_{21}^l\mid q_{22}\right) & \cdots & P_{2t}^l=\left(h_{2t}^l\mid p_{2t},g_{2t}^l\mid q_{2t}\right)\\
\\
\vdots & \vdots & \cdots & \vdots\\
\\
P_{s1}^l=\left(h_{s1}^l\mid p_{s1},g_{s1}^l\mid q_{s1}\right) & P_{s2}^l=\left(h_{s2}^l\mid p_{s2},g_{s2}^l\mid q_{s2}\right) & \cdots & P_{st}^l=\left(h_{st}^l\mid p_{st},g_{st}^l\mid q_{st}\right),
\end{array}\right]
$$

where $P_{ij}^{l}=\left(h_{ij}^{l}\mid p_{ij},g_{ij}^{l}\mid q_{ij}\right)$ indicate the grade value of the alternative A_i relative to the attribute β_j given by the experts.

5.2 COPRAS technique with PDHFMSM

In this subsection, an extended COPRAS method is presented for PDHF information. First, we get the individual PDHF decision matrices from the experts. Under the WPDHFMSM and WPDHFDMSM operators, we get the group decision matrix by combining the separate decision matrices. Taking advantage of the WPDHFMSM or the WPDHFDMSM operator, the collective value of alternatives A_i for attribute β_j is calculated according to the result 2 as follows:

$$P_{ij}=WPDHFMSM^{(m)}\left(P_{ij}^{1},P_{ij}^{2},P_{ij}^{3},\dots,P_{ij}^{n}\right)=\left(\frac{\sum_{1\le i_1\le i_2\le\dots\le i_m\le g}\left(\prod_{l=1}^{n}\left(\lambda_{i_j}.P_{i_j}^{l}\right)\right)}{\binom{n}{m}}\right)^{\frac{1}{m}}$$

$$=\left\{\begin{array}{l}\cup_{\gamma_{i_j}\in h_j}\left(1-\prod_{1\le i_1\le i_2\le\dots\le i_m\le n}\left(1-\prod_{l=1}^{n}\left(1-\left(1-\gamma_{i_j}^{l}\right)^{\lambda_{i_j}}\right)\right)^{\frac{1}{\binom{n}{m}}}\right)^{\frac{1}{m}}\mid\prod_{l=1}^{n}p_{\gamma_{i_j}},\\ \cup_{\eta_{i_j}\in g_j}\left(1-\left(1-\prod_{1\le i_1\le i_2\le\dots\le i_m\le n}\left(1-\prod_{l=1}^{n}\left(1-\left(\eta_{i_j}^{l}\right)^{\lambda_{i_j}}\right)\right)^{\frac{1}{\binom{n}{m}}}\right)^{\frac{1}{m}}\right)\mid\prod_{l=1}^{n}q_{\eta_{i_j}}.\end{array}\right\} \tag{1}$$

If the DMs go for the WPDHFDMSM operator, then the collective value of alternatives A_i with respect to attribute β_j is calculated according to the result 4 as follows:

$$P_{ij}=WPDHFDMSM^{(m)}\left(P_{ij}^{1},P_{ij}^{2},P_{ij}^{3},\dots,P_{ij}^{n}\right)=\frac{1}{m}\prod_{1\le i_1\le i_2\le\dots\le i_m\le n}\left(\sum_{l=1}^{n}\left(P_{i_j}^{l}\right)^{\lambda_{i_j}}\right)^{\binom{n}{m}}$$

$$=\left\{\begin{array}{l}\cup_{\gamma_{i_j}\in h_j}\left(1-\left(1-\prod_{1\le i_1\le i_2\le\dots\le i_m\le n}\left(1-\prod_{l=1}^{n}\left(1-\left(\gamma_{i_j}^{l}\right)^{\lambda_{i_j}}\right)\right)^{\frac{1}{\binom{n}{m}}}\right)^{\frac{1}{m}}\right)\mid\prod_{l=1}^{n}p_{\gamma_{i_j}},\\ \cup_{\eta_{i_j}\in g_j}\left(1-\prod_{1\le i_1\le i_2\le\dots\le i_m\le n}\left(1-\prod_{l=1}^{n}\left(1-\left(1-\eta_{i_j}\right)^{\lambda_{i_j}}\right)\right)^{\frac{1}{\binom{n}{m}}}\right)^{\frac{1}{m}}\mid\prod_{l=1}^{n}q_{\eta_{i_j}}\end{array}\right\} \tag{2}$$

Now, from Eqs. 1 and 2, we can find the group decision matrix C as follows:

$$C=\begin{array}{c}A_1\\ A_2\\ \vdots\\ A_s\end{array}\begin{bmatrix}P_{11}=(h_{11}\mid p_{11},g_{11}\mid q_{11}) & P_{12}=(h_{12}\mid p_{12},g_{12}\mid q_{12}) & \cdots & P_{1t}=(h_{1t}\mid p_{1t},g_{1t}\mid q_{1t})\\ P_{21}=(h_{21}\mid p_{21},g_{21}\mid q_{21}) & P_{22}^{l}=(h_{22}\mid p_{22},g_{21}\mid q_{22}) & \cdots & P_{2t}=(h_{2t}\mid p_{2t},g_{2t}\mid q_{2t})\\ \vdots & \vdots & \cdots & \vdots\\ P_{s1}=(h_{s1}\mid p_{s1},g_{s1}\mid q_{s1}) & P_{s2}=(h_{s2}\mid p_{s2},g_{s2}\mid q_{s2}) & \cdots & P_{st}=(h_{st}\mid p_{st},g_{st}\mid q_{st}).\end{bmatrix}$$

Group decision matrix C represents the collective information of the experts in the form of the PDHF matrix. To categorize the information given in the group decision matrix, we propose two new extended COPRAS methods under the PDHF environment, based on the PDHFMSM and PDHFDMSM operators as shown below:

- If the attribute β_j belongs to the benefit type, then
  - Based on 2 and the WPDHFMSM operator the summation of the benefit attributes

$$\overline{B}_i = s\left(\frac{\sum_{1\le i_1\le i_2\le\ldots\le i_m\le u}\left(\prod_{j=1;j\in N_B}^{m}\left(w_{i_j}.P_{i_j}\right)\right)}{\binom{u}{m}}\right)^{\frac{1}{m}}$$

$$= s\left(\begin{array}{l} \cup_{\gamma_{i_j}\in h_j}\left(1-\prod\limits_{1\le i_1\le i_2\le\ldots\le i_m\le u}\left(1-\prod\limits_{j=1;j\in N_B}^{m}\left(1-\left(1-\gamma_{i_j}\right)^{w_{i_j}}\right)\right)^{\frac{1}{\binom{u}{m}}}\right)^{\frac{1}{m}} \mid \prod_{j=1;j\in N_B}^{m} p_{\gamma_{i_j}}, \\ \cup_{\eta_{i_j}\in g_j}\left(1-\left(1-\prod\limits_{1\le i_1\le i_2\le\ldots\le i_m\le u}\left(1-\prod\limits_{j=1;j\in N_B}^{m}\left(1-\left(\eta_{i_j}\right)^{w_{i_j}}\right)\right)^{\frac{1}{\binom{u}{m}}}\right)^{\frac{1}{m}}\right) \mid \prod_{j=1;j\in N_B}^{m} q_{\eta_{i_j}} \end{array}\right) \tag{3}$$

  - If the WPDHFDMSM operator and 4 are used then the summation of the benefit attributes

$$\overline{B}_i = s\left(\frac{1}{m}\prod_{1\le i_1\le i_2\le\ldots\le i_m\le u}\left(\sum_{j=1;j\in N_B}^{m}\left(P_{i_j}\right)^{w_{i_j}}\right)^{\binom{u}{m}}\right)$$

$$= \left\{\begin{array}{l} \cup_{\gamma_{i_j}\in h_j}\left(1-\left(1-\prod\limits_{1\le i_1\le i_2\le\ldots\le i_m\le u}\left(1-\prod\limits_{j=1;j\in N_B}^{m}\left(1-\left(\gamma_{i_j}\right)^{w_{i_j}}\right)\right)^{\frac{1}{\binom{u}{m}}}\right)^{\frac{1}{m}}\right) \mid \prod_{j=1;j\in N_B}^{m} p_{\gamma_{i_j}}, \\ \cup_{\eta_{i_j}\in g_j}\left(1-\prod\limits_{1\le i_1\le i_2\le\ldots\le i_m\le u}\left(1-\prod\limits_{j=1;j\in N_B}^{m}\left(1-\left(1-\eta_{i_j}\right)^{w_{i_j}}\right)\right)^{\frac{1}{\binom{u}{m}}}\right)^{\frac{1}{m}} \mid \prod\limits_{j=1;j\in N_B}^{m} q_{\eta_{i_j}} \end{array}\right\} \tag{4}$$

- If the attribute β_j belongs to the cost type, then
  - Based on 2 and the WPDHFMSM operator the summation of the cost attributes

$$\overline{S}_i = s\left(\frac{\sum_{1\le i_1\le i_2\le\ldots\le i_m\le t-u}\left(\prod_{j=1;j\in N_C}^{m}\left(w_{i_j}.P_{i_j}\right)\right)}{\binom{t-u}{m}}\right)^{\frac{1}{m}}$$

$$= s\left(\begin{array}{l} \cup_{\gamma_{i_j}\in h_j}\left(1-\prod\limits_{1\le i_1\le i_2\le\ldots\le i_m\le t-u}\left(1-\prod\limits_{j=1;j\in N_C}^{m}\left(1-\left(1-\gamma_{i_j}\right)^{w_{i_j}}\right)\right)^{\frac{1}{\binom{t-u}{m}}}\right)^{\frac{1}{m}} \mid \prod_{j=1;j\in N_C}^{m} p_{\gamma_{i_j}}, \\ \cup_{\eta_{i_j}\in g_j}\left(1-\left(1-\prod\limits_{1\le i_1\le i_2\le\ldots\le i_m\le t-u}\left(1-\prod\limits_{j=1;j\in N_C}^{m}\left(1-\left(\eta_{i_j}\right)^{w_{i_j}}\right)\right)^{\frac{1}{\binom{t-u}{m}}}\right)^{\frac{1}{m}}\right) \mid \prod\limits_{j=1;j\in N_C}^{m} q_{\eta_{i_j}} \end{array}\right) \tag{5}$$

– If the WPDHFDMSM operator and 4 are used then the summation of the cost attributes

$$\overline{S}_i = s\left(\frac{1}{m}\prod_{1\le i_1\le i_2\le\ldots\le i_m\le t-u}\left(\sum_{j=1;j\in N_C}^{m}\left(P_{i_j}\right)^{w_{i_j}}\right)^{\binom{t-u}{m}}\right)$$

$$= \left\{ \begin{array}{l} \cup_{\gamma_{i_j}\in h_j}\left(1-\left(1-\prod_{1\le i_1\le i_2\le\ldots\le i_m\le t-u}\left(1-\prod_{j=1;j\in N_C}^{m}\left(1-\left(\gamma_{i_j}\right)^{w_{i_j}}\right)\right)^{\frac{1}{\binom{t-u}{m}}}\right)^{\frac{1}{m}}\right) \mid \prod_{j=1;j\in N_C}^{m} p_{\gamma_{i_j}}, \\ \cup_{\eta_{i_j}\in g_j}\left(1-\prod_{1\le i_1\le i_2\le\ldots\le i_m\le t-u}\left(1-\prod_{j=1;j\in N_C}^{m}\left(1-\left(1-\eta_{i_j}\right)^{w_{i_j}}\right)\right)^{\frac{1}{\binom{t-u}{m}}}\right)^{\frac{1}{m}} \mid \prod_{j=1;j\in N_C}^{m} q_{\eta_{i_j}} \end{array} \right\} \tag{6}$$

From the Definition of 4 and 5, the score of 3, 4, 5, and 6 can easily be ascertained.

Moreover, the relative significance (RS) R_i of every alternate $A_i(i=1,2,3,\ldots,s)$ can be acquire as follows:

$$R_i = \overline{B}_i + \frac{\sum_{i=1}^{s}\overline{S}_i}{\overline{S}_i\sum_{i=1}^{s}\frac{1}{\overline{S}_i}}. \tag{7}$$

The RS of R_i represents the satisfaction level of each alternative. Mani-festively, the greater the value of R_i will result in the higher the importance of that alternative. Based on 7, we can find the maximal RS value R, i.e.

$$R = \max_{1\le i\le s} R_i. \tag{8}$$

Therefore, the alternative having the highest RS value would be the best among all the alternatives. Furthermore, based on the RS, the utility degree U_i of each alternative can be determined using the formula:

$$U_i = \left(\frac{R_i}{R}\right)\times 100\%. \tag{9}$$

Select the alternatives according to 9 and Make decisions, greater the value of U_i, the optimal the alternative would be.

5.3 Algorithm for decision-making

To resolve the MAGDM problem when the PDHFEs express attributes values, we summarize the COPRAS method using the WPDHFMSM and WPDHFDMSM operators and design an algorithm to manage the decision-making problems. Suppose the experts $e^l(l=1,2,3,\ldots n)$ give the evaluations values in the form of PDHFEs for the discrete set of alternatives $A_i(i=1,2,3,\ldots,s)$ with respect to each attribute $\beta_j(j=1,2,3,\ldots,t)$. Let $w=(w_1,w_2,w_3,\ldots,w_t)^T$ represents the set of attribute weight with conditions $w_j>0$ and $\sum_{j=1}^{t}w_j=1$ and $\lambda=(\lambda_1,\lambda_2,\lambda_3,\ldots,\lambda_n)^T$ are the weights information of the experts with the requirements $w_l>0$ and $\sum_{l=1}^{m}\lambda_l=1$. Then evaluation information is summarize in the decision matrix as $M^l=\left[P_{ij}^l\right]_{s\times t}=\left[\left(h_{ij}^l,g_{ij}^l\right)\right]_{s\times t}$. The following steps are used to obtain the most suitable alternative.

Step 1: In agreement with the value of m, group decision matrix C is obtained after integrate the values of the individual matrices with the help of WPDHFMSM and 1 (or WPDHFDMSM and 2).
Step 2: Interaction with 3 or 4, aggregate the values of benefit criteria by utilizing the group decision matrix C and obtain $\overline{B}_i$.
Step 3: Besides, with the help of Definition 4, 5 and 5 or 6, sum the arguments of the cost criteria and get $\overline{S}_i$.
Step 4: Now find the RS R_i by employing the result 7 for each alternatives A_i and identify the maximum RS value R.
Step 5: At this stage, compute the utility degree of every alternative using 9.
Step 6: Rank all the alternatives based on Definition 9, and sort them in descending order.

6 An illustrative example

This section presents the proposed approach to selecting the best photovoltaic cell under the PDHF context. The effectiveness and benefits of the presented methods are confirmed by analyzing the parameters and comparing them with other techniques.

6.1 Problem description

With a lack of natural resources and environmental protection, renewable energy has become a promise to provide clean and abundant energy. The photovoltaic cell is one of the emerging renewable energies because it has virtually no environmental impact, pollution, heat or noise, and shortage. It is straightforward to wear and tear to maintain any mechanical moving part. Choosing the best photovoltaic cell is essential in maximizing revenue in grid-connected systems (increasing production or efficiency), reducing costs, and delivering at the same time high maturity and reliability. In this context, this section aims to study and analyze the proposed method used in choosing the best photovoltaic cell. As reported by Socorro García-Cascales et al (2011), five different types of potential photovoltaic cells are available as described below:

- A_1 : photovoltaic cells with crystalline silicon (mono-crystalline and poly-crystalline)
- A_2 : photovoltaic cells with inorganic thin layer (amorphous silicon)
- A_3 : photovoltaic cells with inorganic thin layer (cadmium tellurium/cadmium sulfide and copper indium gallium diselenide/cadmium sulfide)
- A_4 : photovoltaic cells with advanced III–V this layer with tracking systems for solar concentration, and
- A_5 : photovoltaic cells with advanced, low cost, thin layers(organic and hybrid cells)

After analyzing the potential photovoltaic cells, the criteria to be considered for diagnosing the decision problem are as follows.

- β_1 : Production cost
- β_2 : Efficiency in energy conversion
- β_3 : Emissions of green house gasses during production
- β_4 : Market share
- β_5 : Energy playback time.

It is observed that β_1, β_3, and β_5 are the cost criteria while β_2 and β_4 are the benefit criteria. Three experts (e^1, e^2, e^3) have been invited to examine the above five possible photovoltaic cells with respect to these five criteria. The weights of the three experts are assumed to be equal, i.e. $\lambda = (\lambda_1, \lambda_2, \lambda_3)^T = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ and the importance degree of the criteria are subjectively given by the experts

Table 1 PDHF decision matrix M^1 prescribed by the expert e^1

	β_1	β_2	β_3	β_4	β_5
A_1	$\{(0.4 \mid 1), (0.1 \mid 0.6, 0.3 \mid 0.4)\}$	$\{(0.6 \mid 1), (0.1 \mid 0.6, 0.3 \mid 0.4)\}$	$\{(0.5 \mid 0.2, 0.6 \mid 0.8), (0.1 \mid 0.6, 0.2 \mid 0.4)\}$	$\{(0.1 \mid 1), (0.2 \mid 0.3, 0.5 \mid 0.7)\}$	$\{(0.2 \mid 0.1, 0.3 \mid 0.9), (0.6 \mid 1)\}$
A_2	$\{(0.6 \mid 0.8, 0.1 \mid 0.2), (0.2 \mid 0.4, 0.3 \mid 0.6)\}$	$\{(0.3 \mid 0.7, 0.4 \mid 0.3), (0.2 \mid 0.8, 0.4 \mid 0.2)\}$	$\{(0.6 \mid 0.8, 0.7 \mid 0.2), (0.1 \mid 0.2, 0.3 \mid 0.8)\}$	$\{(0.5 \mid 0.3, 0.6 \mid 0.7), (0.1 \mid 1)\}$	$\{(0.4 \mid 1), (0.4 \mid 0.7, 0.5 \mid 0.3)\}$
A_3	$\{(0.4 \mid 0.2, 0.6 \mid 0.8), (0.3 \mid 1)\}$	$\{(0.4 \mid 0.6, 0.5 \mid 0.4), (0.2 \mid 0.6, 0.3 \mid 0.4)\}$	$\{(0.4 \mid 0.1, 0.6 \mid 0.9), (0.2 \mid 0.2, 0.3 \mid 0.8)\}$	$\{(0.3 \mid 0.4, 0.5 \mid 0.6), (0.1 \mid 0.6, 0.2 \mid 0.4)\}$	$\{(0.5 \mid 0.7, 0.6 \mid 0.3), (0.2 \mid 0.4, 0.3 \mid 0.6)\}$
A_4	$\{(0.2 \mid 0.5, 0.4 \mid 0.5), (0.4 \mid 0.6, 0.4 \mid 0.4)\}$	$\{(0.2 \mid 0.3, 0.4 \mid 0.7), (0.4 \mid 1)\}$	$\{(0.5 \mid 0.4, 0.6 \mid 0.6), (0.2 \mid 0.6, 0.3 \mid 0.4)\}$	$\{(0.6 \mid 0.5, 0.7 \mid 0.5), (0.2 \mid 1)\}$	$\{(0.4 \mid 0.4, 0.5 \mid 0.6), (0.2 \mid 0.7, 0.3 \mid 0.3)\}$
A_5	$\{(0.1 \mid 0.2.0.4 \mid 0.8), (0.6 \mid 1)\}$	$\{(0.1 \mid 0.2.0.4 \mid 0.8), (0.6 \mid 1)\}$	$\{(0.4 \mid 0.6, 0.5 \mid 0.4), (0.2 \mid 0.2, 0.3 \mid 0.8)\}$	$\{(0.5 \mid 1), (0.2 \mid 1)\}$	$\{(0.2 \mid 0.2.0.3 \mid 0.8), (0.5 \mid 1)\}$

Table 2 PDHF decision matrix M^2 prescribed by the expert e^2

	β_1	β_2	β_3	β_4	β_5
A_1	$\{(0.1 \mid 0.5, 0.2 \mid 0.5), (0.6 \mid 0.6, 0.7 \mid 0.4)\}$	$\{(0.2 \mid 0.2, 0.3 \mid 0.8), (0.4 \mid 0.5, 0.5 \mid 0.5)\}$	$\{(0.6 \mid 0.2, 0.7 \mid 0.8), (0.2 \mid 0.5, 0.3 \mid 0.5)\}$	$\{(0.6 \mid 0.8, 0.3 \mid 0.2), (0.1 \mid 0.3, 0.2 \mid 0.7)\}$	$\{(0.4 \mid 0.3, 0.5 \mid 0.7), (0.1 \mid 0.1, 0.3 \mid 0.9)\}$
A_2	$\{(0.4 \mid 0.4, 0.5 \mid 0.6), (0.2 \mid 0.2, 0.3 \mid 0.8)\}$	$\{(0.4 \mid 0.4, 0.5 \mid 0.6), (0.2 \mid 0.2, 0.3 \mid 0.8)\}$	$\{(0.6 \mid 1), (0.7 \mid 0.1, 0.2 \mid 0.9)\}$	$\{(0.3 \mid 0.7, 0.4 \mid 0.3), (0.4 \mid 0.5, 0.5 \mid 0.5)\}$	$\{(0.2 \mid 0.6, 0.3 \mid 0.4), (0.4 \mid 0.5, 0.5 \mid 0.5)\}$
A_3	$\{(0.6 \mid 0.5, 0.7 \mid 0.5), (0.1 \mid 1)\}$	$\{(0.3 \mid 1), (0.1 \mid 0.5, 0.2 \mid 0.5)\}$	$\{(0.4 \mid 1), (0.1 \mid 0.3, 0.2 \mid 0.7)\}$	$\{(0.3 \mid 0.6, 0.4 \mid 0.4), (0.4 \mid 0.2, 0.5 \mid 0.8)\}$	$\{(0.5 \mid 1), (0.2 \mid 0.2, 0.3 \mid 0.8)\}$
A_4	$\{(0.5 \mid 0.2, 0.6 \mid 0.8), (0.1 \mid 0.2, 0.2 \mid 0.8)\}$	$\{(0.2 \mid 0.2, 0.3 \mid 0.8), (0.4 \mid 1)\}$	$\{(0.6 \mid 0.5, 0.7 \mid 0.5), (0.3 \mid 1)\}$	$\{(0.6 \mid 1), (0.2 \mid 1)\}$	$\{(0.5 \mid 0.3, 0.6 \mid 0.7), (0.1 \mid 1)\}$
A_5	$\{(0.1 \mid 0.2, 0.2 \mid 0.8), (0.5 \mid 0.2, 0.6 \mid 0.8)\}$	$\{(0.5 \mid 1), (0.1 \mid 0.6, 0.2 \mid 0.4)\}$	$\{(0.4 \mid 0.4, 0.5 \mid 0.6), (0.2 \mid 0.5, 0.3 \mid 0.5)\}$	$\{(0.4 \mid 1), (0.3 \mid 0.2, 0.5 \mid 0.8)\}$	$\{(0.7 \mid 0.2, 0.8 \mid 0.8), (0.9 \mid 0.4, 0.1 \mid 0.6)\}$

Table 3 PDHF decision matrix M^3 prescribed by the expert e^3

	β_1	β_2	β_3	β_4	β_5
A_1	$\{(0.6 \mid 0.6, 0.7 \mid 0.4), (0.1 \mid 0.5, 0.2 \mid 0.5)\}$	$\{(0.3 \mid 0.6, 0.4 \mid 0.4), (0.5 \mid 1)\}$	$\{(0.3 \mid 0.6, 0.4 \mid 0.4), (0.1 \mid 1)\}$	$\{(0.3 \mid 0.7, 0.5 \mid 0.3), (0.4 \mid 0.2, 0.6 \mid 0.8)\}$	$\{(0.2 \mid 0.1, 0.3 \mid 0.9), (0.4 \mid 0.4, 0.5 \mid 0.6)\}$
A_2	$\{(0.1 \mid 1), (0.6 \mid 1)\}$	$\{(0.4 \mid 0.6, 0.5 \mid 0.4), (0.2 \mid 0.2, 0.3 \mid 0.8)\}$	$\{(0.5 \mid 1), (0.3 \mid 0.6, 0.4 \mid 0.4)\}$	$\{(0.5 \mid 0.1, 0.7 \mid 0.9), (0.1 \mid 1)\}$	$\{(0.3 \mid 1), (0.4 \mid 0.4, 0.5 \mid 0.6)\}$
A_3	$\{(0.6 \mid 0.6, 0.7 \mid 0.4), (0.1 \mid 0.5, 0.2 \mid 0.5)\}$	$\{(0.8 \mid 1), (0.1 \mid 1)\}$	$\{(0.4 \mid 0.1, 0.5 \mid 0.9), (0.1 \mid 0.5, 0.2 \mid 0.5)\}$	$\{(0.5 \mid 1), (0.3 \mid 0.6, 0.4 \mid 0.4)\}$	$\{(0.3 \mid 0.5, 0.4 \mid 0.5), (0.3 \mid 0.5, 0.4 \mid 0.5)\}$
A_4	$\{(0.5 \mid 0.6, 0.7 \mid 0.4), (0.1 \mid 0.5, 0.2 \mid 0.5)\}$	$\{(0.5 \mid 0.6, 0.7 \mid 0.4), (0.2 \mid 0.3, 0.3 \mid 0.7)\}$	$\{(0.1 \mid 0.1, 0.3 \mid 0.9), (0.4 \mid 0.2, 0.5 \mid 0.8)\}$	$\{(0.6 \mid 0.5, 0.7 \mid 0.5), (0.1 \mid 0.3, 0.2 \mid 0.7)\}$	$\{(0.3 \mid 0.4, 0.4 \mid 0.6), (0.5 \mid 1)\}$
A_5	$\{(0.1 \mid 0.2, 0.3 \mid 0.8), (0.6 \mid 1)\}$	$\{(0.3 \mid 0.6, 0.4 \mid 0.4), (0.7 \mid 0.2, 0.9 \mid 0.8)\}$	$\{(0.3 \mid 1), (0.5 \mid 0.3, 0.6 \mid 0.7)\}$	$\{(0.4 \mid 0.4, 0.5 \mid 0.6), (0.3 \mid 1)\}$	$\{(0.6 \mid 1), (0.1 \mid 0.3, 0.3 \mid 0.7)\}$

as $w = (w_1, w_2, w_3, w_4, w_5)^T = (0.2, 0.4, 0.1, 0.1, 0.2)^T$. The assessment values of the alternatives with respect to each criterion given by the experts are assumed to represent by PDHFS, because of the uncertainty, impression, and incompleteness in the evaluation of these cells. Their assessments are presented in the probabilistic DHF decision matrices $M^l = \left[P^l_{ij}\right]_{5\times5} = \left[\left(h^l_{ij}, g^l_{ij}\right)\right]_{5\times5}$ in Tables 1, 2 and 3.

6.2 The process of deciding the selection problem of photovoltaic cells with WPDHFMSM

In the following, we utilize our developed method in Sect. 5 to solve the selection problem of photovoltaic cells.

Step 1: For the construction of the group decision matrix C, utilize the information given by the experts $e^l(l = 1, 2, 3)$ mentioned in Tables 1, 2 and 3, the WPDHFMSM operator and Eq. 1 to normalize the evaluations values for $m = 2$. Because of the huge number of elements in the decision process, here we only show the detailed calculation of the alternative A_1 with respect to the criteria β_2 as follows:

$$
= \left\{ \begin{array}{c}
\left(\begin{array}{l}
\left(1-\left(1-\left(1-(1-0.6)^{0.33}\right)\times\left(1-(1-0.2)^{0.33}\right)\right)^{\frac{1}{3}}\times\left(1-\left(1-(1-0.6)^{0.33}\right)\times\left(1-(1-0.3)^{0.33}\right)\right)^{\frac{1}{3}}\times\left(1-\left(1-(1-0.2)^{0.33}\right)\times\left(1-(1-0.3)^{0.33}\right)\right)^{\frac{1}{3}}\right)^{\frac{1}{2}} | 1.0\times0.2\times0.6, \\
\left(1-\left(1-\left(1-(1-0.6)^{0.33}\right)\times\left(1-(1-0.2)^{0.33}\right)\right)^{\frac{1}{3}}\times\left(1-\left(1-(1-0.6)^{0.33}\right)\times\left(1-(1-0.4)^{0.33}\right)\right)^{\frac{1}{3}}\times\left(1-\left(1-(1-0.2)^{0.33}\right)\times\left(1-(1-0.4)^{0.33}\right)\right)^{\frac{1}{3}}\right)^{\frac{1}{2}} | 1.0\times0.2\times0.4, \\
\left(1-\left(1-\left(1-(1-0.6)^{0.33}\right)\times\left(1-(1-0.3)^{0.33}\right)\right)^{\frac{1}{3}}\times\left(1-\left(1-(1-0.6)^{0.33}\right)\times\left(1-(1-0.3)^{0.33}\right)\right)^{\frac{1}{3}}\times\left(1-\left(1-(1-0.3)^{0.33}\right)\times\left(1-(1-0.3)^{0.33}\right)\right)^{\frac{1}{3}}\right)^{\frac{1}{2}} | 1.0\times0.8\times0.6, \\
\left(1-\left(1-\left(1-(1-0.6)^{0.33}\right)\times\left(1-(1-0.3)^{0.33}\right)\right)^{\frac{1}{3}}\times\left(1-\left(1-(1-0.6)^{0.33}\right)\times\left(1-(1-0.4)^{0.33}\right)\right)^{\frac{1}{3}}\times\left(1-\left(1-(1-0.3)^{0.33}\right)\times\left(1-(1-0.4)^{0.33}\right)\right)^{\frac{1}{3}}\right)^{\frac{1}{2}} | 1.0\times0.8\times0.4,
\end{array} \right), \\
\left(\begin{array}{l}
1-\left(1-(1-(1-0.1^{0.33})(1-0.4^{0.33}))^{\frac{1}{3}}\times(1-(1-0.1^{0.33})(1-0.5^{0.33}))^{\frac{1}{3}}\times(1-(1-0.4^{0.33})(1-0.5^{0.33}))^{\frac{1}{3}}\right)^{\frac{1}{2}} | 0.6\times0.5\times1.0, \\
1-\left(1-(1-(1-0.1^{0.33})(1-0.5^{0.33}))^{\frac{1}{3}}\times(1-(1-0.1^{0.33})(1-0.5^{0.33}))^{\frac{1}{3}}\times(1-(1-0.5^{0.33})(1-0.5^{0.33}))^{\frac{1}{3}}\right)^{\frac{1}{2}} | 0.6\times0.5\times1.0, \\
1-\left(1-(1-(1-0.3^{0.33})(1-0.4^{0.33}))^{\frac{1}{3}}\times(1-(1-0.3^{0.33})(1-0.5^{0.33}))^{\frac{1}{3}}\times(1-(1-0.4^{0.33})(1-0.5^{0.33}))^{\frac{1}{3}}\right)^{\frac{1}{2}} | 0.4\times0.5\times1.0, \\
1-\left(1-(1-(1-0.3^{0.33})(1-0.5^{0.33}))^{\frac{1}{3}}\times(1-(1-0.3^{0.33})(1-0.5^{0.33}))^{\frac{1}{3}}\times(1-(1-0.5^{0.33})(1-0.5^{0.33}))^{\frac{1}{3}}\right)^{\frac{1}{2}} | 0.4\times0.5\times1.0
\end{array} \right)
\end{array} \right\}
$$

$$
= \{(0.1373|0.12, 0.1544|0.08, 0.1546|0.48, 0.1717|0.32), (0.6796|0.3, 0.7026|0.3, 0.7356|0.2, 0.7556|0.2)\}
$$

All the processed values of alternative A_1 with respect to criteria $\beta_j(j = 1, 2, 3, 4, 5)$ are given in Table 4.

Step 2: Taking into account the collective decision matrix C, we sum the overall values of the benefit attributes with the help of 3 and pick up $\overline{B}_i(i = 1, 2, 3, 4, 5)$. Due to lack of space, we omit detailed results and calculations. After obtaining the collective values of the alternatives, we find that the score of each $\overline{B}_i$ with the help of Definition 4.

Step 3: Similarly, the decision matrix C is used to find the collective values of the cost attributes and determine the values $\overline{S}_i(i = 1, 2, 3, 4, 5)$ with the aid of Eq. 5. Using the Definition 4, we evaluate the score of each alternative.

Step 4: After finding the scores of the alternatives, we find out the RS $R_i(i = 1, 2, 3, 4, 5)$, by using the Eq. 7. According to the Eq. 8, the maximum RS can be determined from the RS R_i and the value of the maximum RS R is given as:

$$R = -1.7572$$

Step 5: Now the utility degree $U_i(i = 1, 2, 3, 4, 5)$ of each alternative can be derived by utilizing the Eq. 9.

Step 6: According to the utility degree $U_i(i = 1, 2, 3, 4, 5)$, we find that the ranking of the alternatives $A_i(i = 1, 2, 3, 4, 5)$ is:

$$U_2 > U_4 > U_5 > U_3 > U_1.$$

Hence the alternatives according to the utility ranking can be categorized as follows:

$$A_2 > A_4 > A_5 > A_3 > A_1.$$

Table 4 Processed values of A_1 by using WPDHFMSM operator

	Membership values	Non-membership values
P_{11}	$(0.1367 \mid 0.3, 0.1519 \mid 0.2, 0.1544 \mid 0.3, 0.1704 \mid 0.2)$	$\begin{pmatrix} 0.6033 \mid 0.18, 0.6440 \mid 0.18, 0.6221 \mid 0.12, 0.6632 \mid 0.12, \\ 0.6746 \mid 0.12, 0.7077 \mid 0.12, 0.6941 \mid 0.08, 0.7266 \mid 0.08 \end{pmatrix}$
P_{12}	$(0.1373 \mid 0.12, 0.1544 \mid 0.08, 0.1546 \mid 0.48, 0.1717 \mid 0.32)$	$(0.6796 \mid 0.3, 0.7026 \mid 0.3, 0.7356 \mid 0.2, 0.7556 \mid 0.2)$
P_{13}	$\begin{pmatrix} 0.1890 \mid 0.024, 0.2065 \mid 0.016, 0.2073 \mid 0.096, 0.2255 \mid 0.064, \\ (0.2072 \mid 0.096, 0.2251 \mid 0.064, 0.2270 \mid 0.384, 0.2453 \mid 0.256) \end{pmatrix}$	$(0.5054 \mid 0.3, 0.5357 \mid 0.3, 0.5460 \mid 0.2, 0.5761 \mid 0.2)$
P_{14}	$(0.1192 \mid 0.56, 0.1541 \mid 0.24, 0.0823 \mid 0.14, 0.1068 \mid 0.06)$	$\begin{pmatrix} 0.6013 \mid 0.18, 0.6440 \mid 0.072, 0.6383 \mid 0.042, 0.6802 \mid 0.168, \\ 0.6796 \mid 0.042, 0.7241 \mid 0.168, 0.7113 \mid 0.098, 0.7525 \mid 0.392 \end{pmatrix}$
P_{15}	$\begin{pmatrix} 0.0959 \mid 0.003, 0.1108 \mid 0.027, 0.1076 \mid 0.007, 0.1238 \mid 0.063, \\ (0.1108 \mid 0.027, 0.1261 \mid 0.243, 0.1238 \mid 0.063, 0.1401 \mid 0.567) \end{pmatrix}$	$(0.7006 \mid 0.04, 0.7241 \mid 0.06, 0.7547 \mid 0.36, 0.7745 \mid 0.54)$

Table 5 Processed values of A_1 using WPDHFDMSM operator

	Membership values	Non-membership values
P_{11}	$(0.7006 \mid 0.3, 0.7206 \mid 0.2, 0.7312 \mid 0.3, 0.7502 \mid 0.2)$	$\begin{pmatrix} 0.0804 \mid 0.18, 0.1008 \mid 0.18, 0.0895 \mid 0.12, 0.1121 \mid 0.12, \\ 0.1192 \mid 0.12, 0.1373 \mid 0.12, 0.1325 \mid 0.08, 0.1517 \mid 0.08 \end{pmatrix}$
P_{12}	$(0.7077 \mid 0.12, 0.7312 \mid 0.08, 0.7329 \mid 0.48, 0.7547 \mid 0.32)$	$(0.1225 \mid 0.3, 0.1381 \mid 0.3, 0.1560 \mid 0.2, 0.1722 \mid 0.2)$
P_{13}	$\begin{pmatrix} 0.7745 \mid 0.024, 0.7935 \mid 0.016, 0.7927 \mid 0.096, 0.8110 \mid 0.064, \\ (0.7932 \mid 0.096, 0.8111 \mid 0.064, 0.8114 \mid 0.384, 0.8283 \mid 0.256) \end{pmatrix}$	$(0.0452 \mid 0.3, 0.0546 \mid 0.3, 0.0580 \mid 0.2, 0.0692 \mid 0.2)$
P_{14}	$(0.6746 \mid 0.56, 0.7241 \mid 0.24, 0.6062 \mid 0.14, 0.6539 \mid 0.06)$	$\begin{pmatrix} 0.0798 \mid 0.18, 0.1008 \mid 0.072, 0.0959 \mid 0.042, 0.1196 \mid 0.168, \\ 0.1225 \mid 0.042, 0.1541 \mid 0.168, 0.1396 \mid 0.098, 0.1717 \mid 0.392 \end{pmatrix}$
P_{15}	$\begin{pmatrix} 0.6383 \mid 0.003, 0.6662 \mid 0.027, 0.6601 \mid 0.007, 0.6277 \mid 0.063, \\ (0.6662 \mid 0.027, 0.6926 \mid 0.243, 0.6877 \mid 0.063, 0.7135 \mid 0.567) \end{pmatrix}$	$(0.1367 \mid 0.04, 0.1541 \mid 0.06, 0.1717 \mid 0.36, 0.1890 \mid 0.54)$

From the above ranking it is clear that the alternative A_2 is the finest photovoltaic cell, while A_1 is the worst among all the photovoltaic cell.

6.3 The process of deciding the selection problem of photovoltaic cells with WPDHFDMSM

Step 1: For the construction of the group decision matrix C, utilize the information given by the experts $e^l (l = 1, 2, 3)$ mentioned in Tables 1, 2 and 3, the WPDHFDMSM operator and Eq. 2 to normalize the evaluations values for $m = 2$. Due to a large number of elements present in the decision process, here we only show the detailed calculation of the alternative A_1 concerning the criteria β_2 as follows:

$$
= \left\{ \begin{array}{c}
\left(\begin{array}{c}
1-\left(1-(1-(1-0.6^{0.333})(1-0.2^{0.333}))^{\frac{1}{3}}\times(1-(1-0.6^{0.333})(1-0.3^{0.333}))^{\frac{1}{3}}\times(1-(1-0.2^{0.333})(1-0.3^{0.333}))^{\frac{1}{3}}\right)^{\frac{1}{2}} \mid 1.0\times 0.2\times 0.6, \\
1-\left(1-(1-(1-0.6^{0.333})(1-0.2^{0.333}))^{\frac{1}{3}}\times(1-(1-0.6^{0.333})(1-0.4^{0.333}))^{\frac{1}{3}}\times(1-(1-0.2^{0.333})(1-0.4^{0.333}))^{\frac{1}{3}}\right)^{\frac{1}{2}} \mid 1.0\times 0.2\times 0.4, \\
1-\left(1-(1-(1-0.6^{0.333})(1-0.3^{0.333}))^{\frac{1}{3}}\times(1-(1-0.6^{0.333})(1-0.3^{0.333}))^{\frac{1}{3}}\times(1-(1-0.3^{0.333})(1-0.3^{0.333}))^{\frac{1}{3}}\right)^{\frac{1}{2}} \mid 1.0\times 0.8\times 0.6, \\
1-\left(1-(1-(1-0.6^{0.333})(1-0.3^{0.333}))^{\frac{1}{3}}\times(1-(1-0.6^{0.333})(1-0.4^{0.333}))^{\frac{1}{3}}\times(1-(1-0.3^{0.333})(1-0.4^{0.333}))^{\frac{1}{3}}\right)^{\frac{1}{2}} \mid 1.0\times 0.8\times 0.4
\end{array} \right), \\
\left(\begin{array}{c}
\left(1-(1-(0.1^{0.333}\times 0.4^{0.333}))^{\frac{1}{3}}\times(1-(0.1^{0.333}\times 0.5^{0.333}))^{\frac{1}{3}}\times(1-(0.4^{0.333}\times 0.5^{0.333}))^{\frac{1}{3}}\right)^{\frac{1}{2}} \mid 0.6\times 0.5\times 1.0, \\
\left(1-(1-(0.1^{0.333}\times 0.5^{0.333}))^{\frac{1}{3}}\times(1-(0.1^{0.333}\times 0.5^{0.333}))^{\frac{1}{3}}\times(1-(0.5^{0.333}\times 0.5^{0.333}))^{\frac{1}{3}}\right)^{\frac{1}{2}} \mid 0.6\times 0.5\times 1.0, \\
\left(1-(1-(0.3^{0.333}\times 0.4^{0.333}))^{\frac{1}{3}}\times(1-(0.3^{0.333}\times 0.5^{0.333}))^{\frac{1}{3}}\times(1-(0.4^{0.333}\times 0.5^{0.333}))^{\frac{1}{3}}\right)^{\frac{1}{2}} \mid 0.4\times 0.5\times 1.0, \\
\left(1-(1-(0.3^{0.333}\times 0.5^{0.333}))^{\frac{1}{3}}\times(1-(0.3^{0.333}\times 0.5^{0.333}))^{\frac{1}{3}}\times(1-(0.5^{0.333}\times 0.5^{0.333}))^{\frac{1}{3}}\right)^{\frac{1}{2}} \mid 0.4\times 0.5\times 1.0,
\end{array} \right)
\end{array} \right\}
$$

$$
= \{(0.7077 \mid 0.12, 0.7312 \mid 0.08, 0.7329 \mid 0.48, 0.7547 \mid 0.32),\ (0.1225 \mid 0.3, 0.1381 \mid 0.3, 0.1560 \mid 0.2, 0.1722 \mid 0.2)\}.
$$

All the processed values of alternative A_1 with respect to criteria $\beta_j (j = 1, 2, 3, 4, 5)$ are given in Table 5.

Step 2: Considering the collective decision matrix C, we sum the collective values of the benefit attributes with the help of 4 and pick up $\overline{B}_i (i = 1, 2, 3, 4, 5)$. Due to lack of space, we omit detailed calculations and results.

Step 3: After obtaining the collective values of the alternatives, we find the score of each $\overline{B}_i$ with the help of Definition 4. Similarly, the decision matrix C is used to find the collective values of the cost attributes and determine the values $\overline{S}_i (i = 1, 2, 3, 4, 5)$ with the aid of Eq. 6.

Step 4: Using the Definition 4, we evaluate the score of each alternative. After finding the scores of the alternatives, we find out the RS $R_i (i = 1, 2, 3, 4, 5)$, by using the Eq. 7. According to the Eq. 8, the maximum RS can be determined from the RS R_i and the value of the maximum RS R is given as

$$R = 1.8792.$$

Step 5: Now the utility degree $U_i (i = 1, 2, 3, 4, 5)$ of each alternative can be derived by utilizing Eq. 9.

Step 6: According to the utility degree $U_i (i = 1, 2, 3, 4, 5)$, we find that the ranking of the alternatives $A_i (i = 1, 2, 3, 4, 5)$ is

$$U_2 > U_4 > U_5 > U_3 > U_1$$

Hence the alternatives according to the utility ranking can be categorized as follows:

$$A_2 > A_4 > A_5 > A_3 > A_1.$$

From the above ranking it is also clear that the alternative A_2 is the finest photovoltaic cell, while A_1 is the worst among all the photovoltaic cell.

Table 6 Ranking results with parameters by utilizing the WPDHFMSM-COPRAS method

Parameter	Utility degree	Ranking
$m = 1$	$U_1 = 99.9490,\ U_2 = 100,\ U_3 = 99.9658,$ $U_4 = 99.9652,\ U_5 = 99.9566$	$A_2 > A_3 > A_4 > A_5 > A_1$
$m = 2$	$U_1 = 97.0031,\ U_2 = 100,\ U_3 = 97.6459,$ $U_4 = 98.2698,\ U_5 = 97.9851$	$A_2 > A_4 > A_5 > A_3 > A_1$
$m = 3$	$U_1 = 80.3857,\ U_2 = 100,\ U_3 = 75.4767,$ $U_4 = 85.7054,\ U_5 = 87.5257$	$A_2 > A_5 > A_4 > A_1 > A_3$

Table 7 Ranking results with parameters by utilizing the WPDHFDMSM-COPRAS method

Parameter	Utility degree	Ranking
$m = 1$	$U_1 = 99.9851,\ U_2 = 100,\ U_3 = 99.9912,$ $U_4 = 99.9949,\ U_5 = 99.9942$	$A_2 > A_4 > A_5 > A_3 > A_1$
$m = 2$	$U_1 = 97.8433,\ U_2 = 100,\ U_3 = 98.3839,$ $U_4 = 99.0464,\ U_5 = 98.8465$	$A_2 > A_4 > A_5 > A_3 > A_1$
$m = 3$	$U_1 = 85.4299,\ U_2 = 100,\ U_3 = 86.8011,$ $U_4 = 90.8863,\ U_5 = 94.7614$	$A_2 > A_5 > A_4 > A_3 > A_1$

Table 8 Comparison of ultimate score values and ranking order of alternatives

Existing approaches	Methods/operators	Score values	Ranking
Hao et al (2017)	$A_1 = 0.0862,\ A_2 = 0.1957,\ A_3 = 0.1567,$ $A_4 = 0.0433,\ A_5 = 0.1105$		$A_2 > A_3 > A_5 > A_1 > A_4$
Garg & Kaur (2018)	$A_1 = 0.0498,\ A_2 = 0.1632,\ A_3 = 0.1528,$ $A_4 = -0.406,\ A_5 = 0.0541$		$A_2 > A_3 > A_5 > A_1 > A_4$
Our proposed method with WPDHFMSM	$U_1 = 97.0031,\ U_2 = 100,\ U_3 = 97.6459,$ $U_4 = 98.2698,\ U_5 = 97.9851$		$A_2 > A_4 > A_5 > A_3 > A_1$
Our proposed method with WPDHFDMSM	$U_1 = 97.8433,\ U_2 = 100,\ U_3 = 98.3839,$ $U_4 = 99.0464,\ U_5 = 98.8465$		$A_2 > A_4 > A_5 > A_3 > A_1$

6.4 Stress analysis

We assumed the parameter $m = 2$ throughout the above decision analysis. The final rankings are affected by using the different parametric values. Therefore, we carefully examine the effects of the additional parameter m on the RS values of the alternatives using the WPDHFMSM-COPRAS and the WPDHFDMSM-COPRAS methods; the final results can be seen in Tables 6 and 7. We also investigate the three different values of m, i.e., $m = 1, 2, 3$. The exploration shows that the RS of each alternative calculated by the proposed COPRAS methods differs per parameter m. Nonetheless, according to the different parameters, the alternative A_2 is the best among all the rankings. Analysis reveals that our proposed method is reliable while changing the parameter m. Furthermore, the parameters m found in the proposed approaches may carefully reflect the decision-maker's risk preference.

6.5 Comparative studies

To confirm the superiority and feasibility of our developed methods, we use various MAGDM techniques to resolve the selection problem of the photovoltaic cell. After that, we perform a comparative analysis by considering the same issue. Under the PDHF framework, Hao et al (2017) proposed the PDHF weighted averaging (PDHFWA) operator. Later, Garg & Kaur (2018) developed the PDHF weighted Einstein average (PDHFWEA) operator to investigate the interdependence between attributes in MAGDM. Here we utilize the PDHFWA and PDHFWEA operators to integrate the multiple-preference values of DMs. We carefully compute the scores of alternatives with the aid of the presented methods and compare them with our proposed approaches. Because of the limited space, we have provided direct results.

Noticeably, the approach proposed by Hao et al (2017) by making use of the PDHFWA operator figures out A_2 as the best alternative remains the same as that of our proposed approach, and the least preferred is A_4. Also, the method defined by Garg & Kaur (2018) by utilizing the PDHFWEA operator makes out A_2 as the best alternative remains the same as that of our proposed approach, and the least preferred is A_2. Furthermore, from Table 8, it is clear that the best alternative selected by the proposed technique remains the same as with existing techniques indicating that the proposed methods are correct. Compared with the PDHFWA and PDHFWEA operators, the superiority of the proposed WPDHFMSM and WPDHFDMSM operators is that they can capture the interdependence between the multiple-preference values. In contrast, the existing operators can only present the interrelationship of two arguments. Our proposed operators are a more flexible and broader range of applications. Moreover, our proposed operators have monotonicity concerning the parameter m and can impact the risk attitude of the DMs. In other words, the proposed operators can provide the chance for DMs to choose the appropriate parameter value based on their risk preferences. This fact verifies that the proposed PDHFMSM and PDHFDMSM operators are reasonable and valid for PDHF decision-making problems.

7 Conclusions

The survey shows that existing aggregation operators have failed to address the interdependence of multiple-preference values under the PDHF environment. To examine the PDHF information given by the DMs and conclude more likely results, we put forward several new aggregation operators to manage complex decision problems by utilizing MSM and DMSM operator, namely PDHFMSM and PDHFDMSM operators to resolve the problem of the interdependence between multiple-preference values in MAGDM. To investigate the situation where the weights of arguments are different, we have also proposed these operators' weighted forms, namely, WPDHFMSM and WPDHFDMSM operators. Furthermore, we developed two novel ranking methods, PDHFMSM-COPRAS and PDHFDMSM-COPRAS, by integrating the weighted operators and COPRAS technique to solve the MAGDM problems under the PDHF framework. The main advantages of this study over other methods are summarized below.

1. Primarily, PDHFS depicts complex information, which examines the preliminary information provided by DMs in all possible aspects. Every membership and non-membership value is considered with its associated probability.
2. According to the proportional assessment given by our presented methods, DMs can make decisions based on two features, namely cost and benefit types.
3. Moreover, the methods based on PDHFMSM and PDHFDMSM operators have better adaptability and generality as they can cover other operators by regulating the parameters. The mechanism presented in this study applies to several complex fields.

In future studies, applying the proposed method to solving practical decision-making problems such as green supplier selection, robot selection, and extensive data analysis will be interesting. Furthermore, we can expand these operators (MSM and DMSM) to various environments, for instance, normal wiggly PHFS (2021), interval-valued PHFS. In addition, we will consider the concept of consensus between DMs and use it in addressing MAGDM issues.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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