Granular Computing (2021) 6:779–795
https://doi.org/10.1007/s41066-020-00233-3

ORIGINAL PAPER

Extension of Einstein geometric operators to multi-attribute decision making under *q*-rung orthopair fuzzy information

Muhammad Akram[1] · Gulfam Shahzadi[1] · Xindong Peng[2]

Received: 15 May 2020 / Accepted: 18 July 2020 / Published online: 1 August 2020

Abstract

Aggregation operators are mathematical functions and essential tools of unifying the several inputs into single valuable output. The purpose of this paper is to analyze the aggregation operators (AOs) under the q-rung orthopair fuzzy environment with the help of Einstein norms operations. This paper presents AOs, namely, q-rung orthopair fuzzy Einstein weighted geometric (q-ROFEWG), q-rung orthopair fuzzy Einstein ordered weighted geometric (q-ROFEOWG), generalized q-rung orthopair fuzzy Einstein weighted geometric (Gq-ROFEWG), generalized q-rung orthopair fuzzy Einstein ordered weighted geometric (Gq-ROFEOWG) operators. Some properties of these operators are explained. An algorithmic model to deal with multi-attribute decision making problems in q-rung orthopair fuzzy(q-ROF) environment using generalized q-ROF Einstein weighted geometric operator is established. These operators can remunerate for the possible asymmetric roles of the attributes that represent the problem. At the end, to prove the validity and feasibility of the proposed model, we give applications for the selection of location of thermal power station and selection of best cardiac surgeon. The comparison analysis with other existing operators shows the reliability of our work.

Keywords Einstein operators · q-rung orthopair fuzzy numbers · Geometric operators · Generalized weighted geometric operators

1 Introduction

Multi-attribute decision making (MADM) performs an important role in finding an excellent alternative from all the appropriate alternatives depending upon certain parameters or attributes. Mostly, the approach of different alternatives and the corresponding weights for various attributes are given in crisp values. But many decisions in real-life problems, for which the objectives and conditions are vague and unclear. To handle such situations, Zadeh (1965) gave the theory of fuzzy sets (FSs) in which the satisfaction of an attribute is represented in terms of membership degree (MD) between zero and one. FSs have become one of the emerging areas in contemporary technologies of information processing (Chen and Chen 2014; Chen et al. 2009; Chen and Niou 2011). To represent the dissatisfaction degree (nonmembership degree) independently, Atanassov (1986) introduced intuitionistic fuzzy sets (IFSs) which are characterized by both the MD μ and the nonmembership degree (NMD) ν in [0, 1], respectively such that the summation of these two values is less than or equal to 1. An IFS has various applications in different fields of life (Chen and Cheng 2016; Chen et al. 2016). Yager(2013a; b) introduced Pythagorean fuzzy sets (PFSs) to broaden the space of IFSs and replaced the condition $\mu + \nu \leq 1$ with $\mu^2 + \nu^2 \leq 1$. PFSs are more applicable to deal the vagueness as compared to IFSs. Moreover, Yager (2016) proposed new notion of q-rung orthopair fuzzy sets (q-ROFSs) in which $\mu^q + \nu^q \leq 1$. In decision making problems, q-ROFSs are more capable than IFSs and PFSs, as shown in Fig. 1.

✉ Xindong Peng
952518336@qq.com

Muhammad Akram
m.akram@pucit.edu.pk

Gulfam Shahzadi
gulfamshahzadi22@gmail.com

[1] Department of Mathematics, University of the Punjab, New Campus, Lahore, Pakistan

[2] School of Information Science and Engineering, Shaoguan University, Shaoguan, China

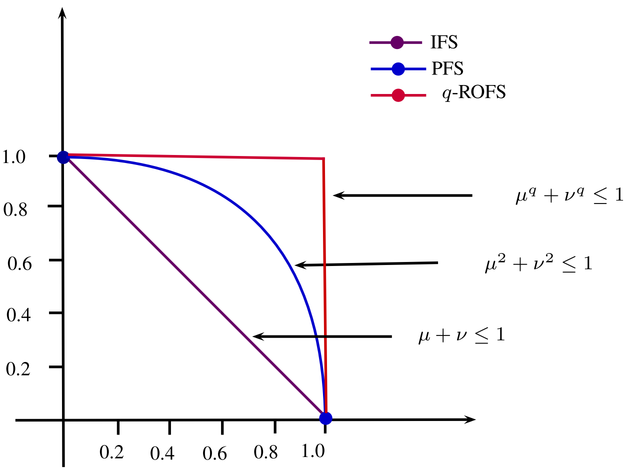

Fig. 1 Comparison of q-ROFNs, PFNs and IFNs spaces

For solving MADM issues and for a single decision in case of collective information, AOs have a key role. The cue of weighted AOs was presented by Yager (1988). The applications of novel geometric AOs for IFSs in MADM were presented by Xu and Yager (2006). The notion of intuitionistic fuzzy AOs was given by Xu (2007). The induced geometric AOs for IFSs were studied by Wei (2010). Wang and Liu (2011) discussed the intuitionistic fuzzy Einstein weighted geometric (IFEWG) operators. Yager (2013a, b) studied both averaging and geometric operators under Pythagorean fuzzy (PF) environment. Peng and Yang (2016; 2016) studied the fundamental properties of PF AOs. Zeng et al. (2016) developed a hybrid method for PF MADM. Garg (2017) developed the generalized PF Einstein weighted geometric AOs. Rahman et al. (2017) discussed the PF Einstein weighted geometric AOs with multiple-attribute group decision making applications. Akram et al. (2019a, b) discussed the Pythagorean Dombi fuzzy AOs. Shahzadi and Akram (2020) introduced the PF Yager AOs. Liu and Wang (2018) expressed q-ROF weighted AOs. Dombi AOs under q-ROF data were handled by Jana et al. (2019). The idea of q-rung orthopair fuzzy power Maclaurin symmetric mean operators was studied by Liu et al. (2018a, b). Some q-rung orthopair fuzzy Bonferroni mean operators were discussed by Liu and Liu (2018). Garg and Chen (2020) introduced the neutrality AOs of q-ROFSs. For other terminologies not discussed in the paper, the readers are referred to (Akram and Adeel 2018; Akram and Ali 2019, 2020; Akram et al. 2019a, b, 2020a, b; Akram and Bashir 2020; Akram and Shahzadi 2020a, b; Bai et al. 2018; Garg and Rani 2019; Joshi and Gegov 2019; Khan et al. 2019; Liu et al. 2018a, b; Peng and Yang 2016; Peng and Yuan 2016; Peng et al. 2018; Wang and Liu 2012; Wei 2017; Wei et al. 2017; Wei and Lu 2018a, b; Yager 1994; Zhang et al. 2014).

The motivations of this article are outlined as follows:

1. The judgement of a perfect alternative in a q-ROF environment is a laborious MADM problem. In existing techniques, assessment information is characterized by IF and PF numbers which promote to do work in q-rung orthopair fuzzy numbers (q-ROFNs).
2. q-ROFNs demonstrate extraordinary execution in providing vague, reliable, and inexact assessment information.
3. Taking into account that Einstein AOs are a straight forward, however ground-breaking, approach for solving decision making issues, this article, in general, aims to define Einstein AOs in the q-ROF context to tackle difficult problems of choice.
4. Einstein AOs make the decision results more precise and exact when applied to real-life MADM based on the q-ROF environment.
5. The proposed operators overcome the restrictions of existing operators.

The main contributions of this article are:

1. The idea of Einstein AOs is extended to q-ROFNs and properties of these operators are discussed.
2. An algorithm is developed to handle complex realistic problems with q-ROF data.
3. Two MADM problems are discussed using proposed operators.
4. At the end, the benefits and characteristics of these operators are discussed by comparison analysis.

The remaining paper is as follows: In Sect. 2, we recalled some basic definitions. In Sect. 3, Einstein operational laws for q-ROFNs are promoted. In Sect. 4 and Sect. 5, q-ROFEWG and q-ROFEOWG operators are proposed, respectively and related results to them. In Sect. 6, the idea of Gq-ROFEWG and Gq-ROFEOWG operators is handled. In Sect. 7, an algorithm for our new model is proposed and two MADM problems, one of which is the selection of location for thermal power station and other one is selection of best cardiac surgeon under these operators are discussed. Section 8 provides the comparison analysis of our model with generalized Pythagorean fuzzy Einstein weighted geometric (GPFEWG) (Garg 2017), intuitionistic fuzzy Einstein weighted geometric (IFEWG) (Wang and Liu 2011), q-rung orthopair fuzzy generalized Maclaurin symmetric mean (q-ROFGMSM) Liu and Wang (2020) operators. In Sect. 9, we have concluded results related to our proposed model.

2 Preliminaries

Definition 1 (Yager 2016) A q-ROFS $\mathfrak{F}$ on non-empty set $\mathcal{X}$ is given by

$$\mathfrak{F} = \{\langle x, \mu_{\mathfrak{F}}(x), \nu_{\mathfrak{F}}(x)\rangle\},$$

where $\mu_{\mathfrak{F}} : \mathcal{X} \to [0,1]$ and $\nu_{\mathfrak{F}} : \mathcal{X} \to [0,1]$ show the MD and NMD of an element $x \in \mathcal{X}$, respectively. $\varpi_{\mathfrak{F}}(x) = \sqrt[q]{1 - (\mu_{\mathfrak{F}}(x))^q + (\nu_{\mathfrak{F}}(x))^q}$ is indeterminacy degree of an element. For easiness, $\mathfrak{F} = \{\langle x, \mu_{\mathfrak{F}}(x), \nu_{\mathfrak{F}}(x)\rangle\}$, called q-ROFN represented by $\mathfrak{F} = \langle \mu_{\mathfrak{F}}, \nu_{\mathfrak{F}}\rangle$.

Definition 2 (Yager 2016) Consider two q-ROFNs $\mathfrak{F}_1 = \langle \mu_{\mathfrak{F}_1}, \nu_{\mathfrak{F}_1}\rangle$ and $\mathfrak{F}_2 = \langle \mu_{\mathfrak{F}_2}, \nu_{\mathfrak{F}_2}\rangle$. The operational laws on q-ROFNs are

1. $\mathfrak{F}_1 \oplus \mathfrak{F}_2 = \left\langle \sqrt[q]{\mu^q_{\mathfrak{F}_1} + \mu^q_{\mathfrak{F}_2} - \mu^q_{\mathfrak{F}_1}\mu^q_{\mathfrak{F}_2}}, \nu_{\mathfrak{F}_1}\nu_{\mathfrak{F}_2} \right\rangle$,
2. $\mathfrak{F}_1 \otimes \mathfrak{F}_2 = \left\langle \mu_{\mathfrak{F}_1}\mu_{\mathfrak{F}_2}, \sqrt[q]{\nu^q_{\mathfrak{F}_1} + \nu^q_{\mathfrak{F}_2} - \nu^q_{\mathfrak{F}_1}\nu^2_{\mathfrak{F}_2}} \right\rangle$,
3. $\gamma\mathfrak{F}_1 = \left\langle \sqrt[q]{1 - (1 - \mu^q_{\mathfrak{F}_1})^\gamma}, \nu^\gamma_{\mathfrak{F}_1} \right\rangle$,
4. $\mathfrak{F}_1^\gamma = \left\langle \mu^\gamma_{\mathfrak{F}_1}, \sqrt[q]{1 - (1 - \nu^q_{\mathfrak{F}_1})^\gamma} \right\rangle$, where $\gamma > 0$.

Definition 3 (Yager 2016) Consider a q-ROFN $\mathfrak{F} = \langle \mu_{\mathfrak{F}}, \nu_{\mathfrak{F}}\rangle$. The score $\mathcal{S}(\mathfrak{F})$ and accuracy functions $\mathcal{A}(\mathfrak{F})$ of $\mathfrak{F}$ are

$$\mathcal{S}(\mathfrak{F}) = \mu^q_{\mathfrak{F}} - \nu^q_{\mathfrak{F}}, \quad \text{where } \mathcal{S}(\mathfrak{F}) \in [-1,1],$$
$$\mathcal{A}(\mathfrak{F}) = \mu^q_{\mathfrak{F}} + \nu^q_{\mathfrak{F}}, \quad \text{where } \mathcal{A}(\mathfrak{F}) \in [0,1].$$

Definition 4 (Yager 2016) Consider two q-ROFNs $\mathfrak{F}_1 = \langle \mu_{\mathfrak{F}_1}, \nu_{\mathfrak{F}_1}\rangle$ and $\mathfrak{F}_2 = \langle \mu_{\mathfrak{F}_2}, \nu_{\mathfrak{F}_2}\rangle$. Then

1. If $\mathcal{S}(\mathfrak{F}_1) < \mathcal{S}(\mathfrak{F}_2)$, then $\mathfrak{F}_1 \prec \mathfrak{F}_2$,
2. If $\mathcal{S}(\mathfrak{F}_1) > \mathcal{S}(\mathfrak{F}_2)$, then $\mathfrak{F}_1 \succ \mathfrak{F}_2$,
3. If $\mathcal{S}(\mathfrak{F}_1) = \mathcal{S}(\mathfrak{F}_2)$, then
   a. If $\mathcal{A}(\mathfrak{F}_1) < \mathcal{A}(\mathfrak{F}_2)$, then $\mathfrak{F}_1 \prec \mathfrak{F}_2$,
   b. If $\mathcal{A}(\mathfrak{F}_1) > \mathcal{A}(\mathfrak{F}_2)$, then $\mathfrak{F}_1 \succ \mathfrak{F}_2$,
   c. If $\mathcal{A}(\mathfrak{F}_1) = \mathcal{A}(\mathfrak{F}_2)$, then $\mathfrak{F}_1 \sim \mathfrak{F}_2$.

Definition 5 (Garg 2016) The Einstein sum $\oplus_\epsilon$ and Einstein product $\otimes_\epsilon$ under the PF environment are defined as

$$S_\epsilon(x,y) = \sqrt{\frac{x^2 + y^2}{1 + x^2 \cdot_\epsilon y^2}},$$
$$T_\epsilon(x,y) = \frac{x \cdot_\epsilon y}{\sqrt{1 + (1-x^2) \cdot_\epsilon (1-y^2)}}.$$

3 *q*-ROFNs under Einstein operational law

The Einstein operations for q-ROFNs are defined as follows:

Definition 6 Let $\mathfrak{F} = \langle \mu, \nu\rangle$, $\mathfrak{F}_1 = \langle \mu_1, \nu_1\rangle$, $\mathfrak{F}_2 = \langle \mu_2, \nu_2\rangle$ be q-ROFNs and $\gamma > 0$, then

(i) $\overline{\mathfrak{F}} = \langle \nu_{\mathfrak{F}}, \mu_{\mathfrak{F}}\rangle$

(ii) $\mathfrak{F}_1 \wedge_\epsilon \mathfrak{F}_2 = \langle \min\{\mu_1, \mu_2\}, \max\{\nu_1, \nu_2\}\rangle$

(iii) $\mathfrak{F}_1 \vee_\epsilon \mathfrak{F}_2 = \langle \max\{\mu_1, \mu_2\}, \min\{\nu_1, \nu_2\}\rangle$

(iv) $\mathfrak{F}_1 \oplus_\epsilon \mathfrak{F}_2 = \left\langle \sqrt[q]{\frac{\mu_1^q + \mu_2^q}{1 + \mu_1^q \cdot_\epsilon \mu_2^q}}, \frac{\nu_1 \cdot_\epsilon \nu_2}{\sqrt[q]{1 + (1-\nu_1^q) \cdot_\epsilon (1-\nu_2^q)}} \right\rangle$

(v) $\mathfrak{F}_1 \otimes_\epsilon \mathfrak{F}_2 = \left\langle \frac{\mu_1 \cdot_\epsilon \mu_2}{\sqrt[q]{1 + (1-\mu_1^q) \cdot_\epsilon (1-\mu_2^q)}}, \sqrt[q]{\frac{\nu_1^q + \nu_2^q}{1 + \nu_1^q \cdot_\epsilon \nu_2^q}} \right\rangle$

(vi) $\gamma \cdot_\epsilon \mathfrak{F} = \left\langle \sqrt[q]{\frac{(1+\mu^q)^\gamma - (1-\mu^q)^\gamma}{(1+\mu^q)^\gamma + (1-\mu^q)^\gamma}}, \frac{\sqrt[q]{2}\nu^\gamma}{\sqrt[q]{(2-\nu^q)^\gamma + (\nu^q)^\gamma}} \right\rangle$

(vii) $\mathfrak{F}^\gamma = \left\langle \frac{\sqrt[q]{2}\mu^\gamma}{\sqrt[q]{(2-\mu^q)^\gamma + (\mu^q)^\gamma}}, \sqrt[q]{\frac{(1+\nu^q)^\gamma - (1-\nu^q)^\gamma}{(1+\nu^q)^\gamma + (1-\nu^q)^\gamma}} \right\rangle$

Theorem 1 *Let $\mathfrak{F} = \langle \mu_{\mathfrak{F}}, \nu_{\mathfrak{F}}\rangle$, $\mathfrak{F}_1 = \langle \mu_1, \nu_1\rangle$ and $\mathfrak{F}_2 = \langle \mu_2, \nu_2\rangle$ be three q-ROFNs; then, $\mathfrak{F}_3 = \mathfrak{F}^\gamma$ and $\mathfrak{F}_4 = \mathfrak{F}_1 \otimes_\epsilon \mathfrak{F}_2$ are also q-ROFNs.*

Proof Since $\gamma > 0$ and $\mathfrak{F}$ be a q-ROFN, therefore, $0 \le \mu_{\mathfrak{F}}(x) \le 1$, $0 \le \nu_{\mathfrak{F}}(x) \le 1$, and $0 \le (\mu_{\mathfrak{F}}(x))^q + (\nu_{\mathfrak{F}}(x))^q \le 1$, then $1 - (\mu_{\mathfrak{F}}(x))^q \ge (\nu_{\mathfrak{F}}(x))^q \ge 0$, $1 - (\nu_{\mathfrak{F}}(x))^q \ge (\mu_{\mathfrak{F}}(x))^q \ge 0$, and $(1 - (\mu_{\mathfrak{F}}(x))^q)^\gamma \ge (\nu_{\mathfrak{F}}(x))^q$, then, we have

$$\frac{\sqrt[q]{2}(\mu_{\mathfrak{F}}(x))^\gamma}{\sqrt[q]{(2 - (\mu_{\mathfrak{F}}(x))^q)^\gamma + (\mu_{\mathfrak{F}}(x)^q)^\gamma}} \le \frac{\sqrt[q]{2}(\mu_{\mathfrak{F}}(x))^\gamma}{\sqrt[q]{(1 + (\nu_{\mathfrak{F}}(x))^q)^\gamma + ((\mu_{\mathfrak{F}}(x))^q)^\gamma}}$$

and

$$\sqrt[q]{\frac{(1 + (\nu_{\mathfrak{F}}(x))^q)^\gamma - (1 - (\nu_{\mathfrak{F}}(x))^q)^\gamma}{(1 + (\nu_{\mathfrak{F}}(x))^q)^\gamma + (1 - (\nu_{\mathfrak{F}}(x))^q)^\gamma}} \le \sqrt[q]{\frac{(1 + (\nu_{\mathfrak{F}}(x))^q)^\gamma - (\mu_{\mathfrak{F}}(x))^q)^\gamma}{(1 + (\nu_{\mathfrak{F}}(x))^q)^\gamma + (\mu_{\mathfrak{F}}(x))^q)^\gamma}}.$$

Thus,

$$\left(\frac{\sqrt[q]{2}(\mu_{\mathfrak{F}}(x))^\gamma}{\sqrt[q]{(2 - (\mu_{\mathfrak{F}}(x))^q)^\gamma + (\mu_{\mathfrak{F}}(x)^q)^\gamma}}\right)^q + \left(\sqrt[q]{\frac{(1 + (\nu_{\mathfrak{F}}(x))^q)^\gamma - (1 - (\nu_{\mathfrak{F}}(x))^q)^\gamma}{(1 + (\nu_{\mathfrak{F}}(x))^q)^\gamma + (1 - (\nu_{\mathfrak{F}}(x))^q)^\gamma}}\right)^q \le 1.$$

Furthermore,

$$\left(\frac{\sqrt[q]{2}(\mu_{\mathfrak{F}}(x))^{\gamma}}{\sqrt[q]{(2-(\mu_{\mathfrak{F}}(x))^{q})^{\gamma}+(\mu_{\mathfrak{F}}(x)^{q})^{\gamma}}}\right)^{q} + \left(\sqrt[q]{\frac{(1+(v_{\mathfrak{F}}(x))^{q})^{\gamma}-(1-(v_{\mathfrak{F}}(x))^{q})^{\gamma}}{(1+(v_{\mathfrak{F}}(x))^{q})^{\gamma}+(1-(v_{\mathfrak{F}}(x))^{q})^{\gamma}}}\right)^{q} = 0$$

iff $\mu_{\mathfrak{F}}(x) = v_{\mathfrak{F}}(x) = 0$ and

$$\left(\frac{\sqrt[q]{2}(\mu_{\mathfrak{F}}(x))^{\gamma}}{\sqrt[q]{(2-(\mu_{\mathfrak{F}}(x))^{q})^{\gamma}+(\mu_{\mathfrak{F}}(x)^{q})^{\gamma}}}\right)^{q} + \left(\sqrt[q]{\frac{(1+(v_{\mathfrak{F}}(x))^{q})^{\gamma}-(1-(v_{\mathfrak{F}}(x))^{q})^{\gamma}}{(1+(v_{\mathfrak{F}}(x))^{q})^{\gamma}+(1-(v_{\mathfrak{F}}(x))^{q})^{\gamma}}}\right)^{q} = 1$$

iff $(\mu_{\mathfrak{F}}(x))^{q} + (v_{\mathfrak{F}}(x))^{q} = 1$.

Thus, $\mathfrak{F}_3 = \mathfrak{F}^{\gamma}$ is a q-ROFN for $\gamma > 0$. □

Theorem 2 *Let* $\gamma, \gamma_1, \gamma_2 \geq 0$, *then*

(i) $\mathfrak{F}_1 \otimes_{\epsilon} \mathfrak{F}_2 = \mathfrak{F}_1 \otimes_{\epsilon} \mathfrak{F}_2$
(ii) $\mathfrak{F}_1 \oplus_{\epsilon} \mathfrak{F}_2 = \mathfrak{F}_2 \oplus_{\epsilon} \mathfrak{F}_1$
(iii) $(\mathfrak{F}_1 \otimes_{\epsilon} \mathfrak{F}_2)^{\gamma} = \mathfrak{F}_1^{\gamma} \otimes_{\epsilon} \mathfrak{F}_2^{\gamma}$
(iv) $\gamma \cdot_{\epsilon} (\mathfrak{F}_1 \oplus_{\epsilon} \mathfrak{F}_2) = \gamma \cdot_{\epsilon} \mathfrak{F}_1 \oplus_{\epsilon} \gamma \cdot_{\epsilon} \mathfrak{F}_2$
(v) $\mathfrak{F}^{\gamma_1} \otimes_{\epsilon} \mathfrak{F}^{\gamma_2} = \mathfrak{F}^{(\gamma_1+\gamma_2)}$
(vi) $\gamma_1 \cdot_{\epsilon} \mathfrak{F} \oplus_{\epsilon} \gamma_2 \cdot_{\epsilon} \mathfrak{F} = (\gamma_1+\gamma_2) \cdot_{\epsilon} \mathfrak{F}$
(vii) $(\mathfrak{F}^{\gamma_1})^{\gamma_2} = (\mathfrak{F})^{\gamma_1 \cdot_{\epsilon} \gamma_2}$
(viii) $\gamma_1 \cdot_{\epsilon} (\gamma_2 \cdot_{\epsilon} \mathfrak{F}) = (\gamma_1 \cdot_{\epsilon} \gamma_2) \cdot_{\epsilon} \mathfrak{F}$

Proof We prove (i), (iii), (v) and similarly others can be verified.

(i)

$$\begin{aligned}\mathfrak{F}_1 \otimes_{\epsilon} \mathfrak{F}_2 &= \left\langle \frac{\mu_1 \cdot_{\epsilon} \mu_2}{\sqrt[q]{1+(1-\mu_1^{q}).\epsilon(1-\mu_2^{q})}}, \sqrt[q]{\frac{v_1^{q}+v_2^{q}}{1+v_1^{q} \cdot_{\epsilon} v_2^{q}}} \right\rangle \\ &= \left\langle \frac{\mu_2 \cdot_{\epsilon} \mu_1}{\sqrt[q]{1+(1-\mu_2^{q}).\epsilon(1-\mu_1^{q})}}, \sqrt[q]{\frac{v_2^{q}+v_1^{q}}{1+v_2^{q} \cdot_{\epsilon} v_1^{q}}} \right\rangle \\ &= \mathfrak{F}_2 \otimes_{\epsilon} \mathfrak{F}_1.\end{aligned}$$

(iii)

$$\mathfrak{F}_1 \otimes_{\epsilon} \mathfrak{F}_2 = \left\langle \frac{\mu_1 \cdot_{\epsilon} \mu_2}{\sqrt[q]{1+(1-\mu_1^{q}).\epsilon(1-\mu_2^{q})}}, \sqrt[q]{\frac{v_1^{q}+v_2^{q}}{1+v_1^{q} \cdot_{\epsilon} v_2^{q}}} \right\rangle,$$

is equivalent to

$$\mathfrak{F}_1 \otimes_{\epsilon} \mathfrak{F}_2 = \left\langle \frac{\sqrt[q]{2}\mu_1 \cdot_{\epsilon} \mu_2}{\sqrt[q]{(2-\mu_1^{q}).\epsilon(2-\mu_2^{q})+\mu_1^{q} \cdot_{\epsilon} \mu_2^{q}}}, \sqrt[q]{\frac{(1+v_1^{q}) \cdot_{\epsilon} (1+v_2^{q}) - (1-v_1^{q}) \cdot_{\epsilon} (1-v_2^{q})}{(1+v_1^{q}) \cdot_{\epsilon} (1+v_2^{q}) + (1-v_1^{q}) \cdot_{\epsilon} (1-v_2^{q})}} \right\rangle.$$

Take $\mathfrak{a} = (1+v_1^{q}) \cdot_{\epsilon} (1+v_2^{q})$, $\mathfrak{b} = (1-v_1^{q}) \cdot_{\epsilon} (1-v_2^{q})$, $\mathfrak{c} = \mu_1^{q} \cdot_{\epsilon} \mu_2^{q}$, and $\mathfrak{d} = (2-\mu_1^{q}).\epsilon(2-\mu_2^{q})$, then

$$\mathfrak{F}_1 \otimes_{\epsilon} \mathfrak{F}_2 = \left\langle \frac{\sqrt[q]{2\mathfrak{c}}}{\sqrt[q]{\mathfrak{d}+\mathfrak{c}}}, \sqrt[q]{\frac{\mathfrak{a}-\mathfrak{b}}{\mathfrak{a}+\mathfrak{b}}} \right\rangle.$$

By the Einstein q-ROF law,

$$\begin{aligned}&(\mathfrak{F}_1 \otimes_{\epsilon} \mathfrak{F}_2)^{\gamma} \\ &= \left\langle \frac{\sqrt[q]{2\mathfrak{c}}}{\sqrt[q]{\mathfrak{d}+\mathfrak{c}}}, \sqrt[q]{\frac{\mathfrak{a}-\mathfrak{b}}{\mathfrak{a}+\mathfrak{b}}} \right\rangle^{\gamma} \\ &= \left\langle \frac{\sqrt[q]{2}.\left(\frac{\sqrt[q]{2\mathfrak{c}}}{\sqrt[q]{\mathfrak{d}+\mathfrak{c}}}\right)^{\gamma}}{\sqrt[q]{\left(2-\frac{2\mathfrak{c}}{\mathfrak{d}+\mathfrak{c}}\right)^{\gamma}+\left(\frac{2\mathfrak{c}}{\mathfrak{d}+\mathfrak{c}}\right)^{\gamma}}}, \sqrt[q]{\frac{\left(1+\frac{\mathfrak{a}-\mathfrak{b}}{\mathfrak{a}+\mathfrak{b}}\right)^{\gamma}-\left(1-\frac{\mathfrak{a}-\mathfrak{b}}{\mathfrak{a}+\mathfrak{b}}\right)^{\gamma}}{\left(1+\frac{\mathfrak{a}-\mathfrak{b}}{\mathfrak{a}+\mathfrak{b}}\right)^{\gamma}+\left(1-\frac{\mathfrak{a}-\mathfrak{b}}{\mathfrak{a}+\mathfrak{b}}\right)^{\gamma}}} \right\rangle \\ &= \left\langle \frac{\sqrt[q]{2\mathfrak{c}^{\gamma}}}{\sqrt[q]{\mathfrak{d}^{\gamma}+\mathfrak{c}^{\gamma}}}, \sqrt[q]{\frac{\mathfrak{a}^{\gamma}-\mathfrak{b}^{\gamma}}{\mathfrak{a}^{\gamma}+\mathfrak{b}^{\gamma}}} \right\rangle \\ &= \left\langle \frac{\sqrt[q]{2}\mu_1^{\gamma} \cdot_{\epsilon} \mu_2^{\gamma}}{\sqrt[q]{(2-\mu_1^{q})^{\gamma}.\epsilon(2-\mu_2^{q})^{\gamma}+(\mu_1^{q})^{\gamma} \cdot_{\epsilon} (\mu_2^{q})^{\gamma}}}, \sqrt[q]{\frac{(1+v_1^{q})^{\gamma} \cdot_{\epsilon} (1+v_2^{q})^{\gamma}-(1-v_1^{q})^{\gamma} \cdot_{\epsilon} (1-v_2^{q})^{\gamma}}{(1+v_1^{q})^{\gamma} \cdot_{\epsilon} (1+v_2^{q})^{\gamma}+(1-v_1^{q})^{\gamma} \cdot_{\epsilon} (1-v_2^{q})^{\gamma}}} \right\rangle.\end{aligned}$$

On the other hand,

$$\begin{aligned}{\mathfrak{F}_1}^{\gamma} &= \left\langle \frac{\sqrt[q]{2}\mu_1^{\gamma}}{\sqrt[q]{(2-\mu_1^{q})^{\gamma}+(\mu_1^{q})^{\gamma}}}, \sqrt[q]{\frac{(1+v_1^{q})^{\gamma}-(1-v_1^{q})^{\gamma}}{(1+v_1^{q})^{\gamma}+(1-v_1^{q})^{\gamma}}} \right\rangle \\ &= \left\langle \frac{\sqrt[q]{2\mathfrak{c}_1}}{\sqrt[q]{\mathfrak{d}_1+\mathfrak{c}_1}}, \sqrt[q]{\frac{\mathfrak{a}_1-\mathfrak{b}_1}{\mathfrak{a}_1+\mathfrak{b}_1}} \right\rangle,\end{aligned}$$

and

$$\begin{aligned}{\mathfrak{F}_2}^{\gamma} &= \left\langle \frac{\sqrt[q]{2}\mu_2^{\gamma}}{\sqrt[q]{(2-\mu_2^{q})^{\gamma}+(\mu_2^{q})^{\gamma}}}, \sqrt[q]{\frac{(1+v_2^{q})^{\gamma}-(1-v_2^{q})^{\gamma}}{(1+v_2^{q})^{\gamma}+(1-v_2^{q})^{\gamma}}} \right\rangle \\ &= \left\langle \frac{\sqrt[q]{2\mathfrak{c}_2}}{\sqrt[q]{\mathfrak{d}_2+\mathfrak{c}_2}}, \sqrt[q]{\frac{\mathfrak{a}_2-\mathfrak{b}_2}{\mathfrak{a}_2+\mathfrak{b}_2}} \right\rangle,\end{aligned}$$

where $\mathfrak{a}_1 = (1+v_1^{q})^{\gamma}$, $\mathfrak{b}_1 = (1-v_1^{q})^{\gamma}$, $\mathfrak{c}_1 = (\mu_1^{q})^{\gamma}$, $\mathfrak{d}_1 = (2-\mu_1^{q})^{\gamma}$, $\mathfrak{a}_2 = (1+v_2^{q})^{\gamma}$, $\mathfrak{b}_2 = (1-v_2^{q})^{\gamma}$, $\mathfrak{c}_2 = (\mu_2^{q})^{\gamma}$, $\mathfrak{d}_2 = (2-\mu_2^{q})^{\gamma}$, therefore,

$$\begin{aligned}\mathfrak{F}_1{}^{\gamma} \otimes_\epsilon \mathfrak{F}_2{}^{\gamma} &= \left\langle \frac{\sqrt[q]{2\mathfrak{c}_1}}{\sqrt[q]{\mathfrak{d}_1+\mathfrak{c}_1}}, \sqrt[q]{\frac{\mathfrak{a}_1-\mathfrak{b}_1}{\mathfrak{a}_1+\mathfrak{b}_1}} \right\rangle \\ &\quad \otimes_\epsilon \left\langle \frac{\sqrt[q]{2\mathfrak{c}_2}}{\sqrt[q]{\mathfrak{d}_2+\mathfrak{c}_2}}, \sqrt[q]{\frac{\mathfrak{a}_2-\mathfrak{b}_2}{\mathfrak{a}_2+\mathfrak{b}_2}} \right\rangle \\ &= \left\langle \frac{2^{\frac{2}{q}} \sqrt[q]{\frac{\mathfrak{c}_1 \cdot_\epsilon \mathfrak{c}_2}{(\mathfrak{d}_1+\mathfrak{c}_1)\cdot_\epsilon(\mathfrak{d}_2+\mathfrak{c}_2)}}}{\sqrt[q]{1+(1-\frac{2\mathfrak{c}_1}{\mathfrak{d}_1+\mathfrak{c}_1})\cdot_\epsilon(1-\frac{2\mathfrak{c}_2}{\mathfrak{d}_2+\mathfrak{c}_2})}}, \right. \\ &\quad \left. \sqrt[q]{\frac{\frac{\mathfrak{a}_1-\mathfrak{b}_1}{\mathfrak{a}_1+\mathfrak{b}_1}+\frac{\mathfrak{a}_2-\mathfrak{b}_2}{\mathfrak{a}_2+\mathfrak{b}_2}}{1+\frac{\mathfrak{a}_1-\mathfrak{b}_1}{\mathfrak{a}_1+\mathfrak{b}_1}\cdot_\epsilon\frac{\mathfrak{a}_2-\mathfrak{b}_2}{\mathfrak{a}_2+\mathfrak{b}_2}}} \right\rangle \\ &= \left\langle \frac{\sqrt[q]{2\mathfrak{c}_1 \cdot_\epsilon \mathfrak{c}_2}}{\sqrt[q]{\mathfrak{d}_1 \cdot_\epsilon \mathfrak{d}_2 + \mathfrak{c}_1 \cdot_\epsilon \mathfrak{c}_2}}, \sqrt[q]{\frac{\mathfrak{a}_1 \cdot_\epsilon \mathfrak{a}_2 - \mathfrak{b}_1 \cdot_\epsilon \mathfrak{b}_2}{\mathfrak{a}_1 \cdot_\epsilon \mathfrak{a}_2 + \mathfrak{b}_1 \cdot_\epsilon \mathfrak{b}_2}} \right\rangle \\ &= \left\langle \frac{\sqrt[q]{2}\mu_1^{\gamma} \cdot_\epsilon \mu_2^{\gamma}}{\sqrt[q]{(2-\mu_1^q)^{\gamma}.\epsilon(2-\mu_2^q)^{\gamma}+(\mu_1^q)^{\gamma}\cdot_\epsilon(\mu_2^q)^{\gamma}}}, \right. \\ &\quad \left. \sqrt[q]{\frac{(1+\nu_1^q)^{\gamma}\cdot_\epsilon(1+\nu_2^q)^{\gamma}-(1-\nu_1^q)^{\gamma}\cdot_\epsilon(1-\nu_2^q)^{\gamma}}{(1+\nu_1^q)^{\gamma}\cdot_\epsilon(1+\nu_2^q)^{\gamma}+(1-\nu_1^q)^{\gamma}\cdot_\epsilon(1-\nu_2^q)^{\gamma}}} \right\rangle.\end{aligned}$$

Hence, $(\mathfrak{F}_1 \otimes_\epsilon \mathfrak{F}_2)^{\gamma} = \mathfrak{F}_1{}^{\gamma} \otimes_\epsilon \mathfrak{F}_2{}^{\gamma}$.

(v) For $\gamma_1, \gamma_2 > 0$.

$$\begin{aligned}\mathfrak{F}^{\gamma_1} &= \left\langle \frac{\sqrt[q]{2}\mu^{\gamma_1}}{\sqrt[q]{(2-\mu^q)^{\gamma_1}+(\mu^q)^{\gamma_1}}}, \sqrt[q]{\frac{(1+\nu^q)^{\gamma_1}-(1-\nu^q)^{\gamma_1}}{(1+\nu^q)^{\gamma_1}+(1-\nu^q)^{\gamma_1}}} \right\rangle \\ &= \left\langle \frac{\sqrt[q]{2\mathfrak{c}_1}}{\sqrt[q]{\mathfrak{d}_1+\mathfrak{c}_1}}, \sqrt[q]{\frac{\mathfrak{a}_1-\mathfrak{b}_1}{\mathfrak{a}_1+\mathfrak{b}_1}} \right\rangle. \\ \mathfrak{F}^{\gamma_2} &= \left\langle \frac{\sqrt[q]{2}\mu^{\gamma_2}}{\sqrt[q]{(2-\mu^q)^{\gamma_2}+(\mu^q)^{\gamma_2}}}, \sqrt[q]{\frac{(1+\nu^q)^{\gamma_2}-(1-\nu^q)^{\gamma_2}}{(1+\nu^q)^{\gamma_2}+(1-\nu^q)^{\gamma_2}}} \right\rangle \\ &= \left\langle \frac{\sqrt[q]{2\mathfrak{c}_2}}{\sqrt[q]{\mathfrak{d}_2+\mathfrak{c}_2}}, \sqrt[q]{\frac{\mathfrak{a}_2-\mathfrak{b}_2}{\mathfrak{a}_2+\mathfrak{b}_2}} \right\rangle,\end{aligned}$$

where $\mathfrak{a}_\mathfrak{j} = (1+\nu^q)^{\gamma_\mathfrak{j}}$, $\mathfrak{b}_\mathfrak{j} = (1-\nu^q)^{\gamma_\mathfrak{j}}$, $\mathfrak{c}_\mathfrak{j} = (\mu^q)^{\gamma_\mathfrak{j}}$, $\mathfrak{d}_\mathfrak{j} = (2-\mu^q)^{\gamma_\mathfrak{j}}$, for $\mathfrak{j} = 1, 2$.

$$\begin{aligned}\mathfrak{F}^{\gamma_1} \otimes_\epsilon \mathfrak{F}^{\gamma_2} &= \left\langle \frac{\sqrt[q]{2\mathfrak{c}_1}}{\sqrt[q]{\mathfrak{d}_1+\mathfrak{c}_1}}, \sqrt[q]{\frac{\mathfrak{a}_1-\mathfrak{b}_1}{\mathfrak{a}_1+\mathfrak{b}_1}} \right\rangle \\ &\quad \otimes_\epsilon \left\langle \frac{\sqrt[q]{2\mathfrak{c}_2}}{\sqrt[q]{\mathfrak{d}_2+\mathfrak{c}_2}}, \sqrt[q]{\frac{\mathfrak{a}_2-\mathfrak{b}_2}{\mathfrak{a}_2+\mathfrak{b}_2}} \right\rangle \\ &= \left\langle \frac{2^{\frac{2}{q}} \sqrt[q]{\frac{\mathfrak{c}_1 \cdot_\epsilon \mathfrak{c}_2}{(\mathfrak{d}_1+\mathfrak{c}_1)\cdot_\epsilon(\mathfrak{d}_2+\mathfrak{c}_2)}}}{\sqrt[q]{1+(1-\frac{2\mathfrak{c}_1}{\mathfrak{d}_1+\mathfrak{c}_1})\cdot_\epsilon(1-\frac{2\mathfrak{c}_2}{\mathfrak{d}_2+\mathfrak{c}_2})}}, \right. \\ &\quad \left. \sqrt[q]{\frac{\frac{\mathfrak{a}_1-\mathfrak{b}_1}{\mathfrak{a}_1+\mathfrak{b}_1}+\frac{\mathfrak{a}_2-\mathfrak{b}_2}{\mathfrak{a}_2+\mathfrak{b}_2}}{1+\frac{\mathfrak{a}_1-\mathfrak{b}_1}{\mathfrak{a}_1+\mathfrak{b}_1}\cdot_\epsilon\frac{\mathfrak{a}_2-\mathfrak{b}_2}{\mathfrak{a}_2+\mathfrak{b}_2}}} \right\rangle \\ &= \left\langle \frac{\sqrt[q]{2\mathfrak{c}_1 \cdot_\epsilon \mathfrak{c}_2}}{\sqrt[q]{\mathfrak{d}_1 \cdot_\epsilon \mathfrak{d}_2 + \mathfrak{c}_1 \cdot_\epsilon \mathfrak{c}_2}}, \sqrt[q]{\frac{\mathfrak{a}_1 \cdot_\epsilon \mathfrak{a}_2 - \mathfrak{b}_1 \cdot_\epsilon \mathfrak{b}_2}{\mathfrak{a}_1 \cdot_\epsilon \mathfrak{a}_2 + \mathfrak{b}_1 \cdot_\epsilon \mathfrak{b}_2}} \right\rangle \\ &= \left\langle \frac{\sqrt[q]{2}\mu^{\gamma_1+\gamma_2}}{\sqrt[q]{(2-\mu^q)^{\gamma_1+\gamma_2}+(\mu_1^q)^{\gamma_1+\gamma_2}}}, \right. \\ &\quad \left. \sqrt[q]{\frac{(1+\nu^q)^{\gamma_1+\gamma_2}-(1-\nu^q)^{\gamma_1+\gamma_2}}{(1+\nu^q)^{\gamma_1+\gamma_2}+(1-\nu^q)^{\gamma_1+\gamma_2}}} \right\rangle \\ &= \mathfrak{F}^{(\gamma_1+\gamma_2)}.\end{aligned}$$

Hence, $\mathfrak{F}^{\gamma_1} \otimes_\epsilon \mathfrak{F}^{\gamma_2} = \mathfrak{F}^{(\gamma_1+\gamma_2)}$. □

Theorem 3 *Let $\mathfrak{F}_1 = \langle \mu_1, \nu_1 \rangle$, $\mathfrak{F}_2 = \langle \mu_2, \nu_2 \rangle$ be two q-ROFNs, then*

(i) $\mathfrak{F}_1^c \wedge_\epsilon \mathfrak{F}_2^c = (\mathfrak{F}_1 \vee_\epsilon \mathfrak{F}_2)^c$
(ii) $\mathfrak{F}_1^c \vee_\epsilon \mathfrak{F}_2^c = (\mathfrak{F}_1 \wedge_\epsilon \mathfrak{F}_2)^c$
(iii) $\mathfrak{F}_1^c \oplus_\epsilon \mathfrak{F}_2^c = (\mathfrak{F}_1 \otimes_\epsilon \mathfrak{F}_2)^c$
(iv) $\mathfrak{F}_1^c \otimes_\epsilon \mathfrak{F}_2^c = (\mathfrak{F}_1 \oplus_\epsilon \mathfrak{F}_2)^c$
(v) $(\mathfrak{F}_1 \vee_\epsilon \mathfrak{F}_2) \oplus_\epsilon (\mathfrak{F}_1 \wedge_\epsilon \mathfrak{F}_2) = \mathfrak{F}_1 \oplus_\epsilon \mathfrak{F}_2$
(vi) $(\mathfrak{F}_1 \vee_\epsilon \mathfrak{F}_2) \otimes_\epsilon (\mathfrak{F}_1 \wedge_\epsilon \mathfrak{F}_2) = \mathfrak{F}_1 \otimes_\epsilon \mathfrak{F}_2$

Proof It is obvious. □

Theorem 4 *Let $\mathfrak{F}_1 = \langle \mu_1, \nu_1 \rangle$, $\mathfrak{F}_2 = \langle \mu_2, \nu_2 \rangle$ and $\mathfrak{F}_3 = \langle \mu_3, \nu_3 \rangle$ be three q-ROFNs, then*

(i) $(\mathfrak{F}_1 \vee_\epsilon \mathfrak{F}_2) \wedge_\epsilon \mathfrak{F}_3 = (\mathfrak{F}_1 \wedge_\epsilon \mathfrak{F}_3) \vee_\epsilon (\mathfrak{F}_2 \wedge_\epsilon \mathfrak{F}_3)$
(ii) $(\mathfrak{F}_1 \wedge_\epsilon \mathfrak{F}_2) \vee_\epsilon \mathfrak{F}_3 = (\mathfrak{F}_1 \vee_\epsilon \mathfrak{F}_3) \wedge_\epsilon (\mathfrak{F}_2 \vee_\epsilon \mathfrak{F}_3)$

(iii) $(\mathfrak{F}_1 \vee_\epsilon \mathfrak{F}_2) \oplus_\epsilon \mathfrak{F}_3 = (\mathfrak{F}_1 \oplus_\epsilon \mathfrak{F}_3) \vee_\epsilon (\mathfrak{F}_2 \oplus_\epsilon \mathfrak{F}_3)$

(iv) $(\mathfrak{F}_1 \wedge_\epsilon \mathfrak{F}_2) \oplus_\epsilon \mathfrak{F}_3 = (\mathfrak{F}_1 \oplus_\epsilon \mathfrak{F}_3) \wedge_\epsilon (\mathfrak{F}_2 \oplus_\epsilon \mathfrak{F}_3)$
(v) $(\mathfrak{F}_1 \vee_\epsilon \mathfrak{F}_2) \otimes_\epsilon \mathfrak{F}_3 = (\mathfrak{F}_1 \otimes_\epsilon \mathfrak{F}_3) \vee_\epsilon (\mathfrak{F}_2 \otimes_\epsilon \mathfrak{F}_3)$
(vi) $(\mathfrak{F}_1 \wedge_\epsilon \mathfrak{F}_2) \otimes_\epsilon \mathfrak{F}_3 = (\mathfrak{F}_1 \otimes_\epsilon \mathfrak{F}_3) \wedge_\epsilon (\mathfrak{F}_2 \otimes_\epsilon \mathfrak{F}_3)$

Proof It is obvious. □

4 *q*-Rung orthopair fuzzy Einstein weighted geometric operators

The Einstein weighted geometric operators under *q*-ROF environment are defined here.

Definition 7 Let $\mathfrak{F}_\mathfrak{j} = \langle \mu_\mathfrak{j}, \nu_\mathfrak{j} \rangle (\mathfrak{j} = 1, 2, \ldots, \mathfrak{s})$ be a collection of *q*-ROFNs and $\varphi_\mathfrak{j}$ is the weight vector (WV) of $\mathfrak{F}_\mathfrak{j}$ with $\varphi_\mathfrak{j} > 0$ and $\sum_{\mathfrak{j}=1}^{\mathfrak{s}} \varphi_\mathfrak{j} = 1$, then *q*-ROFEWG operator is a mapping $\mathcal{Q}^\mathfrak{s} \to \mathcal{Q}$ such that

$$q-\mathrm{ROFEWG}(\mathfrak{F}_1, \mathfrak{F}_2, \ldots, \mathfrak{F}_\mathfrak{s}) = \mathfrak{F}_1{}^{\varphi_1} \otimes_\epsilon \mathfrak{F}_2{}^{\varphi_2} \otimes_\epsilon \ldots \otimes_\epsilon \mathfrak{F}_\mathfrak{s}{}^{\varphi_\mathfrak{s}}. \tag{1}$$

If $\varphi_\mathfrak{j} = 1/\mathfrak{s}$, $\forall \mathfrak{j}$, then *q*-ROFEWG operator becomes *q*-ROFWG operator

$$q-\mathrm{ROFWG}(\mathfrak{F}_1, \mathfrak{F}_2, \ldots, \mathfrak{F}_\mathfrak{s}) = (\mathfrak{F}_1 \otimes_\epsilon \mathfrak{F}_2 \otimes_\epsilon \ldots \otimes_\epsilon \mathfrak{F}_\mathfrak{s})^{\frac{1}{\mathfrak{s}}}.$$

Theorem 5 *Let $\mathfrak{F}_\mathfrak{j} = (\mu_\mathfrak{j}, \nu_\mathfrak{j}) \in$ q-ROFNs, then aggregated value using Equation 1 is*

$$q-\mathrm{ROFEWG}(\mathfrak{F}_1, \mathfrak{F}_2, \ldots, \mathfrak{F}_\mathfrak{s}) = \left\langle \frac{\sqrt[q]{2}\prod_{\mathfrak{j}=1}^{\mathfrak{s}} \mu_\mathfrak{j}^{\varphi_\mathfrak{j}}}{\sqrt[q]{\prod_{\mathfrak{j}=1}^{\mathfrak{s}}(2-\mu_\mathfrak{j}^q)^{\varphi_\mathfrak{j}} + \prod_{\mathfrak{j}=1}^{\mathfrak{s}}(\mu_\mathfrak{j}^q)^{\varphi_\mathfrak{j}}}}, \sqrt[q]{\frac{\prod_{\mathfrak{j}=1}^{\mathfrak{s}}(1+\nu_\mathfrak{j}^q)^{\varphi_\mathfrak{j}} - \prod_{\mathfrak{j}=1}^{\mathfrak{s}}(1-\nu_\mathfrak{j}^q)^{\varphi_\mathfrak{j}}}{\prod_{\mathfrak{j}=1}^{\mathfrak{s}}(1+\nu_\mathfrak{j}^q)^{\varphi_\mathfrak{j}} + \prod_{\mathfrak{j}=1}^{\mathfrak{s}}(1-\nu_\mathfrak{j}^q)^{\varphi_\mathfrak{j}}}} \right\rangle. \tag{2}$$

Proof We prove Eq. 2 by mathematical induction.

When $\mathfrak{s} = 2$,

$$q-ROFEWG(\mathfrak{F}_1, \mathfrak{F}_2) = \mathfrak{F}_1{}^{\varphi_1} \otimes_\epsilon \mathfrak{F}_2{}^{\varphi_2}.$$

From Theorem 1, both $\mathfrak{F}_1{}^{\varphi_1}$ and $\mathfrak{F}_2{}^{\varphi_2}$ are *q*-ROFNs and value of $\mathfrak{F}_1{}^{\varphi_1} \otimes_\epsilon \mathfrak{F}_2{}^{\varphi_2}$ is a *q*-ROFN. Using (vii) in Definition 6,

$$\mathfrak{F}_1{}^{\varphi_1} = \left\langle \frac{\sqrt[q]{2}\mu_1^{\varphi_1}}{\sqrt[q]{(2-\mu_1^q)^{\varphi_1} + (\mu_1^q)^{\varphi_1}}}, \sqrt[q]{\frac{(1+\nu_1^q)^{\varphi_1} - (1-\nu_1^q)^{\varphi_1}}{(1+\nu_1^q)^{\varphi_1} + (1-\nu_1^q)^{\varphi_1}}} \right\rangle.$$

$$\mathfrak{F}_2{}^{\varphi_2} = \left\langle \frac{\sqrt[q]{2}\mu_2^{\varphi_2}}{\sqrt[q]{(2-\mu_2^q)^{\varphi_2} + (\mu_2^q)^{\varphi_2}}}, \sqrt[q]{\frac{(1+\nu_2^q)^{\varphi_2} - (1-\nu_2^q)^{\varphi_2}}{(1+\nu_2^q)^{\varphi_2} + (1-\nu_2^q)^{\varphi_2}}} \right\rangle.$$

Then

$$q-\mathrm{ROFEWG}(\mathfrak{F}_1, \mathfrak{F}_2) = \mathfrak{F}_1{}^{\varphi_1} \otimes_\epsilon \mathfrak{F}_2{}^{\varphi_2}$$

$$= \left\langle \frac{\left(\frac{\sqrt[q]{2}\mu_1^{\varphi_1}}{\sqrt[q]{(2-\mu_1^q)^{\varphi_1}+(\mu_1^q)^{\varphi_1}}}\right) \cdot_\epsilon \left(\frac{\sqrt[q]{2}\mu_2^{\varphi_2}}{\sqrt[q]{(2-\mu_2^q)^{\varphi_2}+(\mu_2^q)^{\varphi_2}}}\right)}{\sqrt[q]{1 + \left(1 - \frac{2\mu_1^{q\varphi_1}}{(2-\mu_1^q)^{\varphi_1}+(\mu_1^q)^{\varphi_1}}\right) \cdot_\epsilon \left(1 - \frac{2\mu_2^{q\varphi_2}}{(2-\mu_2^q)^{\varphi_2}+(\mu_2^q)^{\varphi_2}}\right)}}, \sqrt[q]{\frac{\frac{(1+\nu_1^q)^{\varphi_1}-(1-\nu_1^q)^{\varphi_1}}{(1+\nu_1^q)^{\varphi_1}+(1-\nu_1^q)^{\varphi_1}} + \frac{(1+\nu_2^q)^{\varphi_2}-(1-\nu_2^q)^{\varphi_2}}{(1+\nu_2^q)^{\varphi_2}+(1-\nu_2^q)^{\varphi_2}}}{1 + \left(\frac{(1+\nu_1^q)^{\varphi_1}-(1-\nu_1^q)^{\varphi_1}}{(1+\nu_1^q)^{\varphi_1}+(1-\nu_1^q)^{\varphi_1}}\right) \cdot_\epsilon \left(\frac{(1+\nu_2^q)^{\varphi_2}-(1-\nu_2^q)^{\varphi_2}}{(1+\nu_2^q)^{\varphi_2}+(1-\nu_2^q)^{\varphi_2}}\right)}} \right\rangle$$

$$= \left\langle \frac{\sqrt[q]{2}\mu_1^{\varphi_1} \cdot_\epsilon \mu_2^{\varphi_2}}{\sqrt[q]{(2-\mu_1^q)^{\varphi_1} .\epsilon (2-\mu_2^q)^{\varphi_2} + (\mu_1^q)^{\varphi_1} \cdot_\epsilon (\mu_2^q)^{\varphi_2}}}, \sqrt[q]{\frac{(1+\nu_1^q)^{\varphi_1} \cdot_\epsilon (1+\nu_2^q)^{\varphi_2} - (1-\nu_1^q)^{\varphi_1} \cdot_\epsilon (1-\nu_2^q)^{\varphi_2}}{(1+\nu_1^q)^{\varphi_1} \cdot_\epsilon (1+\nu_2^q)^{\varphi_2} + (1-\nu_1^q)^{\varphi_1} \cdot_\epsilon (1-\nu_2^q)^{\varphi_2}}} \right\rangle.$$

Thus, result holds when $\mathfrak{s} = 2$.

Suppose result holds for $\mathfrak{s} = k$,

$$q-\mathrm{ROFEWG}(\mathfrak{F}_1, \mathfrak{F}_2, \ldots, \mathfrak{F}_k) = \left\langle \frac{\sqrt[q]{2}\prod_{\mathfrak{j}=1}^{k} \mu_\mathfrak{j}^{\varphi_\mathfrak{j}}}{\sqrt[q]{\prod_{\mathfrak{j}=1}^{k}(2-\mu_\mathfrak{j}^q)^{\varphi_\mathfrak{j}} + \prod_{\mathfrak{j}=1}^{k}(\mu_\mathfrak{j}^q)^{\varphi_\mathfrak{j}}}}, \sqrt[q]{\frac{\prod_{\mathfrak{j}=1}^{k}(1+\nu_\mathfrak{j}^q)^{\varphi_\mathfrak{j}} - \prod_{\mathfrak{j}=1}^{k}(1-\nu_\mathfrak{j}^q)^{\varphi_\mathfrak{j}}}{\prod_{\mathfrak{j}=1}^{k}(1+\nu_\mathfrak{j}^q)^{\varphi_\mathfrak{j}} + \prod_{\mathfrak{j}=1}^{k}(1-\nu_\mathfrak{j}^q)^{\varphi_\mathfrak{j}}}} \right\rangle.$$

Now for $\mathfrak{s} = k+1$, $q-\mathrm{ROFEWG}(\mathfrak{F}_1, \mathfrak{F}_2, \ldots, \mathfrak{F}_{k+1})$

$$= \left\langle \frac{\sqrt[q]{2}\prod_{\mathfrak{j}=1}^{k} \mu_\mathfrak{j}^{\varphi_\mathfrak{j}}}{\sqrt[q]{\prod_{\mathfrak{j}=1}^{k}(2-\mu_\mathfrak{j}^q)^{\varphi_\mathfrak{j}} + \prod_{\mathfrak{j}=1}^{k}(\mu_\mathfrak{j}^q)^{\varphi_\mathfrak{j}}}}, \sqrt[q]{\frac{\prod_{\mathfrak{j}=1}^{k}(1+\nu_\mathfrak{j}^q)^{\varphi_\mathfrak{j}} - \prod_{\mathfrak{j}=1}^{k}(1-\nu_\mathfrak{j}^q)^{\varphi_\mathfrak{j}}}{\prod_{\mathfrak{j}=1}^{k}(1+\nu_\mathfrak{j}^q)^{\varphi_\mathfrak{j}} + \prod_{\mathfrak{j}=1}^{k}(1-\nu_\mathfrak{j}^q)^{\varphi_\mathfrak{j}}}} \right\rangle \otimes_\epsilon \left\langle \frac{\sqrt[q]{2}\mu_{k+1}^{\varphi_{k+1}}}{\sqrt[q]{(2-\mu_{k+1}^q)^{\varphi_{k+1}} + (\mu_{k+1}^q)^{\varphi_{k+1}}}}, \sqrt[q]{\frac{(1+\nu_{k+1}^q)^{\varphi_{k+1}} - (1-\nu_{k+1}^q)^{\varphi_{k+1}}}{(1+\nu_{k+1}^q)^{\varphi_{k+1}} + (1-\nu_{k+1}^q)^{\varphi_{k+1}}}} \right\rangle$$

$$= \left\langle \frac{\sqrt[q]{2}\prod_{\mathfrak{j}=1}^{k+1} \mu_\mathfrak{j}^{\varphi_\mathfrak{j}}}{\sqrt[q]{\prod_{\mathfrak{j}=1}^{k+1}(2-\mu_\mathfrak{j}^q)^{\varphi_\mathfrak{j}} + \prod_{\mathfrak{j}=1}^{k+1}(\mu_\mathfrak{j}^q)^{\varphi_\mathfrak{j}}}}, \sqrt[q]{\frac{\prod_{\mathfrak{j}=1}^{k+1}(1+\nu_\mathfrak{j}^q)^{\varphi_\mathfrak{j}} - \prod_{\mathfrak{j}=1}^{k+1}(1-\nu_\mathfrak{j}^q)^{\varphi_\mathfrak{j}}}{\prod_{\mathfrak{j}=1}^{k+1}(1+\nu_\mathfrak{j}^q)^{\varphi_\mathfrak{j}} + \prod_{\mathfrak{j}=1}^{k+1}(1-\nu_\mathfrak{j}^q)^{\varphi_\mathfrak{j}}}} \right\rangle.$$

Thus, result is true for $\mathfrak{s} = k+1$. Hence, Equation 2 holds, $\forall$ $\mathfrak{s}$. □

Lemma 1 *Let $\mathfrak{F}_\mathfrak{j} = \langle \mu_\mathfrak{j}, \nu_\mathfrak{j} \rangle, \varphi_\mathfrak{j} > 0$ and $\sum_{\mathfrak{j}=1}^{\mathfrak{s}} \varphi_\mathfrak{j} = 1$, then*

$$\prod_{j=1}^{s}\mathfrak{F}_j^{\varphi_j} \le \sum_{j=1}^{s}\varphi_j\mathfrak{F}_j,$$

equality holds iff $\mathfrak{F}_1 = \mathfrak{F}_2 = \ldots = \mathfrak{F}_s$.

Theorem 6 *If* $\mathfrak{F}_j = \langle\mu_j, \nu_j\rangle \in$ *q-ROFNs, then*

$$q - ROFEWG(\mathfrak{F}_1, \mathfrak{F}_2, \ldots, \mathfrak{F}_s) \in q - ROFN.$$

Proof Since $\mathfrak{F}_j = \langle\mu_j, \nu_j\rangle \in q$-ROFNs, so $0 \le \mu_j, \nu_j \le 1$ and $0 \le \mu_j^q + \nu_j^q \le 1$. Therefore,

$$\begin{aligned}&\frac{\prod_{j=1}^{s}(1+\nu_j^q)^{\varphi_j} - \prod_{j=1}^{s}(1-\nu_j^q)^{\varphi_j}}{\prod_{j=1}^{s}(1+\nu_j^q)^{\varphi_j} + \prod_{j=1}^{s}(1-\nu_j^q)^{\varphi_j}}\\ &\quad = 1 - \frac{2\prod_{j=1}^{s}(1-\nu_j^q)^{\varphi_j}}{\prod_{j=1}^{s}(1+\nu_j^q)^{\varphi_j} + \prod_{j=1}^{s}(1-\nu_j^q)^{\varphi_j}}\\ &\quad \le 1 - \prod_{j=1}^{s}(1-\nu_j^q)^{\varphi_j} \le 1.\end{aligned} \tag{3}$$

Also,

$$(1+\nu_j^q) \ge (1-\nu_j^q) \Rightarrow \prod_{j=1}^{s}(1+\nu_j^q) - \prod_{j=1}^{s}(1-\nu_j^q) \ge 0.$$

Therefore,

$$\frac{\prod_{j=1}^{s}(1+\nu_j^q)^{\varphi_j} - \prod_{j=1}^{s}(1-\nu_j^q)^{\varphi_j}}{\prod_{j=1}^{s}(1+\nu_j^q)^{\varphi_j} + \prod_{j=1}^{s}(1-\nu_j^q)^{\varphi_j}} \ge 0.$$

Thus, $0 \le \nu_{q-\text{ROFEWG}} \le 1$.

Moreover,

$$\begin{aligned}&\frac{2\prod_{j=1}^{s}(\mu_j^q)^{\varphi_j}}{\prod_{j=1}^{s}(2-\mu_j^q)^{\varphi_j} + \prod_{j=1}^{s}(\mu_j^q)^{\varphi_j}}\\ &\quad \le \frac{2\prod_{j=1}^{s}(1-\nu_j^q)^{\varphi_j}}{\prod_{j=1}^{s}(1+\nu_j^q)^{\varphi_j} + \prod_{j=1}^{s}(1-\nu_j^q)^{\varphi_j}}\\ &\quad \le \prod_{j=1}^{s}(1-\nu_j^q)^{\varphi_j} \le 1.\end{aligned}$$

Also,

$$\prod_{j=1}^{s}(\mu_j^q)^{\varphi_j} \ge 0 \Leftrightarrow \frac{2\prod_{j=1}^{s}(\mu_j^q)^{\varphi_j}}{\prod_{j=1}^{s}(2-\mu_j^q)^{\varphi_j} + \prod_{j=1}^{s}(\mu_j^q)^{\varphi_j}} \ge 0.$$

Thus, $0 \le \mu_{q-\text{ROFEWG}} \le 1$. Moreover,

$$\begin{aligned}&\mu_{q-\text{ROFEWG}}^q + \nu_{q-\text{ROFEWG}}^q\\ &\quad = \frac{2\prod_{j=1}^{s}(\mu_j^q)^{\varphi_j}}{\prod_{j=1}^{s}(2-\mu_j^q)^{\varphi_j} + \prod_{j=1}^{s}(\mu_j^q)^{\varphi_j}}\\ &\quad + \frac{\prod_{j=1}^{s}(1+\nu_j^q)^{\varphi_j} - \prod_{j=1}^{s}(1-\nu_j^q)^{\varphi_j}}{\prod_{j=1}^{s}(1+\nu_j^q)^{\varphi_j} + \prod_{j=1}^{s}(1-\nu_j^q)^{\varphi_j}}\\ &\quad \le \frac{2\prod_{j=1}^{s}(1-\nu_j^q)^{\varphi_j}}{\prod_{j=1}^{s}(1+\nu_j^q)^{\varphi_j} + \prod_{j=1}^{s}(1-\nu_j^q)^{\varphi_j}}\\ &\quad + 1 - \frac{2\prod_{j=1}^{s}(1-\nu_j^q)^{\varphi_j}}{\prod_{j=1}^{s}(1+\nu_j^q)^{\varphi_j} + \prod_{j=1}^{s}(1-\nu_j^q)^{\varphi_j}}\\ &\quad = 1.\end{aligned}$$

Hence, q-ROFEWG $\in [0, 1]$. Therefore,

$$q - \text{ROFEWG}(\mathfrak{F}_1, \mathfrak{F}_2, \ldots, \mathfrak{F}_s) \in q - \text{ROFN}.$$

Corollary 1 *The q-ROFEWG and q-ROFWG operators have the relationship*:
q-ROFEWG $(\mathfrak{F}_1, \mathfrak{F}_2, \ldots, \mathfrak{F}_s) \ge$ *q-ROFWG* $(\mathfrak{F}_1, \mathfrak{F}_2, \ldots, \mathfrak{F}_s)$.

Proof Let q-ROFEWG $(\mathfrak{F}_1, \mathfrak{F}_2, \ldots, \mathfrak{F}_s) = (\mu_{\mathfrak{F}}^{\beta}, \nu_{\mathfrak{F}}^{\beta}) = \mathfrak{F}^{\beta}$ and q-ROFWG $(\mathfrak{F}_1, \mathfrak{F}_2, \ldots, \mathfrak{F}_s) = (\mu_{\mathfrak{F}}, \nu_{\mathfrak{F}}) = \mathfrak{F}$. Since

$$\begin{aligned}&\prod_{j=1}^{s}(1+\nu_j^q)^{\varphi_j} + \prod_{j=1}^{s}(1-\nu_j^q)^{\varphi_j} \le \sum_{j=1}^{s}(1+\nu_j^q)^{\varphi_j}\\ &\quad + \sum_{j=1}^{s}(1-\nu_j^q)^{\varphi_j} = 2,\end{aligned}$$

then from 3, we get

$$\sqrt[q]{\frac{\prod_{j=1}^{s}(1+\nu_j^q)^{\varphi_j} - \prod_{j=1}^{s}(1-\nu_j^q)^{\varphi_j}}{\prod_{j=1}^{s}(1+\nu_j^q)^{\varphi_j} + \prod_{j=1}^{s}(1-\nu_j^q)^{\varphi_j}}} \le \sqrt[q]{1 - \prod_{j=1}^{s}(1-\nu_j^q)^{\varphi_j}} \Leftrightarrow \nu_{\mathfrak{F}}^{\beta} \le \nu_{\mathfrak{F}},$$

equality holds iff $\nu_1 = \nu_2 = \ldots = \nu_s$.

Also,

$$\frac{2\prod_{j=1}^{s}(\mu_j^q)^{\varphi_j}}{\prod_{j=1}^{s}(2-\mu_j^q)^{\varphi_j} + \prod_{j=1}^{s}(\mu_j^q)^{\varphi_j}} \ge \frac{2\prod_{j=1}^{s}(\mu_j^q)^{\varphi_j}}{\sum_{j=1}^{s}\varphi_j(2-\mu_j^q) + \sum_{j=1}^{s}\varphi_j\mu_j^q} \ge \prod_{j=1}^{s}(\mu_j^q)^{\varphi_j}.$$

This implies

$$\sqrt[q]{\frac{2\prod_{j=1}^{s}(\mu_j^q)^{\varphi_j}}{\prod_{j=1}^{s}(2-\mu_j^q)^{\varphi_j} + \prod_{j=1}^{s}(\mu_j^q)^{\varphi_j}}} \ge \prod_{j=1}^{s}\mu_j^{\varphi_j} \Rightarrow \mu_{\mathfrak{F}}^{\beta} \ge \mu_{\mathfrak{F}},$$

equality holds iff $\mu_1 = \mu_2 = \ldots = \mu_s$.

Thus,

$$\mathcal{S}(\mathfrak{F}^{\beta}) = (\mu^{\beta}_{\mathfrak{F}})^q - (v^{\beta}_{\mathfrak{F}})^q \geq (\mu_{\mathfrak{F}})^q - (v_{\mathfrak{F}})^q = \mathcal{S}(\mathfrak{F}).$$

If $\mathcal{S}(\mathfrak{F}^{\beta}) > \mathcal{S}(\mathfrak{F})$, then q-ROFEWG$(\mathfrak{F}_1, \mathfrak{F}_2, \ldots, \mathfrak{F}_{\mathfrak{s}}) >$ q-ROFWA$(\mathfrak{F}_1, \mathfrak{F}_2, \ldots, \mathfrak{F}_{\mathfrak{s}})$. If $\mathcal{S}(\mathfrak{F}^{\beta}) = \mathcal{S}(\mathfrak{F})$, that is, $(\mu^{\beta}_{\mathfrak{F}})^q - (v^{\beta}_{\mathfrak{F}})^q = (\mu_{\mathfrak{F}})^q - (v_{\mathfrak{F}})^q$, then by condition $\mu^{\beta}_{\mathfrak{F}} \geq \mu_{\mathfrak{F}}$ and $v^{\beta}_{\mathfrak{F}} \leq v_{\mathfrak{F}}$; thus, the accuracy function $\mathcal{A}(\mathfrak{F}^{\beta}) = (\mu^{\beta}_{\mathfrak{F}})^q - (v^{\beta}_{\mathfrak{F}})^q = (\mu_{\mathfrak{F}})^q - (v_{\mathfrak{F}})^q = \mathcal{A}(\mathfrak{F})$.

Thus

$$q-\text{ROFEWG}(\mathfrak{F}_1, \mathfrak{F}_2, \ldots, \mathfrak{F}_{\mathfrak{s}}) = q-\text{ROFWG}(\mathfrak{F}_1, \mathfrak{F}_2, \ldots, \mathfrak{F}_{\mathfrak{s}}).$$

Hence,

$$q-\text{ROFEWG}(\mathfrak{F}_1, \mathfrak{F}_2, \ldots, \mathfrak{F}_{\mathfrak{s}}) \geq q-\text{ROFWG}(\mathfrak{F}_1, \mathfrak{F}_2, \ldots, \mathfrak{F}_{\mathfrak{s}}),$$

equality holds iff $\mathfrak{F}_1 = \mathfrak{F}_2 = \ldots = \mathfrak{F}_{\mathfrak{s}}$. □

Example 1 Let $\mathfrak{F}_1 = (0.7, 0.6)$, $\mathfrak{F}_2 = (0.8, 0.4)$, $\mathfrak{F}_3 = (0.5, 0.7)$ and $\mathfrak{F}_4 = (0.7, 0.5)$ be four q-ROFNs and $\varphi = (0.3, 0.2, 0.3, 0.2)^T$, take $q = 3$, then

$$\begin{aligned}
&q-\text{ROFEWG}(\mathfrak{F}_1, \mathfrak{F}_2, \mathfrak{F}_3, \mathfrak{F}_4)\\
&= \left\langle \frac{\sqrt[q]{2}\prod_{\mathfrak{j}=1}^{4}\mu_{\mathfrak{j}}^{\varphi_{\mathfrak{j}}}}{\sqrt[q]{\prod_{\mathfrak{j}=1}^{4}(2-\mu_{\mathfrak{j}}^{q})^{\varphi_{\mathfrak{j}}} + \prod_{\mathfrak{j}=1}^{4}(\mu_{\mathfrak{j}}^{q})^{\varphi_{\mathfrak{j}}}}}, \sqrt[q]{\frac{\prod_{\mathfrak{j}=1}^{4}(1+v_{\mathfrak{j}}^{q})^{\varphi_{\mathfrak{j}}} - \prod_{\mathfrak{j}=1}^{4}(1-v_{\mathfrak{j}}^{q})^{\varphi_{\mathfrak{j}}}}{\prod_{\mathfrak{j}=1}^{4}(1+v_{\mathfrak{j}}^{q})^{\varphi_{\mathfrak{j}}} + \prod_{\mathfrak{j}=1}^{4}(1-v_{\mathfrak{j}}^{q})^{\varphi_{\mathfrak{j}}}}} \right\rangle\\
&= \left\langle \frac{\sqrt[3]{2}\times 0.65}{\sqrt[3]{1.68+0.27}}, \sqrt[3]{\frac{1.20-0.79}{1.20+0.79}} \right\rangle\\
&= \langle 0.66, 0.59 \rangle.
\end{aligned}$$

Now,

$$\begin{aligned}
&q-\text{ROFWG}(\mathfrak{F}_1, \mathfrak{F}_2, \mathfrak{F}_3, \mathfrak{F}_4)\\
&= \left\langle \prod_{\mathfrak{j}=1}^{4}(\mu_{\mathfrak{j}})^{\varphi_{\mathfrak{j}}}, \sqrt[q]{1-\prod_{\mathfrak{j}=1}^{4}(1-v_{\mathfrak{j}}^{q})^{\varphi_{\mathfrak{j}}}} \right\rangle\\
&= \langle 0.65, 0.59 \rangle.
\end{aligned}$$

$\Rightarrow$ q-ROFEWG$(\mathfrak{F}_1, \mathfrak{F}_2, \mathfrak{F}_3, \mathfrak{F}_4) \geq$ q-ROFWG$(\mathfrak{F}_1, \mathfrak{F}_2, \mathfrak{F}_3, \mathfrak{F}_4)$.

Proposition 1 *Let $\mathfrak{F}_{\mathfrak{j}} = \langle \mu_{\mathfrak{j}}, v_{\mathfrak{j}} \rangle \in$ q-ROFNs and $\varphi_{\mathfrak{j}}$ is the weight of $\mathfrak{F}_{\mathfrak{j}}$, such that $\varphi_{\mathfrak{j}} \in [0,1]$ and $\sum_{\mathfrak{j}=1}^{\mathfrak{s}}\varphi_{\mathfrak{j}} = 1$.*

(i) Idempotency: If $\mathfrak{F}_{\mathfrak{j}} = \mathfrak{F}_o = \langle \mu_o, v_o \rangle$ for all $\mathfrak{j}$, then

$$q-ROFEWG(\mathfrak{F}_1, \mathfrak{F}_2, \ldots, \mathfrak{F}_{\mathfrak{s}}) = \mathfrak{F}_o.$$

Proof As $\mathfrak{F}_{\mathfrak{j}} = \langle \mu_o, v_o \rangle \in q$-ROFNs, $\forall\, \mathfrak{j}$, then

$$\begin{aligned}
&q-\text{ROFEWG}(\mathfrak{F}_1, \mathfrak{F}_2, \ldots, \mathfrak{F}_{\mathfrak{s}})\\
&= \left\langle \frac{\sqrt[q]{2}\prod_{\mathfrak{j}=1}^{\mathfrak{s}}\mu_o^{\varphi_{\mathfrak{j}}}}{\sqrt[q]{\prod_{\mathfrak{j}=1}^{\mathfrak{s}}(2-\mu_o^{q})^{\varphi_{\mathfrak{j}}} + \prod_{\mathfrak{j}=1}^{\mathfrak{s}}(\mu_o^{q})^{\varphi_{\mathfrak{j}}}}}, \sqrt[q]{\frac{\prod_{\mathfrak{j}=1}^{\mathfrak{s}}(1+v_o^{q})^{\varphi_{\mathfrak{j}}} - \prod_{\mathfrak{j}=1}^{\mathfrak{s}}(1-v_o^{q})^{\varphi_{\mathfrak{j}}}}{\prod_{\mathfrak{j}=1}^{\mathfrak{s}}(1+v_o^{q})^{\varphi_{\mathfrak{j}}} + \prod_{\mathfrak{j}=1}^{\mathfrak{s}}(1-v_o^{q})^{\varphi_{\mathfrak{j}}}}} \right\rangle\\
&= \left\langle \frac{\sqrt[q]{2}\mu_o^{\sum_{\mathfrak{j}=1}^{\mathfrak{s}}\varphi_{\mathfrak{j}}}}{\sqrt[q]{(2-v_o^{q})^{\sum_{\mathfrak{j}=1}^{\mathfrak{s}}\varphi_{\mathfrak{j}}} + (\mu_o^{q})^{\sum_{\mathfrak{j}=1}^{\mathfrak{s}}\varphi_{\mathfrak{j}}}}}, \sqrt[q]{\frac{(1+v_o^{q})^{\sum_{\mathfrak{j}=1}^{\mathfrak{s}}\varphi_{\mathfrak{j}}} - (1-v_o^{q})^{\sum_{\mathfrak{j}=1}^{\mathfrak{s}}\varphi_{\mathfrak{j}}}}{(1+v_o^{q})^{\sum_{\mathfrak{j}=1}^{\mathfrak{s}}\varphi_{\mathfrak{j}}} + (1-v_o^{q})^{\sum_{\mathfrak{j}=1}^{\mathfrak{s}}\varphi_{\mathfrak{j}}}}} \right\rangle\\
&= \langle \mu_o, v_o \rangle.
\end{aligned}$$

□

(ii) Boundedness: Let $\mathfrak{F}^{-} = (\min_{\mathfrak{j}}(\mu_{\mathfrak{j}}), \max_{\mathfrak{j}}(v_{\mathfrak{j}}))$, $\mathfrak{F}^{+} = (\max_{\mathfrak{j}}(\mu_{\mathfrak{j}}), \min_{\mathfrak{j}}(v_{\mathfrak{j}}))$, then

$$\mathfrak{F}^{-} \leq q-\text{ROFEWG}(\mathfrak{F}_1, \mathfrak{F}_2, \ldots, \mathfrak{F}_{\mathfrak{s}}) \leq \mathfrak{F}^{+}.$$

Proof Consider $f(x) = \frac{1-x}{1+x}, x \in [0,1]$, then $f'(x) = -\frac{2}{(1+x)^2} < 0$, so $f(x)$ is a decreasing function (DF). As $v^q_{\mathfrak{j},min} \leq v^q_{\mathfrak{j}} \leq v^q_{\mathfrak{j},max}, \forall\, \mathfrak{j} = 1, 2, \ldots, \mathfrak{s}$, then $f(v^q_{\mathfrak{j},max}) \leq f(v^q_{\mathfrak{j}}) \leq f(v^q_{\mathfrak{j},min}), \forall\, \mathfrak{j}$, that is,

$$\frac{1-v^q_{\mathfrak{j},max}}{1+v^q_{\mathfrak{j},max}} \leq \frac{1-v^q_{\mathfrak{j}}}{1+v^q_{\mathfrak{j}}} \leq \frac{1-v^q_{\mathfrak{j},min}}{1+v^q_{\mathfrak{j},min}}$$

, $\forall\, \mathfrak{j}$. Let $\varphi_{\mathfrak{j}} \in [0,1]$ and $\sum_{\mathfrak{j}=1}^{\mathfrak{s}}\varphi_{\mathfrak{j}} = 1$, then

$$\left(\frac{1-v_{\mathrm{j,max}}^q}{1+v_{\mathrm{j,max}}^q}\right)^{\varphi_{\mathrm{j}}} \le \left(\frac{1-v_{\mathrm{j}}^q}{1+v_{\mathrm{j}}^q}\right)^{\varphi_{\mathrm{j}}} \le \left(\frac{1-v_{\mathrm{j,min}}^q}{1+v_{\mathrm{j,min}}^q}\right)^{\varphi_{\mathrm{j}}}$$

$$\prod_{\mathrm{j}=1}^{\mathfrak{s}}\left(\frac{1-v_{\mathrm{j},max}^q}{1+v_{\mathrm{j},max}^q}\right)^{\varphi_{\mathrm{j}}} \le \prod_{\mathrm{j}=1}^{\mathfrak{s}}\left(\frac{1-v_{\mathrm{j}}^q}{1+v_{\mathrm{j}}^q}\right)^{\varphi_{\mathrm{j}}} \le \prod_{\mathrm{j}=1}^{\mathfrak{s}}\left(\frac{1-v_{\mathrm{j,min}}^q}{1+v_{\mathrm{j,min}}^q}\right)^{\varphi_{\mathrm{j}}}$$

$$\Leftrightarrow \left(\frac{1-v_{\mathrm{j,max}}^q}{1+v_{\mathrm{j,max}}^q}\right)^{\sum_{\mathrm{j}=1}^{\mathfrak{s}}\varphi_{\mathrm{j}}} \le \prod_{\mathrm{j}=1}^{\mathfrak{s}}\left(\frac{1-v_{\mathrm{j}}^q}{1+v_{\mathrm{j}}^q}\right)^{\varphi_{\mathrm{j}}} \le \left(\frac{1-v_{\mathrm{j},min}^q}{1+v_{\mathrm{j},min}^q}\right)^{\sum_{\mathrm{j}=1}^{\mathfrak{s}}\varphi_{\mathrm{j}}}$$

$$\Leftrightarrow \left(\frac{1-v_{\mathrm{j,max}}^q}{1+v_{\mathrm{j,max}}^q}\right) \le \prod_{\mathrm{j}=1}^{\mathfrak{s}}\left(\frac{1-v_{\mathrm{j}}^q}{1+v_{\mathrm{j}}^q}\right)^{\varphi_{\mathrm{j}}} \le \left(\frac{1-v_{\mathrm{j,min}}^q}{1+v_{\mathrm{j,min}}^q}\right)$$

$$\Leftrightarrow \left(\frac{2}{1+v_{\mathrm{j,max}}^q}\right) \le 1+\prod_{\mathrm{j}=1}^{\mathfrak{s}}\left(\frac{1-v_{\mathrm{j}}^q}{1+v_{\mathrm{j}}^q}\right)^{\varphi_{\mathrm{j}}} \le \left(\frac{2}{1+v_{\mathrm{j,min}}^q}\right)$$

$$\Leftrightarrow \left(\frac{1+v_{\mathrm{j,min}}^q}{2}\right) \le \frac{1}{1+\prod_{\mathrm{j}=1}^{\mathfrak{s}}\left(\frac{1-v_{\mathrm{j}}^q}{1+v_{\mathrm{j}}^q}\right)^{\varphi_{\mathrm{j}}}} \le \left(\frac{1+v_{\mathrm{j,max}}^q}{2}\right)$$

$$\Leftrightarrow \left(1+v_{\mathrm{j,min}}^q\right) \le \frac{2}{1+\prod_{\mathrm{j}=1}^{\mathfrak{s}}\left(\frac{1-v_{\mathrm{j}}^q}{1+v_{\mathrm{j}}^q}\right)^{\varphi_{\mathrm{j}}}} \le \left(1+v_{\mathrm{j,max}}^q\right)$$

$$\Leftrightarrow v_{\mathrm{j,min}}^q \le \frac{2}{1+\prod_{\mathrm{j}=1}^{\mathfrak{s}}\left(\frac{1-v_{\mathrm{j}}^q}{1+v_{\mathrm{j}}^q}\right)^{\varphi_{\mathrm{j}}}} - 1 \le v_{\mathrm{j,max}}^q$$

$$\Leftrightarrow v_{\mathrm{j,min}}^q \le \frac{\prod_{\mathrm{j}=1}^{\mathfrak{s}}(1+v_{\mathrm{j}}^q)^{\varphi_{\mathrm{j}}} - \prod_{\mathrm{j}=1}^{\mathfrak{s}}(1-v_{\mathrm{j}}^q)^{\varphi_{\mathrm{j}}}}{\prod_{\mathrm{j}=1}^{\mathfrak{s}}(1+v_{\mathrm{j}}^q)^{\varphi_{\mathrm{j}}} + \prod_{\mathrm{j}=1}^{\mathfrak{s}}(1-v_{\mathrm{j}}^q)^{\varphi_{\mathrm{j}}}} \le v_{\mathrm{j,max}}^q$$

Thus,

$$v_{\mathrm{j,min}} \le \sqrt[q]{\frac{\prod_{\mathrm{j}=1}^{\mathfrak{s}}(1+v_{\mathrm{j}}^q)^{\varphi_{\mathrm{j}}} - \prod_{\mathrm{j}=1}^{\mathfrak{s}}(1-v_{\mathrm{j}}^q)^{\varphi_{\mathrm{j}}}}{\prod_{\mathrm{j}=1}^{\mathfrak{s}}(1+v_{\mathrm{j}}^q)^{\varphi_{\mathrm{j}}} + \prod_{\mathrm{j}=1}^{\mathfrak{s}}(1-v_{\mathrm{j}}^q)^{\varphi_{\mathrm{j}}}}} \le v_{\mathrm{j,max}}. \qquad (4)$$

Consider $g(y) = \frac{2-y}{y}, y \in (0,1]$, then $g'(y) = -\frac{2}{y^2}$, i.e., $g(y)$ is a DF on (0, 1]. Since $\mu_{\mathrm{j,min}}^q \le \mu_{\mathrm{j}}^q \le \mu_{\mathrm{j,max}}^q, \forall \mathrm{j}$, then $g(\mu_{\mathrm{j,max}}^q) \le g(\mu_{\mathrm{j}}^q) \le g(\mu_{\mathrm{j,min}}^q), \forall \mathrm{j}$, i.e.,

$$\frac{2-\mu_{\mathrm{j,max}}^q}{\mu_{\mathrm{j,max}}^q} \le \frac{2-\mu_{\mathrm{j}}^q}{\mu_{\mathrm{j}}^q} \le \frac{2-\mu_{\mathrm{j,min}}^q}{\mu_{\mathrm{j,min}}^q}.$$

Then,

$$\left(\frac{2-\mu_{\mathrm{j},max}^q}{\mu_{\mathrm{j,max}}^q}\right)^{\varphi_{\mathrm{j}}} \le \left(\frac{2-\mu_{\mathrm{j}}^q}{\mu_{\mathrm{j}}^q}\right)^{\varphi_{\mathrm{j}}} \le \left(\frac{2-\mu_{\mathrm{j,min}}^q}{\mu_{\mathrm{j,min}}^q}\right)^{\varphi_{\mathrm{j}}}$$

$$\prod_{\mathrm{j}=1}^{\mathfrak{s}}\left(\frac{2-\mu_{\mathrm{j,max}}^q}{\mu_{\mathrm{j,max}}^q}\right)^{\varphi_{\mathrm{j}}} \le \prod_{\mathrm{j}=1}^{\mathfrak{s}}\left(\frac{2-\mu_{\mathrm{j}}^q}{\mu_{\mathrm{j}}^q}\right)^{\varphi_{\mathrm{j}}} \le \prod_{\mathrm{j}=1}^{\mathfrak{s}}\left(\frac{2-\mu_{\mathrm{j,min}}^q}{\mu_{\mathrm{j,min}}^q}\right)^{\varphi_{\mathrm{j}}}$$

$$\Rightarrow \left(\frac{2-\mu_{\mathrm{j},max}^q}{\mu_{\mathrm{j,max}}^q}\right)^{\sum_{\mathrm{j}=1}^{\mathfrak{s}}\varphi_{\mathrm{j}}} \le \prod_{\mathrm{j}=1}^{\mathfrak{s}}\left(\frac{2-\mu_{\mathrm{j}}^q}{\mu_{\mathrm{j}}^q}\right)^{\varphi_{\mathrm{j}}} \le \left(\frac{2-\mu_{\mathrm{j,min}}^q}{\mu_{\mathrm{j,min}}^q}\right)^{\sum_{\mathrm{j}=1}^{\mathfrak{s}}\varphi_{\mathrm{j}}}$$

$$\Rightarrow \left(\frac{2-\mu_{\mathrm{j},max}^q}{\mu_{\mathrm{j,max}}^q}\right) \le \prod_{\mathrm{j}=1}^{\mathfrak{s}}\left(\frac{2-\mu_{\mathrm{j}}^q}{\mu_{\mathrm{j}}^q}\right)^{\varphi_{\mathrm{j}}} \le \left(\frac{2-\mu_{\mathrm{j},min}^q}{\mu_{\mathrm{j,min}}^q}\right)$$

$$\Rightarrow \left(\frac{2}{\mu_{\mathrm{j,max}}^q}\right) \le 1+\prod_{\mathrm{j}=1}^{\mathfrak{s}}\left(\frac{2-\mu_{\mathrm{j}}^q}{\mu_{\mathrm{j}}^q}\right)^{\varphi_{\mathrm{j}}} \le \left(\frac{2}{\mu_{\mathrm{j,min}}^q}\right)$$

$$\Rightarrow \left(\frac{\mu_{\mathrm{j,min}}^q}{2}\right) \le \frac{1}{1+\prod_{\mathrm{j}=1}^{\mathfrak{s}}\left(\frac{2-\mu_{\mathrm{j}}^q}{\mu_{\mathrm{j}}^q}\right)^{\varphi_{\mathrm{j}}}} \le \left(\frac{\mu_{\mathrm{j,max}}^q}{2}\right)$$

$$\Rightarrow \left(\mu_{\mathrm{j},min}^q\right) \le \frac{2}{1+\prod_{\mathrm{j}=1}^{\mathfrak{s}}\left(\frac{2-\mu_{\mathrm{j}}^q}{\mu_{\mathrm{j}}^q}\right)^{\varphi_{\mathrm{j}}}} \le \left(\mu_{\mathrm{j,max}}^q\right)$$

$$\Rightarrow \mu_{\mathrm{j,min}}^q \le \frac{2}{1+\prod_{\mathrm{j}=1}^{\mathfrak{s}}\left(\frac{2-\mu_{\mathrm{j}}^q}{\mu_{\mathrm{j}}^q}\right)^{\varphi_{\mathrm{j}}}} \le \mu_{\mathrm{j,max}}^q$$

$$\Rightarrow \mu_{\mathrm{j,min}}^q \le \frac{2\prod_{\mathrm{j}=1}^{\mathfrak{s}}(\mu_{\mathrm{j}}^q)^{\varphi_{\mathrm{j}}}}{\prod_{\mathrm{j}=1}^{\mathfrak{s}}(2-\mu_{\mathrm{j}}^q)^{\varphi_{\mathrm{j}}} + \prod_{\mathrm{j}=1}^{\mathfrak{s}}(\mu_{\mathrm{j}}^q)^{\varphi_{\mathrm{j}}}} \le \mu_{\mathrm{j,max}}^q$$

$$\Rightarrow \mu_{\mathrm{j,min}} \le \frac{\sqrt[q]{2}\prod_{\mathrm{j}=1}^{\mathfrak{s}}\mu_{\mathrm{j}}^{\varphi_{\mathrm{j}}}}{\sqrt[q]{\prod_{\mathrm{j}=1}^{\mathfrak{s}}(2-\mu_{\mathrm{j}}^q)^{\varphi_{\mathrm{j}}} + \prod_{\mathrm{j}=1}^{\mathfrak{s}}(\mu_{\mathrm{j}}^q)^{\varphi_{\mathrm{j}}}}} \le \mu_{\mathrm{j,max}}. \qquad (5)$$

Let q-ROFEWG$(\mathfrak{F}_1, \mathfrak{F}_2, \ldots, \mathfrak{F}_{\mathfrak{s}}) = \mathfrak{F} = \langle \mu_{\mathfrak{F}}, v_{\mathfrak{F}} \rangle$, then from (4) and (5),

$$\mu_{\min} \le \mu_{\mathfrak{F}} \le \mu_{\max}, \;\; v_{\min} \le v_{\mathfrak{F}} \le v_{\max},$$

where $\mu_{\min} = \min_{\mathrm{j}}\{\mu_{\mathrm{j}}\}$, $\mu_{\max} = \max_{\mathrm{j}}\{\mu_{\mathrm{j}}\}$, $v_{\min} = \min_{\mathrm{j}}\{v_{\mathrm{j}}\}$, and $v_{\max} = \max_{\mathrm{j}}\{v_{\mathrm{j}}\}$. So, $\mathcal{S}(\mathfrak{F}) = \mu_{\mathfrak{F}}^q - v_{\mathfrak{F}}^q \le \mu_{\max}^q - v_{\min}^q = \mathcal{S}(\mathfrak{F}^+)$ and $\mathcal{S}(\mathfrak{F}) = \mu_{\mathfrak{F}}^q - v_{\mathfrak{F}}^q \ge \mu_{\min}^q - v_{\max}^q = \mathcal{S}(\mathfrak{F}^-)$. As $S(\mathfrak{F}) < S(\mathfrak{F}^+)$ and $S(\mathfrak{F}) > S(\mathfrak{F}^-)$. So

$$\mathfrak{F}^- \le q - \mathrm{ROFEWG}(\mathfrak{F}_1, \mathfrak{F}_2, \ldots, \mathfrak{F}_{\mathfrak{s}}) \le \mathfrak{F}^+.$$

□

(iii) Monotonicity: When $\mathfrak{F}_{\mathrm{j}} \le \mathcal{P}_{\mathrm{j}}, \forall \mathrm{j}$, then

$$q - \mathrm{ROFEWG}(\mathfrak{F}_1, \mathfrak{F}_2, \ldots, \mathfrak{F}_{\mathfrak{s}}) \le q - \mathrm{ROFEWG}(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_{\mathfrak{s}}).$$

Proof It is similar to (ii), so we omit it. □

(iv) If $\eta = \langle \mu_\eta, v_\eta \rangle$ is a q-ROFN, then

$$q-\text{ROFEWG}(\eta\otimes_\epsilon\mathfrak{F}_1,\eta\otimes_\epsilon\mathfrak{F}_2,\ldots,\eta\otimes_\epsilon\mathfrak{F}_\mathfrak{s}) = \eta\otimes q-ROFEWG(\mathfrak{F}_1,\mathfrak{F}_2,\ldots,\mathfrak{F}_\mathfrak{s}).$$

Proof As R_i and η are q-ROFNs, $\forall\, \mathfrak{j}$,

$$\eta\otimes_\epsilon R_\mathfrak{j} = \left\langle \frac{\sqrt[q]{2}\mu_{R_\mathfrak{j}}\cdot_\epsilon\mu_\eta}{\sqrt[q]{(2-\mu^q_{R_\mathfrak{j}})\cdot_\epsilon(2-\mu^q_\eta)+\mu^q_{R_\mathfrak{j}}\cdot_\epsilon\mu^q_\eta}}, \sqrt{\frac{(1+\nu^q_{R_\mathfrak{j}})\cdot_\epsilon(1+\nu^q_\eta)-(1-\nu^q_{R_\mathfrak{j}})\cdot_\epsilon(1-\nu^q_\eta)}{(1+\nu^q_{R_\mathfrak{j}})\cdot_\epsilon(1+\nu^q_\eta)+(1-\nu^q_{R_\mathfrak{j}})\cdot_\epsilon(1-\nu^q_\eta)}}\right\rangle$$

Therefore, by Theorem 5, $q-\text{ROFEWG}(\eta\otimes_\epsilon\mathfrak{F}_1,\eta\otimes_\epsilon\mathfrak{F}_2,\ldots,\eta\otimes_\epsilon\mathfrak{F}_\mathfrak{s})$

$$=\left\langle \sqrt[q]{\frac{2\prod_{\mathfrak{j}=1}^{\mathfrak{s}}(\alpha)^{\varphi_\mathfrak{j}}}{\prod_{\mathfrak{j}=1}^{\mathfrak{s}}(2-\alpha)^{\varphi_\mathfrak{j}}+\prod_{\mathfrak{j}=1}^{\mathfrak{s}}(\alpha)^{\varphi_\mathfrak{j}}}}, \frac{\sqrt[q]{\prod_{\mathfrak{j}=1}^{\mathfrak{s}}(1+\beta)^{\varphi_\mathfrak{j}}-\prod_{\mathfrak{j}=1}^{\mathfrak{s}}(1-\beta)^{\varphi_\mathfrak{j}}}}{\sqrt[q]{\prod_{\mathfrak{j}=1}^{\mathfrak{s}}(1+\beta)^{\varphi_\mathfrak{j}}+\prod_{\mathfrak{j}=1}^{\mathfrak{s}}(1-\beta)^{\varphi_\mathfrak{j}}}}\right\rangle$$

$$=\left\langle \sqrt[q]{\frac{2\prod_{\mathfrak{j}=1}^{\mathfrak{s}}\left(\mu^q_{R_\mathfrak{j}}\mu^q_\eta\right)^{\varphi_\mathfrak{j}}}{\prod_{\mathfrak{j}=1}^{\mathfrak{s}}\left((2-\mu^q_{R_\mathfrak{j}})\cdot_\epsilon(2-\mu^q_\eta)\right)^{\varphi_\mathfrak{j}}+\prod_{\mathfrak{j}=1}^{\mathfrak{s}}\left(\mu^q_{R_\mathfrak{j}}\mu^q_\eta\right)^{\varphi_\mathfrak{j}}}}, \sqrt[q]{\frac{\prod_{\mathfrak{j}=1}^{\mathfrak{s}}\left((1+\nu^q_{R_\mathfrak{j}})\cdot_\epsilon(1+\nu^q_\eta)\right)^{\varphi_\mathfrak{j}}-\prod_{\mathfrak{j}=1}^{\mathfrak{s}}\left((1+\nu^q_{R_\mathfrak{j}})\cdot_\epsilon(1+\nu^q_\eta)\right)^{\varphi_\mathfrak{j}}}{\prod_{\mathfrak{j}=1}^{\mathfrak{s}}\left((1+\nu^q_{R_\mathfrak{j}})\cdot_\epsilon(1+\nu^q_\eta)\right)^{\varphi_\mathfrak{j}}+\prod_{\mathfrak{j}=1}^{\mathfrak{s}}\left((1+\nu^q_{R_\mathfrak{j}})\cdot_\epsilon(1+\nu^q_\eta)\right)^{\varphi_\mathfrak{j}}}}\right\rangle$$

$$=\left\langle \frac{\sqrt[q]{2}\prod_{\mathfrak{j}=1}^{\mathfrak{s}}(\mu_{R_\mathfrak{j}})^{\varphi_\mathfrak{j}}\cdot_\epsilon\mu_\eta}{\sqrt[q]{\prod_{\mathfrak{j}=1}^{\mathfrak{s}}(2-\mu^q_{R_\mathfrak{j}})^{\varphi_\mathfrak{j}}\cdot_\epsilon(2-\mu^q_\eta)+\prod_{\mathfrak{j}=1}^{\mathfrak{s}}(\mu^q_{R_\mathfrak{j}})^{\varphi_\mathfrak{j}}\cdot_\epsilon\mu^q_\eta}}, \sqrt[q]{\frac{\prod_{\mathfrak{j}=1}^{\mathfrak{s}}(1+\nu^q_{R_\mathfrak{j}})^{\varphi_\mathfrak{j}}\cdot_\epsilon(1+\nu^q_\eta)-\prod_{\mathfrak{j}=1}^{\mathfrak{s}}(1+\nu^q_{R_\mathfrak{j}})^{\varphi_\mathfrak{j}}\cdot_\epsilon(1+\nu^q_\eta)}{\prod_{\mathfrak{j}=1}^{\mathfrak{s}}(1+\nu^q_{R_\mathfrak{j}})^{\varphi_\mathfrak{j}}\cdot_\epsilon(1+\nu^q_\eta)+\prod_{\mathfrak{j}=1}^{\mathfrak{s}}(1+\nu^q_{R_\mathfrak{j}})^{\varphi_\mathfrak{j}}\cdot_\epsilon(1+\nu^q_\eta)}}\right\rangle$$

$$=\langle\mu_\eta,\nu_\eta\rangle\otimes q-\text{ROFEWG}(\mathfrak{F}_1,\mathfrak{F}_2,\ldots,\mathfrak{F}_\mathfrak{s})$$
$$=\eta\otimes q-\text{ROFEWG}(\mathfrak{F}_1,\mathfrak{F}_2,\ldots,\mathfrak{F}_\mathfrak{s}).$$

where

$$\alpha=\frac{2\mu^q_{R_i}\cdot\mu^q_\beta}{(2-\mu^q_{R_\mathfrak{j}})\cdot_\epsilon(2-\mu^q_\beta\cdot)+\mu^q_{R_i}\cdot\mu^q_\beta},$$

$$\beta=\frac{(1+\nu^q_{R_\mathfrak{j}})\cdot_\epsilon(1+\nu^q_\eta)-(1-\nu^q_{R_\mathfrak{j}})\cdot_\epsilon(1-\nu^q_\eta)}{(1+\nu^q_{R_\mathfrak{j}})\cdot_\epsilon(1+\nu^q_\eta)+(1-\nu^q_{R_\mathfrak{j}})\cdot_\epsilon(1-\nu^q_\eta)}.$$

Hence,

$$q-\text{ROFEWG}(\eta\otimes_\epsilon\mathfrak{F}_1,\eta\otimes_\epsilon\mathfrak{F}_2,\ldots,\eta\otimes_\epsilon\mathfrak{F}_\mathfrak{s}) = \eta\otimes q-\text{ROFEWG}(\mathfrak{F}_1,\mathfrak{F}_2,\ldots,\mathfrak{F}_\mathfrak{s}).$$

□

5 *q*-Rung orthopair fuzzy Einstein ordered weighted geometric operators

Definition 8 Let $\mathfrak{F}_\mathfrak{j}=\langle\mu_\mathfrak{j},\nu_\mathfrak{j}\rangle\in q$-ROFNs and $\varphi_\mathfrak{j}$ is the weight of $\mathfrak{F}_\mathfrak{j}$ with $\varphi_\mathfrak{j}>0$ and $\sum_{\mathfrak{j}=1}^{\mathfrak{s}}\varphi_\mathfrak{j}=1$, then q-ROFEOWG operator is a mapping $\mathcal{Q}^\mathfrak{s}\to\mathcal{Q}$ such that

$$q-\text{ROFEOWG}(\mathfrak{F}_1,\mathfrak{F}_2,\ldots,\mathfrak{F}_\mathfrak{s})=\mathfrak{F}^{\varphi_1}_{\varrho(1)}\otimes_\epsilon\mathfrak{F}^{\varphi_2}_{\varrho(2)}\otimes_\epsilon\ldots\otimes_\epsilon\mathfrak{F}^{\varphi_\mathfrak{s}}_{\varrho(\mathfrak{s})},$$

where $(\varrho(1),\varrho(2),\ldots,\varrho(\mathfrak{s}))$ is the permutation of $(\mathfrak{j}=1,2,\ldots,\mathfrak{s})$ such that $\mathfrak{F}_{\varrho(\mathfrak{j}-1)}\geq\mathfrak{F}_{\varrho(\mathfrak{j})},\forall\,\mathfrak{j}=1,2,\ldots,\mathfrak{s}$.

Theorem 7 *Let $\mathfrak{F}_\mathfrak{j}=(\mu_\mathfrak{j},\nu_\mathfrak{j})\in q$-ROFNs, then aggregated value using q-ROFEOWG is a q-ROFN and*

$$q-ROFEOWG(\mathfrak{F}_1,\mathfrak{F}_2,\ldots,\mathfrak{F}_\mathfrak{s}) =\left\langle \frac{\sqrt[q]{2}\prod_{\mathfrak{j}=1}^{\mathfrak{s}}\mu^{\varphi_\mathfrak{j}}_{\varrho(\mathfrak{j})}}{\sqrt[q]{\prod_{\mathfrak{j}=1}^{\mathfrak{s}}(2-\mu^q_{\varrho(\mathfrak{j})})^{\varphi_\mathfrak{j}}+\prod_{\mathfrak{j}=1}^{\mathfrak{s}}(\mu^q_{\varrho(\mathfrak{j})})^{\varphi_\mathfrak{j}}}}, \sqrt[q]{\frac{\prod_{\mathfrak{j}=1}^{\mathfrak{s}}(1+\nu^q_{\varrho(\mathfrak{j})})^{\varphi_\mathfrak{j}}-\prod_{\mathfrak{j}=1}^{\mathfrak{s}}(1-\nu^q_{\varrho(\mathfrak{j})})^{\varphi_\mathfrak{j}}}{\prod_{\mathfrak{j}=1}^{\mathfrak{s}}(1+\nu^q_{\varrho(\mathfrak{j})})^{\varphi_\mathfrak{j}}+\prod_{\mathfrak{j}=1}^{\mathfrak{s}}(1-\nu^q_{\varrho(\mathfrak{j})})^{\varphi_\mathfrak{j}}}}\right\rangle. \quad (6)$$

Proof Using the similar arguments as used in Theorem 5, we can prove it. □

We give some properties without their proofs.

Corollary 2 *The q-ROFEOWG and q-ROFOWG operators have the relation*:

$$q-\text{ROFEOWG}(\mathfrak{F}_1,\mathfrak{F}_2,\ldots,\mathfrak{F}_\mathfrak{s})\geq q-\text{ROFOWG}(\mathfrak{F}_1,\mathfrak{F}_2,\ldots,\mathfrak{F}_\mathfrak{s}).$$

Example 2 Let $\mathfrak{F}_1=(0.6,0.7)$, $\mathfrak{F}_2=(0.8,0.7)$, $\mathfrak{F}_3=(0.6,0.9)$ and $\mathfrak{F}_4=(0.9,0.4)$ be four q-ROFNs and $\varphi=(0.3,0.3,0.2,0.2)^T$, take $q=3$. As

$$\mathcal{S}(\mathfrak{F}_1)=(0.6)^3-(0.7)^3=-0.13,$$
$$\mathcal{S}(\mathfrak{F}_2)=(0.8)^3-(0.7)^3=0.17,$$
$$\mathcal{S}(\mathfrak{F}_3)=(0.6)^3-(0.9)^3=-0.51,$$
$$\mathcal{S}(\mathfrak{F}_4)=(0.9)^3-(0.4)^3=0.67.$$

Since $\mathcal{S}(\mathfrak{F}_4)>\mathcal{S}(\mathfrak{F}_2)>\mathcal{S}(\mathfrak{F}_1)>\mathcal{S}(\mathfrak{F}_3)$, therefore

$$\mathfrak{F}_{\varrho(1)}=\mathfrak{F}_4=(0.9,0.4),$$
$$\mathfrak{F}_{\varrho(2)}=\mathfrak{F}_2=(0.8,0.7),$$
$$\mathfrak{F}_{\varrho(3)}=\mathfrak{F}_1=(0.6,0.7),$$
$$\mathfrak{F}_{\varrho(4)}=\mathfrak{F}_3=(0.6,0.9).$$

By q-ROFEOWG operator, we get

$$q-\text{ROFEOWG}(\mathfrak{F}_1,\mathfrak{F}_2,\mathfrak{F}_3,\mathfrak{F}_4)$$
$$=\left\langle \frac{\sqrt[q]{2}\prod_{j=1}^{4}\mu_{\varrho(j)}^{\varphi_j}}{\sqrt[q]{\prod_{j=1}^{4}(2-\mu_{\varrho(j)}^{q})^{\varphi_j}+\prod_{j=1}^{4}(\mu_{\varrho(j)}^{q})^{\varphi_j}}},\right.$$
$$\left.\sqrt[q]{\frac{\prod_{j=1}^{4}(1+v_{\varrho(j)}^{q})^{\varphi_j}-\prod_{j=1}^{4}(1-v_{\varrho(j)}^{q})^{\varphi_j}}{\prod_{j=1}^{4}(1+v_{\varrho(j)}^{q})^{\varphi_j}+\prod_{j=1}^{4}(1-v_{\varrho(j)}^{q})^{\varphi_j}}}\right\rangle$$
$$=\left\langle \frac{\sqrt[3]{2}\times 0.72}{\sqrt[3]{1.55+0.37}},\sqrt[3]{\frac{1.32-0.61}{1.32+0.61}}\right\rangle$$
$$=\langle 0.73,0.72\rangle.$$

Now,

$$q-\text{ROFOWG}(\mathfrak{F}_1,\mathfrak{F}_2,\mathfrak{F}_3,\mathfrak{F}_4)$$
$$=\left\langle \prod_{j=1}^{4}(\mu_{\varrho(j)})^{\varphi_j},\sqrt[q]{1-\prod_{j=1}^{4}(1-v_{\varrho(j)}^{q})^{\varphi_j}}\right\rangle$$
$$=\langle 0.72,0.73\rangle.$$

$\Rightarrow q\text{-ROFEOWG}(\mathfrak{F}_1,\mathfrak{F}_2,\mathfrak{F}_3,\mathfrak{F}_4) > q\text{-ROFOWG}(\mathfrak{F}_1,\mathfrak{F}_2,\mathfrak{F}_3,\mathfrak{F}_4)$.

Proposition 2 *Let $\mathfrak{F}_j=\langle\mu_j,v_j\rangle\in$ q-ROFNs and φ_j is the WV of $\mathfrak{F}_j$, such that $\varphi_j\in[0,1]$ and $\sum_{j=1}^{\mathfrak{s}}\varphi_j=1$.*

(i) Idempotency: If $\mathfrak{F}_j=\mathfrak{F}_o=\langle\mu_o,v_o\rangle$, $\forall$ j, then

$$q-ROFEOWG(\mathfrak{F}_1,\mathfrak{F}_2,\ldots,\mathfrak{F}_{\mathfrak{s}})=\mathfrak{F}_o.$$

(ii) *Boundedness: Let $\mathfrak{F}^-=(min_j(\mu_j),max_j(v_j))$, $\mathfrak{F}^+=(max_j(\mu_j),min_j(v_j))$, then*

$$\mathfrak{F}^-\leq q-ROFEOWG(\mathfrak{F}_1,\mathfrak{F}_2,\ldots,\mathfrak{F}_{\mathfrak{s}})\leq\mathfrak{F}^+.$$

(iii) *Monotonicity: When $\mathfrak{F}_j\leq\mathcal{P}_j,\forall$ j, then*

$$q-ROFEOWG(\mathfrak{F}_1,\mathfrak{F}_2,\ldots,\mathfrak{F}_{\mathfrak{s}})\leq q-ROFEOWG(\mathcal{P}_1,\mathcal{P}_2,\ldots,\mathcal{P}_{\mathfrak{s}}).$$

(iv) *Shift invariance: If $\eta=\langle\mu_\eta,v_\eta\rangle$ is a q-ROFN, then*

$$q-ROFEOWG(\eta\otimes_\epsilon\mathfrak{F}_1,\eta\otimes_\epsilon\mathfrak{F}_2,\ldots,\eta\otimes_\epsilon\mathfrak{F}_{\mathfrak{s}})$$
$$=\eta\otimes q-ROFEOWG(\mathfrak{F}_1,\mathfrak{F}_2,\ldots,\mathfrak{F}_{\mathfrak{s}}).$$

6 Generalized *q*-rung orthopair fuzzy Einstein weighted geometric operators

Definition 9 Let $\mathfrak{F}_j=\langle\mu_j,v_j\rangle\in q$-ROFNs and φ_j is the WV of $\mathfrak{F}_j$ with $\varphi_j>0$ and $\sum_{j=1}^{\mathfrak{s}}\varphi_j=1$, then G*q*-ROFEWG operator is a mapping $\mathcal{Q}^{\mathfrak{s}}\rightarrow\mathcal{Q}$ such that

$$Gq-\text{ROFEWG}(\mathfrak{F}_1,\mathfrak{F}_2,\ldots,\mathfrak{F}_{\mathfrak{s}})=\frac{1}{\gamma}\cdot_\epsilon\left(\bigotimes_{j=1}^{\mathfrak{s}}(\gamma\cdot_\epsilon\mathfrak{F}_j)^{\varphi_j}\right),$$

where $\gamma>0$.

Particularly,

- If $\gamma=1$, then G*q*-ROFEWG becomes *q*-ROFEWG.
- If $\varphi=(1/\mathfrak{s},1/\mathfrak{s},\ldots,1/\mathfrak{s})$, then G*q*-ROFEWG

$$(\mathfrak{F}_1,\mathfrak{F}_2,\ldots,\mathfrak{F}_{\mathfrak{s}})=\frac{1}{\gamma}\cdot_\epsilon\left(\bigotimes_{j=1}^{\mathfrak{s}}(\gamma\mathfrak{F}_j)^{1/\mathfrak{s}}\right).$$

Theorem 8 *Let $\mathfrak{F}_j=\langle\mu_j,v_j\rangle\in$ q-ROFNs and φ_j is the WV of $\mathfrak{F}_j$ with $\varphi_j>0$ and $\sum_{j=1}^{\mathfrak{s}}\varphi_j=1$, then* G*q-ROFEWG operator is a q-ROFN and*

$$Gq-ROFEWG(\mathfrak{F}_1,\mathfrak{F}_2,\ldots,\mathfrak{F}_{\mathfrak{s}})=$$
$$\left\langle\sqrt[q]{\frac{\left(\prod_{j=1}^{\mathfrak{s}}\{(1+\mu_j^q)^\gamma+3\omega_\gamma\}^{\varphi_j}+3\prod_{j=1}^{\mathfrak{s}}\{(1+\mu_j^q)^\gamma-\omega_\gamma\}^{\varphi_j}\right)^{1/\gamma}-\left(\prod_{j=1}^{\mathfrak{s}}\{(1+\mu_j^q)^\gamma+3\omega_\gamma\}^{\varphi_j}-\prod_{j=1}s\{(1+\mu_j^q)^\gamma-\omega_\gamma\}^{\varphi_j}\right)^{1/\gamma}}{\left(\prod_{j=1}^{\mathfrak{s}}\{(1+\mu_j^q)^\gamma+3\omega_\gamma\}^{\varphi_j}+3\prod_{j=1}^{\mathfrak{s}}\{(1+\mu_j^q)^\gamma-\omega_\gamma\}^{\varphi_j}\right)^{1/\gamma}+\left(\prod_{j=1}^{\mathfrak{s}}\{(1+\mu_j^q)^\gamma+3\omega_\gamma\}^{\varphi_j}-\prod_{j=1}s\{(1+\mu_j^q)^\gamma-\omega_\gamma\}^{\varphi_j}\right)^{1/\gamma}}},\right.$$
$$\left.\frac{\sqrt[q]{2}\left\{\prod_{j=1}^{\mathfrak{s}}\{(2-v_j^q)^\gamma+3(v_j^q)^\gamma\}^{\varphi_j}-\prod_{j=1}^{\mathfrak{s}}\{(2-v_j^q)^\gamma-(v_j^q)^\gamma\}^{\varphi_j}\right\}^{1/q\gamma}}{\sqrt[q]{\left(\prod_{j=1}^{\mathfrak{s}}\{(2-v_j^q)^\gamma+3(v_j^q)^\gamma\}^{\varphi_j}+3\prod_{j=1}^{\mathfrak{s}}\{(2-v_j^q)^\gamma-(v_j^q)^\gamma\}^{\varphi_j}\right)^{1/\gamma}+\left(\prod_{j=1}^{\mathfrak{s}}\{(2-v_j^q)^\gamma+3(v_j^q)^\gamma\}^{\varphi_j}-\prod_{j=1}^{\mathfrak{s}}\{(2-v_j^q)^\gamma-(v_j^q)^\gamma\}^{\varphi_j}\right)^{1/\gamma}}}\right\rangle.$$

where $\omega_\gamma=(1-\mu_j{}^q)^\gamma$.

Proof Since

$$\gamma\cdot_\epsilon\mathfrak{F}_j=\left\langle\sqrt[q]{\frac{(1+\mu_j^q)^\gamma-(1-\mu_j^q)^\gamma}{(1+\mu_j^q)^\gamma+(1-\mu_j^q)^\gamma}},\frac{\sqrt[q]{2}v_j^\gamma}{\sqrt[q]{(2-v_j^q)^\gamma+(v_j^q)^\gamma}}\right\rangle$$

$$\Rightarrow\bigotimes_{j=1}^{\mathfrak{s}}(\gamma\cdot_\epsilon\mathfrak{F}_j)^{\varphi_j}=$$

$$\left\langle\frac{\sqrt[q]{2}\prod_{j=1}^{\mathfrak{s}}\left(\sqrt[q]{\frac{(1+\mu_j^q)^\gamma-(1-\mu_j^q)^\gamma}{(1+\mu_j^q)^\gamma+(1-\mu_j^q)^\gamma}}\right)^{\varphi_j}}{\sqrt[q]{\prod_{j=1}^{\mathfrak{s}}\left(2-\frac{(1+\mu_j^q)^\gamma-(1-\mu_j^q)^\gamma}{(1+\mu_j^q)^\gamma+(1-\mu_j^q)^\gamma}\right)^{\varphi_j}+\prod_{j=1}^{\mathfrak{s}}\left(\frac{(1+\mu_j^q)^\gamma-(1-\mu_j^q)^\gamma}{(1+\mu_j^q)^\gamma+(1-\mu_j^q)^\gamma}\right)^{\varphi_j}}},\right.$$
$$\left.\sqrt[q]{\frac{\prod_{j=1}^{\mathfrak{s}}\left(1+\frac{2v_j^{q\gamma}}{(2-v_j^q)^\gamma+(v_j^q)^\gamma}\right)^{\varphi_j}-\prod_{j=1}^{\mathfrak{s}}\left(1-\frac{2v_j^{q\gamma}}{(2-v_j^q)^\gamma+(v_j^q)^\gamma}\right)^{\varphi_j}}{\prod_{j=1}^{\mathfrak{s}}\left(1+\frac{2v_j^{q\gamma}}{(2-v_j^q)^\gamma+(v_j^q)^\gamma}\right)^{\varphi_j}+\prod_{j=1}^{\mathfrak{s}}\left(1-\frac{2v_j^{q\gamma}}{(2-v_j^q)^\gamma+(v_j^q)^\gamma}\right)^{\varphi_j}}}\right\rangle$$
$$=\left\langle\frac{\sqrt[q]{2}\prod_{j=1}^{\mathfrak{s}}\left(\sqrt[q]{(1+\mu_j^q)^\gamma-\omega_\gamma}\right)^{\varphi j}}{\sqrt[q]{\prod_{j=1}^{\mathfrak{s}}\{(1+\mu_j^q)^\gamma+3\omega_\gamma\}^{\varphi_j}+\prod_{j=1}^{\mathfrak{s}}\{(1+\mu_j^q)^\gamma-\omega_\gamma\}^{\varphi_j}}},\right.$$
$$\left.\sqrt[q]{\frac{\prod_{j=1}^{\mathfrak{s}}\{(2-v_j^q)^\gamma+3(v_j^q)^\gamma\}^{\varphi_j}-\prod_{j=1}^{\mathfrak{s}}\{(2-v_j^q)^\gamma-(v_j^q)^\gamma\}^{\varphi_j}}{\prod_{j=1}^{\mathfrak{s}}\{(2-v_j^q)^\gamma+3(v_j^q)^\gamma\}^{\varphi_j}+\prod_{j=1}^{\mathfrak{s}}\{(2-v_j^q)^\gamma-(v_j^q)^\gamma\}^{\varphi_j}}}\right\rangle.$$

Therefore, $\frac{1}{\gamma}\cdot_\epsilon\left(\bigotimes_{j=1}^{\mathfrak{s}}(\gamma\cdot_\epsilon\mathfrak{F}_j)^{\varphi_j}\right)=$

$$\left\langle \sqrt[q]{\frac{\left(1+\frac{2\prod_{j}^{\mathfrak{s}}\left\{(1+\mu_j^q)^\gamma-\omega_\gamma\right\}^{\varphi_j}}{\prod_{j}^{\mathfrak{s}}\left\{(1+\mu_j^q)^\gamma+3\omega_\gamma\right\}^{\varphi_j}+\prod_{j}^{\mathfrak{s}}\left\{(1+\mu_j^q)^\gamma-\omega_\gamma\right\}^{\varphi_j}}\right)^{1/\gamma}-\left(1-\frac{2\prod_{j=1}^{\mathfrak{s}}\left\{(1+\mu_j^q)^\gamma-\omega_\gamma\right\}^{\varphi_j}}{\prod_{j=1}^{\mathfrak{s}}\left\{(1+\mu_j^q)^\gamma+3\omega_\gamma\right\}^{\varphi_j}+\prod_{j=1}^{\mathfrak{s}}\left\{(1+\mu_j^q)^\gamma-\omega_\gamma\right\}^{\varphi_j}}\right)^{1/\gamma}}{\left(1+\frac{2\prod_{j=1}^{\mathfrak{s}}\left\{(1+\mu_j^q)^\gamma-\omega_\gamma\right\}^{\varphi_j}}{\prod_{j=1}^{\mathfrak{s}}\left\{(1+\mu_j^q)^\gamma+3\omega_\gamma\right\}^{\varphi_j}+\prod_{j=1}^{\mathfrak{s}}\left\{(1+\mu_j^q)^\gamma-\omega_\gamma\right\}^{\varphi_j}}\right)^{1/\gamma}+\left(1-\frac{2\prod_{j=1}^{\mathfrak{s}}\left\{(1+\mu_j^q)^\gamma-\omega_\gamma\right\}^{\varphi_j}}{\prod_{j=1}^{\mathfrak{s}}\left\{(1+\mu_j^q)^\gamma+3\omega_\gamma\right\}^{\varphi_j}+\prod_{j=1}^{\mathfrak{s}}\left\{(1+\mu_j^q)^\gamma-\omega_\gamma\right\}^{\varphi_j}}\right)^{1/\gamma}}}, \right.$$

$$\left. \frac{\sqrt[q]{2\left(\frac{\prod_{j=1}^{\mathfrak{s}}\left\{(2-v_j^q)^\gamma+3(v_j^q)^\gamma\right\}^{\varphi_j}-\prod_{j=1}^{\mathfrak{s}}\left\{(2-v_j^q)^\gamma-(v_j^q)^\gamma\right\}^{\varphi_j}}{\prod_{j=1}^{\mathfrak{s}}\left\{(2-v_j^q)^\gamma+3(v_j^q)^\gamma\right\}^{\varphi_j}+\prod_{j=1}^{\mathfrak{s}}\left\{(2-v_j^q)^\gamma-(v_j^q)^\gamma\right\}^{\varphi_j}}\right)^{1/\gamma}}}{\sqrt[q]{\left(2-\frac{\prod_{j=1}^{\mathfrak{s}}\left\{(2-v_j^q)^\gamma+3(v_j^q)^\gamma\right\}^{\varphi_j}-\prod_{j=1}^{\mathfrak{s}}\left\{(2-v_j^q)^\gamma-(v_j^q)^\gamma\right\}^{\varphi_j}}{\prod_{j=1}^{\mathfrak{s}}\left\{(2-v_j^q)^\gamma+3(v_j^q)^\gamma\right\}^{\varphi_j}+\prod_{j=1}^{\mathfrak{s}}\left\{(2-v_j^q)^\gamma-(v_j^q)^\gamma\right\}^{\varphi_j}}\right)^{1/\gamma}+\left(\frac{\prod_{j=1}^{\mathfrak{s}}\left\{(2-v_j^q)^\gamma+3(v_j^q)^\gamma\right\}^{\varphi_j}-\prod_{j=1}^{\mathfrak{s}}\left\{(2-v_j^q)^\gamma-(v_j^q)^\gamma\right\}^{\varphi_j}}{\prod_{j=1}^{\mathfrak{s}}\left\{(2-v_j^q)^\gamma+3(v_j^q)^\gamma\right\}^{\varphi_j}+\prod_{j=1}^{\mathfrak{s}}\left\{(2-v_j^q)^\gamma-(v_j^q)^\gamma\right\}^{\varphi_j}}\right)^{1/\gamma}}} \right\rangle$$

$$= \left\langle \sqrt[q]{\frac{\left(\prod_{j=1}^{\mathfrak{s}}\{(1+\mu_j^q)^\gamma+3\omega_\gamma\}^{\varphi_j}+3\prod_{j=1}^{\mathfrak{s}}\{(1+\mu_j^q)^\gamma-\omega_\gamma\}^{\varphi_j}\right)^{1/\gamma}-\left(\prod_{j=1}^{\mathfrak{s}}\{(1+\mu_j^q)^\gamma+3\omega_\gamma\}^{\varphi_j}-\prod_{j=1} s\{(1+\mu_j^q)^\gamma-\omega_\gamma\}^{\varphi_j}\right)^{1/\gamma}}{\left(\prod_{j=1}^{\mathfrak{s}}\{(1+\mu_j^q)^\gamma+3\omega_\gamma\}^{\varphi_j}+3\prod_{j=1}^{\mathfrak{s}}\{(1+\mu_j^q)^\gamma-\omega_\gamma\}^{\varphi_j}\right)^{1/\gamma}+\left(\prod_{j=1}^{\mathfrak{s}}\{(1+\mu_j^q)^\gamma+3\omega_\gamma\}^{\varphi_j}-\prod_{j=1} s\{(1+\mu_j^q)^\gamma-\omega_\gamma\}^{\varphi_j}\right)^{1/\gamma}}}, \right.$$

$$\left. \frac{\sqrt[q]{2}\left\{\prod_{j=1}^{\mathfrak{s}}\{(2-v_j^q)^\gamma+3(v_j^q)^\gamma\}^{\varphi_j}-\prod_{j=1}^{\mathfrak{s}}\{(2-v_j^q)^\gamma-(v_j^q)^\gamma\}^{\varphi_j}\right\}^{1/q\gamma}}{\sqrt[q]{\left(\prod_{j=1}^{\mathfrak{s}}\{(2-v_j^q)^\gamma+3(v_j^q)^\gamma\}^{\varphi_j}+3\prod_{j=1}^{\mathfrak{s}}\{(2-v_j^q)^\gamma-(v_j^q)^\gamma\}^{\varphi_j}\right)^{1/\gamma}+\left(\prod_{j=1}^{\mathfrak{s}}\{(2-v_j^q)^\gamma+3(v_j^q)^\gamma\}^{\varphi_j}-\prod_{j=1}^{\mathfrak{s}}\{(2-v_j^q)^\gamma-(v_j^q)^\gamma\}^{\varphi_j}\right)^{1/\gamma}}} \right\rangle.$$

When $\gamma = 1$, then

$$Gq-\mathrm{ROFEWG}(\mathfrak{F}_1,\mathfrak{F}_2,\ldots,\mathfrak{F}_{\mathfrak{s}}) = \left\langle \frac{\sqrt[q]{2}\prod_{j=1}^{\mathfrak{s}}\mu_j^{\varphi_j}}{\sqrt[q]{\prod_{j=1}^{\mathfrak{s}}(2-\mu_j^q)^{\varphi_j}+\prod_{j=1}^{\mathfrak{s}}(\mu_j^q)^{\varphi_j}}}, \sqrt[q]{\frac{\prod_{j=1}^{\mathfrak{s}}(1+v_j^q)^{\varphi_j}-\prod_{j=1}^{\mathfrak{s}}(1-v_j^q)^{\varphi_j}}{\prod_{j=1}^{\mathfrak{s}}(1+v_j^q)^{\varphi_j}+\prod_{j=1}^{\mathfrak{s}}(1-v_j^q)^{\varphi_j}}} \right\rangle.$$

□

6.1 Generalized q-rung orthopair fuzzy Einstein ordered weighted geometric operators

Definition 10 Let $\mathfrak{F}_j = \langle \mu_j, v_j \rangle \in q$-ROFNs and φ_j is the WV of $\mathfrak{F}_j$ with $\varphi_j > 0$ and $\sum_{j=1}^{\mathfrak{s}} \varphi_j = 1$, then G$q$-ROFEOWG operator is a mapping $\mathcal{Q}^{\mathfrak{s}} \to \mathcal{Q}$ such that

$$Gq-\mathrm{ROFEOWG}(\mathfrak{F}_1,\mathfrak{F}_2,\ldots,\mathfrak{F}_{\mathfrak{s}}) = \frac{1}{\gamma}\cdot_\epsilon\left(\bigotimes_{j=1}^{\mathfrak{s}}(\gamma\cdot_\epsilon\mathfrak{F}_j)^{\varphi_j}\right),$$

where $\gamma > 0$.

Theorem 9 *Let $\mathfrak{F}_j = \langle \mu_j, v_j \rangle \in q$-ROFNs and φ_j is the WV of $\mathfrak{F}_j$ with $\varphi_j > 0$ and $\sum_{j=1}^{\mathfrak{s}} \varphi_j = 1$, then* G$q$*-ROFEOWG operator is a q-ROFN and*

$$Gq-ROFEOWG(\mathfrak{F}_1,\mathfrak{F}_2,\ldots,\mathfrak{F}_{\mathfrak{s}}) = \left\langle \sqrt[q]{\frac{\left(\prod_{j=1}^{\mathfrak{s}}\{\rho_\gamma^+ + 3\rho_\gamma\}^{\varphi_j}+3\prod_{j=1} s\{(1+\mu_{\varrho(j)}^q)^\gamma-\rho_\gamma\}^{\varphi_j}\right)^{1/\gamma}-\left(\prod_{j=1}^{\mathfrak{s}}\{(1+\mu_{\varrho(j)}^q)^\gamma+3\rho_\gamma\}^{\varphi_j}-\prod_{j=1}^{\mathfrak{s}}\{(1+\mu_{\varrho(j)}^q)^\gamma-\rho_\gamma\}^{\varphi_j}\right)^{1/\gamma}}{\left(\prod_{j=1}^{\mathfrak{s}}\{(\rho_\gamma^+\}^{\varphi_j}+3\prod_{j=1} s\{\rho_\gamma^+ -(\rho_\gamma\}^{\varphi_j}\right)^{1/\gamma}+\left(\prod_{j=1}^{\mathfrak{s}}\{(1+\mu_{\varrho(j)}^q)^\gamma+3\rho_\gamma\}^{\varphi_j}-\prod_{j=1}^{\mathfrak{s}}\{(1+\mu_{\varrho(j)}^q)^\gamma-\rho_\gamma\}^{\varphi_j}\right)^{1/\gamma}}}, \right.$$

$$\left. \frac{\sqrt[q]{2}\left\{\prod_{j=1}^{\mathfrak{s}}\{(2-v_{\varrho(j)}^q)^\gamma+3(v_{\varrho(j)}^q)^\gamma\}^{\varphi_j}-\prod_{j=1}^{\mathfrak{s}}\{(2-v_{\varrho(j)}^q)^\gamma-(v_{\varrho(j)}^q)^\gamma\}^{\varphi_j}\right\}^{1/q\gamma}}{\sqrt[q]{\left(\prod_{j=1}^{\mathfrak{s}}\{(2-v_{\varrho(j)}^q)^\gamma+3(v_{\varrho(j)}^q)^\gamma\}^{\varphi_j}+3\prod_{j=1}^{\mathfrak{s}}\{(2-v_{\varrho(j)}^q)^\gamma-(v_{\varrho(j)}^q)^\gamma\}^{\varphi_j}\right)^{1/\gamma}+\left(\prod_{j=1}^{\mathfrak{s}}\{(2-v_{\varrho(j)}^q)^\gamma+3(v_{\varrho(j)}^q)^\gamma\}^{\varphi_j}-\prod_{j=1}^{\mathfrak{s}}\{(2-v_{\varrho(j)}^q)^\gamma-(v_{\varrho(j)}^q)^\gamma\}^{\varphi_j}\right)^{1/\gamma}}} \right\rangle.$$

where $\rho_\gamma = (1-\mu_{\varrho(j)}^q)^\gamma$ and $\rho_\gamma^+ = (1+\mu_{\varrho(j)}^q)^\gamma$.

Proof Using the similar arguments as used in Theorem 8, we can prove it. □

When $\gamma = 1$, then

$$Gq-\mathrm{ROFEOWG}(\mathfrak{F}_1,\mathfrak{F}_2,\ldots,\mathfrak{F}_{\mathfrak{s}}) = \left\langle \frac{\sqrt[q]{2}\prod_{j=1}^{\mathfrak{s}}\mu_{\varrho(j)}^{\varphi_j}}{\sqrt[q]{\prod_{j=1}^{\mathfrak{s}}(2-\mu_{\varrho(j)}^q)^{\varphi_j}+\prod_{j=1}^{\mathfrak{s}}(\mu_{\varrho(j)}^q)^{\varphi_j}}}, \sqrt[q]{\frac{\prod_{j=1}^{\mathfrak{s}}(1+v_{\varrho(j)}^q)^{\varphi_j}-\prod_{j=1}^{\mathfrak{s}}(1-v_{\varrho(j)}^q)^{\varphi_j}}{\prod_{j=1}^{\mathfrak{s}}(1+v_{\varrho(j)}^q)^{\varphi_j}+\prod_{j=1}^{\mathfrak{s}}(1-v_{\varrho(j)}^q)^{\varphi_j}}} \right\rangle.$$

7 MADM problems using q-ROF information

We discuss the MADM problems with q-ROF information using q-ROF Einstein geometric operators proposed in the preceding sections. The following suppositions or notations are used to handle the MADM problem for the competent selection of alternative affected by human decision making with q-ROF information. let $\mathcal{L} = \{\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_m\}$ be a set of possible alternatives and $J = \{J_1, J_2, \ldots, J_{\mathfrak{s}}\}$ be a set of possible attributes chosen by the decision maker. Let $\varphi = \{\varphi_1, \varphi_2, \ldots, \varphi_{\mathfrak{s}}\}$ is the WV with $\varphi_j > 0$ and $\sum_{j=1}^{\mathfrak{s}} \varphi_j = 1$. Suppose that $\tilde{\mathcal{E}} = (\mu_{lj}, v_{lj})_{m\times\mathfrak{s}}$ is the q-ROF decision matrix (DM), where μ_{lj} and v_{lj} are the satisfaction and dissatisfaction degrees of the alternative for the attribute, where $0 \le \mu_{lj}^q + v_{lj}^q \le 1$.

For solving a MADM problem, we give the following Algorithm 1.

Algorithm 1: Steps to solve MADM problem by using Gq-ROFEWG operator

1. **Input**:
   Possible alternatives,
   Probable attributes,
   WV for attributes.

2. Use the Gq-ROFEWG operator to evaluate the information in q-ROFDM, find preference values $\mathcal{B}_l, l = 1, 2, ..., m$ of the alternatives $\mathcal{L}_l$.

$$\mathcal{B}_l = Gq - ROFEWG(\mathcal{L}_{l1}, \mathcal{L}_{l2}, \cdots, \mathcal{L}_{l\mathfrak{s}})$$

$$= \left\langle \sqrt[q]{\frac{\left(\prod_{j=1}^{\mathfrak{s}}\{\xi_\gamma + 3\zeta_\gamma\}^{\varphi_j} + 3\prod_{j=1}^{\mathfrak{s}}\{\xi_\gamma - \zeta_\gamma\}^{\varphi_j}\right)^{1/\gamma} - \left(\prod_{j=1}^{\mathfrak{s}}\{(1+\mu_{lj}^q)^\gamma + 3\zeta_\gamma\}^{\varphi_j} - \prod_{j=1}^{\mathfrak{s}}\{(\epsilon_\gamma - \omega_\gamma\}^{\varphi_j}\right)^{1/\gamma}}{\left(\prod_{j=1}^{\mathfrak{s}}\{\xi_\gamma + 3\zeta_\gamma\}^{\varphi_j} + 3\prod_{j=1}^{\mathfrak{s}}\{\xi_\gamma - \zeta_\gamma\}^{\varphi_j}\right)^{1/\gamma} + \left(\prod_{j=1}^{\mathfrak{s}}\{(1+\mu_{lj}^q)^\gamma + 3\zeta_\gamma\}^{\varphi_j} - \prod_{j=1}^{\mathfrak{s}}\{\xi_\gamma - (1-\mu_{lj}^q)^\gamma\}^{\varphi_j}\right)^{1/\gamma}}}, \right.$$

$$\left. \frac{\sqrt[q]{2}\left\{\prod_{j=1}^{\mathfrak{s}}\{\eta_\gamma + 3(\nu_{lj}^q)^\gamma\}^{\varphi_j} - \prod_{j=1}^{\mathfrak{s}}\{\eta_\gamma - (\nu_{lj}^q)^\gamma\}^{\varphi_j}\right\}^{1/q\gamma}}{\sqrt[q]{\left(\prod_{j=1}^{\mathfrak{s}}\{\eta_\gamma + 3(\nu_{lj}^q)^\gamma\}^{\varphi_j} + 3\prod_{j=1}^{\mathfrak{s}}\{\eta_\gamma - (\nu_{lj}^q)^\gamma\}^{\varphi_j}\right)^{1/\gamma} + \left(\prod_{j=1}^{\mathfrak{s}}\{\eta_\gamma + 3(\nu_{lj}^q)^\gamma\}^{\varphi_j} - \prod_{j=1}^{\mathfrak{s}}\{\eta_\gamma - (\nu_{lj}^q)^\gamma\}^{\varphi_j}\right)^{1/\gamma}}} \right\rangle.$$

Where $\xi_\gamma = (1+\mu_{lj}^q)^\gamma$, $\zeta_\gamma = (1-\mu_{lj}^q)^\gamma$, $\eta_\gamma = (2-\nu_{lj}^q)^\gamma$, $\epsilon_\gamma = (1+\mu_{lj}^q)^\gamma$, and $\omega_\gamma = (1-\mu_j^q)^\gamma$.

3. Calculate the score values.
4. Rank the alternatives $\mathcal{L}_l$, $l = 1, 2, ..., m$ according to their score values $S(\mathcal{B}_l), l = 1, 2, ..., m$. For equal score, use the accuracy function for ranking of alternatives.

Output: The alternative with greatest score will be the decision

7.1 Suitable location for thermal power station

A Thermal Power Station (TPS) uses heat energy generated from burning coal to produce electrical energy. Such power stations are broadly used in the world. Certain TPSs are made to produce heat for industrial purposes, for district heating, or desalination of water and for generating electrical power. Therefore, it is necessary to select a location for a TPS, as it needs a massive capacity of land and position to bear the static and dynamic pressure during the whole process. Suppose that the government wants to plant a TPS to fulfill the requirements of electric power. Let $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$ and $\mathcal{L}_4$ be possible locations for TPSs. Let $\mathcal{K}_1$="Availability of coal", $\mathcal{K}_2$="Availability of water", $\mathcal{K}_3$="Transportation facilities" be the three criteria for the judgement of a location.

The framework for selection of location for a TPS is given in Fig. 2.

1. The q-ROFDM is shown in Table 1.
2. The weights assigned by decision maker are

$$\varphi_1 = 0.4, \varphi_2 = 0.3, \varphi_3 = 0.3, \text{and} \sum_{j=1}^{3}\varphi_j = 1.$$

We use the Gq-ROFEWG operator for the selection of TPS location.

Step 1. For performance values $\mathcal{B}_l$ of locations, use the Gq-ROFEWG operator for $q = 3, \gamma = 1$.

$\mathcal{B}_1 = (0.65, 0.34)$,
$\mathcal{B}_2 = (0.51, 0.46)$,
$\mathcal{B}_3 = (0.55, 0.40)$,
$\mathcal{B}_4 = (0.49, 0.45)$.

Step 2. Compute the scores $\mathcal{S}(\mathcal{B}_l)$ of q-ROFNs $\mathcal{B}_l$ and rank the locations.

$\mathcal{S}(\mathcal{B}_1) = 0.24$,
$\mathcal{S}(\mathcal{B}_2) = 0.04$,
$\mathcal{S}(\mathcal{B}_3) = 0.10$,
$\mathcal{S}(\mathcal{B}_4) = 0.03$.

The ranking of locations is

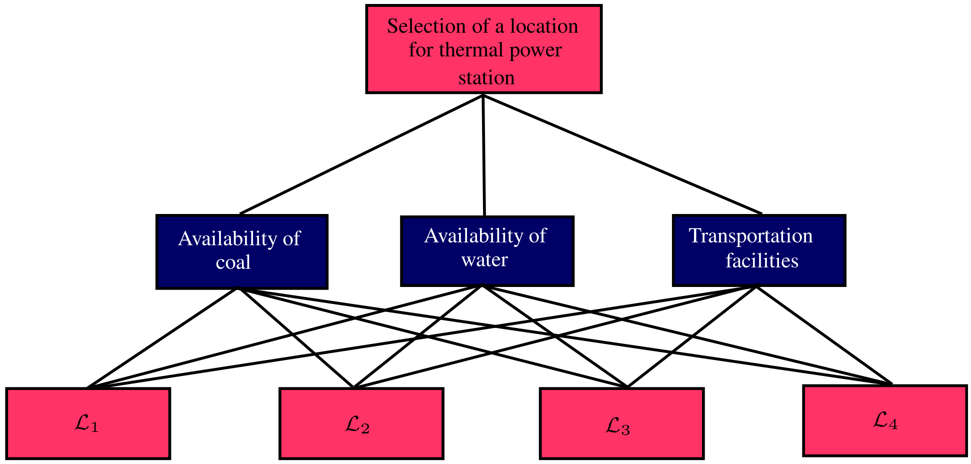

Fig. 2 The framework for location-selection of a TPS

Table 1 q-ROFDM

	$\mathcal{L}_1$	$\mathcal{L}_2$	$\mathcal{L}_3$	$\mathcal{L}_4$
$\mathcal{K}_1$	(0.6, 0.4)	(0.4, 0.6)	(0.5, 0.4)	(0.4, 0.5)
$\mathcal{K}_2$	(0.7, 0.3)	(0.7, 0.2)	(0.8, 0.1)	(0.9, 0.1)
$\mathcal{K}_3$	(0.8, 0.2)	(0.5, 0.4)	(0.4, 0.5)	(0.3, 0.5)

$\mathcal{L}_1 > \mathcal{L}_3 > \mathcal{L}_2 > \mathcal{L}_4.$

Step 3. Therefore, $\mathcal{L}_1$ is the suitable location for TPS.

7.2 Selection of cardiothoracic surgeon

A Cardiothoracic Surgeon (CS) is a medical doctor, who is trained in surgical procedures of heart, lungs and other organs in the chest. Surgery of the heart and chest are performed by CSs. Cardiologists work with surgeons to handle patients and determine whether the patient needs surgery. They also work together to treat irregular heart beat problems. Suppose that a heart patient wants to select a best CS for heart surgery. Let $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$ and $\mathcal{L}_4$ be possible surgeons for heart surgery. Let $\mathcal{K}_1$="Cardiac Surgeon's Experience", $\mathcal{K}_2$="Hospital Quality", $\mathcal{K}_3$="Communication Style" be the three criteria for the judgement of a surgeon.

The framework for selection of best CS is given in Fig. 3.

1. The q-ROFDM is shown in Table 2.
2. The weights assigned by decision maker are

$$\varphi_1 = 0.3, \varphi_2 = 0.4, \varphi_3 = 0.3, \text{and} \sum_{j=1}^{3} \varphi_j = 1.$$

We use the Gq-ROFEWG operator for the selection of best CS.

Step 1. For performance values $\mathcal{B}_l$ of CSs, use the Gq-ROFEWG operator for $q = 3, \gamma = 1$.

$$\mathcal{B}_1 = (0.58, 0.55),$$
$$\mathcal{B}_2 = (0.70, 0.61),$$
$$\mathcal{B}_3 = (0.82, 0.41),$$
$$\mathcal{B}_4 = (0.71, 0.59).$$

Step 2. Calculate the scores $\mathcal{S}(\mathcal{B}_l)$ of q-ROFNs $\mathcal{B}_l$ and rank CSs.

$$\mathcal{S}(\mathcal{B}_1) = 0.03,$$
$$\mathcal{S}(\mathcal{B}_2) = 0.12,$$
$$\mathcal{S}(\mathcal{B}_3) = 0.48,$$
$$\mathcal{S}(\mathcal{B}_4) = 0.15.$$

The ranking is

$\mathcal{L}_3 > \mathcal{L}_4 > \mathcal{L}_2 > \mathcal{L}_1.$

Step 3. Therefore, $\mathcal{L}_3$ is the best CS.

8 Comparison analysis

This section provides a comparison analysis of proposed operators with others operators such as GPFEWG (Garg 2017), IFEWG (Wang and Liu 2011), q-ROFGMSM (Liu and Wang (2020) operators to show the efficacy of our model.

1. For Application 7.1, it is clear from Table 3 that the final rankings by applying the Gq-ROFEWG, GPFEWG, IFEWG, q-ROFGMSM operators are $\mathcal{L}_1 > \mathcal{L}_3 > \mathcal{L}_2 > \mathcal{L}_4$, respectively. However, the final score values are not same. So, the optimal decision, using all these operators are same. This shows that our model is applicable to resolve real life problems.

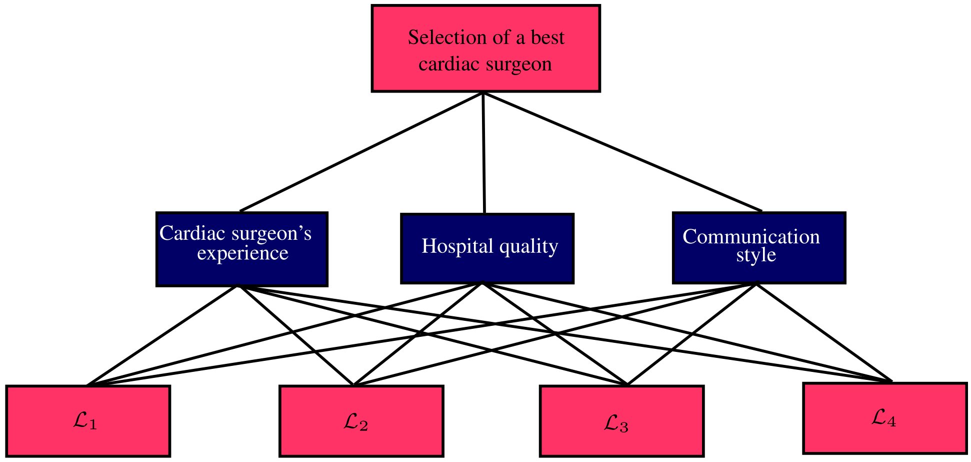

Fig. 3 The framework for selection of best CS

Table 2 q-ROFDM

	$\mathcal{L}_1$	$\mathcal{L}_2$	$\mathcal{L}_3$	$\mathcal{L}_4$
$\mathcal{J}_1$	(0.9, 0.6)	(0.8, 0.4)	(0.7, 0.3)	(0.7, 0.8)
$\mathcal{J}_2$	(0.4, 0.6)	(0.8, 0.7)	(0.9, 0.5)	(0.8, 0.2)
$\mathcal{J}_3$	(0.5, 0.4)	(0.5, 0.6)	(0.8, 0.3)	(0.6, 0.5)

2. The reason behind our proposed model is that IFNs and PFNs can deal only those situations where $\mu + \nu \leq 1$ and $\mu^2 + \nu^2 \leq 1$, respectively but many problems where data exceed by $\mu^2 + \nu^2 \leq 1$, then we need q-ROFS. To show feasibility and attractiveness, we have proposed another application.
3. As the ranking lists obtained from proposed approach and q-ROFGMSM operator are same. The q-ROFGMSM operator is a good approach to solve DM problems but proposed Einstein AOs are more flexible and easy approach.

9 Conclusions and future directions

The notion of q-rung orthopair fuzzy model generalizes the PF model to describe complicated uncertain information more effectively. In this article, we have worked to the progress of MADM with the study of problems in q-ROF environment. For the utilization in decision making, the logical basis of AOs need to be carefully considered. The shortcomings of existing methods and beneficial characteristics of Einstein AOs motivate us to consider their ability to produce suitable combinations of q-ROFNs. Therefore, we have introduced geometric operators to construct q-ROF AOs that closely follow the motivation of Einstein operations. They include the q-ROFEWG, q-ROFEOWG, Gq-ROFEWG and Gq-ROFEOWG operators. The elementary characteristics of these operators are explained so that the experts can select the version that better fits their needs. We have utilized these operators to expand a number of strategies to address MADM problems. The comparison analysis of proposed operators with existing operators is done. Finally, practical examples for the selection of location for TPS and the selection of best cardiac surgeon are given. In these problems, we have applied the concept of Gq-ROFEWG operator to summarize the information corresponding to each alternative. Then, we have derived appropriate results with the help of score functions. These operators allow us to assess the value of each alternative in a comparable fashion. Altogether they build up a procedure and make a case for the pertinence and adequacy of the proposed approach. In the future, we plan to extend our study to (i). q-rung picture fuzzy Einstein hybrid weighted operators. (ii). q-rung orthopair fuzzy soft Einstein hybrid weighted operators.

Table 3 Comparison analysis for Application 7.1 with GPFEWG, IFEWG, q-ROFGMSM operators (suppose $q = 3, \gamma = 1$)

Methods	$\mathcal{S}(\mathcal{B}_1)$	$\mathcal{S}(\mathcal{B}_2)$	$\mathcal{S}(\mathcal{B}_3)$	$\mathcal{S}(\mathcal{B}_4)$	Ranking order
Gq-ROFEWG	0.24	0.04	0.10	0.03	$\mathcal{L}_1 > \mathcal{L}_3 > \mathcal{L}_2 > \mathcal{L}_4$
GPFEWG	0.30	0.06	0.11	0.04	$\mathcal{L}_1 > \mathcal{L}_3 > \mathcal{L}_2 > \mathcal{L}_4$
IFEWG	0.30	0.09	0.12	0.06	$\mathcal{L}_1 > \mathcal{L}_3 > \mathcal{L}_2 > \mathcal{L}_4$
q-ROFGMSM	0.23	0.04	0.9	0.03	$\mathcal{L}_1 > \mathcal{L}_3 > \mathcal{L}_2 > \mathcal{L}_4$

Compliance with ethical standards

Conflict of interest The authors declare no conflicts of interest.

References

Akram M, Adeel A (2018) Novel hybrid decision-making methods based on mF rough information. Granul Comput. https://doi.org/10.1007/s41066-018-00142-6:1-17

Akram M, Ali G (2019) Group decision making approach under multi (Q, N)-soft multi granulation rough model. Granul Comput. https://doi.org/10.1007/s41066-019-00190-6

Akram M, Ali G (2020) Hybrid models for decision making based on rough Pythagorean fuzzy bipolar soft information. Granul Comput 5(1):1–15

Akram M, Bashir A (2020) Complex fuzzy ordered weighted quadratic averaging operators. Granul Comput. https://doi.org/10.1007/s41066-020-00213-7:1-16

Akram M, Shahzadi G (2020) A hybrid decision making model under q-rung orthopair fuzzy Yager aggregation operators. Granul Comput. https://doi.org/10.1007/s41066-020-00229-z

Akram M, Shahzadi G (2020) Decision making approach based on Pythagorean Dombi fuzzy soft graphs. Granul Comput. https://doi.org/10.1007/s41066-020-00224-4

Akram M, Dudek WA, Dar JM (2019) Pythagorean Dombi fuzzy aggregation operators with application in multicriteria decision-making. Int J Intell Syst 34(11):3000–3019

Akram M, Dudek WA, Ilyas F (2019) Group decision-making based on Pythagorean fuzzy TOPSIS method. Int J Intell Syst 34(7):1455–1475

Akram M, Ali G, Shabir M (2020a) A hybrid decision making framework using rough m F bipolar soft environment. Granul Comput. https://doi.org/10.1007/s41066-020-00214-6:1-17

Akram M, Garg H, Ilyas F (2020b) Multi-criteria group decision making based on ELECTRE I method in Pythagorean fuzzy information. Soft Comput 24(5):3425–3453

Atanassov KT (1986) Intuitionistic fuzzy sets. Fuzzy Sets Syst 20(1):87–96

Bai K, Zhu X, Wang J, Zhang R (2018) Some partitioned Maclaurin symmetric mean based on q-rung orthopair fuzzy information for dealing with multi-attribute group decision making. Symmetry 10(9):383

Chen SM, Chen SW (2014) Fuzzy forecasting based on two-factors second-order fuzzy-trend logical relationship groups and the probabilities of trends of fuzzy logical relationships. IEEE Trans Cybern 45(3):391–403

Chen SM, Cheng SH (2016) Fuzzy multi-attribute decision making based on transformation techniques of intuitionistic fuzzy values and intuitionistic fuzzy geometric averaging operators. Inf Sci 352:133–149

Chen SM, Niou SJ (2011) Fuzzy multiple attributes group decision making based on fuzzy preference relations. Expert Syst Appl 38(4):3865–3872

Chen SM, Ko YK, Chang YC, Pan JS (2009) Weighted fuzzy interpolative reasoning based on weighted increment transformation and weighted ratio transformation techniques. IEEE Trans Fuzzy Syst 17(6):1412–1427

Chen SM, Cheng SH, Lan TC (2016) Multi-criteria decision making based on the TOPSIS method and similarity measures between intuitionistic fuzzy values. Inf Sci 367:279–295

Garg H (2016) A new generalized Pythagorean fuzzy information aggregation using Einstein operations and its application to decision making. Int J Intell Syst 31(9):886–920

Garg H (2017) Generalized Pythagorean fuzzy geometric aggregation operators using Einstein t-norm and t-conorm for multi-criteria decision making process. Int J Intell Syst 32(6):597–630

Garg H, Chen SM (2020) Multi-attribute group decision making based on neutrality aggregation operators of q-rung orthopair fuzzy sets. Inf Sci 517:427–447

Garg H, Rani D (2019) Novel aggregation operators and ranking method for complex intuitionistic fuzzy sets and their applications to decision-making process. Artif Intell Rev :1-26

Jana C, Muhiuddin G, Pal M (2020) Some Dombi aggregation of q-rung orthopair fuzzy numbers in multiple-attribute decision making. Int J Intell Syst 34(12):3220–3240

Joshi BP, Gegov A (2020) Confidence levels q-rung orthopair fuzzy aggregation operators and its applications to MCDM problems. Int J Intell Syst 35(1):125–149

Khan AA, Ashraf S, Abdullah S, Qiyas M, Luo J, Khan SU (2019) Pythagorean fuzzy Dombi aggregation operators and their application in decision support system. Symmetry 11(3):383

Liu P, Liu J (2018) Some q-rung orthopair fuzzy Bonferroni mean operators and their application to multi-attribute group decision making. Int J Intell Syst 33(2):315–347

Liu P, Wang P (2018) Some q-rung orthopair fuzzy aggregation operators and their applications to multiple-attribute decision making. Int J Intell Syst 33(2):259–280

Liu P, Wang P (2020) Multiple attribute decision making based on q-rung orthopair fuzzy generalized Maclaurin symmetic mean operators. Inf Sci 518:181–210

Liu P, Chen SM, Wang P (2018) Multiple-attribute group decision making based on q-rung orthopair fuzzy power maclaurin symmetric mean operators. IEEE Trans Syst Man Cybern B. https://doi.org/10.1109/TSMC.2018.2852948

Liu P, Liu J, Chen SM (2018) Some intuitionistic fuzzy Dombi bonferroni mean operators and their application to multi-attribute group decision making. J Operat Res Soc 69(1):1–24

Peng X, Yang Y (2016) Fundamental properties of interval-valued Pythagorean fuzzy aggregation operators. Int J Intell Syst 31(5):444–487

Peng X, Yuan H (2016) Fundamental properties of Pythagorean fuzzy aggregation operators. Fundam Inform 147(4):415–446

Peng X, Dai J, Garg H (2018) Exponential operation and aggregation operator for q-rung orthopair fuzzy set and their decision-making method with a new score function. Int J Intell Syst 33(11):2255–2282

Rahman K, Abdullah S, Ahmed R, Ullah M (2017) Pythagorean fuzzy Einstein weighted geometric aggregation operator and their application to multiple attribute group decision making. J Intell Fuzzy Syst 33(1):635–647

Shahzadi G, Akram M, Al-Kenani AN (2020) Decision making approach under Pythagorean fuzzy Yager weighted operators. Mathematics 8(1):70

Wang W, Liu X (2011) Intuitionistic fuzzy geometric aggregation operators based on Einstein operations. Int J Intell Syst 26(11):1049–1075

Wang W, Liu X (2012) Intuitionistic fuzzy information aggregation using Einstein operations. IEEE Trans Fuzzy Syst 20(5):923–938

Wei G (2010) Some induced geometric aggregation operators with intuitionistic fuzzy information and their application to group decision making. Appl Soft Comput 10(2):423–431

Wei G (2017) Pythagorean fuzzy interaction aggregation operators and their application to multipleattribute decision making. J Intell Fuzzy Syst 33(4):2119–2132

Wei G, Lu M (2018a) Pythagorean fuzzy Maclaurin symmetric mean operators in multiple-attribute decision making. Int J Intell Syst 33(5):1043–1070

Wei G, Lu M (2018b) Pythagorean fuzzy power aggregation operators in multiple-attribute decision making. Int J Intell Syst 33(1):169–186

Wei G, Lu M, Alsaadi FE, Hayat T, Alsaedi A (2017) Pythagorean 2-tuple linguistic aggregation operators in multiple-attribute decision making. J Intell Fuzzy Syst 33(2):1129–1142

Xu Z (2007) Intuitionistic fuzzy aggregation operators. IEEE Trans Fuzzy Syst 15(6):1179–1187

Xu Z, Yager RR (2006) Some geometric aggregation operators based on intuitionistic fuzzy sets. Int J Gen Syst 35(4):417–433

Yager RR (2013) Pythagorean fuzzy subsets. In: 2013 Joint IFSA-World Congress and NAFIPS Annual Meeting (IFSA/NAFIPS). IEEE, pp 57–61

Yager RR (1988) On ordered weighted averaging Aggregation operators in multi-criteria decision-making. IEEE Trans Syst Man Cybern 18(1):183–190

Yager RR (1994) Aggregation operators and fuzzy systems modeling. Fuzzy Set Syst 67(2):129–145

Yager RR (2013) Pythagorean membership grades in multi-criteria decision making. IEEE Trans Fuzzy Syst 22(4):958–965

Yager RR (2016) Generalized orthopair fuzzy sets. IEEE Trans Fuzzy Syst 25(5):1222–1230

Yager RR, Abbasov AM (2013) Pythagorean membership grades, complex numbers, and decision making. Int J Intell Syst 28(5):436–452

Zadeh LA (1965) Fuzzy sets. Inf Control 8(3):338–353

Zeng S, Chen J, Li X (2016) A hybrid method for Pythagorean fuzzy multiple-criteria decision making. Int J Inf Tec Decis 15(02):403–422

Zhang W, Li X, Ju Y (2014) Some aggregation operators based on Einstein operations under interval-valued dual hesitant fuzzy setting and their application. Math Probl Eng. https://doi.org/10.1155/2014/958927:21