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Q-rung orthopair fuzzy Frank aggregation operators and its application in multiple attribute decision-making with unknown attribute weights

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Abstract

In decision-making problems, q-rung orthopair fuzzy sets are considered as a more effective tool than intuitionistic fuzzy sets and Pythagorean fuzzy sets. This article develops some aggregation operators based on Frank t-norm and t-conorm for fusing q-rung orthopair fuzzy (q-ROF) information. Then a multiple attribute decision-making (MADM) approach is introduced based on the proposed operators. The Frank operations of t-norm and t-conorm can have the advantage of good flexibility with the operational parameter. From that point of view, in this paper, we extend the ideas of Frank t-norm and t-conorm to the q-ROF environment and introduce some aggregation operators. Moreover, we illustrate the compatible properties of the proposed operators. Generally, the attribute weights are unknown in the MADM problems. The analytical hierarchy process and entropy methods are efficient tools to handle such MADM problems with unknown attribute weights. So, we present an MADM approach with unknown attribute weights under q-ROF environment using proposed operators. Then to elaborate the flexibility and validity of the proposed model, we discuss and solve a numerical problem concerned with a government project of choosing the best way of industrialization. Next, we show how the involvement of the parameters in our proposed model affects the decision-making results. Finally, to exhibit the superiority of our proposed methodology, the obtained results are compared with the existing ones.

Keywords Q-rung orthopair fuzzy sets · Frank operations · Q-rung orthopair fuzzy Frank aggregation operator · q-ROF entropy method · MADM

1 Introduction

Nowadays, due to the availability of so many choices, finding the most promising alternative is a challenging task. MADM is the latest method of finding the best/most suitable choice from the set of available choices. Therefore, the decision-makers generally prefer to use the MADM technique to make their decisions. However, it is necessary to assess attribute values more accurately and conveniently in a decision-making problem. But, due to insufficiency of

information and proper knowledge of real-world scenarios the decision-makers face the challenges of vagueness and uncertainty. To deal with such kind of vagueness and uncertainty of the decision-making it becomes convenient when the attribute values are expressed by fuzzy sets. However, the fuzzy set is not always sufficient to express vague information because it consist of membership degree only. In many situations, the decision-makers provide the degree of membership (DM) along with the degree of nonmembership (DNM) of the attribute concerning the alternative. Atanassov ([1986](#)) introduced intuitionistic fuzzy set (IFS), consisting of DM and DNM such that their sum is bounded by one. During the last four decades, the IFS received a few achievements with the introduction of several aggregation operators. Seikh and Mandal ([2021](#)) proposed an intuitionistic fuzzy Dombi aggregation operator and applied it to solve MADM problems.

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IFS is a significant improvement of the fuzzy set. However, in some real-problems, when the values of the DM and the DNM lies between 0 and 1 but their sum is greater than 1, the IFSs fail to handle such situations. For dealing with such circumstances, Yager (2014) proposed Pythagorean fuzzy set (PFS), associated with DM and DNM such that the sum of their squares is bounded by one. Several studies have been conducted based on this PFS theory. Ejegwa (2019) explored the concept of PFS and applied it in carrier placements based on academic performance with the help of max-min-max composition. Yang and Hussain (2018) proposed new entropies of PFS and extended the concept to σ-entropy to solve multi-criteria decision-making (MCDM) problems. Yager and Abbasov (2013) introduced weighted averaging/geometric operators under the Pythagorean fuzzy environment.

Later, Yager (2017) combined IFS and PFS to generalize them into q-rung orthopair fuzzy set (q-ROFS) which consist of DM and DNM such that the sum of the qth powers of the DM and DNM is less than or equal to 1. The q-ROFSs can capture the uncertain information more precisely and flexibly than IFS and PFS due to the presence of the parameter q. The development and analysis of the properties of q-ROFS have acquired great interest from the researchers (Gao et al. 2019; Peng and Liu 2019; Wang et al. 2020; Shu et al. 2019; Akram and Shahzadi 2020; Akram et al. 2021; Feng et al. 2021). To combine q-ROF information several aggregation operators have also been developed such as Maclaurin symmetric mean operator (Wei et al. 2018), Muirhead mean operator (Wang et al. 2019a) and Hamy mean operator (Wang et al. 2019b).

A fascinating generalization of probabilistic and Lukasiewicz t-norm and t-conorm (Wang and He 2009) are Frank t-norm (FTN) and Frank t-conorm (FTCN) (Frank 1979), which form an ordinary and adequately flexible family of the continuous triangular norms. The employment of a certain parameter not only makes the FTN and FTCN more flexible but also equips to design the pattern of practical decision-making problems. Exploring the additive generating function of FTN, Yager (2004) launched a framework in approximate reasoning with FTN. Comparing the FTNs and the Hamacher t-norms up to an extent, Sarkoci (2005) concluded that the same family contains two different t-norms. Casasnovas and Torrens (2003) proposed an axiomatic approach to the scalar cardinality of FTNs. Several researchers proposed aggregation operators using FTN and FTCN in different fuzzy environments to consolidate uncertain information, for instance, intuitionistic fuzzy environment (Zhang et al. 2015), Pythagorean fuzzy environment (Xing et al. 2018; Yi et al. 2018), hesitant fuzzy environment (Qin et al. 2016), interval-valued neutrosophic fuzzy environment (Zhou et al. 2019) and

triangular interval type-2 fuzzy environment (Qin and Liu 2014).

Frank has developed the operations of t-norm and t-conorm, which are more generally known as Frank operations. These operations will have a better field of applications when presented in a new form of flexibility. Again, the q-ROFS can handle the uncertain information more deliberately. In the above literature survey, we have noticed that the Frank operations are not applied yet, to exhibit q-ROFS in an appropriate form and shape. Motivated by the theory of Frank operations, in the present article, we introduce new operational rules of q-ROFNs based on Frank operations. Then, we propose a series of aggregation operators to aggregate q-ROF information with the assistance of the proposed operational rules. Frank aggregation operators make the information aggregation process more flexible due to the involvement of a parameter. Then we introduce an MADM approach based on the proposed operators with unknown attribute weights.

As each attribute possesses some different aspects, all of them cannot be leveled with the same weight. Therefore, finding the appropriate weight for every attribute is the major aspect in an MADM problem. Generally, in the MADM problems, the attribute weights are considered to be completely known. But, this is not the case in real-world MADM problems. To determine attribute weights for these types of MADM problems several methodologies have been introduced. The most popular among those methods are the analytical hierarchy process (AHP) (Saaty 2008) and the entropy method (Shannon 1948). Here, we utilize AHP and entropy methods to determine the attribute weights. Then, the alternatives of the presented numerical example are to be ranked based on the decision-maker's opinion, where Frank aggregation operators can be used to aggregate the information given by the decision-maker.

The contributions of this paper are summarized as follows:

- Some novel operational laws for q-ROFNs based on FTN and FTCN are introduced.
- Using the proposed FTN and FTCN operational rules a series of aggregation operators of q-ROFNs are defined.
- An MADM technique with q-ROF data and unknown attribute weights is deliberated based on the proposed aggregation operators.
- The validity of the proposed model is verified using it to solve a numerical problem concerned with a government project of choosing the best way of industrialization.
- Our proposed model is compared with several existing methods to show its superiority.

The remaining paper is organized in the following sequence. In Sect. 2, some basic definitions and

prerequisite concepts are recalled. In Sect. 3, some new operational rules for q-ROFNs grounded on FTN and FTCN are proposed. Further, some novel aggregation operators with the help of those operational rules are defined. In Sect. 4, some novel approaches with the help of the q-ROFFWA and q-ROFFWG operators are built, for handling q-ROF MADM problems. In Sect. 5, the validity and accuracy of our proposed method is thoroughly investigated by discussing a real-life situation concerned with a government project of distributing the socio-economic growth among people from place to place through industrialization. The impacts of the parameters on the decision-making results are carefully analysed in Sect. 6. In Sect. 7, the comparisons between our proposed model and other significant well-known approaches is presented. Finally, Sect. 8 concludes the manuscript.

2 Preliminaries

This section is enriched with basic definitions and some prerequisite concepts.

Definition 2.1 (*Yager* 2017). The q-ROFS h in the universal set $\mathbb{X}$ is described as

$$h = \{ <s, \mu_h(s), v_h(s) > | s \in \mathbb{X} \},$$

where $\mu_h : \mathbb{X} \to [0,1]$ and $v_h : \mathbb{X} \to [0,1]$ are called the DM and the DNM to the set h, respectively, with the condition $(\mu_h(s))^q + (v_h(s))^q \leq 1, q \geq 1, \quad \forall s \in \mathbb{X}$. Also, $\sqrt[q]{1 - (\mu_h(s))^q - (v_h(s))^q}$ represents the indeterminacy degree of $s \in \mathbb{X}$. For simplicity, $h = (\mu_h, v_h)$ can be disclosed as a q-rung orthopair fuzzy number (q-ROFN) (Liu and Wang 2018).

Definition 2.2 (*Liu and Wang* 2018). Let $h = (\mu_h, v_h)$ be a q-ROFN. Then the score function $\mathbb{S}$ and the accuracy function $\mathbb{A}$ of h are defined as

$$\mathbb{S}(h) = \frac{1}{2}(1 + \mu_h^q - v_h^q) \tag{1}$$

$$\mathbb{A}(h) = \mu_h^q + v_h^q, \tag{2}$$

where both $\mathbb{S}(h)$ and $\mathbb{A}(h)$ belong to $[0,1]$.

Let $h_1 = (\mu_{h_1}, v_{h_1})$ and $h_2 = (\mu_{h_2}, v_{h_2})$ be two q-ROFNs. Then from Eqs. (1) and (2), we have

1. $h_1 > h_2$ if $\mathbb{S}(h_1) > \mathbb{S}(h_2)$,

2. $h_1 < h_2$ if $\mathbb{S}(h_1) < \mathbb{S}(h_2)$,

3. If $\mathbb{S}(h_1) = \mathbb{S}(h_2)$, and

   - If $\mathbb{A}(h_1) > \mathbb{A}(h_2)$, then $h_1 > h_2$,
   - If $\mathbb{A}(h_1) = \mathbb{A}(h_2)$, then $h_1 = h_2$.

Definition 2.3 (Liu and Wang 2018) The q-ROFNs $h = (\mu_h, v_h)$, $h_1 = (\mu_{h_1}, v_{h_1})$ and $h_2 = (\mu_{h_2}, v_{h_2})$ satisfies the following operational laws:

(1) $h_1 \cup h_2 = (\max\{\mu_{h_1}, \mu_{h_2}\}, \min\{v_{h_1}, v_{h_2}\})$.

(2) $h_1 \cap h_2 = (\min\{\mu_{h_1}, \mu_{h_2}\}, \max\{v_{h_1}, v_{h_2}\})$.

(3) $h^c = (v_h, \mu_h)$.

(4) $h_1 \oplus h_2 = \left(\sqrt[q]{\mu_{h_1}^q + \mu_{h_2}^q - \mu_{h_1}^q \mu_{h_2}^q}, v_{h_1} v_{h_2} \right)$.

(5) $h_1 \otimes h_2 = \left(\mu_{h_1} \mu_{h_2}, \sqrt[q]{v_{h_1}^q + v_{h_2}^q - v_{h_1}^q v_{h_2}^q} \right)$.

(6) $\lambda h = \left(\sqrt[q]{1 - (1 - \mu_h^q)^\lambda}, v_h^\lambda \right), \lambda > 0$.

(7) $h^\lambda = \left(\mu_h^\lambda, \sqrt[q]{1 - (1 - v_h^q)^\lambda} \right), \lambda > 0$.

The FTN and FTCN are stated in the following.

Definition 2.4 (*Frank* 1979). For two real numbers r and s in $[0, 1]$ FTN and FTCN are defined as

$$Fra(r,s) = \log_\tau \left(1 + \frac{(\tau^r - 1)(\tau^s - 1)}{\tau - 1} \right)$$

$$Fra'(r,s) = 1 - \log_\tau \left(1 + \frac{(\tau^{1-r} - 1)(\tau^{1-s} - 1)}{\tau - 1} \right),$$

respectively, where $(r,s) \in [0,1] \times [0,1]$ and $\tau \neq 1$.

Using limit theory, we can obtain the following results (Wang and He 2009):

1. If $\tau \to 1$, then $Fra'(r,s) \to r + s - rs$ and $Fra(r,s) \to rs$, the Frank sum and Frank product reduce to the probabilistic sum and probabilistic product.

2. If $\tau \to \infty$, then $Fra'(r,s) \to min\{r + s, 1\}$ and $Fra(r,s) \to max\{0, r + s - 1\}$, the Frank sum and Frank product transform into the Lukasiewicz sum and Lukasiewicz product.

Example 1 Let $r = 0.60$, $s = 0.40$ and $\tau = 2$, then FTN and FTCN are,

$Fra(0.60, 0.40)$

$$= \log_2\left(1 + \frac{(2^{0.60} - 1)(2^{0.40} - 1)}{2 - 1}\right) = 0.2200 \text{ and}$$

$Fra'(0.60, 0.40)$

$$= 1 - \log_2\left(1 + \frac{(2^{1-0.60} - 1)(2^{1-0.40} - 1)}{2 - 1}\right) = 0.7799.$$

3 q-ROF Frank aggregation operators

We interpret some novel operations based on FTN and FTCN concerning q-ROFNs in the following. Further, we suggest some aggregation operators, namely, q-rung orthopair fuzzy Frank weighted averaging (q-ROFFWA) operator, q-rung orthopair fuzzy Frank order weighted averaging (q-ROFFOWA) operator, q-rung orthopair fuzzy Frank weighted geometric (q-ROFFWG) operator, and q-rung orthopair fuzzy Frank order weighted geometric (q-ROFFOWG) operator.

Definition 3.1 Let λ and τ be two positive real numbers with $\tau \neq 1$. Then, for three q-ROFNs $h = (\mu_h, v_h)$, $h_1 = (\mu_{h_1}, v_{h_1})$ and $h_2 = (\mu_{h_2}, v_{h_2})$ the FTN and FTCN operations are defined as

1.

$h_1 \oplus h_2$

$$= \left(\sqrt[q]{1 - \log_\tau\left(1 + \frac{(\tau^{1-\mu_{h_1}^q} - 1)(\tau^{1-\mu_{h_2}^q} - 1)}{\tau - 1}\right)},\right.$$
$$\left.\sqrt[q]{\log_\tau\left(1 + \frac{(\tau^{v_{h_1}^q} - 1)(\tau^{v_{h_2}^q} - 1)}{\tau - 1}\right)}\right).$$

2.

$h_1 \otimes h_2$

$$= \left(\sqrt[q]{\log_\tau\left(1 + \frac{(\tau^{\mu_{h_1}^q} - 1)(\tau^{\mu_{h_2}^q} - 1)}{\tau - 1}\right)},\right.$$
$$\left.\sqrt[q]{1 - \log_\tau\left(1 + \frac{(\tau^{1-v_{h_1}^q} - 1)(\tau^{1-v_{h_2}^q} - 1)}{\tau - 1}\right)}\right).$$

3.

λh

$$= \left(\sqrt[q]{1 - \log_\tau\left(1 + \frac{(\tau^{1-\mu_h^q} - 1)^\lambda}{(\tau - 1)^{\lambda-1}}\right)},\right.$$
$$\left.\sqrt[q]{\log_\tau\left(1 + \frac{(\tau^{v_h^q} - 1)^\lambda}{(\tau - 1)^{\lambda-1}}\right)}\right).$$

4.

h^λ

$$= \left(\sqrt[q]{\log_\tau\left(1 + \frac{(\tau^{\mu_h^q} - 1)^\lambda}{(\tau - 1)^{\lambda-1}}\right)},\right.$$
$$\left.\sqrt[q]{1 - \log_\tau\left(1 + \frac{(\tau^{1-v_h^q} - 1)^\lambda}{(\tau - 1)^{\lambda-1}}\right)}\right).$$

Example 2 Let $h_1 = (0.85, 0.74)$ and $h_2 = (0.90, 0.66)$ be two q-ROFNs, using Definition 3.1 for $\tau = 2$, $q = 4$ and $\lambda = 3$ we get

1.

$h_1 \oplus h_2$

$$= \left(\sqrt[4]{1 - \log_2\left(1 + \frac{(2^{1-(0.85)^4} - 1)(2^{1-(0.90)^4} - 1)}{2 - 1}\right)},\right.$$
$$\left.\sqrt[4]{\log_2\left(1 + \frac{(2^{(0.74)^4} - 1)(2^{(0.66)^4} - 1)}{2 - 1}\right)}\right)$$
$$= (0.9995, 0.083).$$

2.

$h_1 \otimes h_2$

$$= \left(\sqrt[4]{\log_2\left(1 + \frac{(2^{(0.85)^4} - 1)(2^{(0.90)^4} - 1)}{2 - 1}\right)},\right.$$
$$\left.\sqrt[4]{1 - \log_2\left(1 + \frac{(2^{1-(0.74)^4} - 1)(2^{1-(0.66)^4} - 1)}{2 - 1}\right)}\right)$$
$$= (0.388, 0.962).$$

3.

$$3h_1 = \left(\sqrt[4]{1 - \log_2\left(1 + \frac{(2^{1-(0.85)^4} - 1)^3}{(2 - 1)^{3-1}}\right)},\right.$$
$$\left.\sqrt[4]{\log_2\left(1 + \frac{(2^{(0.74)^4} - 1)^3}{(2 - 1)^{3-1}}\right)}\right) = (0.978, 0.364).$$

4.

$$h_1{}^3 = \left(\sqrt[4]{\log_2 \left(1 + \frac{(2^{(0.85)^4} - 1)^3}{(2-1)^{3-1}} \right)}, \right.$$
$$\left. \sqrt[4]{1 - \log_2 \left(1 + \frac{(2^{1-(0.74)^4} - 1)^3}{(2-1)^{3-1}} \right)} \right) = (0.582, 0.909).$$

Theorem 3.1 *If* $h = (\mu_h, v_h)$, $h_1 = (\mu_{h_1}, v_{h_1})$ *and* $h_2 = (\mu_{h_2}, v_{h_2})$ *be any three q-ROFNs,* $\tau > 1$ *then considering* $\lambda, \lambda_1, \lambda_2$ *as positive real numbers we get*:

1. $h_1 \oplus h_2 = h_2 \oplus h_1$;
2. $h_1 \otimes h_2 = h_2 \otimes h_1$;
3. $\lambda(h_1 \oplus h_2) = \lambda h_1 \oplus \lambda h_2$;
4. $(\lambda_1 + \lambda_2)h = \lambda_1 h \oplus \lambda_2 h$;
5. $(h_1 \otimes h_2)^\lambda = h_1{}^\lambda \otimes h_2{}^\lambda$;
6. $h^{\lambda_1} \otimes h^{\lambda_2} = h^{\lambda_1 + \lambda_2}$.

Proof Using Definition 3.1, we get

1.

$$h_1 \oplus h_2 = \left(\sqrt[q]{1 - \log_\tau \left(1 + \frac{(\tau^{1-\mu_{h_1}^q} - 1)(\tau^{1-\mu_{h_2}^q} - 1)}{\tau - 1} \right)}, \right.$$
$$\left. \sqrt[q]{\log_\tau \left(1 + \frac{(\tau^{v_{h_1}^q} - 1)(\tau^{v_{h_2}^q} - 1)}{\tau - 1} \right)} \right)$$
$$= \left(\sqrt[q]{1 - \log_\tau \left(1 + \frac{(\tau^{1-\mu_{h_2}^q} - 1)(\tau^{1-\mu_{h_1}^q} - 1)}{\tau - 1} \right)}, \right.$$
$$\left. \sqrt[q]{\log_\tau \left(1 + \frac{(\tau^{v_{h_2}^q} - 1)(\tau^{v_{h_1}^q} - 1)}{\tau - 1} \right)} \right)$$
$$= h_2 \oplus h_1.$$

2.

$$h_1 \otimes h_2 = \left(\sqrt[q]{\log_\tau \left(1 + \frac{(\tau^{\mu_{h_1}^q} - 1)(\tau^{\mu_{h_2}^q} - 1)}{\tau - 1} \right)}, \right.$$
$$\left. \sqrt[q]{1 - \log_\tau \left(1 + \frac{(\tau^{1-v_{h_1}^q} - 1)(\tau^{1-v_{h_2}^q} - 1)}{\tau - 1} \right)} \right)$$
$$= \left(\sqrt[q]{\log_\tau \left(1 + \frac{(\tau^{\mu_{h_2}^q} - 1)(\tau^{\mu_{h_1}^q} - 1)}{\tau - 1} \right)}, \right.$$
$$\left. \sqrt[q]{1 - \log_\tau \left(1 + \frac{(\tau^{1-v_{h_2}^q} - 1)(\tau^{1-v_{h_1}^q} - 1)}{\tau - 1} \right)} \right)$$
$$= h_2 \otimes h_1.$$

3.

$$\lambda(h_1 \oplus h_2)$$
$$= \lambda \left(\sqrt[q]{1 - \log_\tau \left(1 + \frac{(\tau^{1-\mu_{h_1}^q} - 1)(\tau^{1-\mu_{h_2}^q} - 1)}{\tau - 1} \right)}, \right.$$
$$\left. \sqrt[q]{\log_\tau \left(1 + \frac{(\tau^{v_{h_1}^q} - 1)(\tau^{v_{h_2}^q} - 1)}{\tau - 1} \right)} \right)$$
$$= \left(\sqrt[q]{1 - \log_\tau \left(1 + \frac{(\tau^{1-\mu_{h_1}^q} - 1)^\lambda (\tau^{1-\mu_{h_2}^q} - 1)}{(\tau - 1)^{2\lambda - 1}} \right)^\lambda}, \right.$$
$$\left. \sqrt[q]{\log_\tau \left(1 + \frac{(\tau^{v_{h_1}^q} - 1)^\lambda (\tau^{v_{h_2}^q} - 1)^\lambda}{(\tau - 1)^{2\lambda - 1}} \right)} \right).$$

Now,

$\lambda h_1 \oplus \lambda h_2$

$$= \left(\sqrt[q]{1 - \log_\tau \left(1 + \frac{(\tau^{1-\mu_{h_1}^q} - 1)^\lambda}{(\tau - 1)^\lambda} \right)}, \right.$$

$$\left. \sqrt[q]{\log_\tau \left(1 + \frac{(\tau^{v_{h_1}^q} - 1)^\lambda}{(\tau - 1)^\lambda} \right)} \right) \oplus$$

$$\left(\sqrt[q]{1 - \log_\tau \left(1 + \frac{(\tau^{1-\mu_{h_2}^q} - 1)^\lambda}{(\tau - 1)^\lambda} \right)}, \right.$$

$$\left. \sqrt[q]{\log_\tau \left(1 + \frac{(\tau^{v_{h_2}^q} - 1)^\lambda}{(\tau - 1)^\lambda} \right)} \right)$$

$$= \left(\sqrt[q]{1 - \log_\tau \left(1 + \frac{(\tau^{1-\mu_{h_1}^q} - 1)^\lambda (\tau^{1-\mu_{h_2}^q} - 1)^\lambda}{(\tau - 1)^{2\lambda - 1}} \right)}, \right.$$

$$\left. \sqrt[q]{\log_\tau \left(1 + \frac{(\tau^{v_{h_1}^q} - 1)^\lambda (\tau^{v_{h_2}^q} - 1)^\lambda}{(\tau - 1)^{2\lambda - 1}} \right)} \right).$$

Therefore, $\lambda(h_1 \oplus h_2) = \lambda h_1 \oplus \lambda h_2$.

4.

$\lambda_1 h \oplus \lambda_2 h$

$$= \left(\sqrt[q]{1 - \log_\tau \left(1 + \frac{(\tau^{1-\mu_h^q} - 1)^{\lambda_1}}{(\tau - 1)^{\lambda}} \right)}, \right.$$

$$\left. \sqrt[q]{\log_\tau \left(1 + \frac{(\tau^{v_h^q} - 1)^{\lambda_1}}{(\tau - 1)^{\lambda}} \right)} \right) \oplus$$

$$\left(\sqrt[q]{1 - \log_\tau \left(1 + \frac{(\tau^{1-\mu_h^q} - 1)^{\lambda_2}}{(\tau - 1)^{\lambda_2}} \right)}, \right.$$

$$\left. \sqrt[q]{\log_\tau \left(1 + \frac{(\tau^{v_h^q} - 1)^{\lambda_2}}{(\tau - 1)^{\lambda_2}} \right)} \right)$$

$$= \left(\sqrt[q]{1 - \log_\tau \left(1 + \frac{(\tau^{1-\mu_h^q} - 1)^{\lambda_1 + \lambda_2}}{(\tau - 1)^{\lambda_1 + \lambda_2}} \right)}, \right.$$

$$\left. \sqrt[q]{\log_\tau \left(1 + \frac{(\tau^{v_h^q} - 1)^{\lambda_1 + \lambda_2}}{(\tau - 1)^{\lambda_1 + \lambda_2}} \right)} \right)$$

$$= (\lambda_1 + \lambda_2) h.$$

5.

$(h_1 \otimes h_2)^\lambda$

$$= \left(\sqrt[q]{\log_\tau \left(1 + \frac{(\tau^{\mu_{h_1}^q} - 1)(\tau^{\mu_{h_2}^q} - 1)}{\tau - 1} \right)}, \right.$$

$$\left. \sqrt[q]{1 - \log_\tau \left(1 + \frac{(\tau^{1-v_{h_1}^q} - 1)(\tau^{1-v_{h_2}^q} - 1)}{\tau - 1} \right)} \right)^\lambda$$

$$= \left(\sqrt[q]{\log_\tau \left(1 + \frac{((\tau^{\mu_{h_1}^q} - 1)(\tau^{\mu_{h_2}^q} - 1))^\lambda}{(\tau - 1)^{2\lambda - 1}} \right)}, \right.$$

$$\left. \sqrt[q]{1 - \log_\tau \left(1 + \frac{((\tau^{1-v_{h_1}^q} - 1)(\tau^{1-v_{h_2}^q} - 1))^\lambda}{(\tau - 1)^{2\lambda - 1}} \right)} \right)$$

$$= \left(\sqrt[q]{\log_\tau \left(1 + \frac{(\tau^{\mu_{h_1}^q} - 1)^\lambda}{(\tau - 1)^\lambda} \right)}, \right.$$

$$\left. \sqrt[q]{1 - \log_\tau \left(1 + \frac{(\tau^{1-v_{h_1}^q} - 1)^\lambda}{(\tau - 1)^\lambda} \right)} \right)$$

$$\otimes \left(\sqrt[q]{\log_\tau \left(1 + \frac{(\tau^{\mu_{h_2}^q} - 1)^\lambda}{(\tau - 1)^\lambda} \right)}, \right.$$

$$\left. \sqrt[q]{1 - \log_\tau \left(1 + \frac{(\tau^{1-v_{h_2}^q} - 1)^\lambda}{(\tau - 1)^\lambda} \right)} \right) = h_1^\lambda \otimes h_2^\lambda.$$

6.

$h^{\lambda_1} \otimes h^{\lambda_2}$

$$= \left(\sqrt[q]{\log_\tau \left(1 + \frac{(\tau^{\mu_h^q} - 1)^{\lambda_1}}{(\tau - 1)^{\lambda_1 - 1}} \right)}, \right.$$

$$\left. \sqrt[q]{1 - \log_\tau \left(1 + \frac{(\tau^{1-v_h^q} - 1)^{\lambda_1}}{(\tau - 1)^{\lambda_1 - 1}} \right)} \right)$$

$$\otimes \left(\sqrt[q]{\log_\tau \left(1 + \frac{(\tau^{\mu_h^q} - 1)^{\lambda_2}}{(\tau - 1)^{\lambda_2 - 1}} \right)}, \right.$$

$$\left. \sqrt[q]{1 - \log_\tau \left(1 + \frac{(\tau^{1-v_h^q} - 1)^{\lambda_2}}{(\tau - 1)^{\lambda_2 - 1}} \right)} \right)$$

$$= \left(\sqrt[q]{\log_\tau \left(1 + \frac{(\tau^{\mu_h^q} - 1)^{\lambda_1 + \lambda_2}}{(\tau - 1)^{\lambda_1 + \lambda_2 - 1}} \right)}, \right.$$

$$\left. \sqrt[q]{1 - \log_\tau \left(1 + \frac{(\tau^{1-v_h^q} - 1)^{\lambda_1 + \lambda_2}}{(\tau - 1)^{\lambda_1 + \lambda_2 - 1}} \right)} \right) = h^{\lambda_1 + \lambda_2}.$$

$\square$

Next, we propose arithmetic aggregation operators utilizing FTN and FTCN.

3.1 q-ROF Frank arithmetic aggregation operators

Definition 3.2 Let $h_x = (\mu_{h_x}, v_{h_x})(x = 1, 2, \ldots, m)$ be a number of q-ROFNs with their associated weight $\partial_x(x = 1, 2, \ldots, m)$, satisfying $\partial_x \in [0, 1]$ and $\sum_{x=1}^{m} \partial_x = 1$. Then the operator q-ROFFWA : $h^m \to h$ is defined as

$$q\text{-ROFFWA}(h_1, h_2, \ldots, h_m) = \bigoplus_{x=1}^{m} \partial_x h_x.$$

Hence, we get consequential theorem which obeys Frank operations on q-ROFNs.

Theorem 3.2 *The aggregated value of a number of q-ROFNs* $h_x = (\mu_{h_x}, v_{h_x})(x = 1, 2, \ldots, m)$ *utilizing* q-*ROFFWA operator is still a q-ROFN, and*

$$q\text{-ROFFWA}(h_1, h_2, \ldots, h_m)$$
$$= \bigoplus_{x=1}^{m} \partial_x h_x$$
$$= \left(\sqrt[q]{1 - \log_\tau \left(1 + \prod_{x=1}^{m} (\tau^{1-\mu_{h_x}^q} - 1)^{\partial_x} \right)}, \right.$$
$$\left. \sqrt[q]{\log_\tau \left(1 + \prod_{x=1}^{m} (\tau^{v_{h_x}^q} - 1)^{\partial_x} \right)} \right).$$

Proof We establish this theorem with the help of mathematical induction method.

For $m = 2$, using Definition 3.2, we get

$$q\text{-ROFFWA}(h_1, h_2)$$
$$= \bigoplus_{x=1}^{2} \partial_x h_x = \partial_1 h_1 \oplus \partial_2 h_2$$
$$= \left(\sqrt[q]{1 - \log_\tau \left(1 + \frac{(\tau^{1-\mu_{h_1}^q} - 1)^{\partial_1}}{(\tau - 1)^{\partial_1 - 1}} \right)}, \right.$$
$$\left. \sqrt[q]{\log_\tau \left(1 + \frac{(\tau^{v_{h_1}^q} - 1)^{\partial_1}}{(\tau - 1)^{\partial_1 - 1}} \right)} \right)$$
$$\oplus \left(\sqrt[q]{1 - \log_\tau \left(1 + \frac{(\tau^{1-\mu_{h_2}^q} - 1)^{\partial_2}}{(\tau - 1)^{\partial_2 - 1}} \right)}, \right.$$
$$\left. \sqrt[q]{\log_\tau \left(1 + \frac{(\tau^{v_{h_2}^q} - 1)^{\partial_2}}{(\tau - 1)^{\partial_2 - 1}} \right)} \right)$$
$$= \left(\sqrt[q]{1 - \log_\tau \left(1 + \prod_{x=1}^{2} (\tau^{1-\mu_{h_x}^q} - 1)^{\partial_x} \right)}, \right.$$
$$\left. \sqrt[q]{\log_\tau \left(1 + \prod_{x=1}^{2} (\tau^{v_{h_x}^q} - 1)^{\partial_x} \right)} \right) \left[\because \sum_{x=1}^{2} \partial_x = 1 \right].$$

Therefore, for $m = 2$, the result is true. Suppose the result is valid for $m = s$; i.e., we assume

$$q\text{-ROFFWA}(h_1, h_2, \ldots, h_s)$$
$$= \bigoplus_{x=1}^{s} \partial_x h_x$$
$$= \left(\sqrt[q]{1 - \log_\tau \left(1 + \prod_{x=1}^{s} (\tau^{1-\mu_{h_x}^q} - 1)^{\partial_x} \right)}, \right.$$
$$\left. \sqrt[q]{\log_\tau \left(1 + \prod_{x=1}^{s} (\tau^{v_{h_x}^q} - 1)^{\partial_x} \right)} \right).$$

Now, for $m = s + 1$,

q-ROFFWA$(h_1, h_2, \ldots, h_s, h_{s+1})$

$$= \bigoplus_{x=1}^{s+1} \eth_x h_x = \bigoplus_{x=1}^{s} \eth_x h_x \bigoplus \eth_{s+1} h_{s+1}$$

$$= \left(\sqrt[q]{1 - \log_\tau \left(1 + \frac{\prod\limits_{x=1}^{s} (\tau^{1-\mu_{h_x}^q} - 1)^{\eth_x}}{(\tau - 1)^{\sum\limits_{x=1}^{s} \eth_x - 1}} \right)}, \right.$$

$$\left. \sqrt[q]{\log_\tau \left(1 + \frac{\prod\limits_{x=1}^{s} (\tau^{v_{h_x}^q} - 1)^{\eth_x}}{(\tau - 1)^{\sum\limits_{x=1}^{s} \eth_x - 1}} \right)} \right)$$

$$\bigoplus \left(\sqrt[q]{1 - \log_\tau \left(1 + \frac{(\tau^{1-\mu_{h_{s+1}}^q} - 1)^{\eth_{s+1}}}{(\tau - 1)^{\eth_{s+1} - 1}} \right)}, \right.$$

$$\left. \sqrt[q]{\log_\tau \left(1 + \frac{(\tau^{v_{h_{s+1}}^q} - 1)^{\eth_{s+1}}}{(\tau - 1)^{\eth_{s+1} - 1}} \right)} \right)$$

$$= \left(\sqrt[q]{1 - \log_\tau \left(1 + \prod_{x=1}^{s+1} (\tau^{1-\mu_{h_x}^q} - 1)^{\eth_x} \right)}, \right.$$

$$\left. \sqrt[q]{\log_\tau \left(1 + \prod_{x=1}^{s+1} (\tau^{v_{h_x}^q} - 1)^{\eth_x} \right)} \right) \left[\because \sum_{x=1}^{s+1} \eth_x = 1 \right].$$

Therefore, if the statement is true for $m = s$, then it will be true for its successor, $s + 1$.

Hence, by the induction hypothesis the given result is valid for all positive integers m. $\qquad\square$

Theorem 3.3 (*Idempotency Property*). *If the q-ROFNs* $h_x = (\mu_{h_x}, v_{h_x})(x = 1, 2, \ldots, m)$ *are identical, i.e.,* $h_x = h$ *for all* x, *where* $h = (\mu_h, v_h)$, *then*

q-ROFFWA$(h_1, h_2, \ldots, h_m) = h$.

Proof As $h_x = h$, for all x, then we obtain

q-ROFFWA$(h_1, h_2, \ldots, h_m)$

$$= \left(\sqrt[q]{1 - \log_\tau \left(1 + \prod_{x=1}^{m} (\tau^{1-\mu_{h_x}^q} - 1)^{\eth_x} \right)}, \right.$$

$$\left. \sqrt[q]{\log_\tau \left(1 + \prod_{x=1}^{m} (\tau^{v_{h_x}^q} - 1)^{\eth_x} \right)} \right)$$

$$= \left(\sqrt[q]{1 - \log_\tau \left(1 + \prod_{x=1}^{m} (\tau^{1-\mu_{h}^q} - 1)^{\eth_x} \right)}, \right.$$

$$\left. \sqrt[q]{\log_\tau \left(1 + \prod_{x=1}^{m} (\tau^{v_{h}^q} - 1)^{\eth_x} \right)} \right)$$

$$= \left(\sqrt[q]{1 - \log_\tau \left(1 + (\tau^{1-\mu_{h}^q} - 1)^{\sum\limits_{x=1}^{m} \eth_x} \right)}, \right.$$

$$\left. \sqrt[q]{\log_\tau \left(1 + (\tau^{v_{h}^q} - 1)^{\sum\limits_{x=1}^{m} \eth_x} \right)} \right)$$

$$= \left(\sqrt[q]{1 - \log_\tau \left(1 + (\tau^{1-\mu_{h}^q} - 1) \right)}, \right.$$

$$\left. \sqrt[q]{\log_\tau \left(1 + (\tau^{v_{h}^q} - 1) \right)} \right)$$

$$= \left(\sqrt[q]{\mu_h^q}, \sqrt[q]{v_h^q} \right) = (\mu_h, v_h) = h.$$

Hence, the result follows. $\qquad\square$

Theorem 3.4 (*Boundedness property*). *Let* $h_x = (\mu_{h_x}, v_{h_x})(x = 1, 2, \ldots, m)$ *be a number of q-ROFNs. If* $h^- = min\{h_1, h_2, \ldots, h_m\}$ *and* $h^+ = max\{h_1, h_2, \ldots, h_m\}$, *then*

$h^- \leq q$-ROFFWA$(h_1, h_2, \ldots, h_m) \leq h^+$.

Proof Let $h^- = (\mu^-, v^-)$ and $h^+ = (\mu^+, v^+)$. Therefore, we have $\mu^- = \min_x\{\mu_{h_x}\}$, $v^- = \max_x\{v_{h_x}\}$, $\mu^+ = \max_x\{\mu_{h_x}\}$ and $v^+ = \min_x\{v_{h_x}\}$.

Now,

$$\sqrt[q]{1 - \log_\tau \left(1 + \prod_{x=1}^{m} (\tau^{1-(\mu^-)^q} - 1)^{\eth_x} \right)}$$

$$\leq \sqrt[q]{1 - \log_\tau \left(1 + \prod_{x=1}^{m} (\tau^{1-\mu_{h_x}^q} - 1)^{\eth_x} \right)}$$

$$\leq \sqrt[q]{1 - \log_\tau \left(1 + \prod_{x=1}^{m} (\tau^{1-(\mu^+)^q} - 1)^{\eth_x} \right)}$$

and

$$\sqrt[q]{\log_\tau \left(1 + \prod_{x=1}^{m} (\tau^{(v^+)^q} - 1)^{\partial_x}\right)}$$

$$\leq \sqrt[q]{\log_\tau \left(1 + \prod_{x=1}^{m} (\tau^{v_{h_x}^q} - 1)^{\partial_x}\right)}$$

$$\leq \sqrt[q]{\log_\tau \left(1 + \prod_{x=1}^{m} (\tau^{(v^-)^q} - 1)^{\partial_x}\right)}.$$

Therefore,

$$h^- \leq q\text{-ROFFWA}(h_1, h_2, \ldots, h_m) \leq h^+.$$

$\square$

Theorem 3.5 (*Monotonicity property*) *Let* $\{h_x | x = 1, 2, \ldots, m\}$ *and* $\{h'_x | x = 1, 2, \ldots, m\}$ *be two sets of q-ROFNs, where* $h_x = (\mu_{h_x}, v_{h_x})$ *and* $h'_x = (\mu'_{h_x}, v'_{h_x})$ *for* $x = 1, 2, \ldots, m$. *If* $\mu_{h_x} \leq \mu'_{h_x}$ *and* $v_{h_x} \geq v'_{h_x}$ *for all x, then, q-ROFFWA*$(h_1, h_2, \ldots, h_m) \leq$ *q-ROFFWA*$(h'_1, h'_2, \ldots, h'_m)$.

Proof Since $\mu_{h_x} \leq \mu'_{h_x}$ and $v_{h_x} \geq v'_{h_x}$ for all $x = 1, 2, \ldots, m$, then

$$(\tau^{1-\mu_{h_x}^q} - 1)^{\partial_x}$$

$$\geq (\tau^{1-\mu'^q_{h_x}} - 1)^{\partial_x} \Rightarrow \log_\tau \left(1 + \prod_{x=1}^{m} (\tau^{1-\mu_{h_x}^q} - 1)^{\partial_x}\right)$$

$$\geq \log_\tau \left(1 + \prod_{x=1}^{m} (\tau^{1-\mu'^q_{h_x}} - 1)^{\partial_x}\right)$$

$$\Rightarrow \sqrt[q]{1 - \log_\tau \left(1 + \prod_{x=1}^{m} (\tau^{1-\mu_{h_x}^q} - 1)^{\partial_x}\right)}$$

$$\leq \sqrt[q]{1 - \log_\tau \left(1 + \prod_{x=1}^{m} (\tau^{1-\mu'^q_{h_x}} - 1)^{\partial_x}\right)}.$$

Similarly, it can be shown that

$$\sqrt[q]{\log_\tau \left(1 + \prod_{x=1}^{m} (\tau^{v_{h_x}^q} - 1)^{\partial_x}\right)} \geq$$

$$\sqrt[q]{\log_\tau \left(1 + \prod_{x=1}^{m} (\tau^{v'^q_{h_x}} - 1)^{\partial_x}\right)}.$$

Thus,

$$\left(\sqrt[q]{1 - \log_\tau \left(1 + \prod_{x=1}^{m} (\tau^{1-\mu_{h_x}^q} - 1)^{\partial_x}\right)}\right)^q$$

$$- \left(\sqrt[q]{\log_\tau \left(1 + \prod_{x=1}^{m} (\tau^{v_{h_x}^q} - 1)^{\partial_x}\right)}\right)^q$$

$$\leq \left(\sqrt[q]{1 - \log_\tau \left(1 + \prod_{x=1}^{m} (\tau^{1-\mu'^q_{h_x}} - 1)^{\partial_x}\right)}\right)^q$$

$$- \left(\sqrt[q]{\log_\tau \left(1 + \prod_{x=1}^{m} (\tau^{v'^q_{h_x}} - 1)^{\partial_x}\right)}\right)^q.$$

Let $h = q\text{-ROFFWA}(h_1, h_2, \ldots, h_m)$ and $h' = q\text{-ROFFWA}(h'_1, h'_2, \ldots, h'_m)$. Then by Definition 2.2, we have $\mathbb{S}(h) \leq \mathbb{S}(h')$.

(I) If $\mathbb{S}(h) < \mathbb{S}(h')$ then we have,
$h < h'$
i.e.,$q\text{-ROFFWA}(h_1, h_2, \ldots, h_m) < q\text{-ROFFWA}(h'_1, h'_2, \ldots, h'_m)$.

(II) If $\mathbb{S}(h) = \mathbb{S}(h')$ then, we get

$$\left(\sqrt[q]{1 - \log_\tau \left(1 + \prod_{x=1}^{m} (\tau^{1-\mu_{h_x}^q} - 1)^{\partial_x}\right)}\right)^q$$

$$- \left(\sqrt[q]{\log_\tau \left(1 + \prod_{x=1}^{m} (\tau^{v_{h_x}^q} - 1)^{\partial_x}\right)}\right)^q$$

$$= \left(\sqrt[q]{1 - \log_\tau \left(1 + \prod_{x=1}^{m} (\tau^{1-\mu'^q_{h_x}} - 1)^{\partial_x}\right)}\right)^q$$

$$- \left(\sqrt[q]{\log_\tau \left(1 + \prod_{x=1}^{m} (\tau^{v'^q_{h_x}} - 1)^{\partial_x}\right)}\right)^q,$$

then, by the condition $\mu_{h_x} \leq \mu'_{h_x}$ and $v_{h_x} \geq v'_{h_x}$ for all $x = 1, 2, \ldots, m$, we have

$$\left(\sqrt[q]{1 - \log_\tau \left(1 + \prod_{x=1}^{m} (\tau^{1-\mu_{h_x}^q} - 1)^{\partial_x}\right)}\right)^q$$

$$= \left(\sqrt[q]{1 - \log_\tau \left(1 + \prod_{x=1}^{m} (\tau^{1-\mu'^q_{h_x}} - 1)^{\partial_x}\right)}\right)^q$$

and

$$\left(\sqrt[q]{\log_\tau \left(1 + \prod_{x=1}^{m} (\tau^{v_{h_x}^q} - 1)^{\partial_x}\right)}\right)^q$$

$$= \left(\sqrt[q]{\log_\tau \left(1 + \prod_{x=1}^{m} (\tau^{v'^q_{h_x}} - 1)^{\partial_x}\right)}\right)^q.$$

So, from Eq. (2), we have

$$\mathbb{A}(h) = \left(\sqrt[q]{1 - \log_\tau \left(1 + \prod_{x=1}^{m} (\tau^{1-\mu_{h_x}^q} - 1)^{\partial_x}\right)}\right)^q$$

$$+ \left(\sqrt[q]{\log_\tau \left(1 + \prod_{x=1}^{m} (\tau^{\mu_{h_x}^q} - 1)^{\partial_x}\right)}\right)^q$$

$$= \left(\sqrt[q]{1 - \log_\tau \left(1 + \prod_{x=1}^{m} (\tau^{1-\mu'^q_{h_x}} - 1)^{\partial_x}\right)}\right)^q$$

$$+ \left(\sqrt[q]{\log_\tau \left(1 + \prod_{x=1}^{m} (\tau^{v'^q_{h_x}} - 1)^{\partial_x}\right)}\right)^q = \mathbb{A}(h').$$

Therefore, from (I) and (II), we get

q-ROFFWA$(h_1, h_2, \ldots, h_m) \leq q$-ROFFWA
$(h'_1, h'_2, \ldots, h'_m)$.

$\square$

Next, we develop q-ROF Frank geometric geometric aggregation operators utilizing FTN and FTCN as follows:

3.2 q-ROF Frank geometric aggregation operators

Definition 3.3 Let $h_x = (\mu_{h_x}, v_{h_x})(x = 1, 2, \ldots, m)$ be a number of q-ROFNs with their associated weight vectors $\partial_x(x = 1, 2, \ldots, m)$, satisfying $\partial_x \in [0, 1]$ and $\sum_{x=1}^{m} \partial_x = 1$. Then the operator q-ROFFWG $: h^m \to h$ is defined as

q-ROFFWG$(h_1, h_2, \ldots, h_m) = \bigotimes_{x=1}^{m}(h_x)^{\partial_x}$.

Hence, we get the consequential theorem that follows the Frank operations on q-ROFNs.

Theorem 3.6 *The aggregated value of a number of q-ROFNs $h_x = (\mu_{h_x}, v_{h_x})(x = 1, 2, \ldots, m)$ utilizing q-ROFFWG operator is still a q-ROFN, and*

q-ROFFWG$(h_1, h_2, \ldots, h_m)$
$$= \bigotimes_{x=1}^{m}(h_x)^{\partial_x}$$
$$= \left(\sqrt[q]{\log_\tau \left(1 + \prod_{x=1}^{m} (\tau^{\mu_{h_x}^q} - 1)^{\partial_x} \right)}, \right.$$
$$\left. \sqrt[q]{1 - \log_\tau \left(1 + \prod_{x=1}^{m} (\tau^{1 - v_{h_x}^q} - 1)^{\partial_x} \right)} \right).$$

Proof The proof of this theorem emulates from Theorem 3.2. $\square$

The following properties can be easily proved for q-ROFFWG operator.

Theorem 3.7 (*Idempotency property*). *If the q-ROFNs $h_x = (\mu_{h_x}, v_{h_x})(x = 1, 2, \ldots, m)$ be identical, i.e., $h_x = h$ for all x, where $h = (\mu_h, v_h)$, then q-ROFFWG$(h_1, h_2, \ldots, h_m) = h$.*

Theorem 3.8 (*Boundedness Property*). *Let $h_x = (\mu_{h_x}, v_{h_x})(x = 1, 2, \ldots, m)$ be a number of q-ROFNs. If $h^- = min\{h_1, h_2, \ldots, h_m\}$ and $h^+ = max\{h_1, h_2, \ldots, h_m\}$, then*

$h^- \leq q$-ROFFWG$(h_1, h_2, \ldots, h_m) \leq h^+$.

Theorem 3.9 (*Monotonicity property*). *Let $\{h_x | x = 1, 2, \ldots, m\}$ and $\{h'_x | x = 1, 2, \ldots, m\}$ be two sets of q-ROFNs, where $h_x = (\mu_{h_x}, v_{h_x})$ and $h'_x = (\mu'_{h_x}, v'_{h_x})$ for $x = 1, 2, \ldots, m$. If $\mu_{h_x} \leq \mu'_{h_x}$ and $v_{h_x} \geq v'_{h_x}$ for all x, then q-ROFFWG$(h_1, h_2, \ldots, h_m) \leq q$-ROFFWG$(h'_1, h'_2, \ldots, h'_m)$.*

3.3 q-ROF order weighted averaging/geometric aggregation operators

Now, we would like to introduce q-ROFFOWA and q-ROFFOWG operators.

Definition 3.4 Let $h_x = (\mu_{h_x}, v_{h_x})(x = 1, 2, \ldots, m)$ be a number of q-ROFNs and $\partial_x(x = 1, 2, \ldots, m)$ be the associated weight vectors satisfying $\partial_x \in [0, 1]$ and $\sum_{x=1}^{m} \partial_x = 1$. Then q-ROFFOWA and q-ROFFOWG operators are the functions from h^m to h defined by,

q-ROFFOWA$(h_1, h_2, \ldots, h_m) = \bigoplus_{x=1}^{m} \partial_x h_{\alpha(x)}$
q-ROFFOWG$(h_1, h_2, \ldots, h_m) = \bigotimes_{x=1}^{m} (h_{\alpha(x)})^{\partial_x}$,

respectively, where $(\alpha(1), \alpha(2), \ldots, \alpha(m))$ is the permutation of $(1, 2, \ldots, m)$, satisfying $h_{\alpha(x-1)} \geq h_{\alpha(x)}$ for all $x = 1, 2, \ldots, m$.

The following theorem is constructed on the ground of Frank operations of q-ROFNs.

Theorem 3.10 *The aggregated value of a number of q-ROFNs $h_x = (\mu_{h_x}, v_{h_x})(x = 1, 2, \ldots, m)$ utilizing q-ROFFOWA and q-ROFFOWG operators are still a q-ROFNs, and*

q-ROFFOWA$(h_1, h_2, \ldots, h_m) = \bigoplus_{x=1}^{m} \partial_x h_{\alpha(x)} =$
$$\left(\sqrt[q]{1 - \log_\tau \left(1 + \prod_{x=1}^{m} (\tau^{1 - \mu_{h_{\alpha(x)}}^q} - 1)^{\partial_x} \right)}, \right.$$
$$\left. \sqrt[q]{\log_\tau \left(1 + \prod_{x=1}^{m} (\tau^{v_{h_{\alpha(x)}}^q} - 1)^{\partial_x} \right)} \right)$$

q-ROFFOWG$(h_1, h_2, \ldots, h_m) = \bigotimes_{x=1}^{m} (h_{\alpha(x)})^{\partial_x} =$
$$\left(\sqrt[q]{\log_\tau \left(1 + \prod_{x=1}^{m} (\tau^{\mu_{h_{\alpha(x)}}^q} - 1)^{\partial_x} \right)}, \right.$$
$$\left. \sqrt[q]{1 - \log_\tau \left(1 + \prod_{x=1}^{m} (\tau^{1 - v_{h_{\alpha(x)}}^q} - 1)^{\partial_x} \right)} \right).$$

Proof The proof of this theorem emulates from Theorem 3.2. $\square$

The following properties can be ascertained easily with the aid of q-ROFFOWA and q-ROFFOWG operators.

Theorem 3.11 (*Idempotency property*). *If all the q-ROFNs* $h_x = (\mu_{h_x}, v_{h_x})(x = 1, 2, \ldots, m)$ *be identical, i.e.,* $h_x = h$ *for all* x, *where* $h = (\mu_h, v_h)$, *then* q -ROFFOWA $(h_1, h_2, \ldots, h_m) = h$ *and* q -ROFFOWG$(h_1, h_2, \ldots, h_m) = h.$

Theorem 3.12 (*Boundedness property*). *Let* $h_x = (\mu_{h_x}, v_{h_x})(x = 1, 2, \ldots, m)$ *be a number of q-ROFNs. If* $h^- = min\{h_1, h_2, \ldots, h_m\}$ *and* $h^+ = max\{h_1, h_2, \ldots, h_m\}$, *then*

$$h^- \leq q\text{-ROFFOWA}(h_1, h_2, \ldots, h_m) \leq h^+, \text{and}$$

$$h^- \leq q\text{-ROFFOWG}(h_1, h_2, \ldots, h_m) \leq h^+.$$

Theorem 3.13 (*Monotonicity property*). *Let* $\{h_x | x = 1, 2, \ldots, m\}$ *and* $\{h'_x | x = 1, 2, \ldots, m\}$ *be two sets of q-ROFNs, where* $h_x = (\mu_{h_x}, v_{h_x})$ *and* $h'_x = (\mu'_{h_x}, v'_{h_x})$ *for* $x = 1, 2, \ldots, m$. *If* $\mu_{h_x} \leq \mu'_{h_x}$ *and* $v_{h_x} \geq v'_{h_x}$ *for all* x, *then,* q -ROFFOWA$(h_1, h_2, \ldots, h_m) \leq q$-ROFFOWA$(h'_1, h'_2, \ldots, h'_m)$, *and* q-ROFFOWG$(h_1, h_2, \ldots, h_m) \leq q$ -ROFFOWG$(h'_1, h'_2, \ldots, h'_m).$

Theorem 3.14 (*Commutative property*). *Let* $\{h_x | x = 1, 2, \ldots, m\}$ *and* $\{h'_x | x = 1, 2, \ldots, m\}$ *be two sets of q-ROFNs, then* q-ROFFOWA$(h_1, h_2, \ldots, h_m) = q$-ROFFOWA $(h'_1, h'_2, \ldots, h'_m)$, *and* q-ROFFOWG $(h_1, h_2, \ldots, h_m) = q$-ROFFOWG$(h'_1, h'_2, \ldots, h'_m)$ *where* h'_x *is any permutation of* $h_x(x = 1, 2, \ldots, m).$

4 Application to MADM with q-ROF information

Here, we discuss an approach for solving MADM problems using our proposed aggregation operators, where the attribute values are expressed by q-ROFNs and attribute weights are unknown. We utilize q-ROF information and manipulate q-ROFFWA and q-ROFFWG operators. Let $\mathcal{V} = \{\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_l\}$ be a discrete set of l alternatives available to the decision-makers to be chosen and $R =$ $\{R_1, R_2, \ldots, R_n\}$ be the set of attributes to be considered. The attribute weights are determined by the AHP and entropy methods. The procedure of finding attribute weights using the AHP and the entropy method is discussed later. Let us consider that the weight vector corresponding to the attribute $R_b(b = 1, 2, \ldots, n)$ be $\partial_b(b = 1, 2, 3, \ldots, n)$, where $\partial_b > 0$ and $\sum_{b=1}^{n} \partial_b = 1$. Suppose that $P = ((\eta_{ab}))_{l \times n} = ((\mu_{ab}, v_{ab}))_{l \times n}$ is the decision matrix, where η_{ab} is the evaluation value of alternative $\mathcal{V}_a$ with respect to attribute R_b. Here η_{ab} is a q-ROFN where $(\mu_{ab})^q + (v_{ab})^q \leq 1$ and $\mu_{ab} \in [0, 1], v_{ab} \in [0, 1]$.

4.1 Analytical hierarchy process

The AHP method developed by Saaty ([2008](#)), is a useful tool in a variety of decision-making situations. The AHP helps decision-makers to trace the one, which fulfils their goals and meets their understanding about the problem. In AHP, a pairwise comparison among the attributes, selected by the decision-makers, is made to prioritize among the attributes. The AHP model consists of the following steps:

Step 1: Construct pairwise comparison matrix B.

$$B = \begin{pmatrix} 1 & r_{11} & r_{12} & \ldots & r_{1p} \\ r_{21} & 1 & r_{23} & \ldots & r_{2p} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r_{p1} & 1 & r_{p3} & \ldots & 1 \end{pmatrix}$$

The matrix B is a square matrix of order p, when p is equal to the total number of attributes selected by the decision-maker. In the above matrix, the entry r_{ab} stands for the significance of ath attribute relative to the bth attribute. $r_{ab} < 1$ if the ath attribute is less important than b^{th} attribute, on the contrary $r_{ab} > 1$ if the ath attribute is more important than b^{th} attribute. Again $r_{ab} = 1$, when two attributes are equally important. The comparative importance of any two attributes (taken from the collection of p attributes) is measured with a scale from 1 to 9, which is exhibited in Table 1.

Step:2 Determine the normalized pairwise comparison matrix $\hat{B} = (\hat{r}_{ab})_{p \times p}$, where $\hat{r}_{ab}$ is calculated using Eq. (3)

$$\hat{r}_{ab} = \frac{r_{ab}}{\sum_{a=1}^{p} r_{ab}}. \tag{3}$$

Step:3 Weights of the attributes $\partial_b(b = 1, 2, \ldots, p)$ are calculated, utilizing Eq. (4)

Table 1 Scale of importance

Intensity of importance	Definition
1	Same importance
3	Medium importance
5	High importance
7	Very high importance
9	Extreme importance
2,4,6,8	Intermediate values
$\frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}$	Value for inverse comparison

$$\eth_b = \frac{\sum_{b=1}^{p} \hat{r}_{ab}}{p}. \tag{4}$$

4.2 q-ROF entropy method

Entropy is the measure of randomness and disorder in the universe. The entropy method proposed by Shannon (1948) is applied for defining the unknown weights of the attributes when the information about the decision matrix is known. Here, we extended Shannon's entropy method under q-ROF information.

Step 1: Constitute the q-ROF decision matrix $P = ((\tau_{ab}))_{l \times n} = ((\mu_{ab}, v_{ab}))_{l \times n}$, where l and n are the number of alternatives and attributes, respectively and μ_{ab} and v_{ab} denote the DM and DNM, respectively.

Step 2: The attribute weights can be calculated through the formula presented in Eq. (5), using the q-ROF decision matrix B

$$\eth_b = \frac{1 + \frac{1}{l}\sum_{a=1}^{n}(\mu_{ab}log(\mu_{ab}) + v_{ab}log(v_{ab}))}{\sum_{b=1}^{n}\left(1 + \frac{1}{l}\sum_{a=1}^{n}(\mu_{ab}log(\mu_{ab}) + v_{ab}log(v_{ab}))\right)}. \tag{5}$$

4.3 Algorithm of solving MADM problem

In the following, the procedure of solving MADM problem using proposed q-ROFFWA and q-ROFFWG operators is described. The steps are:

Step 1: Based on the decision-maker's opinion, constitute the decision matrix P with the q-ROF data, where $P = ((\tau_{ab}))_{l \times n} = ((\mu_{ab}, v_{ab}))_{l \times n}$.

Step 2: Determine the attribute weights by the AHP and the q-ROF entropy methods, described in Sects. 4.1 and 4.2, respectively.

Step 3: Transform the matrix $P = (\tau_{ab})_{l \times n} = ((\mu_{ab}, v_{ab}))_{l \times n}$ into a normalized q-ROF matrix $P' = (\tau'_{ab})_{l \times n} = ((\mu'_{ab}, v'_{ab}))_{l \times n}$ by Eq. (6)

$$\tau'_{ab} = \begin{cases} (\mu_{ab}, v_{ab}), & \text{if } R_b \text{ is benefit attribute;} \\ (v_{ab}, \mu_{ab}), & \text{if } R_b \text{ is cost attribute.} \end{cases} \tag{6}$$

Note that, this step can be omitted if all the attributes are of benefit type.

Step 4: Utilizing our proposed q-ROFFWA operator and q-ROFFWG operator, the collective value β_a of the alternative $\mathcal{V}_a$ is calculated by Eqs. (7) and (8), respectively.

$$\begin{aligned} \beta_a &= q\text{-ROFFWA } (\tau'_{a1}, \tau'_{a2}, \ldots, \tau'_{an}) = \bigoplus_{b=1}^{n}(\eth_b \tau'_{ab}) \\ &= \left(\sqrt[q]{1 - \log_\tau\left(1 + \prod_{b=1}^{n}(\tau^{1-\mu'^q_{h_{ab}}} - 1)^{\eth_b}\right)}, \right. \\ &\quad \left. \sqrt[q]{\log_\tau\left(1 + \prod_{b=1}^{n}(\tau^{v'^q_{h_{ab}}} - 1)^{\eth_b}\right)} \right) \end{aligned} \tag{7}$$

and

$$\begin{aligned} \beta_a &= q\text{-ROFFWG } (\tau'_{a1}, \tau'_{a2}, \ldots, \tau'_{an}) = \bigotimes_{b=1}^{n}(\tau'_{ab})^{\eth_b} \\ &= \left(\sqrt[q]{\log_\tau\left(1 + \prod_{b=1}^{n}(\tau^{\mu'^q_{h_{ab}}} - 1)^{\eth_b}\right)}, \right. \\ &\quad \left. \sqrt[q]{1 - \log_\tau\left(1 + \prod_{b=1}^{n}(\tau^{1-v'^q_{h_{ab}}} - 1)^{\eth_b}\right)} \right). \end{aligned} \tag{8}$$

Step 5: Calculate the score value $\mathbb{S}(\alpha_a)(a = 1, 2, \ldots, l)$ for each aggregated value $\beta_a(a = 1, 2, \ldots, l)$ using Eq. (1).

Step 6: If the score value for one alternative is maximum, then this would be the most preferable, i.e., the best alternative is $\mathcal{V}_a$ if $\mathbb{S}(\beta_a) = \max_{1 \le a \le l}\{\mathbb{S}(\beta_a)\}$. Otherwise, we have to find out accuracy function using Eq. (2) for all the alternatives whose score values are simultaneously maximum. Now,

- If the accuracy value be maximum for only one of those alternatives, then this would be the best alternative.
- If the accuracy value also be maximum for two or more of those alternatives, then any of them could be selected to be the best alternative.

5 Practical implication

To better express the proposed method and its benefits, we are eager to explain a numerical example concerned with the balanced development of the socio-economic condition of a state.

5.1 An example to balancing the socio-economic condition of a state

West Bengal is an eastern state of India. It possesses 6th rank in GDP in India and at the same time, 23rd rank in GDP per-capita. So, the easy conclusion about the economy West Bengal, is that here less number of people handles the maximum share of the economy and economical growth, while a large portion of the society suffers from poverty. In West Bengal, also we can see that maximum industrial growth is centralized in some districts, namely, Kolkata, Howrah, Hooghly, etc., and the agricultural development has been centralized to the various river basins only (mainly basin of the Ganges). The rest of this state is economically underdeveloped. Now, the priority of the finance ministry of the state government is to step for the proper distribution of wealth among people.

Keeping such problems of unequal distributions in mind, the finance ministry of the state is eager to initiate a 'decentralization of industrialization program'. For this purpose, a committee has been selected by the finance ministry and they have to suggest such a way of industrialization which will develop the economy of even the interior areas (the districts of Bankura, Purulia, 24 Parganas, North and South Dinajpur, etc.) of the state and involve a large number of people into it giving them an equal share of profit. The committee has raised some alternative choices and some attributes to judge the alternatives. The alternatives are

- $\mathcal{V}_1$: Taking initiatives to expand livestocks (e.g., poultry, fishery, animal farms, etc.).
- $\mathcal{V}_2$: Encouraging agricultural industries (e.g., rice mills, dry food industry, cold storage industry, food packaging, etc.).
- $\mathcal{V}_3$: Establishing new hardware or machinery industries (e.g., still plant, large scale industries, etc.).
- $\mathcal{V}_4$: Initiating software industries (e.g., animation, information technology, etc.).
- $\mathcal{V}_5$: Initiating business hubs containing small independent businesses.

The attributes are

- R_1 : Investment.
- R_2 : Involvement of common people.
- R_3 : Usage of local wealth.
- R_4 : Necessity of infrastructure.
- R_5 : Necessity of skilled labor.

The attribute R_1 is a cost attribute because less investment makes an alternative superior. R_2 is a benefit attribute as it provides employment to common people. The alternative which uses more amount of raw material available at the locality is more preferable so the attribute R_3 is of benefit type. R_4 is a cost attribute because the alternative which requires fewer infrastructure facilities is treated as a better alternative. R_5 is a cost attribute as less necessity of skilled labor is more preferable as it includes the maximum amount of unskilled labor of local people.

Now, we see that each alternative optimizes different attributes to a different extent. So, our proposed method will assist the committee to access such a MADM problem. Under the preferences and suggestions of the committee, the weight vectors corresponding to the above attributes are calculated by AHP and entropy methods as described in Sect. 4.

Let the evaluation of the alternative $\mathcal{V}_a$ relative to the attribute R_b is expressed by q-ROFN (μ_{ab}, v_{ab}). Furthermore, a q-ROF decision matrix $((\mu_{ab}, v_{ab}))_{5 \times 5}$ is obtained. The appraisement for the alternatives is exhibited in Table 2.

5.2 Solution

Now, we utilize our proposed MADM model to determine the best way of industrialization, which will distribute its share among people in a balanced way.

Step 1: Express the evaluations of alternatives through a q-ROF decision matrix exhibited in Table 2.

Step 2: Determine the weights of the five attributes by the AHP and q-ROF entropy methods.

- **Attribute weights obtained by AHP:**
   Here, we utilized the AHP method discussed in Sect. 4.1 to determine the attribute weights.
   Initially, the pairwise comparison matrix B is constructed using scale of importance given in Table 1.

Table 2 Assessments of the alternatives with q-ROF data

	R_1	R_2	R_3	R_4	R_5
$\mathcal{V}_1$	(0.3,0.8)	(0.7,0.3)	(0.6,0.2)	(0.2,0.6)	(0.2,0.7)
$\mathcal{V}_2$	(0.2,0.7)	(0.7,0.2)	(0.4,0.2)	(0.2,0.5)	(0.3,0.4)
$\mathcal{V}_3$	(0.4,0.5)	(0.5,0.6)	(0.2,0.7)	(0.3,0.8)	(0.4,0.5)
$\mathcal{V}_4$	(0.3,0.4)	(0.5,0.2)	(0.7,0.3)	(0.5,0.8)	(0.9,0.6)
$\mathcal{V}_5$	(0.7,0.5)	(0.7,0.5)	(0.7,0.6)	(0.2,0.3)	(0.7,0.8)

$$R_1 R_2 R_3 R_4 R_5$$

$$B = \begin{array}{c} R_1 \\ R_2 \\ R_3 \\ R_4 \\ R_5 \end{array} \begin{pmatrix} 1 & \frac{1}{7} & \frac{1}{5} & \frac{1}{3} & \frac{1}{5} \\ 7 & 1 & 3 & 5 & 3 \\ 5 & \frac{1}{3} & a1 & 3 & 5 \\ 3 & \frac{1}{5} & \frac{1}{3} & 1 & \frac{1}{3} \\ 5 & \frac{1}{3} & \frac{1}{5} & 3 & 1 \end{pmatrix}$$

Normalized pairwise comparison matrix $\hat{B}$ is computed through Eq. (3).

$$R_1 R_2 R_3 R_4 R_5$$

$$\hat{B} = \begin{array}{c} R_1 \\ R_2 \\ R_3 \\ R_4 \\ R_5 \end{array} \begin{pmatrix} 0.0476 & 0.0700 & 0.0422 & 0.0267 & 0.0209 \\ 0.3333 & 0.5000 & 0.6342 & 0.4055 & 0.3147 \\ 0.2380 & 0.1650 & 0.2114 & 0.2433 & 0.5246 \\ 0.1428 & 0.1000 & 0.0697 & 0.0811 & 0.0346 \\ 0.2380 & 0.1650 & 0.0422 & 0.2433 & 0.1049 \end{pmatrix}$$

Utilizing Eq. (4) attribute weights are computed as

$$\partial_b = (0.04148, 0.43754, 0.27646, \\ 0.08564, 0.15868). \tag{9}$$

- **Attribute weights obtained by q-ROF entropy method:**

    The attribute weights can be calculated through the formula presented in Eq. (5) using the q-ROF information exhibited in Table 2. Attribute weights are calculated as

$$\partial_b = (0.1888, 0.1937, \\ 0.1982, 0.1993, 0.2200). \tag{10}$$

Step 3: Normalizing the decision matrix using Eq. (6), we obtain:

$$N = \begin{pmatrix} (0.8,0.3) & (0.7,0.3) & (0.6,0.2) & (0.6,0.2) & (0.7,0.2) \\ (0.7,0.2) & (0.7,0.2) & (0.4,0.2) & (0.5,0.2) & (0.4,0.3) \\ (0.5,0.4) & (0.5,0.6) & (0.2,0.7) & (0.8,0.3) & (0.5,0.4) \\ (0.4,0.3) & (0.5,0.2) & (0.7,0.3) & (0.8,0.5) & (0.6,0.9) \\ (0.5,0.7) & (0.7,0.5) & (0.7,0.6) & (0.3,0.2) & (0.8,0.7) \end{pmatrix}$$

Step 4: We consider $\tau = 2, q = 4$ and exploit the q-ROFFWA and q-ROFFWG operators to determine complete aggregate values $\beta_a (a = 1, 2, 3, 4, 5)$ of alternatives $\mathcal{V}_a$ using Eqs. (7) and (8) respectively. The aggregated results using AHP and q-ROF entropy are shown in Table 3.

Step 5: The score values $\mathbb{S}(\beta_a)(a = 1, 2, 3, 4, 5)$ using Eq. (1) and rankings of the alternatives are exhibited in Table 4.

Step 6: So, based on the committee's advice, the government will take initiatives for the 'decentralization of industrialization project' with the most suitable alternative i.e., $\mathcal{V}_1$.

From the above, we can see that the ranking orders of the alternatives utilizing the q-ROFFWA and q-ROFFWG operators are different but the best alternative remains the same. The best alternative is $\mathcal{V}_1$ i.e., the government will take initiatives to expand livestock (like poultry, fishery, animal farms, etc.) for decentralizing the economical growth of the state and for making balanced distribution of the economy among people, from place to place. To provide a better view of the ranking order among the alternatives deduced by our suggested model, we show the result in Figs. 1 and 2.

It can be easily seen from Fig. 1 that the ranking of the five alternatives, obtained by proposed q-ROFFWA operator based method using AHP and entropy weights is quite similar. The best alternative obtained using AHP and entropy weights is the same, namely $\mathcal{V}_1$. The main difference is just the ranking order between $\mathcal{V}_4$ and $\mathcal{V}_5$. We obtain $\mathcal{V}_5 \succ \mathcal{V}_4$ using AHP weights, whereas $\mathcal{V}_4 \succ \mathcal{V}_5$ using entropy weights.

Again from Fig. 2, we have seen that the ranking of the five alternatives, obtained by proposed q-ROFFWG operator based method using AHP and entropy weights is quite

Table 3 Aggregated values with the help of q-ROFFWA and q-ROFFWG operators

Aggregation operator	Attribute weight	β_1	β_2	β_3	β_4	β_5
q-ROFFWA	AHP	(0.6746,0.2429)	(0.6047,0.2133)	(0.5333,0.5460)	(0.6287,0.3154)	(0.7034,0.5218)
q-ROFFWA	Entropy	(0.6951,0.2335)	(0.5875,0.2186)	(0.6008,0.4581)	(0.6550,0.3958)	(0.6752,0.4988)
q-ROFFWG	AHP	(0.6663,0.2620)	(0.5355,0.2265)	(0.4058,0.6024)	(0.5844,0.6225)	(0.6588,0.5806)
q-ROFFWG	Entropy	(0.6763,0.2528)	(0.5199,0.2346)	(0.4609,0.5446)	(0.5886,0.6745)	(0.5758,0.6117)

Table 4 Score values and ranking of the alternatives

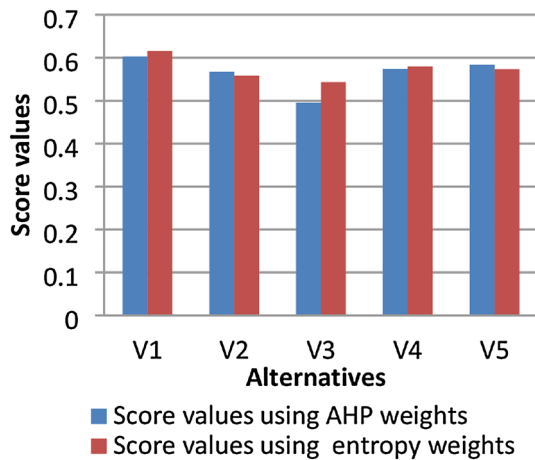| Aggregation operator	Attribute weight	$\mathbb{S}(\beta_1)$	$\mathbb{S}(\beta_2)$	$\mathbb{S}(\beta_3)$	$\mathbb{S}(\beta_4)$	$\mathbb{S}(\beta_5)$	Ranking
q-ROFFWA	AHP	0.6029	0.5670	0.4960	0.5731	0.5835	$\mathcal{V}_1 > \mathcal{V}_5 > \mathcal{V}_4 > \mathcal{V}_2 > \mathcal{V}_3$
q-ROFFWA	Entropy	0.6152	0.5584	0.5431	0.5797	0.5729	$\mathcal{V}_1 > \mathcal{V}_4 > \mathcal{V}_5 > \mathcal{V}_2 > \mathcal{V}_3$
q-ROFFWG	AHP	0.5962	0.5398	0.4476	0.4832	0.5363	$\mathcal{V}_1 > \mathcal{V}_2 > \mathcal{V}_5 > \mathcal{V}_4 > \mathcal{V}_3$
q-ROFFWG	Entropy	0.6025	0.5350	0.4785	0.4565	0.4849	$\mathcal{V}_1 > \mathcal{V}_2 > \mathcal{V}_5 > \mathcal{V}_3 > \mathcal{V}_4$

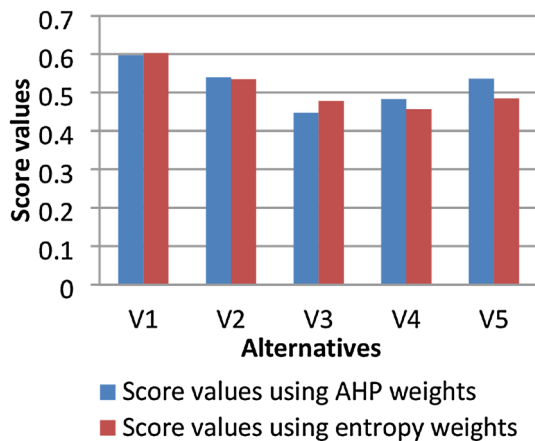**Fig. 1** Comparison of score values based on q-ROFFWA operator

Fig. 2 Comparison of score values based on q-ROFFWG operator

similar. We shall always get the same alternative, namely $\mathcal{V}_1$, as the best choice using both of AHP and entropy weights. The main difference is just the ranking order between $\mathcal{V}_3$ and $\mathcal{V}_4$. We obtain $\mathcal{V}_4 \succ \mathcal{V}_3$ using AHP weights, whereas $\mathcal{V}_4 \succ \mathcal{V}_3$ using entropy weights.

An inspection of the effectiveness of the parameter $\tau \in [2, 10]$ to rank the given alternatives for instances of q-ROFFWA as well as q-ROFFWG operators, is demonstrated in Sect. 6.

6 Analysis of the effect of the parameters τ and q on decision-making

In this section, we show how the decision-making results be influenced by parameters τ and q.

6.1 Effects of the parameter τ

We take different values of the parameter τ in q-ROFFWA operator and q-ROFFWG operator to explore the flexibility and benefits of the parameter τ. Also, we determine the collective information for each alternative and rank them. For $2 \leq \tau \leq 10$ and $q = 4$ we execute the ranking results of each alternative.

At first, we use AHP weights and q-ROF entropy weights for making decision concerned with q-ROFFWA operator. The results are delimited in Table 5.

If we utilize AHP weights, then Table 5 shows that the optimal ranking among available alternatives is $\mathcal{V}_1 > \mathcal{V}_5 > \mathcal{V}_4 > \mathcal{V}_2 > \mathcal{V}_3$ and the best alternative is always $\mathcal{V}_1$ while the worst alternative is always $\mathcal{V}_3$, irrespective of different choices of τ. Again, if we utilize entropy weights, then from Table 5, we can find out that the optimal ranking order of these five alternatives is $\mathcal{V}_1 > \mathcal{V}_4 > \mathcal{V}_5 > \mathcal{V}_2 > \mathcal{V}_3$ and the best alternative is always $\mathcal{V}_1$, while the worst alternative is always $\mathcal{V}_3$, irrespective of different choices of τ. The ranking results obtained by q-ROFFWA operator for $q = 4$ and $\tau \in [2, 10]$ using AHP and entropy weights are also demonstrated in Figs. 3 and 4 respectively.

Now, we use AHP weights and q-ROF entropy weights to obtain the outcomes of the MADM with q-ROFFWG operator. Table 6 exhibits the results.

If we utilize the AHP weights, then from Table 6, it can be seen that for $2 \leq \tau \leq 8$, the ranking order of these five alternatives is $\mathcal{V}_1 > \mathcal{V}_5 > \mathcal{V}_4 > \mathcal{V}_2 > \mathcal{V}_3$ and the best alternative is always $\mathcal{V}_1$ while the worst alternative is always $\mathcal{V}_3$. But, for $\tau = 9, 10$ the ranking among the alternative choices differs slightly and the ranking order is $\mathcal{V}_1 > \mathcal{V}_5 > \mathcal{V}_2 > \mathcal{V}_4 > \mathcal{V}_3$ and the best and the worst alternatives are $\mathcal{V}_1$ and $\mathcal{V}_3$ respectively. Again, if we utilize the entropy weights, then from Table 6, we can find out

Table 5 Effect of the parameter τ over the outcomes of the MADM concerning q-ROFFWA operator

τ	Type of attribute weight	$\mathbb{S}(\beta_1)$	$\mathbb{S}(\beta_2)$	$\mathbb{S}(\beta_3)$	$\mathbb{S}(\beta_4)$	$\mathbb{S}(\beta_5)$	Ranking order	Optimal alternative
2	AHP	0.60292	0.56703	0.49601	0.57318	0.58536	$\mathcal{V}_1 > \mathcal{V}_5 > \mathcal{V}_4 > \mathcal{V}_2 > \mathcal{V}_3$	$\mathcal{V}_1$
	Entropy	0.6152	0.5584	0.5431	0.5797	0.5729	$\mathcal{V}_1 > \mathcal{V}_4 > \mathcal{V}_5 > \mathcal{V}_2 > \mathcal{V}_3$	$\mathcal{V}_1$
3	AHP	0.60277	0.56666	0.49518	0.57260	0.58471	$\mathcal{V}_1 > \mathcal{V}_5 > \mathcal{V}_4 > \mathcal{V}_2 > \mathcal{V}_3$	$\mathcal{V}_1$
	Entropy	0.6148	0.5580	0.5420	0.5786	0.5716	$\mathcal{V}_1 > \mathcal{V}_4 > \mathcal{V}_5 > \mathcal{V}_2 > \mathcal{V}_3$	$\mathcal{V}_1$
4	AHP	0.60267	0.56641	0.49463	0.57222	0.58427	$\mathcal{V}_1 > \mathcal{V}_5 > \mathcal{V}_4 > \mathcal{V}_2 > \mathcal{V}_3$	$\mathcal{V}_1$
	Entropy	0.6146	0.5578	0.5413	0.5778	0.5707	$\mathcal{V}_1 > \mathcal{V}_4 > \mathcal{V}_5 > \mathcal{V}_2 > \mathcal{V}_3$	$\mathcal{V}_1$
5	AHP	0.60260	0.56624	0.49423	0.57194	0.58394	$\mathcal{V}_1 > \mathcal{V}_5 > \mathcal{V}_4 > \mathcal{V}_2 > \mathcal{V}_3$	$\mathcal{V}_1$
	Entropy	0.6144	0.5576	0.5408	0.5772	0.5701	$\mathcal{V}_1 > \mathcal{V}_4 > \mathcal{V}_5 > \mathcal{V}_2 > \mathcal{V}_3$	$\mathcal{V}_1$
6	AHP	0.60254	0.56611	0.49391	0.57173	0.58368	$\mathcal{V}_1 > \mathcal{V}_5 > \mathcal{V}_4 > \mathcal{V}_2 > \mathcal{V}_3$	$\mathcal{V}_1$
	Entropy	0.6142	0.5575	0.5404	0.5768	0.5696	$\mathcal{V}_1 > \mathcal{V}_4 > \mathcal{V}_5 > \mathcal{V}_2 > \mathcal{V}_3$	$\mathcal{V}_1$
7	AHP	0.60250	0.56601	0.49366	0.57155	0.58347	$\mathcal{V}_1 > \mathcal{V}_5 > \mathcal{V}_4 > \mathcal{V}_2 > \mathcal{V}_3$	$\mathcal{V}_1$
	Entropy	0.6141	0.5574	0.5401	0.5764	0.5692	$\mathcal{V}_1 > \mathcal{V}_4 > \mathcal{V}_5 > \mathcal{V}_2 > \mathcal{V}_3$	$\mathcal{V}_1$
8	AHP	0.60246	0.56593	0.49344	0.57140	0.58329	$\mathcal{V}_1 > \mathcal{V}_5 > \mathcal{V}_4 > \mathcal{V}_2 > \mathcal{V}_3$	$\mathcal{V}_1$
	Entropy	0.6140	0.5573	0.5398	0.5761	0.5689	$\mathcal{V}_1 > \mathcal{V}_4 > \mathcal{V}_5 > \mathcal{V}_2 > \mathcal{V}_3$	$\mathcal{V}_1$
9	AHP	0.60243	0.56586	0.49325	0.57128	0.58313	$\mathcal{V}_1 > \mathcal{V}_5 > \mathcal{V}_4 > \mathcal{V}_2 > \mathcal{V}_3$	$\mathcal{V}_1$
	Entropy	0.6139	0.5573	0.5396	0.5758	0.5686	$\mathcal{V}_1 > \mathcal{V}_4 > \mathcal{V}_5 > \mathcal{V}_2 > \mathcal{V}_3$	$\mathcal{V}_1$
10	AHP	0.60240	0.56580	0.49309	0.57117	0.58300	$\mathcal{V}_1 > \mathcal{V}_5 > \mathcal{V}_4 > \mathcal{V}_2 > \mathcal{V}_3$	$\mathcal{V}_1$
	Entropy	0.6139	0.5572	0.5394	0.5756	0.5683	$\mathcal{V}_1 > \mathcal{V}_4 > \mathcal{V}_5 > \mathcal{V}_2 > \mathcal{V}_3$	$\mathcal{V}_1$

Fig. 3 Score of alternatives based on q-ROFFWA operator using AHP weights
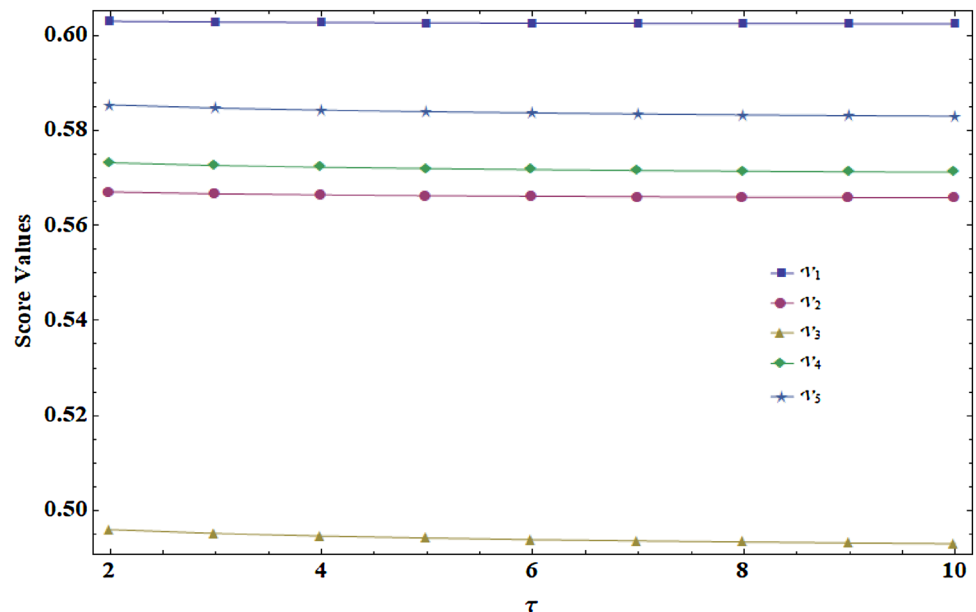
that for $2 \leq \tau \leq 10$, the ranking order of these five alternatives is $\mathcal{V}_1 > \mathcal{V}_2 > \mathcal{V}_5 > \mathcal{V}_3 > \mathcal{V}_4$ and the best alternative is always $\mathcal{V}_1$, while the worst alternative is always $\mathcal{V}_4$. The ranking results obtained by q-ROFFWG operator for $q = 4$ and $\tau \in [2, 10]$, using AHP and entropy weights are also demonstrated in Figs. 5 and 6 respectively.

From the above discussion, we can say that, if we take $q = 4$ then for $2 \leq \tau \leq 10$ the ranking among the alternatives differs slightly but the optimal choice remains the same.

6.2 Effects of the parameter *q*

The proposed q-ROFFWA operator and q-ROFFWG operator-based method is a general method that gives the decision-makers the freedom to extend their decision

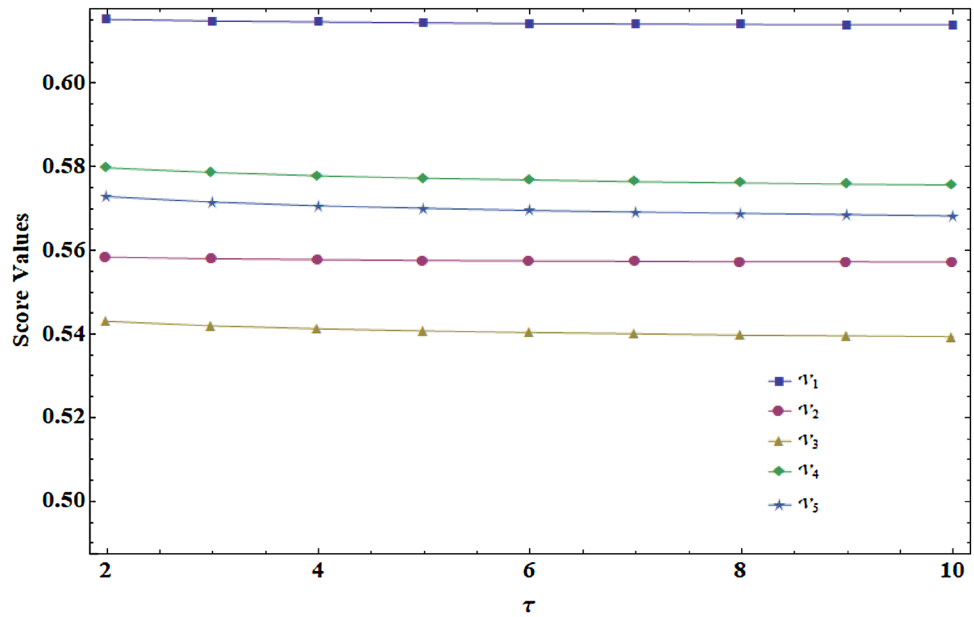**Fig. 4** Score of alternatives based on q-ROFFWA operator using entropy weights

Table 6 Effect of the parameter τ over the outcomes of the MADM concerning q-ROFFWG operator

τ	Type of attribute weight	$\mathbb{S}(\beta_1)$	$\mathbb{S}(\beta_2)$	$\mathbb{S}(\beta_3)$	$\mathbb{S}(\beta_4)$	$\mathbb{S}(\beta_5)$	Ranking order	Optimal alternative
2	AHP	0.59621	0.53980	0.44767	0.48323	0.53734	$\mathcal{V}_1 > \mathcal{V}_2 > \mathcal{V}_5 > \mathcal{V}_4 > \mathcal{V}_3$	$\mathcal{V}_1$
	Entropy	0.6025	0.5350	0.4785	0.4565	0.4849	$\mathcal{V}_1 > \mathcal{V}_2 > \mathcal{V}_5 > \mathcal{V}_3 > \mathcal{V}_4$	$\mathcal{V}_1$
3	AHP	0.59636	0.54017	0.44803	0.48699	0.53848	$\mathcal{V}_1 > \mathcal{V}_2 > \mathcal{V}_5 > \mathcal{V}_4 > \mathcal{V}_3$	$\mathcal{V}_1$
	Entropy	0.6030	0.5353	0.4792	0.4608	0.4862	$\mathcal{V}_1 > \mathcal{V}_2 > \mathcal{V}_5 > \mathcal{V}_3 > \mathcal{V}_4$	$\mathcal{V}_1$
4	AHP	0.59642	0.54045	0.44827	0.48886	0.53926	$\mathcal{V}_1 > \mathcal{V}_2 > \mathcal{V}_5 > \mathcal{V}_4 > \mathcal{V}_3$	$\mathcal{V}_1$
	Entropy	0.6032	0.5355	0.4796	0.4636	0.4872	$\mathcal{V}_1 > \mathcal{V}_2 > \mathcal{V}_5 > \mathcal{V}_3 > \mathcal{V}_4$	$\mathcal{V}_1$
5	AHP	0.59657	0.54067	0.44845	0.49041	0.53986	$\mathcal{V}_1 > \mathcal{V}_2 > \mathcal{V}_5 > \mathcal{V}_4 > \mathcal{V}_3$	$\mathcal{V}_1$
	Entropy	0.6035	0.5357	0.4800	0.4656	0.4879	$\mathcal{V}_1 > \mathcal{V}_2 > \mathcal{V}_5 > \mathcal{V}_3 > \mathcal{V}_4$	$\mathcal{V}_1$
6	AHP	0.59665	0.54086	0.44860	0.49158	0.54033	$\mathcal{V}_1 > \mathcal{V}_2 > \mathcal{V}_5 > \mathcal{V}_4 > \mathcal{V}_3$	$\mathcal{V}_1$
	Entropy	0.6036	0.5358	0.4802	0.4671	0.4884	$\mathcal{V}_1 > \mathcal{V}_2 > \mathcal{V}_5 > \mathcal{V}_3 > \mathcal{V}_4$	$\mathcal{V}_1$
7	AHP	0.59672	0.54101	0.44871	0.49252	0.54073	$\mathcal{V}_1 > \mathcal{V}_2 > \mathcal{V}_5 > \mathcal{V}_4 > \mathcal{V}_3$	$\mathcal{V}_1$
	Entropy	0.6038	0.5360	0.4804	0.4683	0.4889	$\mathcal{V}_1 > \mathcal{V}_2 > \mathcal{V}_5 > \mathcal{V}_3 > \mathcal{V}_4$	$\mathcal{V}_1$
8	AHP	0.59678	0.54115	0.44881	0.49329	0.54106	$\mathcal{V}_1 > \mathcal{V}_2 > \mathcal{V}_5 > \mathcal{V}_4 > \mathcal{V}_3$	$\mathcal{V}_1$
	Entropy	0.6039	0.5361	0.4806	0.4693	0.4893	$\mathcal{V}_1 > \mathcal{V}_2 > \mathcal{V}_5 > \mathcal{V}_3 > \mathcal{V}_4$	$\mathcal{V}_1$
9	AHP	0.59683	0.54127	0.44890	0.49394	0.54136	$\mathcal{V}_1 > \mathcal{V}_5 > \mathcal{V}_2 > \mathcal{V}_4 > \mathcal{V}_3$	$\mathcal{V}_1$
	Entropy	0.6040	0.5362	0.4808	0.4702	0.4897	$\mathcal{V}_1 > \mathcal{V}_2 > \mathcal{V}_5 > \mathcal{V}_3 > \mathcal{V}_4$	$\mathcal{V}_1$
10	AHP	0.59687	0.54138	0.44897	0.49449	0.54161	$\mathcal{V}_1 > \mathcal{V}_5 > \mathcal{V}_2 > \mathcal{V}_4 > \mathcal{V}_3$	$\mathcal{V}_1$
	Entropy	0.6041	0.5363	0.4809	0.4709	0.4900	$\mathcal{V}_1 > \mathcal{V}_2 > \mathcal{V}_5 > \mathcal{V}_3 > \mathcal{V}_4$	$\mathcal{V}_1$

evaluation space grounded on the parameter q. The parameter q plays a vital role in this method and it significantly affects the decision results. Suppose q assumes different values in the practical example to explore the sensitivity and flexibility of the parameter q. For $\tau = 4$, the outcomes of the ranking with the help of q-ROFFWA and q-ROFFWG operators using AHP and q-ROF entropy weights are exhibited in Table 7 and Table 8 respectively.

From Table 7, we find that for different values of the parameter q, the score values and ranking order of the alternatives are different while using q-ROFFWA operator with respect to AHP and entropy weights. If we use AHP weights, then for $q = 3, 4, 5$ the optimal alternative is $\mathcal{V}_1$ and for $q = 8, 10, 12, 16, 20$ the optimal choice is $\mathcal{V}_5$. Again, If we use entropy weights, then for $q = 3, 4, 5, 8, 10$

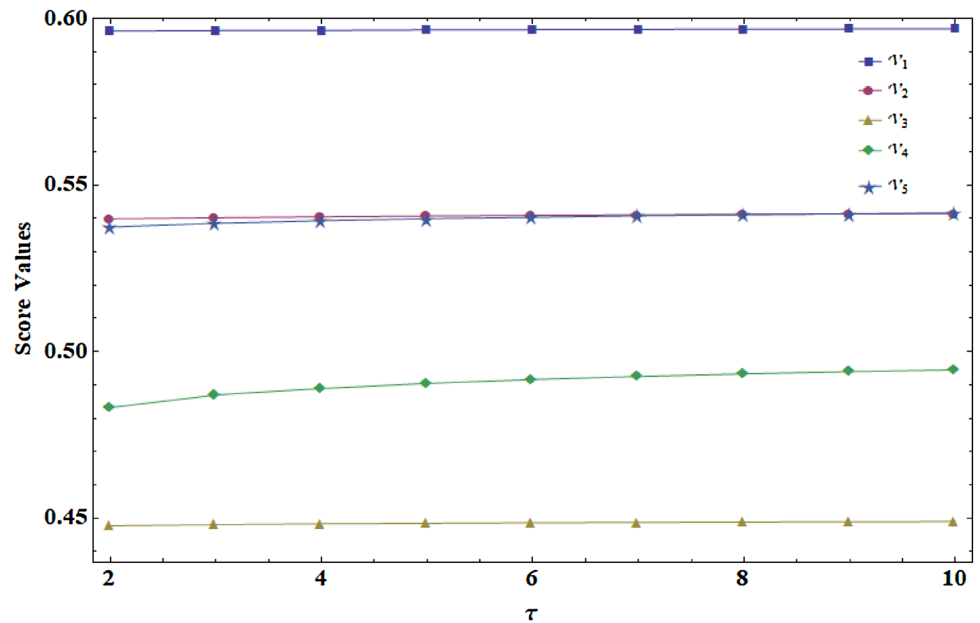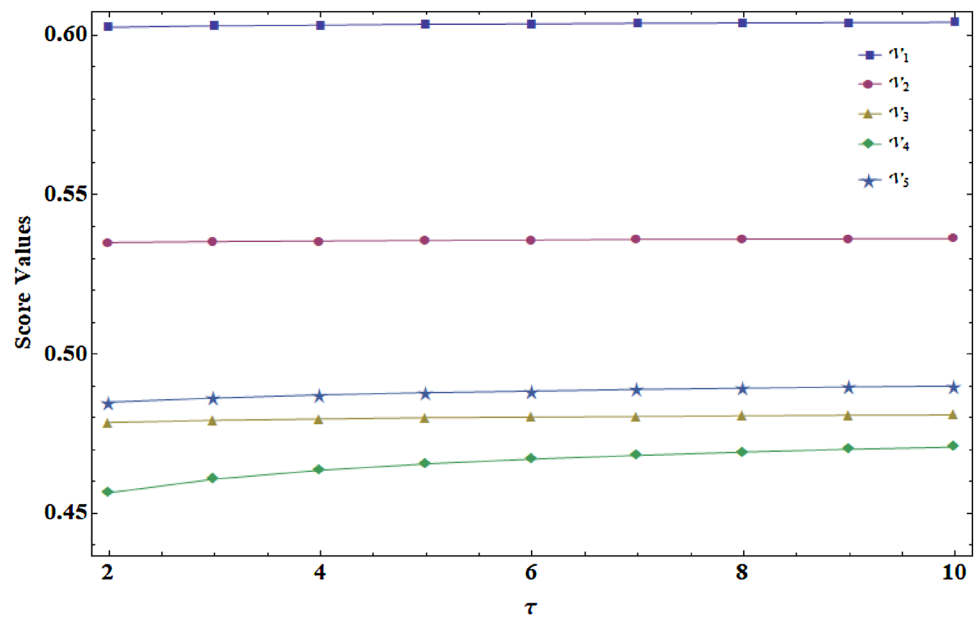**Fig. 5** Score of alternatives based on q-ROFFWG operator using AHP weights

Fig. 6 Score of alternatives based on q-ROFFWG operator using entropy weights

the optimal alternative is $\mathcal{V}_1$ and for $q = 12, 16, 20$ the optimal choice is $\mathcal{V}_5$.

From Table 8, we can find that for $\tau = 4$ and for different values of the parameter q in the q-ROFFWG operator with respect to AHP and entropy weights, the ranking results and ranking order of the alternatives are different but, the best choice is always $\mathcal{V}_1$.

7 Comparison analysis

In the following, we will prove the competency and superiority of our proposed method by comparing it with some existing well-known methods. Our proposed aggregation operators are the generalization of several existing aggregation operators through Frank t-conorm and t-norm. Also, our proposed operators have a wide range of applicability due to the presence of the parameter q. Therefore, our proposed method of solving the MADM problem utilizing q-ROFFWA and q-ROFFWG operators, develops advanced authenticity in the real-life decision-making

Table 7 Impact of the parameter q over the outcomes of the MADM concerning q-ROFFWA operator

q	Type of attribute weight	$\mathbb{S}(\beta_1)$	$\mathbb{S}(\beta_2)$	$\mathbb{S}(\beta_3)$	$\mathbb{S}(\beta_4)$	$\mathbb{S}(\beta_5)$	Ranking order	Optimal alternative
3	AHP	0.64608	0.59983	0.48337	0.60206	0.59763	$\mathcal{V}_1 > \mathcal{V}_4 > \mathcal{V}_2 > \mathcal{V}_5 > \mathcal{V}_3$	$\mathcal{V}_1$
	Entropy	0.6586	0.5890	0.5466	0.5992	0.5798	$\mathcal{V}_1 > \mathcal{V}_4 > \mathcal{V}_2 > \mathcal{V}_5 > \mathcal{V}_3$	$\mathcal{V}_1$
4	AHP	0.60267	0.56641	0.49463	0.57222	0.58427	$\mathcal{V}_1 > \mathcal{V}_5 > \mathcal{V}_4 > \mathcal{V}_2 > \mathcal{V}_3$	$\mathcal{V}_1$
	Entropy	0.6146	0.5578	0.5413	0.5778	0.5707	$\mathcal{V}_1 > \mathcal{V}_4 > \mathcal{V}_5 > \mathcal{V}_2 > \mathcal{V}_3$	$\mathcal{V}_1$
5	AHP	0.57114	0.54447	0.50086	0.55022	0.56855	$\mathcal{V}_1 > \mathcal{V}_5 > \mathcal{V}_4 > \mathcal{V}_2 > \mathcal{V}_3$	$\mathcal{V}_1$
	Entropy	0.5821	0.5378	0.5343	0.5589	0.5588	$\mathcal{V}_1 > \mathcal{V}_4 > \mathcal{V}_5 > \mathcal{V}_2 > \mathcal{V}_3$	$\mathcal{V}_1$
8	AHP	0.52389	0.51427	0.50489	0.51765	0.53163	$\mathcal{V}_5 > \mathcal{V}_1 > \mathcal{V}_4 > \mathcal{V}_2 > \mathcal{V}_3$	$\mathcal{V}_5$
	Entropy	0.5314	0.5116	0.5175	0.5249	0.5288	$\mathcal{V}_1 > \mathcal{V}_5 > \mathcal{V}_4 > \mathcal{V}_3 > \mathcal{V}_2$	$\mathcal{V}_1$
10	AHP	0.51185	0.50691	0.50393	0.50936	0.51808	$\mathcal{V}_5 > \mathcal{V}_1 > \mathcal{V}_4 > \mathcal{V}_2 > \mathcal{V}_3$	$\mathcal{V}_5$
	Entropy	0.5173	0.5055	0.5110	0.5144	0.5171	$\mathcal{V}_1 > \mathcal{V}_5 > \mathcal{V}_4 > \mathcal{V}_3 > \mathcal{V}_2$	$\mathcal{V}_1$
12	AHP	0.50602	0.50339	0.50278	0.50518	0.51031	$\mathcal{V}_5 > \mathcal{V}_1 > \mathcal{V}_4 > \mathcal{V}_2 > \mathcal{V}_3$	$\mathcal{V}_5$
	Entropy	0.5098	0.5026	0.5069	0.5085	0.5102	$\mathcal{V}_5 > \mathcal{V}_1 > \mathcal{V}_4 > \mathcal{V}_3 > \mathcal{V}_2$	$\mathcal{V}_5$
16	AHP	0.50168	0.50085	0.50124	0.50175	0.50347	$\mathcal{V}_5 > \mathcal{V}_4 > \mathcal{V}_1 > \mathcal{V}_3 > \mathcal{V}_2$	$\mathcal{V}_5$
	Entropy	0.5034	0.5006	0.5028	0.5031	0.5037	$\mathcal{V}_5 > \mathcal{V}_1 > \mathcal{V}_4 > \mathcal{V}_3 > \mathcal{V}_2$	$\mathcal{V}_5$
20	AHP	0.50054	0.50025	0.50055	0.50066	0.50126	$\mathcal{V}_5 > \mathcal{V}_4 > \mathcal{V}_3 > \mathcal{V}_1 > \mathcal{V}_2$	$\mathcal{V}_5$
	Entropy	0.50126	0.5001	0.5011	0.50123	0.5014	$\mathcal{V}_5 > \mathcal{V}_1 > \mathcal{V}_4 > \mathcal{V}_3 > \mathcal{V}_2$	$\mathcal{V}_5$

**Table 8** Impact of the parameter q over the outcomes of the MADM concerning q-ROFFWG operator

q	Type of attribute weight	$\mathbb{S}(\beta_1)$	$\mathbb{S}(\beta_2)$	$\mathbb{S}(\beta_3)$	$\mathbb{S}(\beta_4)$	$\mathbb{S}(\beta_5)$	Ranking order	Optimal alternative
3	AHP	0.63974	0.57287	0.43005	0.51275	0.55210	$\mathcal{V}_1 > \mathcal{V}_2 > \mathcal{V}_5 > \mathcal{V}_4 > \mathcal{V}_3$	$\mathcal{V}_1$
	Entropy	0.6479	0.5655	0.4778	0.4816	0.4917	$\mathcal{V}_1 > \mathcal{V}_2 > \mathcal{V}_5 > \mathcal{V}_4 > \mathcal{V}_3$	$\mathcal{V}_1$
4	AHP	0.59648	0.54045	0.44827	0.48886	0.53926	$\mathcal{V}_1 > \mathcal{V}_2 > \mathcal{V}_5 > \mathcal{V}_4 > \mathcal{V}_3$	$\mathcal{V}_1$
	Entropy	0.6032	0.5355	0.4796	0.4636	0.4872	$\mathcal{V}_1 > \mathcal{V}_2 > \mathcal{V}_5 > \mathcal{V}_3 > \mathcal{V}_4$	$\mathcal{V}_1$
5	AHP	0.56514	0.52182	0.46360	0.47557	0.52790	$\mathcal{V}_1 > \mathcal{V}_5 > \mathcal{V}_2 > \mathcal{V}_4 > \mathcal{V}_3$	$\mathcal{V}_1$
	Entropy	0.5706	0.5187	0.4834	0.4543	0.4864	$\mathcal{V}_1 > \mathcal{V}_2 > \mathcal{V}_5 > \mathcal{V}_3 > \mathcal{V}_4$	$\mathcal{V}_1$
8	AHP	0.51933	0.50324	0.48852	0.46800	0.50844	$\mathcal{V}_1 > \mathcal{V}_5 > \mathcal{V}_2 > \mathcal{V}_3 > \mathcal{V}_4$	$\mathcal{V}_1$
	Entropy	0.5218	0.5026	0.4934	0.4537	0.4919	$\mathcal{V}_1 > \mathcal{V}_2 > \mathcal{V}_3 > \mathcal{V}_5 > \mathcal{V}_4$	$\mathcal{V}_1$
10	AHP	0.50852	0.50087	0.49474	0.47191	0.50352	$\mathcal{V}_1 > \mathcal{V}_5 > \mathcal{V}_2 > \mathcal{V}_3 > \mathcal{V}_4$	$\mathcal{V}_1$
	Entropy	0.5099	0.5007	0.4967	0.4606	0.4954	$\mathcal{V}_1 > \mathcal{V}_2 > \mathcal{V}_3 > \mathcal{V}_5 > \mathcal{V}_4$	$\mathcal{V}_1$
12	AHP	0.50374	0.50020	0.49755	0.47670	0.50139	$\mathcal{V}_1 > \mathcal{V}_5 > \mathcal{V}_2 > \mathcal{V}_3 > \mathcal{V}_4$	$\mathcal{V}_1$
	Entropy	0.5045	0.5001	0.4984	0.4676	0.4975	$\mathcal{V}_1 > \mathcal{V}_2 > \mathcal{V}_3 > \mathcal{V}_5 > \mathcal{V}_4$	$\mathcal{V}_1$
16	AHP	0.50168	0.49996	0.49941	0.48470	0.50115	$\mathcal{V}_1 > \mathcal{V}_5 > \mathcal{V}_2 > \mathcal{V}_3 > \mathcal{V}_4$	$\mathcal{V}_1$
	Entropy	0.5009	0.5000	0.4996	0.4789	0.4993	$\mathcal{V}_1 > \mathcal{V}_2 > \mathcal{V}_3 > \mathcal{V}_5 > \mathcal{V}_4$	$\mathcal{V}_1$
20	AHP	0.50008	0.49994	0.49982	0.49905	0.49996	$\mathcal{V}_1 > \mathcal{V}_5 > \mathcal{V}_2 > \mathcal{V}_3 > \mathcal{V}_4$	$\mathcal{V}_1$
	Entropy	0.5001	0.5000	0.4999	0.4863	0.4998	$\mathcal{V}_1 > \mathcal{V}_2 > \mathcal{V}_3 > \mathcal{V}_5 > \mathcal{V}_4$	$\mathcal{V}_1$

process. Under the q-ROF environment, we elaborate a strict comparison between our proposed Frank aggregation operators with the existing ones. We calculate the decision-making results of our proposed method using intuitionistic fuzzy Frank power average (IFFPA) operator (Zhang et al. 2015), intuitionistic fuzzy Frank power weighted average (IFFPWA) operator (Zhang et al. 2015), Pythagorean fuzzy Frank weighted averaging (PFFWA) operator (Yi et al. 2018), Pythagorean fuzzy Frank weighted geometric(PFFWG) operator (Yi et al. 2018), q-ROF weighted averaging (q-ROFWA) operator (Liu and Wang 2018), q-ROF weighted geometric (q-ROFWG) operator (Liu and Wang 2018). The outcomes of the comparison are illustrated in Table 9.

Table 9 Comparison table

Aggregation operators	The score values	Ranking order	Optimal alternative
IFFPA (Zhang et al. 2015)	Cannot be determined	No	No
IFFPWA (Zhang et al. 2015)	Cannot be determined	No	No
PFFWA (Yi et al. 2018)	Cannot be determined	No	No
PFFWG (Yi et al. 2018)	Cannot be determined	No	No
q-ROFWA (Liu and Wang 2018)	$\mathbb{S}(\beta_1)$=0.6289, $\mathbb{S}(\beta_2)$=0.5674, $\mathbb{S}(\beta_3)$=0.5644, $\mathbb{S}(\beta_4)$= 0.5871, $\mathbb{S}(\beta_5)$=0.5624	$\mathcal{V}_1 > \mathcal{V}_4 > \mathcal{V}_2 > \mathcal{V}_3 > \mathcal{V}_5$	$\mathcal{V}_1$
q-ROFWG (Liu and Wang 2018)	$\mathbb{S}(\beta_1)$=0.6112, $\mathbb{S}(\beta_2)$=0.5423,$\mathbb{S}(\beta_3)$= 0.5039, $\mathbb{S}(\beta_4)$=0.4466, $\mathbb{S}(\beta_1)$=0.4637	$\mathcal{V}_1 > \mathcal{V}_2 > \mathcal{V}_3 > \mathcal{V}_5 > \mathcal{V}_4$	$\mathcal{V}_1$
Proposed [q-ROFFWA] [$\tau = 2, q = 4$]	$\mathbb{S}(\beta_1)$=0.6279,$\mathbb{S}(\beta_2)$=0.5666,$\mathbb{S}(\beta_3)$=0.5621, $\mathbb{S}(\beta_4)$=0.5771, $\mathbb{S}(\beta_5)$=0.5596	$\mathcal{V}_1 > \mathcal{V}_4 > \mathcal{V}_2 > \mathcal{V}_3 > \mathcal{V}_5$	$\mathcal{V}_1$
Proposed[q-ROFFWG] [$\tau = 2, q = 4$]	$\mathbb{S}(\beta_1)$=0.6573,$\mathbb{S}(\beta_2)$=0.5758,$\mathbb{S}(\beta_3)$=0.5143, $\mathbb{S}(\beta_4)$=0.4683,$\mathbb{S}(\beta_5)$=0.4633	$\mathcal{V}_1 > \mathcal{V}_2 > \mathcal{V}_3 > \mathcal{V}_4 > \mathcal{V}_5$	$\mathcal{V}_1$
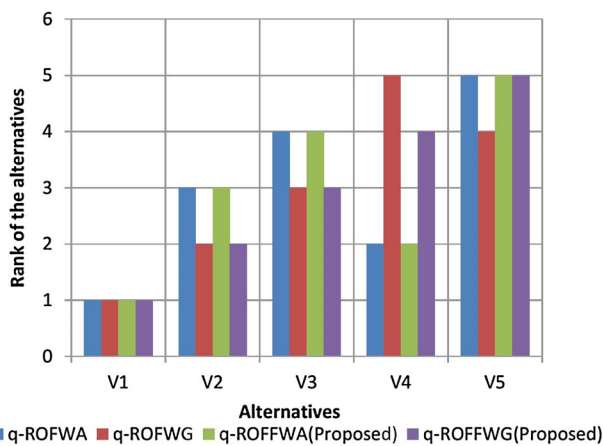
Fig. 7 Pictorial representation of the ranking of the alternatives

To provide a better view of the comparison results, we show the ranking results of the alternatives in Fig. 7. It can be easily seen from Fig. 7 that the ranking of the five alternatives using q-ROFWA (Liu and Wang 2018) operator and proposed q-ROFFWA operator are same. The best alternative using these two operators is $\mathcal{V}_1$. But, the ranking of the five alternatives using q-ROFWG (Liu and Wang 2018) operator and proposed q-ROFFWG operator are slightly different. The best alternative obtained using these two operators is the same, namely, $\mathcal{V}_1$, and the difference is just the ranking order between $\mathcal{V}_4$ and $\mathcal{V}_5$. Using q-ROFWG operator we get $\mathcal{V}_5 > \mathcal{V}_4$, while using q-ROFFWG operator we get $\mathcal{V}_4 > \mathcal{V}_5$.

The procedure of calculation for the IFFPA (Zhang et al. 2015) operator and IFFPWA (Zhang et al. 2015) operator is not complicated, but it is applicable to a confined range. It can deal with only those MADM problems in which an intuitionistic fuzzy number (IFN) expresses the decision information. So, IFFPA and IFFPWA operators fail to deal with our introduced MADM problem as it contains evaluation values other than IFN.

Now PFFWA and PFFWG operators are capable to deal with only those MADM problems where a Pythagorean fuzzy number (PFN) expresses the evaluation information. But only when the sum of the squares of DM and DNM be bounded by 1, the evaluation information can be elicited by the PFNs. Since, in our proposed MADM problem, there exists some decision information for which the sum of the squares of the DM and DNM is greater than 1, so PFFWA and PFFWG operators cannot handle our proposed MADM problem. Our proposed q-ROFFWA and q-ROFFWG operators show the maximum capability of handling decision fuzzy information using q-ROFNs since they enhance the flexibility of the information aggregation procedure with the parameter q and τ.

According to the increment of the parameter q, the range of the expressible decision information will be expanded. Hence, our proposed method can be established to be superior to existing methods associated with IFFPA, IFFPWA, PFFWA, and PFFWG operators.

From Definition 2.4 we can see that when $\tau \rightarrow 1$, FTCN and FTN are transformed into probabilistic sum and probabilistic product. Therefore, we can show that the q-ROFWA or q-ROFWG operators introduced by Liu and Wang (2018) are nothing but a special case of our proposed q-ROFFWA and q-ROFFWG operators respectively, when the parameter $\tau \rightarrow 1$. Therefore, certainly, our introduced procedures are more generalized and nourished. Also, our proposed operators present the Lukasiewicz product and Lukasiewicz sum (Wang and He 2009) when the parameter $\tau \rightarrow \infty$. Therefore, we have decided that all of the arithmetic and geometric aggregation operators for q-ROFNs

are contained in q-ROF Frank aggregation operators, concerning the different values of τ.

If we modify the value of the parameter τ in the problem, we get various ranking results for the alternatives. For example, if we modify the value of the parameter τ from 2 to 15, then using q-ROFFWG operator we obtain the score values of the alternatives as $\mathbb{S}(\beta_1) = 0.6148$, $\mathbb{S}(\beta_2) = 0.5499$, $\mathbb{S}(\beta_3) = 0.5088$, $\mathbb{S}(\beta_4) = 0.4713$ and $\mathbb{S}(\beta_5) = 0.4707$. It is worth noticing that the ranking position of the alternative $\mathcal{V}_4$ changed from a bad position to a good position. But the q-ROFWA and the q-ROFWG operators are independent of the parameter τ. So, the ranking order with the help of those operators remains the same.

Based on the above comparison analysis, the approach in the present study is proved to be more flexible, capable, and reliable than other existing procedures in monitoring q-ROF environment based on MADM problems.

8 Conclusions

The q-ROF models are more potential and suitable than intuitionistic and Pythagorean fuzzy models since it deals with larger space, characterized with vagueness and uncertainty. q-ROFSs show more flexibility to express the information for the presence of the parameter q. Moreover, Frank operators make the process of information fusion more flexible and potential because they contain a parameter. So, here, we introduce Frank operations on q-ROFSs and consecrated a series of new aggregation operators concerning q-ROFNs, viz. q-ROFFWA operator, q-ROFFOWA operator, q-ROFFWG operator, and q-ROF-FOWG operator. Also, we have discussed some valuable properties of these operators. Furthermore, with the assistance of the q-ROFFWA operator and q-ROFFWG operator, we present two approaches to solve an MADM problem within the q-ROF environment. Then, we have discussed a real-world socio-economic MADM problem of 'balancing the share of economical growth in the society' to demonstrate the applicability, feasibility, and superiority of our proposed methodology. Also, to determine the attribute weights, we have used the AHP method and the q-ROF entropy method. We have used these weights to solve the presented numerical example. The illustrated example shows that our proposed decision-making procedure is more flexible and fruitful than other existing methods for the presence of some operational parameters, τ and q, appointed by the decision-maker's preference.

In future research, we can define such aggregation operators over some advancement of q-ROFS such as complex q-ROFS (Liu et al. 2020). Our introduced operators can be further extended up to the larger field of various types of MADM processes, e.g., the COPRAS

method (Krishankumar et al. 2019), TOPSIS method (Riaz et al. 2020), EDAS method (Li et al. 2020), VIKOR method (Mi et al. 2019), etc. Also, we can prolong our proposed operators to deal with interval-valued fuzzy (Chen 1997; Chen et al. 1997; Chen and Hsiao 2000; Chen et al. 2012; Turksen 1986) MADM problems. Also, we may exert them to deal with some real-life decision-making problems, such as domestic airlines evaluation (Ma and Xu 2016), company investment decision (Garg 2016), etc.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

References

Akram M, Shahzadi G (2020) A hybrid decision making model under q-Rung orthopair fuzzy Yager aggregation operators. Granul Comput. https://doi.org/10.1007/s41066-020-00229-z

Akram M, Shahzadi G, Alcantud JCR (2021) Multi-attribute decision-making with q-rung picture fuzzy information. Granul Comput. https://doi.org/10.1007/s41066-021-00260-8

Atanassov KT (1986) Intuitionistic fuzzy sets. Fuzzy Sets Syst 20(1):87–96

Casasnovas J, Torrens J (2003) An axiomatic approach to fuzzy cardinalities of finite fuzzy sets. Fuzzy Sets Syst 133(2):193–209

Chen SM (1997) Interval-valued fuzzy hypergraph and fuzzy partition. IEEE Trans Syst Man Cybern B 27(4):725–733

Chen SM, Hsiao WH (2000) Bidirectional approximate reasoning for rule-based systems using interval-valued fuzzy sets. Fuzzy Sets Syst 113(2):185–203

Chen SM, Hsiao WH, Jong WT (1997) Bidirectional approximate reasoning based on interval-valued fuzzy sets. Fuzzy Sets Syst 91(3):339–353

Chen SM, Chang YC, Pan JS (2012) Fuzzy rules interpolation for sparse fuzzy rule-based systems based on interval type-2 Gaussian fuzzy sets and genetic algorithms. IEEE Trans Fuzzy Syst 21(1):412–425

Ejegwa PA (2019) Pythagorean fuzzy set and its application in career placements based on academic performance using max-min-max composition. Complex Intell Syst 5(2):165–175

Feng F, Zheng Y, Sun B, Akram M (2021) Novel score functions of generalized orthopair fuzzy membership grades with application to multiple attribute decision making. Granul Comput. https://doi.org/10.1007/s41066-021-00253-7

Frank MJ (1979) On the simultaneous associativity of $F(x,y)$ and $x + y - F(x,y)$. Aequ Math 19(1):194–226

Gao J, Liang Z, Xu Z (2019) Additive integrals of q-rung orthopair fuzzy functions. IEEE Trans Cybern 50(10):4406–4419

Garg H (2016) A novel accuracy function under interval-valued Pythagorean fuzzy environment for solving multicriteria decision making problem. J Intell Fuzzy Syst 31(1):529–540

Krishankumar R, Ravichandran KS, Kar S, Cavallaro F, Zavadskas EK, Mardani A (2019) Scientific decision framework for

evaluation of renewable energy sources under q-rung orthopair fuzzy set with partially known weight information. Sustainability. https://doi.org/10.3390/su11154202

Li Z, Wei G, Wang R, Wu J, Wei C, Wei Y (2020) EDAS method for multiple attribute group decision making under q-rung orthopair fuzzy environment. Technol Econ Dev Econ 26(1):86–102

Liu P, Wang P (2018) Some q-rung orthopair fuzzy aggregation operators and their application to multiple attribute decision making. Int J Intell Syst 33(2):259–280

Liu P, Mahmood T, Ali Z (2020) Complex q-rung orthopair fuzzy aggregation operators and their application in multi-attribute decision making. Information. https://doi.org/10.3390/info11010005

Ma Z, Xu Z (2016) Symmetric Pythagorean fuzzy weighted geometric/averaging operators and their application in multicriteria decision-making problems. Int J Intell Syst 31(12):1198–1219

Mi X, Li J, Liao H, Zavadskas EK, Al-Barakati A, Barnawi A, Taylan O, Herrera-Viedma E (2019) Hospitality brand management by a score-based q-rung orthopair fuzzy VIKOR method integrated with the best worst method. Econ Res-Ekon Istraz 32(1):3272–3301

Peng X, Liu L (2019) Information measures for q-rung orthopair fuzzy sets. Int J Intell Syst 34(8):1795–1834

Qin J, Liu X (2014) Frank aggregation operators for triangular interval type-2 fuzzy set and its application to multiple attribute group decision making. J Appl Math 2014:1–24

Qin J, Liu X, Pedrycz W (2016) Frank aggregation operators and their application to hesitant fuzzy multiple attribute decision making. Appl Soft Comput 41:428–452

Riaz M, Farid H, Karaaslan F, Hashmi M (2020) Some q-rung orthopair fuzzy hybrid aggregation operators and TOPSIS method for multi-attribute decision-making. J Intell Fuzzy Syst 39(1):1227–1241

Saaty TL (2008) Decision making with the analytic hierarchy process. Int J Serv Sci 1(1):83–98

Sarkoci P (2005) Domination in the families of Frank and Hamacher t-norms. Kybernetika 41(3):349–360

Seikh MR, Mandal U (2021) Intuitionistic fuzzy Dombi aggregation operators and their application to multiple attribute decision-making. Granul Comput 6(3):473–488

Shannon CE (1948) A mathematical theory of communication. Bell Syst Tech 27(3):379–423

Shu X, Ai Z, Xu Z, Ye J (2019) Integration of q-rung orthopair fuzzy continuous information. IEEE Trans Fuzzy Syst. https://doi.org/10.1109/TFUZZ.2019.2893205

Turksen IB (1986) Interval valued fuzzy sets based on normal forms. Fuzzy Sets Syst 20(2):191–210

Wang WS, He HC (2009) Research on flexible probability logic operator based on Frank T/S norms. Acta Electron Sin 37(5):1141–1145

Wang J, Zhang R, Zhu X, Zhou Z, Shang X, Li W (2019a) Some q-rung orthopair fuzzy Muirhead means with their application to multi-attribute group decision making. J Intell Fuzzy Syst 36(2):1614–1699

Wang J, Wei G, Lu J, Alasaadi FE, Hayat T, Wei C, Zhang Y (2019b) Some q-rung orthopair fuzzy hammy mean operators in multiple attribute decision-making and their application to enterprise resource planning system selection. Int J Intell Syst 34(10):2429–2458

Wang J, Wei G, Wei C, Wei Y (2020) MABAC method for multiple attribute group decision making under q-rung orthopair fuzzy environment. Def Technol 16:208–216

Wei G, Wei C, Wang J, Gao H, Wei Y (2018) Some q-rung orthopair fuzzy maclaurin symmetric mean operators and their applications to potential evaluation of emerging technology commercialization. Int J Intell Syst 34(1):50–81

Xing Y, Zhang R, Wang J, Zhu X (2018) Some new Pythagorean fuzzy Choquet-Frank aggregation operators for multi-attribute decision making. Int J Intell Syst 33(11):2189–2215

Yager RR (2004) On some new classes of implication operators and their role in approximate reasoning. Inf Sci 167(1):193–216

Yager RR (2014) Pythagorean membership grades in multicriteria decision making. IEEE Trans Fuzzy Syst 22(4):958–965

Yager RR (2017) Generalized orthopair fuzzy sets. IEEE Trans Fuzzy Syst 25(5):1222–1230

Yager RR, Abbasov AM (2013) Pythagorean membership grades, complex numbers, and decision making. Int J Intell Syst 28(5):436–452

Yang M, Hussain Z (2018) Fuzzy entropy for Pythagorean fuzzy sets with application to multicriterion decision making. Complexity. https://doi.org/10.1155/2018/2832839

Yi Y, Li Y, Heng D, Qian G, Lyu H (2018) The Pythagorean fuzzy Frank aggregation operators based on isomorphism Frank t-norm and s-norm and their application. Control Decis 33(8):1471–1480

Zhang X, Liu P, Wang Y (2015) Multiple attribute group decision making methods based on intuitionistic fuzzy Frank power aggregation operators. J Intell Fuzzy Syst 29(5):2235–2246

Zhou L, Dong J, Wan S (2019) Two new approaches for multi-attribute group decision-making with interval-valued neutrosophic Frank aggregation operators and incomplete weights. IEEE Access 7(1):102727–102750