



Bipolarity in multi-way fuzzy context and its analysis using m -way granulation

Prem Kumar Singh¹

Received: 25 March 2021 / Accepted: 22 June 2021 / Published online: 8 July 2021
 © The Author(s), under exclusive licence to Springer Nature Switzerland AG 2021

Abstract

Recently, most of the researchers focused on dealing with uncertainty and unclear data beyond the three-way fuzzy space and its graphical visualization. An issue arises when the bipolar attributes exist beyond the three-way or m -way fuzzy space. One of suitable examples is colors consist of several opposite and non-opposite side of red–green–blue to produce a new color. To deal with these types of bipolarity exists in multi-valued attribute, a method is proposed in this paper using the connection of bipolar fuzzy set with multi-fuzzy set. Same time graphical structure visualization of bipolar multi-fuzzy set is also introduced for precise analysis of knowledge discovery and representation tasks. In addition, an another method is proposed to refine the bipolar multi-fuzzy context at user defined multi-granulation with an illustrative example. The analysis derived from the proposed method is also compared with recently available approaches on bipolar multi-fuzzy set.

Keywords Bipolar fuzzy concept · Bipolar multi-fuzzy set · Concept lattice · Knowledge representation · m -polar fuzzy concept lattice · Granular computing

List of symbols

L	Scale of truth degree	\wedge	Infimum
\mathbf{L}	Residuated lattice	\vee	Supremum
(X, Y, R)	Formal fuzzy context-F	m	m -polarity
(X, Y, \tilde{R})	Bipolar multi-fuzzy context-F	k	Total number of attributes
\otimes	Multiplication	n	Total number of objects
\rightarrow	Residuum	t	tuple
a, b, c	Elements in L	I and J	bipolar fuzzy set
(\uparrow, \downarrow)	Galois connection	\tilde{I} and \tilde{J}	bipolar multi-fuzzy set
\prod	Projection operator	V	Vertex set
A	Extent	E	Edges set
B	Intent	v_1, v_2, v_3	Vertex of graph
A_{s_i}	Set of m -polar object set	$v_1 v_2, v_2 v_3, v_3 v_1$	Edges of fuzzy graph
B_{s_j}	Set of m -polar attribute set	x, y, u, v	Elements
L^X	L -set of objects	x_1, x_2, \dots, x_n	n th objects
L^Y	L -set of attributes	y_1, y_2, \dots, y_k	k th attributes
\cup	Union	$\ $	Cardinality
\cap	Intersection		
$\mu^P(z)$	Positive membership degree		
$\mu^N(z)$	Negative membership degree		

1 Introduction

Knowledge discovery and representation from a given data set is considered as one of the major issues as discussed in Singh et al. (2016). In this regard, a mathematical model based on calculus of applied abstract algebra was developed by Wille (1982) at early eighties. This model is

✉ Prem Kumar Singh
 premsingh.csjm@gmail.com; premsingh.csjm@yahoo.com

¹ Department of Computer Science and Engineering, Gandhi Institute of Technology and Management, Visakhapatnam, Andhra Pradesh 530045, India

known as formal concept analysis (FCA) which is applied in various fields for knowledge discovery and representation tasks (Ganter and Wille 1999). This mathematical model receives data input in matrix format (X, Y, R) where $R (\subseteq X \times Y)$ and provides set of patterns based on the objects X and their common attributes Y set. It generates pattern in form of concepts having a set of objects $(A \subseteq X)$ and their common attributes $(B \subseteq Y)$ closed with Galois connection as discussed by Wille (1982). Same time the calculus of FCA provides super and sub concept hierarchical order visualization among the formal concepts as well as attribute implications which helps a lot in knowledge processing tasks. The graphical structure connection of concept lattice provided a platform to incorporate the FCA in other dimensions (Berry and Sigayret 2004). Same time theory of fuzzy set (Zadeh 1965) given an alternative way to compute with words like tall and young which helped a lot in various applications (Chen et al. 1990; Chen 1996; Chen and Jong 1997; Chen and Huang 2003). Due to that, the mathematical algebra of FCA is emerged with fuzzy setting by Burusco and Fuentes-Gonzalez (1994). Same time dealing with partial ignorance the properties of interval-valued fuzzy concept analysis is introduced (Burusco and Fuentes-Gonzalez 2001; Djouadi and Prade 2009). These extensions able to represent acceptance and rejection values of any fuzzy attributes in the defined unipolar space. The problem arises in case of handling bipolar fuzzy attributes where positive and negative values exist simultaneously. To compute with these type of uncertainty Singh and Kumar (2014a, b) introduced bipolar fuzzy concept lattice using the calculus of bipolar lattice (Bloch 2011; Zhang 1994) and its graph (Akram 2011; Yang et al. 2013). The problem arises while dealing with hesitant part or indeterminacy exists in fuzzy attributes excluding its acceptance and negation part. It become more complex when the bipolarity exists beyond the unipolar space and contains several positive and negative values. These types of condition used to found in dynamic system while investigation of homoclinic¹ or heteroclinic² orbit. In this case, the event may contain m -number of saddle points rather than equilibrium. In these types of dynamic cases, the hesitant part plays major role in discovering the pattern as discussed by Singh (2019c, 2020). Some other researchers tried to represent them as three-way decision space (Huang et al. 2017; Li et al. 2017; Singh 2017; Yao 2021a, b) whereas some of them tried beyond the three m -polar fuzzy space (Kroonberg 2007; Li 2017; Mesiarova-Zemankova and Ahmad 2014; Mesiarova-Zemankova and Hycko 2015; Singh 2018c, d, 2019a). The problem arises when bipolarity exists beyond the

three-way fuzzy space (Singh 2019b, c) or other spaces (Singh 2021; Yao 2021a, b). This creates two possible notions: (1) the existence of bipolarity in m -way fuzzy space and (2) the existence of m -tuples in bipolar fuzzy attributes. This paper focused on generalized representation of bipolar fuzzy attributes in m -polar fuzzy space to deal with existence of bipolar fuzzy attributes in m -polar fuzzy space as shown in Table 1. To achieve this goal, current paper focuses on bipolar multi-fuzzy context analysis and its graphical structure visualization using hybridization of bipolar fuzzy set and m -polar fuzzy graph as explained in Fig. 1.

The m -polar fuzzy attributes (Y) can be found both in dark or soft data sets as discussed by Singh (2018c, 2018d, 2021). One of the suitable examples is democratic country like India where opinion of people used to depends on 29-distinct states. Each of the 29 states used to accept a political party or reject the political party, independently. The positive side can be represented precisely using m -polar fuzzy set and its mapping $f : Y \rightarrow [0, 1]^m$ where $(0, 0, \dots, 0)$ denotes the least and $(1, 1, \dots, 1)$ as the greatest element as discussed by Chen et al. (2014). In case, the bipolarity exists in m -polar fuzzy space then the given theory require an extension or hybridization. It is indeed requirement to deal with multi-valued attributes data set like win and loss in tug of war (Ghorai and Pal 2015), positive and negative position of a person in a democratic country (Singh 2018c, d), safety or non-safety of a mankind (Kapoor and Singh 2020). In some cases the dimension of multi-valued attributes may be bounded within a given space as for example the color of a pixel (Red, Green, Blue) or the price (high, medium, low) of a commodity. In this case, the considered dimensions of multi-fuzzy attributes can be represented as an ordered sequence of membership function i.e. $f : y_m \in Y \rightarrow [0, 1]$ where $0 \leq \sum \mu(y_m) \leq 1$ (Sebastian and Ramakrishnan 2011). It can be observed that, this set uses an ordinary single-valued membership fuzzy set as a building block to represent the acceptance or rejection of multi-valued fuzzy attributes. In this case, a problem arises at the time of analyzing bipolarity existing in each of the defined building blocks of m -polar fuzzy space (Dubois and Prade 2012a, b; Mechelen and Smilde 2011; Sadaaki 2001; Zadeh 2011; Zhang et al. 2016). The bipolarity in m -fuzzy space represents positive or negative side of given multi-fuzzy attributes, independently. It means each building block of m -polar fuzzy space may contain a positive $(0, 1]$ and negative side $[-1, 0)$ to represent the bipolar information precisely. The membership value 0 means the information is neutral to the given property. Recently, some of the researchers tried to represent this type of attributes using the bipolar multi-fuzzy set with its applications in multi-

¹ https://en.wikipedia.org/wiki/Homoclinic_orbit.

² https://en.wikipedia.org/wiki/Heteroclinic_orbit.

Table 1 Some of the possible conditions existing in data with bipolar fuzzy attributes

Conditions	Objects	Attributes	Bipolar fuzzy relation
a	Binary	m -polar or three-way	Bipolar multi fuzzy set
b	m -polar or three-way	Binary	Bipolar multi fuzzy set
c	Bipolar	m -polar three-way	Bipolar multi fuzzy set
d	m -polar three-way	Bipolar	Bipolar multi fuzzy set
e	m -polar or three-way	m -polar or three-way	Bipolar multi fuzzy set

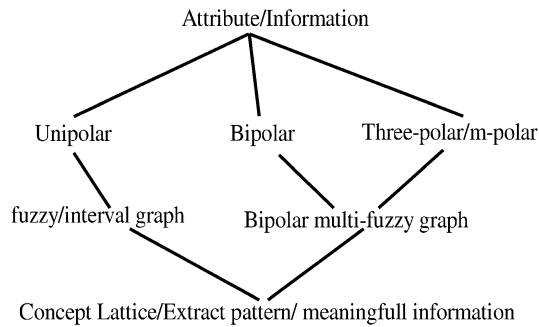


Fig. 1 The existence of bipolarity beyond three-way fuzzy space

decision making process (Yang et al. 2014). This paper focuses on bipolar multi-fuzzy concept lattice visualization and its analysis at user defined multi-granulation.

The bipolar multi-fuzzy set and its applications can be understood from the tug of war game. In this game, people used to pull the rope in different directions. The person who pulls the rope in their direction used to win the game. In this process, several time the rope used to move from positive to negative side from the chosen direction the center of rope. Let us suppose, m -number of people are participating in the game then the strength used by m th-people can be represented using the bipolar multi-fuzzy set. The reason is it contains the positive and negative membership value as homoclinic or heteroclinic orbit. In this way, the strength at m th-block and its fluctuation can be defined via m -polar fuzzy space. The position of an object in a space can be defined through its positive and negative membership value computed for the defined m th-block. Even the opinion of people in a democratic country like India is based on 29 opposite and non-opposite side of opinion which arises from the 29 states. In similar way, the doctor used to give group of medicines to cure a disease which may be effective or destructive can be represented using precisely using the bipolar multi-fuzzy set. The reviewer feedback and rating used to be bipolar based on several metric as discussed by Kamaci and Petchimuthu (2020). This type of bipolarity and its existence beyond three-way fuzzy space is mathematically expensive in representation. Due to which, polyadic context and its analysis is introduced by Voutsadakis (2002) without considering the bipolar fuzzy attributes. Some of the

researchers tried to analyze the bipolar fuzzy attributes using the two-relations (Alcalde et al. 2015), bi-directional (Kumar et al. 2015), independent positive and negative relation (Singh and Kumar 2014a, b), dependent positive and negative relations (Singh 2018b), multi-fuzzy attribute (Chen et al. 2014; Coppi 1994; Singh 2018a, b, c), bipolar multi-fuzzy attributes (Singh 2019c; Sebastian and Ramakrishnan 2011; Yang et al. 2013) at defined multi-granules (Pedrycz and Chen 2015; Singh and Kumar 2014a, b; Singh and Abdullah 2015; Singh 2018a; Skowron et al. 2016; Yao 2004, 2016). The reason is bipolarity cannot be excluded from the three-way or multi-fuzzy space as shown in Table 1. Due to which, this paper focuses on hybridization of bipolar fuzzy set and m -polar fuzzy set. The motive is to deal with bipolarity and its existence beyond the three-way or m -polar fuzzy space for handling the soft data set. The objective is to investigate some of the useful patterns in bipolar multi-fuzzy contexts for knowledge processing tasks and other decision making process. To achieve this goal, two methods are proposed in this paper : (1) a bipolar multi-fuzzy matrix representation of soft data set is introduced, (2) a method is proposed to extract bipolar multi-fuzzy pattern using Next Neighbor algorithm motivated from Lindig (2000), (3) another method is proposed to extract some useful information based on required multi-granulation (α_m, β_m) , (4) to validate the results analysis obtained from the proposed methods are compared with Yang et al. (2013).

Rest part of the paper is organized as follows: Sect. 2 provides a brief background about FCA in the fuzzy setting and some properties of bipolar multi-fuzzy set. Section 3 provides the proposed method for generating the bipolar multi-fuzzy concepts and its reduction at user defined multi-granulation. Section 4 contains the illustration of the proposed methods using an illustrative example. Section 5 contains discussion followed by conclusions, acknowledgements and references.

2 Preliminaries

In this section, some basic definition about FCA in the fuzzy setting and bipolar multi-fuzzy graph is given to analyze the data with bipolar multi-fuzzy attributes.

2.1 FCA in the fuzzy setting

FCA with fuzzy setting starts the analysis from a given fuzzy matrix i.e. called as formal fuzzy context. Let L be a scale of truth degrees of a complete residuated lattice \mathbf{L} (defined below). A formal fuzzy context is a triplet $F = (X, Y, R)$, where X is a finite set of objects, Y is a finite set of attributes and R is an L -relation among X and Y as follows: $X \times Y \rightarrow L$ (Burusco and Fuentes-Gonzales 2001). Each relation $R(x, y) \in L$ represents the membership value at which the object $x \in X$ has the attribute $y \in Y$ in $[0, 1]$ (L is a support set of some complete residuated lattice \mathbf{L}).

A residuated lattice $\mathbf{L} = (L, \wedge, \vee, \otimes, \rightarrow, 0, 1)$ is the basic structure of truth degrees, where 0 and 1 represent least and greatest elements, respectively. \mathbf{L} is a complete residuated lattice iff (Pollandt 1997):

1. $(L, \wedge, \vee, 0, 1)$ is a complete lattice.
2. $(L, \otimes, 1)$ is commutative monoid.
3. \otimes and \rightarrow are adjoint operators (called multiplication and residuum, respectively), that is $a \otimes b \leq c$ iff $a \leq b \rightarrow c, \forall a, b, c \in L$.

The operators \otimes and \rightarrow are defined distinctly by Lukasiewicz, Godel, and Goguen t-norms and their residua as given below (Belohlavek 2004):

Lukasiewicz:

- $a \otimes b = \max(a + b - 1, 0)$,
- $a \rightarrow b = \min(1 - a + b, 1)$.

Godel:

- $a \otimes b = \min(a, b)$,
- $a \rightarrow b = 1$ if $a \leq b$, otherwise b .

Goguen:

- $a \otimes b = a \cdot b$,
- $a \rightarrow b = 1$ if $a \leq b$, otherwise b/a .

Classical logic of FCA is an example of a complete residuated lattice which is represented as $(\{0, 1\}, \wedge, \vee, \otimes, \rightarrow, 0, 1)$.

For any \mathbf{L} -set $A \in L^X$ of objects, and $B \in L^Y$ of attributes an \mathbf{L} -set $A^\uparrow \in L^Y$ of attributes and \mathbf{L} -set $B^\downarrow \in L^X$ of objects can be defined as follows (Belohlavek 2004):

- (1) $A^\uparrow(y) = \bigwedge_{x \in X} (A(x) \rightarrow R(x, y))$,
- (2) $B^\downarrow(x) = \bigwedge_{y \in Y} (B(y) \rightarrow R(x, y))$.

$A^\uparrow(y)$ is interpreted as the \mathbf{L} -set of attribute $y \in Y$ shared by all objects from A . Similarly, $B^\downarrow(x)$ is interpreted as the \mathbf{L} -set of all objects $x \in X$ having the same attributes from B in common. The formal fuzzy concept is a pair of $(A, B) \in L^X \times L^Y$ satisfying $A^\uparrow = B$ and $B^\downarrow = A$, where fuzzy set of objects A is called an extent and fuzzy set of attributes B is called an intent. The operators (\uparrow, \downarrow) are known

as Galois connection which was extended in several ways (Bloch 2011; Djouadi 2011; Djouadi and Prade 2009; Dubois and Prade 2012a, b) for knowledge processing tasks in fuzzy settings (Glodeanu 2014).

The down-operator (\downarrow) of Galois connection is applied on the fuzzy set defined on the given subset of the attributes. This provides a fuzzy set of objects having maximal membership value while integrating the information from the given subset of attributes. Consequently, the up-operator (\uparrow) of Galois connection is applied on these obtained fuzzy set of constituted covered objects set. It seize the fuzzy set of attributes with their maxim membership value while integrating the information from the constituted objects set. Furthermore extra fuzzy set of objects (attributes) cannot be discovered which make the constituted fuzzy set of attributes (objects) bigger then it forms a formal fuzzy concept. It means a formal fuzzy concept is a maximal rectangle of a given formal fuzzy context (F) filled with membership value between $[0, 1]$, which is an ordered pair of two sets (A, B) , where A is called (fuzzy) extent, and B is called (fuzzy) intent. The set of formal fuzzy concepts C , generated from a given fuzzy context defines the partial ordering principle i.e. $(A_1, B_1) \leq (A_2, B_2) \iff A_1 \subseteq A_2 (\iff B_2 \subseteq B_1)$ for every formal fuzzy concept. Together with this ordering, in the complete lattice there exist an infimum and a supremum for some formal concepts as follows (Ganter and Wille 1999):

- $\bigwedge_{j \in J} (A_j, B_j) = (\bigcap_{j \in J} A_j, (\bigcup_{j \in J} B_j)^{\uparrow\downarrow})$,
- $\bigvee_{j \in J} (A_j, B_j) = ((\bigcup_{j \in J} A_j)^{\uparrow\downarrow}, \bigcap_{j \in J} B_j)$.

Extracting all the formal fuzzy concepts and their specialization and generalization visualization in the concept lattice is the main concern for the practical applications of FCA in various research fields discussed by Singh et al. (2016). In this process, a problem is addressed while handling fuzzy attributes beyond the bipolar fuzzy space (Singh 2019c). The reason is some time the attributes may contain the bipolarity for each of the building blocks of m -polar fuzzy space (Yang et al. 2013). To deal with this types of attributes, the current paper introduces bipolar multi-fuzzy set and its properties for knowledge processing tasks. In the next section, some of its required properties are given for the depth analysis of data with bipolar multi-fuzzy attributes.

2.2 Bipolar-multi fuzzy graph

This section provides some basic definitions related to bipolar multi-fuzzy set and its graphical structure visualization.

Definition 1 (Bipolar fuzzy set) (Zhang 1994, 2021) Let Z be a non-empty set (i.e. there exists $z \in Z$). A bipolar fuzzy

set J in Z is an object having the form $J = \{z, \mu^P(z), \mu^N(z) \mid z \in Z\}$ where $\mu^P(z): Z \rightarrow [0, 1]$ and $\mu^N(z): Z \rightarrow [-1, 0]$ are mappings. We use the positive membership degree $\mu^P(z)$ to denote the satisfaction degree of an element z to the property corresponding to a bipolar fuzzy set J , and the negative membership degree $\mu^N(z)$ to denote the satisfaction degree of an element z to some implicit counter-property corresponding to a bipolar fuzzy set J .

For every two bipolar fuzzy sets $I = (\mu_I^P, \mu_I^N)$ and $J = (\mu_J^P, \mu_J^N)$ following can be computed (Lee 2000):

- (i) $(I \cap J)(z) = (\min(\mu_I^P(z), \mu_J^P(z)), \max(\mu_I^N(z), \mu_J^N(z)))$,
- (ii) $(I \cup J)(z) = (\max(\mu_I^P(z), \mu_J^P(z)), \min(\mu_I^N(z), \mu_J^N(z)))$.

Definition 2 (*Bipolar multi-fuzzy set*) (Yang et al. 2013) A bipolar multi-fuzzy set is an extensive representation of bipolar fuzzy set to m -space. It is a set on X can be represented as a mapping $f: X \rightarrow ([-1, 0] \times [0, 1])^m$. The set of all bipolar multi-fuzzy sets can be represented as $\tilde{M}(X)$ as given below:

$$\tilde{M}(X) = \{((\mu_1^P, \mu_1^N), (\mu_2^P, \mu_2^N), \dots, (\mu_m^P, \mu_m^N))/x\}.$$

Here the set $\{((0, 1), (0, 1), \dots, (0, 1)_m)/x\}$ represents corresponding property and $\{((-1, 0), (-1, 0), \dots, (-1, 0)_m)/x\}$ represents implicit counter property of the given information. The set $\{((0, 0), (0, 0), \dots, (0, 0)_m)/x\}$ represents irrelevant property to the given information i.e. null bipolar fuzzy set of dimension m . The set $\{((1, -1), (1, -1), \dots, (1, -1)_m)/x\}$ represents absolute bipolar multi-fuzzy set of dimension m . The bipolar multi-fuzzy set is an extension of bipolar fuzzy set which can be visualized in the concept lattice as bipolar fuzzy set is visualized (Singh and Kumar 2014a, b; Singh 2018b).

Definition 3 (*Bipolar multi-fuzzy relation*) (Yang et al. 2014; Zhang et al. 2016) Bipolar multi-fuzzy relation is an extensive representation of bipolar fuzzy relation. In this case all bipolar multi-fuzzy set membership degrees will be represented in interval $[-1, 0] \times [0, 1]$ i.e. the bipolar multi-fuzzy relation can be mapped in the interval $[-1, 1]^m$ as follows: $\mu_m^R(x_i, y_i) \rightarrow [-1, 1]^m$. It can be represented as follows for the set X and Y as given below:

$$\tilde{R}(x_i, y_i) = \{((\mu_1^P, \mu_1^N), (\mu_2^P, \mu_2^N), \dots, (\mu_m^P, \mu_m^N))/(x_i, y_i)\}$$

where positive membership degree $\mu^P(x_i, y_i)$ to denote the satisfaction degree of relation (x_i, y_i) to the property corresponding, and the negative membership degree $\mu^N(x_i, y_i)$ to denote implicit counter-property corresponding to the given context. Let $\tilde{I}(X)$ and $\tilde{J}(X)$ two bipolar-multi fuzzy set then following operations can be computed:

- (i) $(\tilde{I} \cap \tilde{J})_m(x) = (\min(\mu_{1,\dots,m}^P \tilde{I}, \mu_{1,\dots,m}^P \tilde{J}), \max(\mu_{1,\dots,m}^N \tilde{I}, \mu_{1,\dots,m}^N \tilde{J}))(x)$,
- (ii) $(\tilde{I} \cup \tilde{J})_m(x) = (\max(\mu_{1,\dots,m}^P \tilde{I}, \mu_{1,\dots,m}^P \tilde{J}), \min(\mu_{1,\dots,m}^N \tilde{I}, \mu_{1,\dots,m}^N \tilde{J}))(x)$.

In this way an infimum and a supremum of a bipolar multi-fuzzy formal concepts can be computed.

Definition 4 (*Bipolar fuzzy graph*) (Akram 2011; Yang et al. 2014) A bipolar fuzzy graph with an underlying sets V and $E \subseteq V \times V$ is defined to be a pair $G = (I, J)$ where $I = (\mu_I^P, \mu_I^N)$ is a bipolar fuzzy set in V and $J = (\mu_J^P, \mu_J^N)$ is a bipolar fuzzy set of edges- E such that:

$$\begin{aligned} \mu_J^P(\{v_1, v_2\}) &\leq \min(\mu_I^P(v_1), \mu_I^P(v_2)) (\forall (v_1, v_2) \in V \times V) \\ \mu_J^N(\{v_1, v_2\}) &\geq \max(\mu_I^N(v_1), \mu_I^N(v_2)) (\forall (v_1, v_2) \in V \times V) \\ &\text{and} \\ \mu_J^N(\{v_1, v_2\}) &= \mu_J^N(\{v_1, v_2\}) = 0 \quad \forall (v_1, v_2) \in (V \times V \setminus E). \end{aligned}$$

Example 1 Let us suppose, an expert provide an opinion about three drawing papers (v_1, v_2, v_3) to accept or reject them for the production. The expert opinion can be represented through positive and negative membership value of a defined bipolar fuzzy set as shown in Table 2. Table 3 represents the corresponding relationship among the given drawing papers. This numerical representation of expert feedback can be visualization in compact format considering Tables 2 and 3 as vertices (V) and edges (E) of a defined bipolar fuzzy graph as shown in Fig. 2.

Definition 5 (*m-polar fuzzy graph*) (Akram 2019; Chen et al. 2014; Singh 2018a) An m -polar fuzzy graph with an underlying pair (V, E) where $E \subseteq V \times V$ is symmetric i.e. $(v_1, v_2) \in E \Leftrightarrow (v_2, v_1) \in E$. Then it is defined to be a pair $G = (I, J)$ where $I: V \rightarrow [0, 1]^m$ i.e. it is an m -polar fuzzy set on V . Similarly, $J: E \rightarrow [0, 1]^m$ i.e. m -polar fuzzy set on E . It follows $J(v_1 v_2) \leq \inf\{I(v_1), I(v_2)\}$ for each m -polar space. The given m -polar fuzzy graph is is strong iff $J(v_1 v_2) = \inf\{I(v_1), I(v_2)\}$. The union and intersection of the m -polar fuzzy graph can be computed as per bipolar fuzzy graph. The m -polar fuzzy graph and its properties can be studied extensively by Akram (2019).

Table 2 A bipolar fuzzy subset of V for Example 1

	v_1	v_2	v_3
μ_I^P	0.5	0.6	0.4
μ_I^N	-0.7	-0.5	-0.7

Table 3 A bipolar fuzzy subset of E for Example 1

	v_1v_2	v_2v_3	v_3v_1
μ_J^P	0.4	0.3	0.3
μ_J^N	-0.3	-0.2	-0.1

Table 5 A m -polar set of E for Example 2

	v_1v_2	v_2v_3	v_3v_1
μ_1	0.45	0.45	0.49
μ_2	0.42	0.40	0.40
μ_3	0.51	0.54	0.51

	v_1v_2	v_2v_3	v_3v_1
μ_J^P	0.4	0.3	0.3
μ_J^N	-0.3	-0.2	-0.1

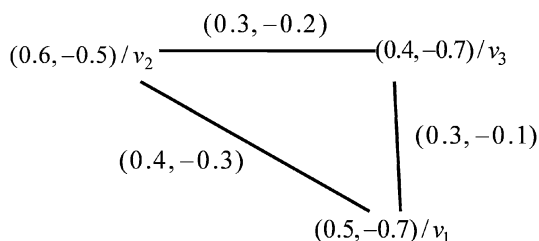


Fig. 2 A bipolar fuzzy graph representation of Example 1

Example 2 Let us extend the Example 1 as the given expert wants to analyze the three given engineering drawing paper based on their color for its precise applications. It is known that, the attribute color is multi-valued which contain combination of red, green, and blue (RGB) simultaneously. In this case, the color for each of the drawing papers (v_1, v_2, v_3) can be measured based on the percentage of red, green, blue as shown in Table 4 whereas the Table 5 represents the corresponding relationship among them. The Tables 4 and 5 can be considered as vertices (V) and edges (E) of a defined multi-fuzzy graph as shown in Fig. 3.

Definition 6 (*Bipolar multi-fuzzy graph*) (Yang et al. (2013); Singh (2019c)): A bipolar multi-fuzzy graph with an underlying sets V and $E \subseteq V \times V$ is defined to be a pair $G = (\tilde{I}, \tilde{J})$ where $\tilde{I} = (\mu_{1,\dots,m}^P(\tilde{I}), \mu_{1,\dots,m}^N(\tilde{I}))$ is a bipolar multi-

Table 4 A m -polar set of V for Example 2

	v_1	v_2	v_3
μ_1	0.49	0.45	0.50
μ_2	0.46	0.42	0.40
μ_3	0.51	0.59	0.54

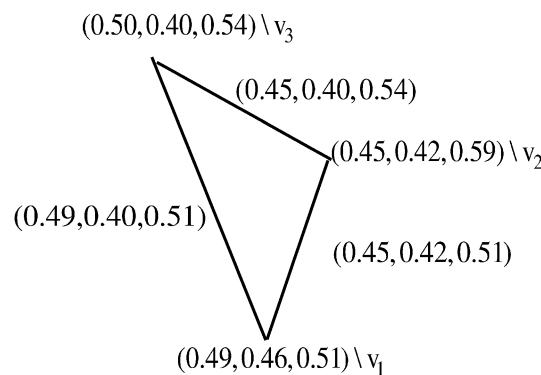


Fig. 3 A m -polar fuzzy graph representation of Example 2

fuzzy set on vertices- V , and $\tilde{J} = (\mu_{1,\dots,m}^P(\tilde{J}), \mu_{1,\dots,m}^N(\tilde{J}))$ is a bipolar multi-fuzzy set of edges- E such that:

$$\mu_J^P(\{v_1, v_2\})_m \leq \min_{1,\dots,m}(\mu_{I_1}^P(v_1), \mu_{I_1}^P(v_2)) \quad (\forall (v_1, v_2) \in V \times V)$$

$$\mu_J^N(\{v_1, v_2\})_m \geq \max_{1,\dots,m}(\mu_{I_1}^N(v_1), \mu_{I_1}^N(v_2)) \quad (\forall (v_1, v_2) \in V \times V)$$

and

$$\mu_J^P(\{v_1, v_2\})_m = \mu_J^N(\{v_1, v_2\})_m = 0 \quad \forall (v_1, v_2) \in (V \times V \setminus E).$$

In the case when equality holds then it is called as bipolar multi-fuzzy complete graph.

Example 3 Let us connect, the Examples 1 and 2 given in this paper. Suppose that, the expert gives opinion (positive and negative) to about drawing papers (v_1, v_2, v_3) based on their Color (Red, Green, Blue). In this case, the expert will give an evidence to accept or reject color of drawing paper based on positive and negative membership value for red, green and blue simultaneously. The expert says that, the drawing paper v_1 build using 50% red colore, 70% green, 80% blue colour whereas the 70% opposite of red color, 30% opposite of greed color and 50% opposite of blue color. This type of complex mixture is require to build the drawing paper for civil engineering students. It can be represented through a defined bipolar multi-fuzzy set as shown in Table 6 whereas Table 7 represents correspond- ing relationship among them. The Tables 6 and 7 can be considered as vertices (V) and Edges (E) of a defined bipolar multi-fuzzy graph as shown in Fig. 4. Now, to analyze these type of context for knowledge processing tasks a method is proposed in the next section.

Table 6 A bipolar multi-fuzzy set of V for Example 3

	v_1	v_2	v_3
$\mu_1^P(I)$	0.5	0.6	0.4
$\mu_1^N(I)$	-0.7	-0.5	-0.7
$\mu_2^P(I)$	0.7	0.5	0.6
$\mu_2^N(I)$	-0.3	-0.3	-0.3
$\mu_3^P(I)$	0.8	0.7	0.4
$\mu_3^N(I)$	-0.5	-0.2	-0.3

Table 7 A bipolar multi-fuzzy set of E for Example 3

	v_1v_2	v_2v_3	v_3v_1
$\mu_1^P(J)$	0.5	0.4	0.4
$\mu_1^N(J)$	-0.7	-0.7	-0.7
$\mu_2^P(J)$	0.5	0.5	0.6
$\mu_2^N(J)$	-0.3	-0.3	-0.3
$\mu_3^P(J)$	0.7	0.4	0.4
$\mu_3^N(j)$	-0.5	-0.3	-0.5

The above backgrounds validates that bipolar multi-fuzzy graph can be used to visualized the data with bipolar multi-fuzzy attributes as other graph theory has been incorporated (Ghosh et al. 2010; Singh and Kumar 2014a, b; Singh 2018b, c). In this case, the bipolar multi-fuzzy concepts can be interpreted as given below.

Definition 7 (*Bipolar multi-fuzzy concepts*) Let us suppose, a pair of bipolar multi-fuzzy set (A, B) where A is set of bipolar-multi fuzzy set of objects as follows:

$$(A) = \{x_i, (\mu_1^P(x_i), \mu_1^N(x_i)), (\mu_2^P(x_i), \mu_2^N(x_i)), \dots, (\mu_m^P(x_i), \mu_m^N(x_i)))\}$$

and B is bipolar multi-fuzzy set of their common attributes i.e.:

$$(B) = \{y_j, ((\mu_1^P(y_j), \mu_1^N(y_j)), (\mu_2^P(y_j), \mu_2^N(y_j)), \dots, (\mu_m^P(y_j), \mu_m^N(y_j))))\}$$

The pair (A, B) is called as a bipolar multi-fuzzy concept iff:

$$(A, ((\mu_1^P(x_i), \mu_1^N(x_i)), (\mu_2^P(x_i), \mu_2^N(x_i)), \dots, (\mu_m^P(x_i), \mu_m^N(x_i))))^\uparrow = B \text{ and,}$$

$$(B, ((\mu_1^P(y_j), \mu_1^N(y_j)), (\mu_2^P(y_j), \mu_2^N(y_j)), \dots, (\mu_m^P(y_j), \mu_m^N(y_j))))^\downarrow = A$$

It can be interpreted as bipolar multi-fuzzy set of objects (A) having maximal positive and minimal negative membership value with respect to integrating the information from their common attributes set (B) closed with Galois connection. After that, none of the objects or attributes can be find which can make the positive or negative membership value of the obtained multi-fuzzy set of attributes (objects) bigger, if the pair (A, B) forms a formal concept. The generated bipolar-multi fuzzy formal concepts and their corresponding relationship can be visualized through vertices and edges of a defined bipolar multi-fuzzy graph as shown in Fig. 4. However, generating the bipolar multi-fuzzy concepts is one of the major issues for knowledge processing tasks. To overcome from this issue a method is proposed in the next section to discover all the bipolar

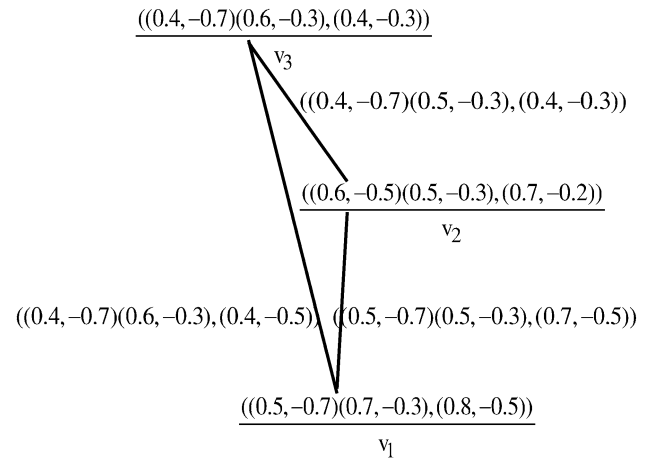


Fig. 4 A bipolar multi-fuzzy graph representation of Example 3

multi-fuzzy concepts and its selection at user defined multi-granulation.

3 Proposed method

In this section, two methods are proposed for analysis of data with bipolar multi-fuzzy attributes using the properties of bipolar multi-fuzzy set and concept lattice.

3.1 A proposed method for generating bipolar-multi formal fuzzy concepts

This section introduces an algorithm to discover all the bipolar-multi fuzzy concepts from a given bipolar-multi fuzzy context $\mathbf{F} = (X, Y, \tilde{R})$ where, $|X| = n$, $|Y| = k$ and, R represents bipolar multifuzzy relationship among X and Y . The bipolar multi-fuzzy concepts can be generated as follows via setting the attribute values in $(0, 1)$:

Step 1: The first concept can be generated using all the bipolar multi-fuzzy set of objects $(X, (\mu_m^P(X), \mu_m^N(X)))$ and finding its covering attributes using the \uparrow

$$(X, (\mu_m^P(X), \mu_m^N(X)))^\uparrow = (y_j, (\mu_m^P(y_j), \mu_m^N(y_j))).$$

The bipolar multi-fuzzy membership value for the obtained attributes can be computed using the properties of bipolar fuzzy set as follows:

$$\min (y_j, \mu_m^P(y_j)) \text{ and,} \\ \max (y_j, \mu_m^N(y_j)).$$

Similarly, \downarrow can be applied to find its covering objects to find the maximal pair of object and attributes closed with Galois connection.

Step 2: The Lower Neighbor of this concept can be discovered using the uncovered attributes from the given set of attributes : $S_k = y_m - y_j$ where $j \leq m$ and $k \leq |m - j|$.

Step 3: The set (s_k) can be considered as Lower Neighbor iff it contains exactly one extra uncovered attributes from the given attributes set i.e. s_k for $k = 1$ to $k = |m - j|$ i.e. $y_j \cup s_k$. The covering objects set can be found using the Galois connection (\downarrow) i.e. $(y_k, (\mu_m^P(y_k), \mu_m^N(y_k))) \downarrow = (x_i, (\mu_m^P(x_i), \mu_m^N(x_i)))$. The membership-value for the obtained objects set can be computed as follows:

$$\min (x_i, \mu_m^P(x_i)) \text{ and } \max (x_i, \mu_m^N(x_i))$$

Step 4: Now, apply the operator (\uparrow) on the obtained objects set as follows:

$$(x_i, (\mu_m^P(x_i), \mu_m^N(x_i))) \uparrow.$$

It will provide maximal covering bipolar multi-fuzzy attributes set. If the membership-value of the obtained attributes set have equal bipolar multi-fuzzy set to the initially considered subset of attributes i.e. $(y_k, (\mu_m^P(y_k), \mu_m^N(y_k)))$. In this case obtain pair of objects and attributes set represents a Lower Neighbor of given node. Similarly, other Lower Neighbors can be computed.

Step 5: The Next Neighbor is the distinct concepts from all the generated Lower Neighbor with maximal bipolar fuzzy membership value.

Step 6: Draw the edges among the current concepts and its Next Neighbor.

Step 7: Similarly, compute the Next Neighbor for each of the concepts.

Step 8: Stop the algorithm when all the unmarked attributes are covered.

Step 9: Draw the edges among each of the generated Next Neighbor which gives the hierarchically ordered visualization of bipolar fuzzy concepts in a concept lattice structure.

The pseudo code for the proposed algorithm is shown in Table 8. The proposed algorithm generates the bipolar multi-fuzzy concept based on its Lower Neighbor as shown in Step 1. The Lower Neighbor can be discovered via joining one of the uncovered attributes in the previous concepts as shown in Steps 2–4. The Lower Neighbors having distinct and maximal bipolar multi-fuzzy membership value can be considered as Next Neighbor concepts as

shown in Step 5. The generated Next Neighbor can be connected with its parent concepts via edges of defined bipolar fuzzy graph as shown in Table 6. Similarly, all the Next Neighbor can be generated and added to the graph based concept lattice as shown in Steps 6–9. This is one of the major advantages of the proposed method while building the bipolar concept lattice and extracting its Lower Neighbor concepts for the practical applications.

Complexity Let, the number of objects $(|X|) = n$ and the number of attributes $(|Y|) = k$ in the given m -polar fuzzy context. To generate the bipolar concepts the proposed algorithm finds Lower Neighbor which takes $O(n k)$ time for each component of m -polar fuzzy attributes. The Lower Neighbors can be generated using each m -polar fuzzy attributes i.e. $O(n k^2 m)$ complexity. In this way, the total complexity for the proposed method is $O(|C| n k^2 m)$ where C represents number of Lower Neighbors. It can be observed that, the proposed method generated the bipolar multi-fuzzy concepts beyond crisply. The reason is some time expert require the possession of other attributes for precise measurement of opinion. In near future, author will focus on fixing the membership values as $(1, 0)$ to generate the concepts and its comparative study. However, in any case the interpretation of bipolar multi-fuzzy concepts is difficult to analyze. Some time the expert need the concepts based on his/her required granulation for precise analysis of given pattern. To fulfil this need, the current paper introduces a method in the next section to decompose the bipolar multi-fuzzy context using extensive properties of granulation computing.

3.2 A method for decomposition of bipolar multi-fuzzy context at user defined multi-granulation

The bipolar multi-fuzzy context contains complex representation of data which takes much time for finding some of the interested concepts. To deal with this problem properties of granular computing can be useful to handle them based on small information of multi-granules. The calculus of granular computing (Yao 2004; Pedrycz and Chen 2015) is utilized to process the data with fuzzy attributes (Kang et al. 2012; Singh and Kumar 2012), interval-valued attributes (Li 2017; Singh 2018a), bipolar fuzzy attributes (Singh and Kumar 2014b; Singh 2019b), vague context (Singh 2018b) for finding corresponding crisp order relation (Singh 2015). The motive is to investigate some useful or interactive pattern in the given context based on user requirement. In this way, granular computing is an important tool to process the data in small chunk of variable for cognitive concept learning (Li et al. 2015). The information granule includes one or another

Table 8 The proposed algorithm for generating the bipolar multi-fuzzy concepts using Next Neighbor

Input: A bipolar fuzzy context $\mathbf{F} = (X, Y, \tilde{R})$
 where $(|X|)=n, (|Y|)=m$.

Output: The bipolar multi-fuzzy concepts (C) and its lattice structure

1. Apply operator (\uparrow) on given object set(X) :
 - (i) $(X, (\mu_m^P(X), \mu_m^N(X)))^\uparrow = (y_j, (\mu_m^P(y_j), \mu_m^N(y_j)))$.
 - (ii) Compute the bipolar membership for the obtained attributes:
 $\min (y_j, \mu_m^P(y_j))$ and $\max (y_j, \mu_m^N(y_j))$
 - (iii) Apply the operator (\downarrow) on obtained attribute set:
 $((y_j, (\mu_m^P(y_j), \mu_m^N(y_j))))^\downarrow = (x_i, (\mu_m^P(x_i), \mu_m^N(x_i)))$
 - (iv) If its provides the initially considered objects set (X) then it is concept.
2. Find its Lower Neighbor:
3. **for** ($k=0$ to m)
 - $S_k=Y - y_j$ where $0 \leq j \leq m$
 - (i). New attribute set: $y_k= y_j \cup s_k$
 - (ii). Set (0,1) positive membership for each attributes.
 - (iii). Apply the operator (\downarrow) on attributes
 $(y_k, (\mu_m^P(y_k), \mu_m^N(y_k)))^\downarrow = (x_i, (\mu_m^P(x_i), \mu_m^N(x_i)))$
 - (iv). Compute the bipolar fuzzy membership value for the extent:
 $\min (x_i, \mu_m^P(x_i))$ and $\max (x_i, \mu_m^N(x_i))$
 - (v). Apply the operator (\uparrow) on extent:
 $(x_i, (\mu_m^P(x_i), \mu_m^N(x_i)))^\uparrow = (y_k, (\mu_m^P(y_k), \mu_m^N(y_k)))$.

End for.
4. List out all the generated Lower Neighbor concepts C .
5. Mark the distinct and maximal intent in C as Next Neighbor.
6. Draw the edges with its parent concept.
7. Similarly compute the Next Neighbor of other concepts.
8. Stop when all the unmarked attributes are covered.

way to quantify the uncertainty in numeric precision computed by different methods. In this way, the calculus of granular computing conquer the complexity of any given problem in small module for its precise analysis. To achieve this goal, current paper proposed a method to refine some of the important bipolar multi-fuzzy concepts based on user defined information granules as given below:

Step 1: Let us suppose, a bipolar multi-fuzzy context $\mathbf{F} = (X, Y, \tilde{R}_m)$ where, $|X| = n, |Y| = m$ and, $(\tilde{R}_m = (P_{\tilde{R}_m(x_i, y_j)}, N_{\tilde{R}_m(x_i, y_j)}))$ represents a bipolar multi-fuzzy relationship among them.

Step 2: In general, user required some of the concepts based on his/her required information granules to process the context. In this case he/she can set maximal positive membership value and minimal negative membership value for each of the bipolar multi-fuzzy attributes i.e. (α_m, β_m) .

Step 3: Now, the given bipolar multi-fuzzy context can be converted into binary context in the m -polar fuzzy space iff: $\mu_m^P \tilde{R}(x_i, y_j) \geq \alpha_m$ and $\mu_m^N \tilde{R}(x_i, y_j) \leq \beta_m$. The obtained

context for the chosen information granules can be considered as $\mathbf{F}_{\alpha_m, \beta_m}$.

Step 4: The obtained context for the chosen granules $\mathbf{F}_{\alpha_m, \beta_m}$ follow the following equality: $\mathbf{F} = \bigcup_{\alpha_m, \beta_m} \mathbf{F}_{\alpha_m, \beta_m}, \forall \alpha_m, \beta_m$ defined in m -polar fuzzy space for its positive and negative information.

Step 5: The obtained context also satisfies the subset properties i. e. $\mathbf{F}_{\alpha_1, \beta_1} \subseteq \mathbf{F}_{\alpha_2, \beta_2}$ when $\alpha_1 \geq \alpha_2, \beta_1 \leq \beta_2$ in given m -polar fuzzy space.

Step 6: In this way, an user or expert can control the number of bipolar multi-fuzzy concepts and their hierarchical order visualization in the concept lattice for precise analysis of given data set with bipolar multi-fuzzy set.

Step 7: Table 9 shows the above-mentioned steps of the proposed method in form of an algorithm.

Complexity The proposed method provides an alternative way to analyze the given bipolar multi-fuzzy context at user-defined multi-granulation for the context $\mathbf{F} = (X, Y, \tilde{R}_m)$, where $|X|=n, |Y|=k$ and $(\tilde{R}_m = (P_{\tilde{R}_m(x_i, y_j)}, N_{\tilde{R}_m(x_i, y_j)}))$. To achieve this goal the proposed method compute (α_m, β_m) for their positive and negative membership value in m -

polar fuzzy space which can take maximum $O(n * k^2 * m)$ computational time. In this way the proposed method reduced the time complexity for processing the bipolar multi-fuzzy context.

4 Illustrations

In this section, both of the proposed methods are demonstrated with an illustrative example. In addition, the analysis derived from them are compared with Yang et al. (2013).

4.1 Illustrations of the proposed method

Extracting the interesting pattern from any given set of data is one of major concern for the research communities for its practical applications in various research fields as discussed by Singh et al. (2016). The complexity increases when the given data set contains bipolar (Singh and Kumar 2014a, b), three-polar (Huang et al. 2017; Singh 2017; Yao 2021a) or multi-polar information (Chen et al. 2014; Mechelen and Smilde 2011; Singh 2018c). It becomes more complex when the bipolarity existing in each building blocks of m -polar fuzzy space. To deal with these type of data set properties of bipolar multi-fuzzy set (Yang et al. 2013) is utilized in this paper for multi-decision making process. To achieve this goal, a method is for handling data with bipolar multi-fuzzy attributes is proposed in Sect. 3.1 which is illustrate as given below:

Example 4 Let us extent the previous examples as, a company manufactures five kind of color drawing papers- $\{x_1, x_2, x_3, x_4, x_5\}$ for engineering based on following

parameters (attributes)- $\{y_1, y_2, y_3\}$. The multi-attributes y_1 represents ‘Thickness’ which includes thick, average and thin. The multi-attribute y_2 represents ‘Color’ which involves combination of red, green and blue, independently. Subsequently, the multi-attribute y_3 represents ‘Ingredients’ which involves cellulose, hemicellulose, lignin. The experts write their opinion based on its acceptance and rejection membership value for the given drawing papers- $\{x_1, x_2, x_3, x_4, x_5\}$ as shown in Tables 10, 11, 12, 13, 14, respectively. Table 15 represents the constituted bipolar multi-fuzzy context where, attributes represents the experts opinion, and objects represents the drawing papers. Now the issue for the company is to find one of the most preferable drawing papers based on constituted bipolar multi-fuzzy context shown in Table 15. To achieve this goal of company a method is proposed in this paper as shown in Table 7. To generate some of the interested pattern based on object attribute of the given context as illustrated below:

The generated bipolar multi-fuzzy concepts from the context shown in Table 15 are as follows using the proposed algorithm in Sect. 3.1:

1. Extent : $((0, 1), (0, 1), (0, 1))/x_1 + ((0, 1), (0, 1), (0, 1))/x_2 + ((0, 1), (0, 1), (0, 1))/x_3 + ((0, 1), (0, 1), (0, 1))/x_4 + ((0, 1), (0, 1), (0, 1))/x_5$.
 Intent : $((0.13, - 0.31), (0.13, - 0.26), (0.03, - 0.16))/y_1 + ((0.15, - 0.31), (0.20, - 0.07), (0.24, - 0.11))/y_2 + ((0.14, - 0.16), (0.20, - 0.29), (0.18, - 0.06))/y_3$.
2. Extent : $((0.22, - 0.27), (0.42, - 0.37), (0.36, - 0.24))/x_1 + ((0.52, - 0.13), (0.31, - 0.35), (0.16, - 0.48))/x_2 + ((0.13, - 0.46), (0.31, - 0.27), (0.53, - 0.18))/x_3 + ((0.57, - 0.34), (0.22, - 0.26), (0.13,$

Table 9 Proposed algorithm to analyze the bipolar multi-fuzzy context at user defined multi-granulation

Input: A bipolar multi-granulation (α_m, β_m)
 where $|X|=n, |Y|=m$ and $(\tilde{R}_m = (P_{R_m(x_i, y_j)}, N_{R_m(x_i, y_j)}))$.

Output: The set of context at user defined multi-granules $\mathbf{F}_{(\alpha_m, \beta_m)}$.

1. Let us suppose bipolar multi-fuzzy context $\mathbf{F} = (X, Y, \tilde{R}_m)$.
2. Define the multi-granulation for positive and negative membership-value i.e. (α_m, β_m) .
3. This provides following binary multi-context $\mathbf{F}_{\alpha_m, \beta_m}$:
4. if $(\mu_m^P \tilde{R}(x_i, y_j) \geq \alpha_m$ and $\mu_m^N \tilde{R}(x_i, y_j) \leq \beta_m)$ then represent 1.0 at the place of chosen bipolar multi-fuzzy relation. else 0.0 at the place of chosen bipolar multi-fuzzy relation.
5. The obtained bipolar multi-context follows the equality:

$$\mathbf{F} = \bigcup_{\alpha_m, \beta_m} \mathbf{F}_{\alpha_m, \beta_m}, \forall \alpha_m, \beta_m.$$

$$\mathbf{F}_{\alpha_1, \beta_1} \subseteq \mathbf{F}_{\alpha_2, \beta_2} \text{ when } \alpha_1 \geq \alpha_2, \beta_1 \leq \beta_2.$$
6. Generate the concepts from the obtained bipolar multi-context.
7. Build the bipolar multi-concept lattice.
8. Discover the knowledge from the obtain concept lattice.

Table 10 The opinion of expert about drawing paper x_1 based on given multi-valued attributes

Pencils	Thickness (y_1)	Color (y_2)	Ingredients (y_3)
$\mu_1^P(x_1)$	0.22	0.31	0.30
$\mu_1^N(x_1)$	- 0.33	- 0.47	- 0.27
$\mu_2^P(x_1)$	0.42	0.24	0.45
$\mu_2^N(x_1)$	- 0.37	- 0.14	- 0.35
$\mu_3^P(x_1)$	0.36	0.45	- 0.25
$\mu_3^N(x_1)$	- 0.24	- 0.37	- 0.38

Table 11 The opinion of expert about drawing paper x_2 based on given multi-valued attributes

Pencils	Thickness (y_1)	Color (y_2)	Ingredients (y_3)
$\mu_1^P(x_2)$	0.52	0.30	0.14
$\mu_1^N(x_2)$	- 0.13	- 0.38	- 0.57
$\mu_2^P(x_2)$	0.31	0.42	0.61
$\mu_2^N(x_2)$	- 0.35	- 0.25	- 0.16
$\mu_3^P(x_2)$	0.16	0.24	0.24
$\mu_3^N(x_2)$	- 0.48	- 0.33	- 0.24

Table 12 The opinion of expert about drawing paper x_3 based on given multi-valued attributes

Pencils	Thickness (y_1)	Color (y_2)	Ingredients (y_3)
$\mu_1^P(x_3)$	0.13	0.15	0.35
$\mu_1^N(x_3)$	- 0.46	- 0.44	- 0.46
$\mu_2^P(x_3)$	0.31	0.38	0.20
$\mu_2^N(x_3)$	- 0.27	- 0.16	0.29
$\mu_3^P(x_3)$	0.57	0.41	0.37
$\mu_3^N(x_3)$	- 0.18	- 0.39	- 0.17

Table 13 The opinion of expert about drawing paper x_4 based on given multi-valued attributes

Pencils	Thickness (y_1)	Color (y_2)	Ingredients (y_3)
$\mu_1^P(x_4)$	0.57	0.36	0.47
$\mu_1^N(x_4)$	- 0.39	- 0.50	- 0.16
$\mu_2^P(x_4)$	0.22	0.28	0.34
$\mu_2^N(x_4)$	- 0.26	- 0.07	- 0.50
$\mu_3^P(x_4)$	0.13	0.30	0.18
$\mu_3^N(x_4)$	- 0.32	- 0.39	- 0.27

Table 14 The opinion of expert about drawing paper x_5 based on given multi-valued attributes

Pencils	Thickness (y_1)	Color (y_2)	Ingredients (y_3)
$\mu_1^P(x_5)$	0.27	0.47	0.17
$\mu_1^N(x_5)$	- 0.34	- 0.31	- 0.32
$\mu_2^P(x_5)$	0.62	0.20	0.29
$\mu_2^N(x_5)$	- 0.46	- 0.56	- 0.56
$\mu_3^P(x_5)$	0.03	0.31	0.53
$\mu_3^N(x_5)$	- 0.16	- 0.11	- 0.06

Table 15 A compact representation of bipolar multi-fuzzy context shown in Tables 10, 11, 12, 13 and 14

Pencils	Thickness (y_1)	Color(y_2)	Ingredients (y_3)
x_1	$((0.22, - 0.33), (0.42, - 0.37), (0.36, - 0.24))$	$((0.31, - 0.47), (0.24, - 0.14), (0.45, - 0.37))$	$((0.30, - 0.27), (0.45, - 0.35), (0.25, - 0.38))$
x_2	$((0.52, - 0.13), (0.31, - 0.35), (0.16, - 0.48))$	$((0.30, - 0.38), (0.42, - 0.25), (0.24, - 0.33))$	$((0.14, - 0.57), (0.61, - 0.16), (0.24, - 0.24))$
x_3	$((0.13, - 0.46), (0.31, - 0.27), (0.53, - 0.18))$	$((0.15, - 0.44), (0.38, - 0.16), (0.41, - 0.39))$	$((0.35, - 0.46), (0.20, - 0.29), (0.37, - 0.17))$
x_4	$((0.57, - 0.39), (0.22, - 0.26), (0.13, - 0.32))$	$((0.36, - 0.50), (0.28, - 0.07), (0.30, - 0.39))$	$((0.47, - 0.16), (0.34, - 0.50), (0.18, - 0.27))$
x_5	$((0.27, - 0.34), (0.62, - 0.46), (0.03, - 0.16))$	$((0.47, - 0.31), (0.20, - 0.56), (0.31, - 0.11))$	$((0.17, - 0.32), (0.29, - 0.56), (0.53, - 0.06))$

- 0.32))/ x_4 +((0.27, - 0.34), (0.62, - 0.46), (0.03, - 0.16))/ x_5 .

Intent : $((0, 1), (0, 1), (0, 1))/y_1+((0.15, - 0.31), (0.20, - 0.07), (0.24, - 0.11))/y_2+((0.14, - 0.16), (0.20, - 0.27), (0.18, - 0.06))/y_3$.

3. Extent : $((0.31, - 0.47), (0.24, - 0.14), (0.45, - 0.37))/x_1+ ((0.30, - 0.38), (0.42, - 0.25), (0.24, - 0.33))/x_2+((0.51, - 0.44), (0.38, - 0.16), (0.41, - 0.39))/x_3+((0.36, - 0.50), (0.28, - 0.07), (0.30, - 0.39))/x_4+((0.47, - 0.31), (0.20, - 0.56), (0.33, - 0.11))/x_5$.

Intent : $((0.13, - 0.13), (0.20, - 0.026), (0.003, - 0.16))/y_1+((0, 1), (0, 1), (0, 1))/y_2+((0.14, - 0.16), (0.20, - 0.27), (0.18, - 0.06))/y_3$.

4. Extent : $((0.30, - 0.27), (0.45, - 0.35), (0.25, - 0.38))/x_1+ ((0.14, - 0.57), (0.61, - 0.16), (0.24, - 0.24))/x_2+((0.35, - 0.46), (0.20, - 0.24), (0.37,$

− 0.16))/x₃+((0.47, − 0.16), (0.34, − 0.57), (0.18, − 0.27))/x₄+((0.17, − 0.32), (0.29, − 0.56), (0.33, − 0.06))/x₅.

Intent : ((0.13, − 0.13), (0.20, − 0.026), (0.003, − 0.16))/y₁+((0.15, − 0.31), (0.20, − 0.07), (0.24, − 0.11))/y₂+((0, 1), (0, 1), (0, 1))/y₃.

5. Extent : ((0.22, − 0.33), (0.24, − 0.14), (0.36, − 0.24))/x₁+ ((0.30, − 0.13), (0.31, − 0.25), (0.16, − 0.33))/x₂+((0.13, − 0.44), (0.31, − 0.16), (0.41, − 0.18))/x₃+((0.36, − 0.34), (0.22, − 0.07), (0.13, − 0.32))/x₄+((0.27, − 0.34), (0.20, − 0.46), (0.03, − 0.11))/x₅.

Intent : ((0, 1), (0, 1), (0, 1))/y₁+((0, 1), (0, 1), (0, 1))/y₂+((0.14, − 0.16), (0.26, − 0.27), (0.18, − 0.06))/y₃.

6. Extent : ((0.22, − 0.27), (0.42, − 0.35), (0.25, − 0.24))/x₁+ ((0.14, − 0.13), (0.31, − 0.16), (0.16, − 0.24))/x₂+((0.13, − 0.46), (0.20, − 0.27), (0.37, − 0.17))/x₃+((0.47, − 0.16), (0.22, − 0.026), (0.13, − 0.27))/x₄+((0.17, − 0.32), (0.20, − 0.46), (0.03, − 0.06))/x₅.

Intent : ((0, 1), (0, 1), (0, 1))/y₁+((0.15, − 0.31), (0.20, − 0.07), (0.24, − 0.11))/y₂+((0, 1), (0, 1), (0, 1))/y₃.

7. Extent : ((0.30, − 0.27), (0.24, − 0.14), (0.25, − 0.37))/x₁+ ((0.14, − 0.38), (0.42, − 0.16), (0.24, − 0.24))/x₂+((0.15, − 0.44), (0.20, − 0.16), (0.37, − 0.17))/x₃+((0.36, − 0.16), (0.28, − 0.07), (0.18, − 0.27))/x₄+((0.17, − 0.31), (0.20, − 0.56), (0.31, − 0.06))/x₅.

Intent : ((0, 1), (0, 1), (0, 1))/y₁+((0, 1), (0, 1), (0, 1))/y₂+((0.13, − 0.13), (0.22, − 0.26), (0.03, − 0.16))/y₃.

8. Extent : ((0.22, − 0.27), (0.24, − 0.14), (0.25, − 0.24))/x₁+ ((0.14, − 0.13), (0.31, − 0.16), (0.16, − 0.24))/x₂+((0.13, − 0.44), (0.20, − 0.16), (0.37, − 0.17))/x₃+((0.36, − 0.16), (0.22, − 0.07), (0.13, − 0.27))/x₄+((0.17, − 0.31), (0.20, − 0.46), (0.03, − 0.06))/x₅.

Intent : ((0, 1), (0, 1), (0, 1))/y₁+((0, 1), (0, 1), (0, 1))/y₂+((0, 1), (0, 1), (0, 1))/y₃.

The above generated concepts and their hierarchical order visualization through bipolar multi-fuzzy graph representation of concept lattice is shown in Fig. 5. The generalized concept number 1 shows that each drawing paper is made from each of the given parameters. The specialized concept number 8 can be interpreted as follows:

8. Extent : ((0.22, − 0.27), (0.24, − 0.14), (0.25, − 0.24))/x₁+ ((0.14, − 0.13), (0.31, − 0.16), (0.16, − 0.24))/x₂+((0.13, − 0.44), (0.20, − 0.16), (0.37, − 0.17))/x₃+((0.36, − 0.16), (0.22, − 0.07), (0.13, − 0.27))/x₄+((0.17, − 0.31), (0.20, − 0.46), (0.03, − 0.06))/x₅.

Intent : ((0, 1), (0, 1), (0, 1))/y₁+((0, 1), (0, 1), (0, 1))/y₂+((0, 1), (0, 1), (0, 1))/y₃.

The acceptance of each drawing paper based on concepts number 8 can be computed as given below:

$$8 \text{ (i). } ((0.22, - 0.27), (0.24, - 0.14), (0.25, - 0.24))/x_1 = ((0.22 - 0.27)+(0.24 - 0.14)+(0.25 - 0.24)) = (- 0.05+0.10+0.01)=0.06.$$

$$8 \text{ (ii). } ((0.14, - 0.13), (0.31, - 0.16), (0.16, - 0.24))/x_2 = ((0.14 - 0.13)+(0.31 - 0.16)+(0.16 - 0.24)) = (0.01+0.15 - 0.08)=0.08.$$

$$8 \text{ (iii). } ((0.13, - 0.44), (0.20, - 0.16), (0.37, - 0.17))/x_3 = ((0.13 - 0.44)+(0.20 - 0.16)+(0.37 - 0.17)) = (- 0.31+0.04+0.20)= - 0.07$$

$$8 \text{ (iv). } ((0.36, - 0.16), (0.22, - 0.07), (0.13, - 0.27))/x_4 = ((0.36 - 0.16)+(0.22 - 0.07)+(0.13 - 0.27)) = (0.20+0.13 - 0.20)=0.13.$$

$$8 \text{ (v). } ((0.17, - 0.31), (0.20, - 0.46), (0.03, - 0.06))/x_5 = ((0.17 - 0.31)+(0.20 - 0.46)+(0.03 - 0.06)) = (- 0.14 - 0.26 - 0.03)= - 0.43$$

It can be observed that, the x_2 and x_4 have positive acceptance due to that both will be preferred by user. In case company want to analyze most suitable among x_2 and x_4 . The company can utilize the expert opinion shown in Table 11 and 13 for x_2 and x_4 , respectively, which shows x_2 is more suitable. This conclusions from the proposed method corresponds to Yang et al. (2013) with compact display of bipolar multi-fuzzy concept. This is one of the major advantages of the proposed method. In the next section illustration of bipolar multi-fuzzy context is shown based on user defined information granules.

4.2 Bipolar multi-fuzzy concepts at user defined multi-granulation

Recently, attention has been paid to extract some interesting knowledge from a given context using the properties of granulation (Kumar and Srinivas 2010; Kumar 2012;

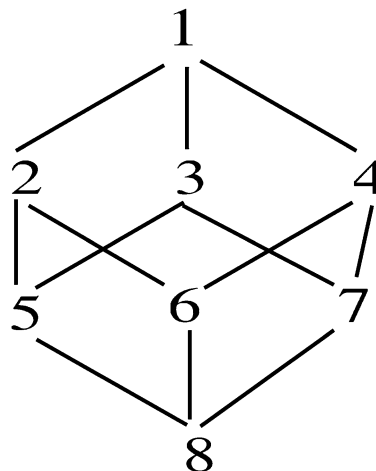


Fig. 5 A bipolar multi-fuzzy concept lattice for the context shown in Table 15

Dias et al. 2020). The calculus of granulation provides an alternative way to analyze the given context based on small or chunk of information granules. Due to this properties, it is applied in various fields for knowledge processing tasks (Pedrycz and Chen 2015). Recently, the calculus of granular computing is utilized in formal fuzzy context (Kang et al. 2012; Singh and Kumar 2012), interval-valued fuzzy context (Singh and Kumar 2018a), bipolar fuzzy context (Singh and Kumar 2014b; Singh 2019b) to control the size of concept lattice visualization (Singh et al. 2015a). These recent studies motivated to utilize the calculus of multi-granular model for analyzing the data with bipolar multi-fuzzy context based on its chunk of variables. To extract the multi-level conceptual knowledge from a given bipolar multi-valued data set. To achieve this goal, a method is proposed in Sect. 3.2 of this paper which can be illustrated on the bipolar multi-fuzzy context shown in Table 15. The context shown in this table represents the opinion of an expert based on three attributes (y_1 -Thickness, y_2 -Color, y_3 -Ingredients) to prefer the suitable engineering drawing papers among them (x_1, x_2, x_3, x_4, x_5). Now the goal is to find some of the suitable drawing paper based on each of the parameters for the manufacturing. This problem can be resolved based on multi-level granulation shown in Table 16 as illustrated in Examples 5 and 6 given below:

Example 5 Let us suppose, the company want to analyze the most interested drawing paper for manufacturing based on given preference by an expert. In this case, the Level-4 shown in Table 16 can be considered to analyze the bipolar multi-fuzzy context shown in Table 15 as given below:

The preference of drawing paper x_1 based on multi-valued attribute y_1 using Level-4 can be computed as given below:

1. $\mu_1(x_1, y_1) = (0.22, - 0.33)$ means $\mu_1^P(x_1, y_1) = 0.22$ and $\mu_1^N(x_1, y_1) = - 0.33$. The positive membership-value $\mu_1^P(x_1, y_1) = 0.22 \leq 0.4$. Hence, $(0.4, - 0.2)$ -cut of $\mu_1(x_1, y_1)=0$.

2. $\mu_2(x_1, y_1) = (0.42, - 0.37)$ means $\mu_2^P(x_1, y_1) = 0.42$ and $\mu_2^N(x_1, y_1) = - 0.37$. The positive membership-value $\mu_2^P(x_1, y_1) = 0.42 \geq 0.4$ whereas negative membership-

value $\mu_2^N(x_1, y_1) = 0.37 \geq 0.2$. Hence, $(0.4, - 0.2)$ -cut of $\mu_1(x_1, y_1)=0$.

3. $\mu_3(x_1, y_1) = (0.36, - 0.24)$ means $\mu_3^P(x_1, y_1) = 0.36$ and $\mu_3^N(x_1, y_1) = - 0.24$. The positive membership-value $\mu_3^P(x_1, y_1) = 0.36 \leq 0.4$. Hence, $(0.4, -0.2)$ -cut of $\mu_3(x_1, y_1)=0$.

It gives $(0, 0, 0)$ preference value using Level-4 as depicted in Table 17.

Similarly, the preference of drawing paper x_1 can be computed based on multi-valued attribute y_2 using Level-4 as given below:

4. $\mu_1(x_1, y_2) = (0.31, - 0.47)$ means $\mu_1^P(x_1, y_2) = 0.31$ and $\mu_1^N(x_1, y_2) = - 0.47$. The positive membership-value $\mu_1^P(x_1, y_2) = 0.31 \leq 0.4$. Hence, $(0.4, - 0.2)$ -cut of $\mu_1(x_1, y_2)=0$.

5. $\mu_2(x_1, y_2) = (0.24, - 0.14)$ means $\mu_2^P(x_1, y_2) = 0.24$ and $\mu_2^N(x_1, y_2) = - 0.14$. The positive membership-value $\mu_2^P(x_1, y_2) = 0.24 \leq 0.4$. Hence, $(0.4, - 0.2)$ -cut of $\mu_2(x_1, y_2)=0$.

6. $\mu_3(x_1, y_2) = (0.45, - 0.37)$ means $\mu_3^P(x_1, y_2) = 0.45$ and $\mu_3^N(x_1, y_2) = - 0.37$. The positive membership-value $\mu_3^P(x_1, y_2) = 0.45 \geq 0.4$ whereas negative membership-value $\mu_3^N(x_1, y_2) = 0.37 \geq 0.2$. Hence, $(0.4, - 0.2)$ -cut of $\mu_3(x_1, y_2)=0$.

It gives $(0, 0, 0)$ preference value using Level-4 as depicted in Table 17.

Similarly, the preference of drawing paper x_1 can be computed based on multi-valued attribute y_3 using Level-4 can be computed as given below:

7. $\mu_1(x_1, y_3) = (0.30, - 0.27)$ means $\mu_1^P(x_1, y_3) = 0.30$ and $\mu_1^N(x_1, y_3) = - 0.27$. The positive membership-value $\mu_1^P(x_1, y_3) = 0.30 \leq 0.4$. Hence, $(0.4, - 0.2)$ -cut of $\mu_1(x_1, y_3)=0$.

8. $\mu_2(x_1, y_3) = (0.45, - 0.37)$ means $\mu_2^P(x_1, y_3) = 0.45$ and $\mu_2^N(x_1, y_3) = - 0.37$. The positive membership-value $\mu_2^P(x_1, y_3) = 0.45 \geq 0.4$ whereas negative membership-value $\mu_2^N(x_1, y_3) = 0.37 \geq 0.2$. Hence, $(0.4, - 0.2)$ -cut of $\mu_1(x_1, y_3)=0$.

9. $\mu_3(x_1, y_3) = (0.25, - 0.38)$ means $\mu_3^P(x_1, y_3) = 0.25$ and $\mu_3^N(x_1, y_3) = - 0.38$. The positive membership-value

Table 16 Some of the interested granules based on user required preference

Granulation	Interested	Measurement	Bipolar membership
Level-1	Highly interested	Highly positive	$(0.8, - 0.2)$
Level-2	Very very interested	Very positive	$(0.7, - 0.2)$
Level-3	Very interested	Absolute positive	$(0.5, - 0.2)$
Level-4	Interested	Positive	$(0.4, - 0.2)$
Level-5	Not interested	Not positive	$(0.4, - 0.6)$

Table 17 Three-way decomposition of Table 15 using (0.4, − 0.2)-bipolar multi-granulation

	y_1	y_2	y_3
x_1	(0, 0, 0)	(0, 0, 0)	(0, 0, 0)
x_2	(1, 0, 0)	(0, 1, 0)	(0, 1, 0)
x_3	(0, 0, 1)	(0, 0, 0)	(0, 0, 0)
x_4	(0, 0, 0)	(0, 0, 0)	(1, 0, 0)
x_5	(0, 0, 0)	(0, 0, 0)	(0, 0, 1)

$\mu_5^p(x_1, y_3) = 0.25 \leq 0.4$. Hence, (0.4, − 0.2)-cut of $\mu_3(x_1, y_3)=0$.

It gives (0, 0, 0) preference value using Level-4 as depicted in Table 17.

Similarly, the preference of other drawing papers (x_2, x_3, x_4, x_5) can be computed using the user defined Level-4. This provides a multi-way binary context shown in Table 17 for further processing using the properties of FCA.

The formal concept generated from Table 17 are as follows:

1. $\{(1, 1, 1)/x_1 + (1, 1, 1)/x_2 + (1, 1, 1)/x_3 + (1, 1, 1)/x_4 + (1, 1, 1)/x_5, \emptyset\}$,
2. $\{(1, 1, 1)/x_1 + (1, 1, 1)/x_3 + (1, 1, 1)/x_4 + (1, 1, 1)/x_5, (0, 0, 0)/y_2\}$,
3. $\{(1, 1, 1)/x_2, (1, 0, 0)/y_1 + (0, 1, 0)/y_2 + (0, 1, 0)/y_3\}$,
4. $\{(1, 1, 1)/x_1 + (1, 1, 1)/x_3, (0, 0, 0)/y_2 + (0, 0, 0)/y_3\}$,
5. $\{(1, 1, 1)/x_1 + (1, 1, 1)/x_4 + (1, 1, 1)/x_5, (0, 0, 0)/y_1 + (0, 0, 0)/y_2\}$,
6. $\{(1, 1, 1)/x_5, (0, 0, 0)/y_1 + (0, 0, 0)/y_2 + (0, 0, 1)/y_3\}$,
7. $\{(1, 1, 1)/x_1, (0, 0, 0)/y_1 + (0, 0, 0)/y_2 + (0, 0, 0)/y_3\}$,
8. $\{(1, 1, 1)/x_3, (0, 0, 1)/y_1 + (0, 0, 0)/y_2 + (0, 0, 0)/y_3\}$,
9. $\{(1, 1, 1)/x_4, (0, 0, 0)/y_1 + (0, 0, 0)/y_2, (1, 0, 0)/y_3\}$,
10. $\{\emptyset, (1, 0, 0)/y_1 + (0, 0, 1)/y_1 + (0, 1, 0)/y_2 + (1, 0, 0)/y_3 + (0, 1, 0)/y_3 + (0, 0, 1)/y_3\}$

where \emptyset represents null set.

The concept lattice for the above generated concept is shown in Fig. 6. From this figure following information can be extracted:

- Concept number-1. $\{(1, 1, 1)/x_1 + (1, 1, 1)/x_2 + (1, 1, 1)/x_3 + (1, 1, 1)/x_4 + (1, 1, 1)/x_5, \emptyset\}$ represents that none of the attribute is general on user required which covers all the drawing papers.

- Concept number-3. $\{(1, 1, 1)/x_2, (1, 0, 0)/y_1 + (0, 1, 0)/y_2 + (0, 1, 0)/y_3\}$ represents that drawing paper x_2 cover each of the given attributes at user defined granules. Hence, this drawing paper will be most suitable when compare to others. This derived analysis is resembled with Yang et al. (2013) as well the bipolar multi-fuzzy concept lattice shown in Sect. 4.1 of this paper.
- Concept number-8. $\{(1, 1, 1)/x_3, (0, 0, 1)/y_1 + (0, 0, 0)/y_2 + (0, 0, 0)/y_3\}$ represents that drawing paper x_3 covers attribute y_1 as per user required information granules. Hence this drawing paper will be preferred when user is interested on Thickness (y_1) .
- Concept number-9. $\{(1, 1, 1)/x_4, (0, 0, 0)/y_1 + (0, 0, 0)/y_2, (1, 0, 0)/y_3\}$ represents that drawing paper x_4 covers the attributes y_3 as per user required information granules. Hence this drawing paper will be preferred when user is interested on Ingredients (y_3).
- Concept number-10. $\{\emptyset, (1, 0, 0)/y_1 + (0, 0, 1)/y_1 + (0, 1, 0)/y_2 + (1, 0, 0)/y_3 + (0, 1, 0)/y_3 + (0, 0, 1)/y_3\}$ represents that none of the drawing paper is specific which covers each attributes maximally at user requirement.

It can be observed that the analysis derived from multi-granulation (0.4, − 0.2) is resembled with Yang et al. (2013) and its bipolar multi-fuzzy concepts shown in Sect. 4.1. In addition, the multi-granulation provides a depth way to refine the knowledge from given bipolar multi-fuzzy context at multi-level knowledge extraction with less computational cost. To understand its working process another multi-level information granules i.e. Level 3 is chosen in the next example as given below:

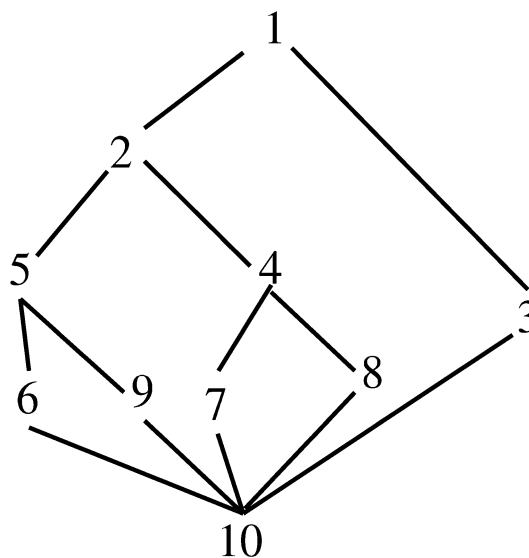


Fig. 6 A concept lattice generated from context shown in Table 17

Example 6 Let us suppose, the company want to analyze the very interested drawing papers based on user required information. This problem can be resolved using the Level-3 i.e. (0.5, − 0.2) shown in Table 16. The obtained multi-level context based on this multi-granulation is shown in Table 18.

The following formal concepts can be generated from the context shown in Table 18:

1. $\{(1, 1, 1)/x_1 + (1, 1, 1)/x_2 + (1, 1, 1)/x_3 + (1, 1, 1)/x_4 + (1, 1, 1)/x_5, (0, 0, 0)/y_2\}$,
2. $\{(1, 1, 1)/x_1 + (1, 1, 1)/x_3 + (1, 1, 1)/x_5, (0, 0, 0)/y_2 + (0, 0, 0)/y_5\}$,
3. $\{(1, 1, 1)/x_1 + (1, 1, 1)/x_4 + (1, 1, 1)/x_5, (0, 0, 0)/y_1 + (0, 0, 0)/y_2\}$,
4. $\{(1, 1, 1)/x_2, (1, 0, 0)/y_1 + (0, 0, 0)/y_2 + (0, 1, 0)/y_3\}$,
5. $\{(1, 1, 1)/x_1 + (1, 1, 1)/x_4, (0, 0, 0)/y_1 + (0, 0, 0)/y_2 + (0, 0, 0)/y_3\}$,
6. $\{(1, 1, 1)/x_3, (0, 0, 1)/y_1 + (0, 0, 0)/y_2, (0, 0, 0)/y_3\}$,
7. $\{(1, 1, 1)/x_5, (0, 0, 0)/y_1 + (0, 0, 0)/y_2 + (0, 0, 1)/y_3\}$,
8. $\{(1, 1, 1)/x_1, (0, 0, 0)/y_1 + (0, 0, 0)/y_2 + (0, 0, 0)/y_3\}$,
9. $\{\emptyset, (1, 0, 0)/y_1 + (0, 0, 1)/y_1 + (0, 0, 0)/y_2 + (0, 0, 0)/y_3 + (0, 1, 0)/y_3 + (0, 0, 1)/y_3\}$.

where \emptyset represents null set.

Concept lattice for the above generated bipolar multi fuzzy concept is shown in Fig. 7. From this following information can be extracted:

- Concept number-1. $\{(1, 1, 1)/x_1 + (1, 1, 1)/x_2 + (1, 1, 1)/x_3 + (1, 1, 1)/x_4 + (1, 1, 1)/x_5, (0, 0, 0)/y_2\}$ none of the attribute is general which should available in each drawing paper as per user requirement for the given information granules.
- Concept number-4. $\{(1, 1, 1)/x_2, (1, 0, 0)/y_1 + (0, 0, 0)/y_2 + (0, 1, 0)/y_3\}$ represents that drawing paper x_2 covers maximal parameters as per user defined granulation when compare to others. In this case it will be most suitable drawing. This derived analysis is resembled with Yang et al. (2013), bipolar multi-fuzzy

concepts shown in Sect. 4.1 as well as (0.4, − 0.2) multi-granulation shown in Fig. 7 of this paper.

- Concept number-6. $\{(1, 1, 1)/x_3, (0, 0, 1)/y_1 + (0, 0, 0)/y_2, (0, 0, 0)/y_3\}$ represents that drawing paper x_3 covers attributes y_1 maximally as per user required information granules. In this case it will be preferred when user interest on Thickness.
- Concept number-9. $\{\emptyset, (1, 0, 0)/y_1 + (0, 0, 1)/y_1 + (0, 0, 0)/y_2 + (0, 0, 0)/y_3 + (0, 1, 0)/y_3 + (0, 0, 1)/y_3\}$ represents that none of the drawing paper is specific as per user required information granulation.

It can be observed that the analysis derived from multi-granulation (0.5, − 0.2) is resembled with Yang et al. (2013) as well as its bipolar multi-fuzzy concepts.

5 Discussions

The precise measurement of bipolarity and its graphical visualization beyond three-way fuzzy space is considered as one of the major tasks for the researchers. One of the suitable examples is electron (−), proton (+), neutron (0) and now positron (+, −) which contains both positive and negative side simultaneously (Zhang 2021). This given a new challenge for data science researchers to deal with acceptance and rejection part exists beyond the unipolar space. It becomes more complex when the group of people support some thing and reject something simultaneously. It is measured recently that many students shown opposite and non-opposite sides of online teaching, traditional teaching and hybrid. This type is issue used to measured

Table 18 Multi-granulation (0.5, − 0.2) decomposition of bipolar multi-fuzzy set shown in Table 15

	y_1	y_2	y_3
x_1	(0, 0, 0)	(0, 0, 0)	(0, 0, 0)
x_2	(1, 0, 0)	(0, 0, 0)	(0, 1, 0)
x_3	(0, 0, 1)	(0, 0, 0)	(0, 0, 0)
x_4	(0, 0, 0)	(0, 0, 0)	(0, 0, 0)
x_5	(0, 0, 0)	(0, 0, 0)	(0, 0, 1)

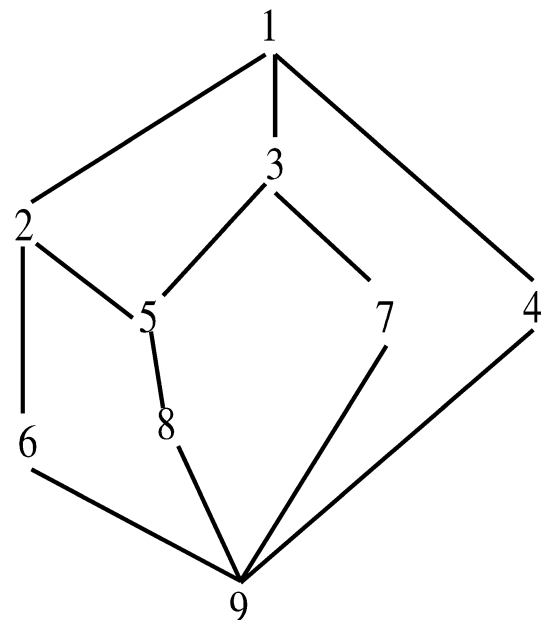


Fig. 7 A concept lattice generated from context shown in Table 18

frequently in democratic like India where people of 29 states keep several opposite and non-opposite sides. To deal these types of information, 29 states can be considered as m -block of a defined m -polar fuzzy concepts lattice (Singh 2018c, d). In case, the author wants to deal them crisply based on object and attribute can be also done via projection operator as discussed by Singh (2019a). The problem arises when 29 states people provides several opposite and non-opposite of opinion with precise bipolar membership-values to accept or reject a particular party. It is indeed requirement in case of civil engineering, building drawing paper or representing z -matrix of an atom in chemistry³ (Zadeh 2011). In this case, an alternative way is to utilize the algebra of bipolar multi-fuzzy context to extract some meaningful information.

To fulfil the above need, the mathematics of Formal Concept Analysis (FCA) has been considered as one of the most suitable mathematical model by the research communities (Singh et al. 2016). This theory is exclusively extended in bipolar fuzzy space (Singh and Kumar 2014a, b; Singh 2019b), three-way fuzzy space (Huang et al. 2017; Singh 2017; Yao 2021a) as well as m -polar fuzzy space (Singh 2018c, d). The problem arises while dealing with hybrid of them which creates two possible notions: (1) the existence of bipolarity in m -way fuzzy space and (2) the existence of m -tuples in bipolar fuzzy attributes. The first case discussed the generalized representation of bipolar fuzzy attributes in m -polar fuzzy space whereas the second case is specialized case of bipolar fuzzy space which contains m -truth and false values. This paper focused on solving the first case to generalize the bipolar fuzzy attributes and its representation beyond the m -polar fuzzy space. The motivation is deal with multi-fuzzy attributes precisely in m -polar fuzzy space based on its acceptance and negation part independently. The objective is to provide a compact graphical visualization for knowledge processing tasks. To achieve this goal, a problem arises due to insufficient mathematical background and graphical representation for data with bipolar multi-fuzzy attributes. To fill this backdrop, the current paper aimed at extracting some of the meaningful information from the data with bipolar multi-fuzzy attributes.

To understand the necessity of the proposed methodology, some important literature related to the current study are shown in Table 19 based on their potential outputs. This table shows that, less attention has been paid towards data with bipolar multi-fuzzy attributes except its numerical representation. None of the mathematical approaches are available to analyze the data with bipolar multi-fuzzy attributes based on some (interesting) formal concepts, their compact lattice visualization or implications as *

shown in Table 19. This research is at infancy stage which needs more attention for descriptive analysis of uncertainty and vagueness. To fill this backdrop following proposals are made in this paper:

1. A method is proposed to discover all the hidden pattern in data with bipolar multi-fuzzy attributes in Sect. 3.1,
2. A method is proposed to analyze the bipolar multi-fuzzy context at user defined multi-granulation in Sect. 3.2,
3. One application of the proposed methods are shown with step by step demonstration,
4. In addition the analysis derived from the both of the proposed methods are compared with Yang et al. (2013). It is shown that the obtained results echo with each other whereas the proposed methods accomplish this tasks in less computational time.

The bipolar multi-fuzzy attributes can be found while handling biconcepts in m -polar fuzzy space (Chen et al. 2014), hexagonal organization of concepts (Dubois and Prade 2012a, b) as well as three-way fuzzy space (Singh 2017). These type of data set can be represented using the properties of bipolar multi-fuzzy set as discussed by Yang et al. (2013). This opaque numerical representation of bipolar multi-fuzzy attributes gives an way to visualize them in compact way for knowledge processing tasks. To achieve this goal, current paper provides a hierarchical order visualization of bipolar multi-fuzzy attributes in the concept lattice for accurate analysis of knowledge processing tasks. In addition, another method is proposed to analyze the bipolar multi-fuzzy attribute data set based on small chunk of context at user defined multi-granulation. This properties of the proposed method helps in finding some meaningful information at multi-level information extraction within $O(n k m^2)$ time complexity. In near future, the author will focus on (1, 0)-bipolar multi-fuzzy concepts generation and its comparative study with the proposed method. Same time the application of the proposed method will be focused on handling bipolar opinion beyond the three-way dimensions for handling human cognition and other data sets⁴ with their saddle or perturbation points.

6 Conclusions

This paper contribute the attention towards handling the bipolar information exists in each building block of the m -polar fuzzy space. The motivation is to span the limited information represented by bipolar information using the m -polar fuzzy space and its generalization. To achieve this

³ [https://en.wikipedia.org/wiki/Z-matrix_\(chemistry\)](https://en.wikipedia.org/wiki/Z-matrix_(chemistry)).

⁴ <https://www.confjuly.org/conference/MLPRIS/1367s8602.html>.

Table 19 Some important papers on data with unipolar, bipolar, multi-polar and bipolar-multi fuzzy attributes

	Unipolar attributes	Bipolar attributes	Multi-polar attributes	Bipolar multi fuzzy attributes
Formal context	Burusco and Fuentes-Gonzalez (1994)	Alcalde et al. (2015)	Chen et al. (2014)	Yang et al. (2013)
Formal concept	Belohlavek (2004)	Singh and Kumar (2014a, 2014b)	Singh (2018c)	*
Lattice	Pollandt (1997) Djouadi (2011)	Bloch (2011)	Mesiarova-Zemankova and Ahmad (2014)	*
Graph	Berry and Sigayret (2004)	Akram (2011) Singh (2019b)	Ghorai and Pal (2015)	Kroonberg (2007)
Concept lattice	Djouadi and Prade (2009)	Singh and Kumar (2014a, 2014b)	Singh (2017)	*
Reduction of concepts	Kumar and Srinivas (2010) Dias et al. (2020)	Singh (2019b)	Singh (2017, 2018a)	*
Granular computing	Yao (2004)	Singh (2019b)	Li et al. (2015)	*
Application	Singh et al. (2016)	Kumar et al. (2015)	Lee (2000)	Zhang et al. (2016)

goal, the properties of bipolar multi-fuzzy set and its graphical structure is introduced for bipolar multi-fuzzy concepts generation and its lattice visualization. In this process, the properties of Next Neighbor algorithm is used which takes $O(|C| n k^2 m)$ time complexity for generating the bipolar multi-fuzzy concepts. In addition, another method is proposed to refine the knowledge based on user defined multi-granulation with an illustrative example. It is shown that, the analysis derived from both of the proposed methods are corresponds to Yang et al. (2013) within $O(n k m^2)$ where n and k represents number of objects and attributes, respectively. The future work will be focused on (1, 0)-bipolar multi-fuzzy concepts generation and its comparison with the proposed method.

Acknowledgements Author would like to thank the anonymous reviewers and editor for providing a chance to revise this paper and its quality.

Declarations

Conflict of interest Author declares that there is no conflict of interest.

Ethical approval This article does not contain any studies with human or animals participants.

References

- Akram M (2011) Bipolar fuzzy graphs. *Inf Sci* 181(24):5548–5564
- Akram M (2019) m -Polar fuzzy graphs. *Stud Fuzziness Soft Comput.* <https://doi.org/10.1007/978-3-030-03751-2>
- Alcalde C, Burusco A, Fuentes-Gonzales R (2015) The use of two relations in L-fuzzy contexts. *Inf Sci* 301:1–12
- Belohlavek R (2004) Concept lattices and order in fuzzy logic. *Ann Pure Appl Logic* 128(1–3):277–298
- Berry A, Sigayret A (2004) Representing concept lattice by a graph. *Discrete Appl Math* 144(1–2):27–42
- Bloch I (2011) Lattices of fuzzy sets and bipolar fuzzy sets, and mathematical morphology. *Inf Sci* 181(10):2002–2015
- Burusco A, Fuentes-Gonzales R (2001) The study on interval-valued contexts. *Fuzzy Sets Syst* 121(3):439–452
- Burusco A, Fuentes-Gonzalez R (1994) The study of the L-fuzzy concept lattice. *Mathware Soft Comput* 1(3):209–218
- Chen SM (1996) A fuzzy reasoning approach for rule-based systems based on fuzzy logics. *IEEE Trans Syst Man Cybern Part B Cybern* 26(5):769–778
- Chen SM, Huang CM (2003) Generating weighted fuzzy rules from relational database systems for estimating null values using genetic algorithms. *IEEE Trans Fuzzy Syst* 11(4):495–506
- Chen SM, Jong WT (1997) Fuzzy query translation for relational database systems. *IEEE Trans Syst Man Cybern Part B Cybern* 27(4):714–721
- Chen SM, Ke JS, Chang JF (1990) Knowledge representation using fuzzy Petri nets. *IEEE Trans Knowl Data Eng* 2(3):311–319
- Chen J, Li J, Ma S, Wang X (2014) m -Polar fuzzy sets: an extension of bipolar fuzzy sets. *Sci World J.* <https://doi.org/10.1155/2014/416530> (Article ID 416530)
- Coppi R (1994) An introduction to multiway data and their analysis. *Comput Stat Data Anal* 18:3–13
- Dias SM, Zarate EL, Song MAJ, Vieira NJ, Kumar CA (2020) Extraction of qualitative behavior rules for industrial processes from reduced concept lattice. *Intell Data Anal* 24(3):643–663
- Djouadi Y (2011) Extended Galois derivation operators for information retrieval based on fuzzy formal concept lattice. In: Benferhal S, Goant J (eds) SUM 2011. Springer, LNAI 6929, pp 346–358
- Djouadi Y, Prade H (2009) Interval-valued fuzzy formal concept analysis. In: Rauch et al (eds) ISMIS 2009. Springer, LNAI 5722, pp 592–601
- Dubois D, Prade H (2012a) Possibility theory and formal concept analysis: characterizing independent sub-contexts. *Fuzzy Sets Syst* 196:4–16

- Dubois D, Prade H (2012b) From Blanche's hexagonal organization of concepts to formal concept analysis and possibility theory. *Log Univers* 6:149–169
- Ganter B, Wille R (1999) Formal concept analysis: mathematical foundation. Springer, Berlin
- Ghorai G, Pal M (2015) On some operations and density of m -polar fuzzy graphs. *Pac Sci Rev A Nat Sci Eng* 17:14–22
- Ghosh P, Kundu K, Sarkar D (2010) Fuzzy graph representation of a fuzzy concept lattice. *Fuzzy Sets Syst* 161(12):1669–1675
- Glodeanu CV (2014) Exploring user's preferences in a fuzzy setting. *Electron Notes Theor Comput Sci* 303:37–57
- Huang C, Li JH, Mei C, Wu WZ (2017) Three-way concept learning based on cognitive operators: an information fusion viewpoint. *Int J Approx Reason* 83(2017):218–242
- Kamaci H, Petchimuthu S (2020) Bipolar N-soft set theory with applications. *Soft Comput* 24:16727–16743
- Kang X, Li D, Wang S, Qu K (2012) Formal concept analysis based on fuzzy granularity base for different granulation. *Fuzzy Sets Syst* 203:33–48
- Kapoor P, Singh PK (2020) Multidimensional crime dataset analysis. In: Abraham A, Cherukuri AS, Melin P, Gandhi N (eds) *Intelligent systems design and applications (ISDA) 2018. Advances in intelligent systems and computing*, vol 940. Springer, Cham. https://doi.org/10.1007/978-3-030-16657-1_7
- Kroonberg KM (2007) Applied multiway data analysis. Wiley, New York
- Kumar CA (2012) Fuzzy clustering-based formal concept analysis for association rules mining. *Appl Artif Intell* 26(3):274–301
- Kumar CA, Srinivas S (2010) Concept lattice reduction using fuzzy K-Means clustering. *Expert Syst Appl* 37(3):2696–2704
- Kumar CA, Ishwarya MS, Loo CK (2015) Formal concept analysis approach to cognitive functionalities of bidirectional associative memory. *Biol Inspired Cogn Archit* 12:20–33
- Lee KM (2000) Bipolar-valued fuzzy sets and their operations. *Proc Int Conf Intell Technol* 2000:307–312
- Li L (2017) Multi-level interval-valued fuzzy concept lattices and their attribute reduction. *Int J Mach Learn Cybern* 8(1):45–56
- Li JH, Mei C, Xu W, Qian Y (2015) Concept learning via granular computing: a cognitive viewpoint. *Inf Sci* 298:447–467
- Li JH, Huang C, Qi J, Qian Y, Liu W (2017) Three-way cognitive concept learning via multi-granularity. *Inf Sci* 378(1):244–263
- Lindig C (2000) Fast concept analysis. In: Ganter B, Mineau GW (eds) *ICCS 2000. LNCS*, vol 1867. Springer, Heidelberg, pp 152–161
- Mechelen IV, Smilde AK (2011) Comparability problems in the analysis of multiway data. *Chemom Intell Lab Syst* 106:11–22
- Mesiarova-Zemankova A, Ahmad K (2014) Extended multi-polarity and multi-polar-valued fuzzy sets. *Fuzzy Sets Syst* 234:61–78
- Mesiarova-Zemankova A, Hyccko M (2015) Aggregation on Boolean multi-polar space: knowledge-based vs category-based ordering. *Inf Sci* 309:163–179
- Pedrycz W, Chen SM (2015) Information granularity, big data, and computational intelligence. Springer, Heidelberg (ISBN: 978-3-319-08254-7)
- Pollandt S (1997) *Fuzzy Begriffe*. Springer, Berlin, p 1997
- Sadaaki M (2001) Fuzzy multisets and their generalizations. *Lecture Notes Comput Sci* 2235(2001):225–236
- Sebastian S, Ramakrishnan TV (2011) Multi-fuzzy sets: an extension of fuzzy sets. *Fuzzy Inf Eng* 1:35–43
- Singh PK (2017) Three-way fuzzy concept lattice representation using neutrosophic set. *Int J Mach Learn Cybern* 8(1):69–79
- Singh PK (2018a) Interval-valued neutrosophic graph representation of concept lattice and its (α, β, γ) -decomposition. *Arabian J Sci Eng* 43(2):723–740
- Singh PK (2018b) Similar vague concepts selection using their Euclidean distance at different granulation. *Cogn Comput* 10(2):228–241
- Singh PK (2018c) m -polar fuzzy graph representation of concept lattice. *Eng Appl Artif Intell* 67:52–62
- Singh PK (2018d) Concept lattice visualization of data with m -polar fuzzy attribute. *Granul Comput* 3(2):123–137
- Singh PK (2019a) Object and attribute oriented m -polar fuzzy concept lattice using the projection operator. *Granul Comput* 4(3):545–558
- Singh PK (2019b) Bipolar fuzzy concept learning using next neighbor and Euclidean distance. *Soft Comput* 23(12):4503–4520
- Singh PK (2019c) Multi-granulation based graphical analytics of three-way bipolar neutrosophic contexts. *Cogn Comput* 11(4):513–528
- Singh PK (2020) Bipolar δ -equal complex fuzzy concept lattice with its application. *Neural Comput Appl* 32(7):2405–2422
- Singh PK (2021) Complex multi-fuzzy context analysis at different granulation. *Granul Comput* 6(1):191–206
- Singh PK, Abdullah G (2015) Fuzzy concept lattice reduction using Shannon entropy and Huffman coding. *J Appl Nonclass Log* 25(2):101–119
- Singh PK, Kumar CA (2012) A method for decomposition of fuzzy formal context. *Procedia Eng* 38:1852–1857
- Singh PK, Kumar CA (2014a) A note on bipolar fuzzy graph representation of concept lattice. *Int J Comput Sci Math* 5(4):381–393
- Singh PK, Kumar CA (2014b) Bipolar fuzzy graph representation of concept lattice. *Inf Sci* 288:437–448
- Singh PK, Kumar CA (2015) A note on computing crisp order context of a fuzzy formal context for knowledge reduction. *J Inf Process Syst* 11(2):184–204
- Singh PK, Kumar CA, Gani A (2016) A comprehensive survey on formal concept analysis, its research trends and applications. *Int J Appl Math Comput Sci* 26(2):495–516
- Skowron A, Jankowski A, Dutta S (2016) Interactive granular computing. *Granul Comput* 1(2):95–113
- Voutsadakis G (2002) Polyadic concept analysis. *Order* 19:295–304
- Wille R (1982) Restructuring lattice theory: an approach based on hierarchies of concepts. In: Rival I (eds) *Ordered sets, NATO advanced study institutes series (Series C —Mathematical and Physical Sciences)*, vol 83. Springer, Dordrecht. https://doi.org/10.1007/978-94-009-7798-3_15
- Yang HL, Li SG, Wang WH, Lu Y (2013) Notes on bipolar fuzzy graphs. *Inf Sci* 242:113–121
- Yang Y, Tan X, Meng C (2013) The multi-fuzzy soft set and its application in decision making. *Appl Math Model* 37(7):4915–4923
- Yang Y, Peng X, Chen H, Zeng L (2014) A decision making approach based on bipolar multi-fuzzy soft set theory. *J Intell Fuzzy Syst* 27:1861–1872
- Yao Y (2004) Granular computing. In: *Proceedings of The 4th Chinese National Conference on rough sets and soft computing 2004, computer science (Ji Suan Ji Ke Xue)* vol 31. pp 1–5
- Yao Y (2016) A triarchic theory of granular computing. *Granul Comput* 1(2):145–157
- Yao Y (2021a) Set-theoretic models of three-way decision. *Granul Comput* 6:133–148
- Yao Y (2021b) The geometry of three-way decision. *Appl Intell*. <https://doi.org/10.1007/s10489-020-02142-z>
- Zadeh LA (1965) Fuzzy sets. *Inf Control* 8(3):338–353
- Zadeh LA (2011) A note on Z-numbers. *Inf Sci* 181(14):2923–2932
- Zhang HD, Liao SL, Ma WY (2016) Generalized multi-fuzzy soft set and its application in decision making. In: Cao BY, Liu ZL, Zhong YB, Mi HH (eds) *Fuzzy systems and operations research*

- and management. *Advances in intelligent systems and computing*, vol 367. pp 31–42
- Zhang WR (1994) Bipolar fuzzy sets and relations: a computational framework for cognitive modeling and multiagent decision analysis. In: *Proceedings of the first international joint conference of the North American fuzzy information processing society biannual conference 1994*. pp 305–309
- Zhang WR (2021) Ground-0 axioms vs. first principles and second law: from the geometry of light and logic of photon to mind-light-matter unity-AI and QI. *IEEE/CAA J Autom Sin* 8(3):534–553

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.