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ORIGINAL PAPER

Decision-making approach based on Pythagorean Dombi fuzzy soft graphs

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Abstract

A Pythagorean fuzzy set model is more useful than intuitionistic fuzzy set model to handle the imprecise information involving both membership and nonmembership degrees, and a soft set is an other parameterized point of view for handling the vagueness. A Pythagorean fuzzy soft graph is considered more capable than intuitionistic fuzzy soft graph for representing the parametric relationships between objects, and the Dombi operators with operational parameters have creditable extensibility. Based on these two notions, we propose the concept of Pythagorean Dombi fuzzy soft graph (PDFSG). We describe certain concepts of graph theory under Pythagorean Dombi fuzzy soft environment. Further, we define the degree sequence and degree set in PDFSG, and the concept of edge regular PDFSG with consequential properties. Moreover, we illustrate the examples in decision making including selection of suitable ETL software for a business intelligence project and evaluation of electronics companies. Finally, we present the comparison analysis of our proposed model to show the superiority than existing model.

Keywords PDFSG · Regularity of PDFSG · Strongly regular PDFSG · Bipartite and biregular PDFSG

1 Introduction

Several operators were interpolated and most important among them are min–max, Einstein, Hamacher, Frank, product, Lukasiewicz, Azcel-Alsina and Dombi operators that appeared in different graphs with fuzzy logic. The product and minimum operators were used by Zadeh (1965) to characterize fuzzy set. The rational format of disjunctive and conjunctive operators in accordance with Kuwagaki's results (1952) was obtained by Hamacher (1978). Then, many researchers studied more generic form, i.e., triangular norms (*t*-norms) and triangular conorms (*t*-conorms). Menger (1942) revived *t*-norms and *t*-conorms within probabilistic metric framework. Many postulates and results relevant to *t*-norms and *t*-conorms were made by Schweizer and Sklar (1983). Klement et al. (2013) introduced a lot of extensions and summarizations of beneficial outcomes of *T*-operators for the similar cause. In many decision-making applications, Zadeh's min and max operators have been broadly adapted. Especially in decision-making problems, other *T*-operators may work better in some cases such as product operator may tend to choose over min operator (Dubois et al. 2000).

To delineate the imprecision and obscureness in different fields, Zadeh (1965) introduced fuzzy set (FS) theory. But sometimes the membership function of the FS is not sufficient to declare the complexity of data. To overcome this difficulty, Atanassov (1986) extended FS to an intuitionistic fuzzy set (IFS) by adding a nonmembership function and a hesitancy function. As a development of IFS, Yager (2013) recommended the notion of Pythagorean fuzzy set (PFS) satisfying the condition $\mu^2 + \nu^2 \leq 1$. The space of PFSs membership degree is greater than the space of IFSs membership degree. For instance, when a decision maker gives the evaluation information whose membership degree is 0.5 and nonmembership degree is 0.7, then the IFN fails to evaluate this situation because $0.5 + 0.7 > 1$. However, $(0.5)^2 + (0.7)^2 \leq 1$. To handle fuzziness, Akram

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et al. (2019, 2020) proved that PFS has much stronger ability in multi-criteria group decision-making problems.

For the parameterized point of view, Molodtsov (1999) proposed soft set (SS) for uncertainty modeling and soft computing. For the hybrid models, Feng et al. (2011a, b) joined the SSs with rough sets (RSs) and FSs. Ali et al. (2009) studied many new operations in SS theory. Som (2006) popularized the idea of soft relation and fuzzy soft relation. Fuzzy soft sets (FSSs) were defined by Maji et al. (2001a) and using this concept in decision-making problems, Roy and Maji (2007) established many applications. The idea of FSs and FSSs induced by SSs was handled by Ali (2011). Maji et al. (2001b) elaborated the theory of intuitionistic fuzzy soft set (IFSS). A graph is a pictorial representation that bonds the items together. To handle the haziness occurring in these bonding, graph can be considered as fuzzy graph (FG). The formation of FGs was introduced by Rosenfeld (1975) using min and max operators. Parvathi and Karunambigai (2006) imported the view of intuitionistic fuzzy graphs (IFGs). Akram and Davvaz (2012) described IFGs. Naz et al. (2018) presented the notion of Pythagorean fuzzy graphs (PFGs). Thumbakara and George (2014) studied the soft graphs. The idea of fuzzy soft graphs (FSGs) was established by Akram and Nawaz (2016). Shahzadi and Akram (2017) illustrated the concept of IFSGs.

In 1982, Dombi (1982) initiated Dombi operator with flexible operational parameter. For different values of operational parameters, different results can be made in decision-making problems, depending upon the requirement. Chen and Ye (2017), Jana et al. (2019), Shi and Ye (2018) used Dombi operations and presented MCDM problem in single-valued neutrosophic, neutrosophic cubic and bipolar fuzzy environments, respectively. Liu et al. (2018) presented MCGDM problem using Dombi Bonferroni mean operator on IFSs. In graph theory, use of Dombi operator is rare. Ashraf et al. (2018) introduced the idea of Dombi fuzzy graphs (DFGs). Akram et al. (2019) introduced the PDFGs. For other terminologies not discussed in the paper, the readers are suggested to Liu and Wang (2020), Liu et al. (2017), Mishra et al. (2020), Chen (1996), Bai and Chen (2008a, b), Chen et al. (2013, 2016a, b), Chen and Cheng (2016), Zhang and Xu (2014) and Akram and Ali (2019, 2020).

The motivations of this article are outlined as follows:

1. The judgment of a perfect alternative in a Pythagorean fuzzy environment corresponding to various parameters is a laborious problem. In existing techniques, assessment information is characterized by IF and PF environments which promote to do work in Pythagorean fuzzy soft environment to discuss pairwise relationship.
2. PDFSGs expose extraordinary decapitation in giving vague and imprecise assessment information.
3. With the help of Dombi operators and score function, we get actual and correct decision.
4. The proposed work overcomes the restrictions of existing work.

The main contributions of this article are:

1. The concept of Dombi operators under Pythagorean fuzzy soft environment describes the relationship between objects and investigates its properties.
2. To handle complex realistic problems, an algorithm is developed when data are given in Pythagorean fuzzy environment corresponding to different parameters.
3. At the end, the benefits and characteristics of proposed approach are discussed by comparison analysis.

The organization of this research article is as follows:

In Sect. 2, we propose various terms including PDFSG, complement of PDFSG, the concept of homomorphism between two PDFSGs, regular, totally regular and strongly regular PDFSG, bipartite PDFSG, biregular PDFSG, edge regular and totally edge regular PDFSG. We propose the results related to these terms. In Sect. 3, we provide the decision-making problems and discuss the comparison analysis to show the importance of proposed model. In Sect. 4, we have concluded our results.

Definition 1.1 (Peng et al. 2015) Let $\mathcal{X}$ be a universe of discourse and $\mathcal{W}$ be the set of all parameters, $\mathcal{U} \subseteq \mathcal{W}$. $P(\mathcal{X})$ denotes the set of all Pythagorean fuzzy subsets. $(\tilde{\mathcal{M}}, \mathcal{U})$ is called an PFSS over $\mathcal{X}$, where PF approximation function is given by $\tilde{\mathcal{M}} = (\tilde{\mathcal{M}_{\mu}}, \tilde{\mathcal{M}_{\nu}}) : \mathcal{U} \to P(\mathcal{X})$.

- The Dombi's t-norm $\frac{1}{1+[(\frac{1-a}{a})^{\gamma}+(\frac{1-b}{b})^{\gamma}]^{\frac{1}{\gamma}}}, \gamma > 0.$
- The Dombi's t-conorm $\frac{1}{1+[(\frac{1-a}{a})^{-\gamma}+(\frac{1-b}{b})^{-\gamma}]^{\frac{-1}{\gamma}}}, \gamma > 0.$

One more set of T-operators is $T(a,b) = \frac{ab}{a+b-ab}$ and $T^{*}(a,b) = \frac{a+b-ab}{1-ab}$, which can be obtained by substituting $\gamma = 1$ in Dombi's t-norm and t-conorm. Also, $\mathbb{P}(a,b) \leq \frac{ab}{a+b-ab} \leq \mathbb{M}(a,b)$ and $\mathbb{M}^{*}(a,b) \leq \frac{a+b-ab}{1-ab} \leq \mathbb{P}^{*}(a,b)$.

Definition 1.2 (2018) A Pythagorean fuzzy preference relation (PFPR) on the set $\mathcal{X} = \{a_1, a_2, \ldots, a_n\}$ is denoted by a matrix $O = (o_{il})_{n \times n}$, where $o_{jl} = (a_j a_l, \mu(a_j a_l), \nu(a_j a_l))$ for all $j, l = 1, 2, \ldots, n$. For easiness, let $o_{jl} = (\mu_{jl}, \nu_{jl})$ where μ_{jl} indicates the degree to which the object a_j is preferred to the object a_l, ν_{jl} denotes the degree to which the object a_j is not preferred to the object a_l and

$\pi_{jl} = \sqrt{1 - \mu_{jl}^2 - v_{jl}^2}$ is interpreted as a hesitancy degree, with the following condition:

$$\mu_{jl}, v_{jl} \in [0,1], \mu_{jl}^2 + v_{jl}^2 \leq 1, \mu_{jl} = v_{lj}, \mu_{jj} = v_{jj} = 0.5, \text{for all} j, l = 1, 2, \ldots, .n$$

2 Pythagorean Dombi fuzzy soft graphs

Definition 2.1 A PDFSG on $\mathcal{X}$ is a tuple $P_D = (\tilde{\mathcal{M}}, \tilde{\mathcal{N}}, \mathcal{U})$ such that

1. $\mathcal{U}$ is a nonempty set of parameters,
2. $(\tilde{\mathcal{M}}, \mathcal{U})$ is a PFSS subset over $\mathcal{X}$,
3. $(\tilde{\mathcal{N}}, \mathcal{U})$ is a PFSS subset over $\mathcal{E} \subseteq \mathcal{X} \times \mathcal{X}$,
4. $(\tilde{\mathcal{M}}(u_i), \tilde{\mathcal{N}}(u_i))$ is a PDF subgraph for all $u_i \in \mathcal{U}$, $i = 1, 2, \ldots, m$, that is,

$$\tilde{\mathcal{N}}_\mu(u_i)(ab) \leq \frac{(\tilde{\mathcal{M}}_\mu(u_i)(a))(\tilde{\mathcal{M}}_\mu(u_i)(b))}{\tilde{\mathcal{M}}_\mu(u_i)(a) + \tilde{\mathcal{M}}_\mu(u_i)(b) - (\tilde{\mathcal{M}}_\mu(u_i)(a))(\tilde{\mathcal{M}}_\mu(u_i)(b))},$$

$$\tilde{\mathcal{N}}_v(u_i)(ab) \leq \frac{\tilde{\mathcal{M}}_v(u_i)(a) + \tilde{\mathcal{M}}_v(u_i)(b) - 2(\tilde{\mathcal{M}}_v(u_i)(a))(\tilde{\mathcal{M}}_v(u_i)(b))}{1 - (\tilde{\mathcal{M}}_v(u_i)(a))(\tilde{\mathcal{M}}_v(u_i)(b))},$$

and $0 \leq \tilde{\mathcal{N}}_\mu^2(u_i)(ab) + \tilde{\mathcal{N}}_v^2(u_i)(ab) \leq 1, \forall u_i \in \mathcal{U}$, $a, b \in \mathcal{X}$.

The PDF subgraph $(\tilde{\mathcal{M}}(u_i), \tilde{\mathcal{N}}(u_i))$ is denoted by $\tilde{\mathbf{H}}(u_i) = (\tilde{\mathbf{H}}_\mu(u_i), \tilde{\mathbf{H}}_v(u_i))$.

Remark $(\tilde{\mathcal{M}}, \mathcal{U})$ is the Pythagorean Dombi fuzzy soft (PDFS) vertex set of P_D and $(\tilde{\mathcal{N}}, \mathcal{U})$ the PDFS edge set of P_D.

Example 2.2 Consider nonempty sets $\mathcal{X} = \{a_1, a_2, a_3, a_4\}$ and $\mathcal{E} = \{a_1a_2, a_1a_3\} \subseteq \mathcal{X} \times \mathcal{X}$. Let $\mathcal{U} = \{u_1, u_2\}$ be a parameter set and $(\tilde{\mathcal{M}}, \mathcal{U})$ be a PDFS vertex set over $\mathcal{X}$ given by

$$\tilde{\mathcal{M}}(u_1) = \{(a_1, 0.3, 0.8), (a_2, 0.9, 0.2), (a_3, 0.8, 0.5)\},$$
$$\tilde{\mathcal{M}}(u_2) = \{(a_1, 0.6, 0.6), (a_2, 0.7, 0.5), (a_3, 0.6, 0.7)\}.$$

Let $(\tilde{\mathcal{N}}, \mathcal{U})$ be a PDFS edge set over $\mathcal{E}$ with PDF approximation function $\tilde{\mathcal{N}} : \mathcal{U} \to P(\mathcal{X})$ given by

$$\tilde{\mathcal{N}}(u_1) = \{(a_1a_2, 0.28, 0.80), (a_1a_3, 0.25, 0.82)\},$$
$$\tilde{\mathcal{N}}(u_2) = \{(a_1a_2, 0.47, 0.70), (a_1a_3, 0.42, 0.70)\}.$$

Clearly, $\tilde{\mathbf{H}}(u_1) = (\tilde{\mathcal{M}}(u_1), \tilde{\mathcal{N}}(u_1))$, $\tilde{\mathbf{H}}(u_2) = (\tilde{\mathcal{M}}(u_2), \tilde{\mathcal{N}}(u_2))$ are PDFGs to the attributes u_1 and u_2, respectively, as shown in Fig. 1.

Hence $P_D = \{\tilde{\mathbf{H}}(u_1), \tilde{\mathbf{H}}(u_2)\}$ is a PDFSG.

Definition 2.3 The complement of a PDFSG $P_D = (\tilde{\mathcal{M}}, \tilde{\mathcal{N}}, \mathcal{U})$ is a PDFSG $\overline{P}_D = (\overline{\tilde{\mathcal{M}}}, \overline{\tilde{\mathcal{N}}}, \overline{\mathcal{U}})$ which is defined by

1. $\overline{\mathcal{U}} = \mathcal{U}$.
2. $\overline{\tilde{\mathcal{M}}_\mu(u_i)(a)} = \tilde{\mathcal{M}}_\mu(u_i)(a)$ and $\overline{\tilde{\mathcal{M}}_v(u_i)(a)} = \tilde{\mathcal{M}}_v(u_i)(a)$.
3.

$$\overline{\tilde{\mathcal{N}}_\mu(u_i)(ab)} = \begin{cases} \dfrac{(\tilde{\mathcal{M}}_\mu(u_i)(a))(\tilde{\mathcal{M}}_\mu(u_i)(b))}{\tilde{\mathcal{M}}_\mu(u_i)(a) + \tilde{\mathcal{M}}_\mu(u_i)(b) - (\tilde{\mathcal{M}}_\mu(u_i)(a))(\tilde{\mathcal{M}}_\mu(u_i)(b))}, & \text{if } \tilde{\mathcal{N}}_\mu(u_i)(ab) = 0 \\ \dfrac{(\tilde{\mathcal{M}}_\mu(u_i)(a))(\tilde{\mathcal{M}}_\mu(u_i)(b))}{\tilde{\mathcal{M}}_\mu(u_i)(a) + \tilde{\mathcal{M}}_\mu(u_i)(b) - (\tilde{\mathcal{M}}_\mu(u_i)(a))(\tilde{\mathcal{M}}_\mu(u_i)(b))} - \tilde{\mathcal{N}}_\mu(u_i)(ab), & \text{if } 0 \leq \tilde{\mathcal{N}}_\mu(u_i)(ab) \leq 1. \end{cases}$$

$$\overline{\tilde{\mathcal{N}}_v(u_i)(ab)} = \begin{cases} \dfrac{\tilde{\mathcal{M}}_v(u_i)(a) + \tilde{\mathcal{M}}_v(u_i)(b) - 2(\tilde{\mathcal{M}}_v(u_i)(a))(\tilde{\mathcal{M}}_v(u_i)(b))}{1 - (\tilde{\mathcal{M}}_v(u_i)(a))(\tilde{\mathcal{M}}_v(u_i)(b))}, & \text{if } \tilde{\mathcal{N}}_v(u_i)(ab) = 0 \\ \dfrac{\tilde{\mathcal{M}}_v(u_i)(a) + \tilde{\mathcal{M}}_v(u_i)(b) - 2(\tilde{\mathcal{M}}_v(u_i)(a))(\tilde{\mathcal{M}}_v(u_i)(b))}{1 - (\tilde{\mathcal{M}}_v(u_i)(a))(\tilde{\mathcal{M}}_v(u_i)(b))} - \tilde{\mathcal{N}}_v(u_i)(ab), & \text{if } 0 \leq \tilde{\mathcal{N}}_v(u_i)(ab) \leq 1. \end{cases}$$

Example 2.4 Consider nonempty sets $\mathcal{X} = \{a_1, a_2, a_3, a_4, a_5\}$ and $\mathcal{E} = \{a_1a_2, a_1a_3, a_1a_4, a_1a_5, a_2a_3\} \subseteq \mathcal{X} \times \mathcal{X}$. Let $\mathcal{U} = \{u_1\}$ be a parameter set and $(\tilde{\mathcal{M}}, \mathcal{U})$ be a PDFS vertex set over $\mathcal{X}$ given by

$$\tilde{\mathcal{M}}(u_1) = \{(a_1, 0.6, 0.6), (a_2, 0.5, 0.7), (a_3, 0.7, 0.6), (a_4, 0.8, 0.4), (a_5, 0.7, 0.7)\}.$$

Let $(\tilde{\mathcal{N}}, \mathcal{U})$ be a PDFS edge set over $\mathcal{E}$ with PDF approximation function $\tilde{\mathcal{N}} : \mathcal{U} \to P(\mathcal{X})$ given by

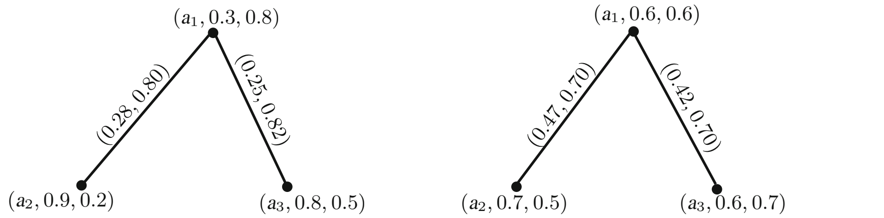
$$\tilde{\mathcal{N}}(u_1) = \{(a_1a_2, 0.30, 0.75), (a_1a_3, 0.40, 0.70), (a_1a_4, 0.50, 0.60), (a_1a_5, 0.45, 0.72), (a_2a_3, 0.40, 0.70)\}.$$

$\tilde{\mathbf{H}}(u_1)$ is the PDFG as shown in Fig. 2. Hence, P_D is a PDFSG. Now the complement of PDFSG is the complement of PFSG $\tilde{\mathbf{H}}(u_1)$, which is shown in Fig. 3.

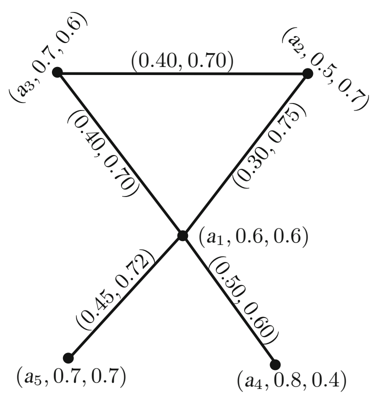
Theorem 2.5 *If* $P_D = (\tilde{\mathcal{M}}, \tilde{\mathcal{N}}, \mathcal{U})$ *is a PDFSG, then* $\overline{\overline{P}}_D = P_D$.

Proof Suppose that P_D is a PDFSG. Then by complement of PDFSG,

Fig. 1 PDFSG $P_D = \{\tilde{\mathbf{H}}(u_1), \tilde{\mathbf{H}}(u_2)\}$

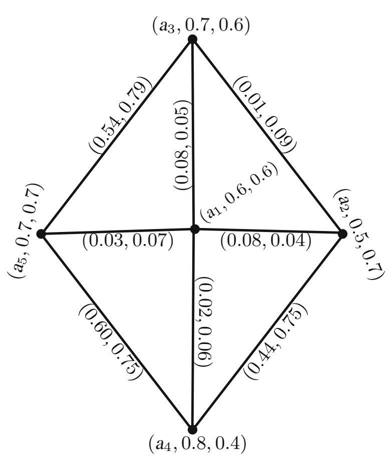


$\tilde{\mathbf{H}}(u_1)$ corresponding to the parameter u_1 $\tilde{\mathbf{H}}(u_2)$ corresponding to the parameter u_2



$\tilde{\mathbf{H}}(u_1)$ corresponding to parameter u_1

Fig. 2 PDFG $\tilde{\mathbf{H}}(u_1)$



$\overline{\tilde{\mathbf{H}}(u_1)} = (\overline{\tilde{\mathcal{M}}(u_1)(a)}, \overline{\tilde{\mathcal{N}}(u_1)(a)})$

Fig. 3 Complement $\overline{\tilde{\mathbf{H}}(u_1)}$

$$\overline{\overline{\mathcal{U}}} = \overline{\mathcal{U}} = \mathcal{U}, \overline{\overline{\tilde{\mathcal{M}}_\mu(u_i)(a)}} = \overline{\tilde{\mathcal{M}}_\mu(u_i)(a)} = \tilde{\mathcal{M}}_\mu(u_i)(a)$$

and $\overline{\overline{\tilde{\mathcal{M}}_v(u_i)(a)}} = \overline{\tilde{\mathcal{M}}_v(u_i)(a)} = \tilde{\mathcal{M}}_v(u_i)(a)$, $\forall u_i \in \mathcal{U}, a \in \mathcal{X}$.

If $\tilde{\mathcal{N}}_\mu(u_i)(ab) = 0$ and $\tilde{\mathcal{N}}_v(u_i)(ab) = 0$, then

$$\begin{aligned}
\overline{\overline{\tilde{\mathcal{N}}_\mu(u_i)(ab)}} &= \frac{(\overline{\tilde{\mathcal{M}}_\mu(u_i)(a)})(\overline{\tilde{\mathcal{M}}_\mu(u_i)(b)})}{\overline{\tilde{\mathcal{M}}_\mu(u_i)(a)} + \overline{\tilde{\mathcal{M}}_\mu(u_i)(b)} - (\overline{\tilde{\mathcal{M}}_\mu(u_i)(a)})(\overline{\tilde{\mathcal{M}}_\mu(u_i)(b)})} \\
&= \frac{(\tilde{\mathcal{M}}_\mu(u_i)(a))(\tilde{\mathcal{M}}_\mu(u_i)(b))}{\tilde{\mathcal{M}}_\mu(u_i)(a) + \tilde{\mathcal{M}}_\mu(u_i)(b) - (\tilde{\mathcal{M}}_\mu(u_i)(a))(\tilde{\mathcal{M}}_\mu(u_i)(b))} \\
&= \tilde{\mathcal{N}}_\mu(u_i)(ab), \\
\overline{\overline{\tilde{\mathcal{N}}_v(u_i)(ab)}} &= \frac{\overline{\tilde{\mathcal{M}}_v(u_i)(a)} + \overline{\tilde{\mathcal{M}}_v(u_i)(b)} - 2(\overline{\tilde{\mathcal{M}}_v(u_i)(a)})(\overline{\tilde{\mathcal{M}}_v(u_i)(b)})}{1 - (\overline{\tilde{\mathcal{M}}_v(u_i)(a)})(\overline{\tilde{\mathcal{M}}_v(u_i)(b)})} \\
&= \frac{\tilde{\mathcal{M}}_v(u_i)(a) + \tilde{\mathcal{M}}_v(u_i)(b) - 2(\tilde{\mathcal{M}}_v(u_i)(a))(\tilde{\mathcal{M}}_v(u_i)(b))}{1 - (\tilde{\mathcal{M}}_v(u_i)(a))(\tilde{\mathcal{M}}_v(u_i)(b))} \\
&= \tilde{\mathcal{N}}_v(u_i)(ab).
\end{aligned}$$

If $0 \le \tilde{\mathcal{N}}_\mu(u_i)(ab), \tilde{\mathcal{N}}_v(u_i)(ab) \le 1$, then

$$\begin{aligned}
\overline{\overline{\tilde{\mathcal{N}}_\mu(u_i)(ab)}} &= \frac{(\overline{\tilde{\mathcal{M}}_\mu(u_i)(a)})(\overline{\tilde{\mathcal{M}}_\mu(u_i)(b)})}{\overline{\tilde{\mathcal{M}}_\mu(u_i)(a)} + \overline{\tilde{\mathcal{M}}_\mu(u_i)(b)} - (\overline{\tilde{\mathcal{M}}_\mu(u_i)(a)})(\overline{\tilde{\mathcal{M}}_\mu(u_i)(b)})} \\
&\quad - \overline{\tilde{\mathcal{N}}_\mu(u_i)(ab)} \\
&= \frac{(\tilde{\mathcal{M}}_\mu(u_i)(a))(\tilde{\mathcal{M}}_\mu(u_i)(b))}{\tilde{\mathcal{M}}_\mu(u_i)(a) + \tilde{\mathcal{M}}_\mu(u_i)(b) - (\tilde{\mathcal{M}}_\mu(u_i)(a))(\tilde{\mathcal{M}}_\mu(u_i)(b))} \\
&\quad - \left(\frac{(\tilde{\mathcal{M}}_\mu(u_i)(a))(\tilde{\mathcal{M}}_\mu(u_i)(b))}{\tilde{\mathcal{M}}_\mu(u_i)(a) + \tilde{\mathcal{M}}_\mu(u_i)(b) - (\tilde{\mathcal{M}}_\mu(u_i)(a))(\tilde{\mathcal{M}}_\mu(u_i)(b))}\right. \\
&\quad \left. - \tilde{\mathcal{N}}_\mu(u_i)(ab)\right) \\
&= \tilde{\mathcal{N}}_\mu(u_i)(ab), \\
\overline{\overline{\tilde{\mathcal{N}}_v(u_i)(ab)}} &= \frac{\overline{\tilde{\mathcal{M}}_v(u_i)(a)} + \overline{\tilde{\mathcal{M}}_v(u_i)(b)} - 2(\overline{\tilde{\mathcal{M}}_v(u_i)(a)})(\overline{\tilde{\mathcal{M}}_v(u_i)(b)})}{1 - (\overline{\tilde{\mathcal{M}}_v(u_i)(a)})(\overline{\tilde{\mathcal{M}}_v(u_i)(b)})} \\
&\quad - \overline{\tilde{\mathcal{N}}_v(u_i)(ab)} \\
&= \frac{\tilde{\mathcal{M}}_v(u_i)(a) + \tilde{\mathcal{M}}_v(u_i)(b) - 2(\tilde{\mathcal{M}}_v(u_i)(a))(\tilde{\mathcal{M}}_v(u_i)(b))}{1 - (\tilde{\mathcal{M}}_v(u_i)(a))(\tilde{\mathcal{M}}_v(u_i)(b))} \\
&\quad - \left(\frac{\tilde{\mathcal{M}}_v(u_i)(a) + \tilde{\mathcal{M}}_v(u_i)(b) - 2(\tilde{\mathcal{M}}_v(u_i)(a))(\tilde{\mathcal{M}}_v(u_i)(b))}{1 - (\tilde{\mathcal{M}}_v(u_i)(a))(\tilde{\mathcal{M}}_v(u_i)(b))}\right. \\
&\quad \left. - \tilde{\mathcal{N}}_v(u_i)(ab)\right) \\
&= \tilde{\mathcal{N}}_v(u_i)(ab).
\end{aligned}$$

$\forall u_i \in \mathcal{U}, a, b \in \mathcal{X}$. Hence, $\overline{\overline{P}}_D = P_D$. □

Definition 2.6 A homomorphism $Q : P_{D_1} \to P_{D_2}$ of two PDFSGs $P_{D_1} = (\tilde{\mathcal{M}}_1, \tilde{\mathcal{N}}_1, \mathcal{U})$ and $P_{D_2} = (\tilde{\mathcal{M}}_2, \tilde{\mathcal{N}}_2, \mathcal{U})$ is a mapping $Q : \mathcal{X}_1 \to \mathcal{X}_2$ satisfying

1. $\tilde{\mathcal{M}}_{1\mu}(u_i)(a) \leq \tilde{\mathcal{M}}_{2\mu}(u_i)(Q(a))$, $\tilde{\mathcal{M}}_{1\nu}(u_i)(a) \leq \tilde{\mathcal{M}}_{2\nu}(u_i)(Q(a))$.
2. $\tilde{\mathcal{N}}_{1\mu}(u_i)(ab) \leq \tilde{\mathcal{N}}_{2\mu}(u_i)(Q(a)Q(b))$, $\tilde{\mathcal{N}}_{1\nu}(u_i)(ab) \leq \tilde{\mathcal{N}}_{2\nu}(u_i)(Q(a)Q(b)), \quad \forall\, a, b \in \mathcal{X},\ ab \in \mathcal{E}_1,\ u_i \in \mathcal{U},\ i = 1, 2, \ldots, m$.

Definition 2.7 An isomorphism $Q : P_{D_1} \to P_{D_2}$ of two PDFSGs $P_{D_1} = (\tilde{\mathcal{M}}_1, \tilde{\mathcal{N}}_1, \mathcal{U})$ and $P_{D_2} = (\tilde{\mathcal{M}}_2, \tilde{\mathcal{N}}_2, \mathcal{U})$ is a bijective mapping $Q : \mathcal{X}_1 \to \mathcal{X}_2$ satisfying

1. $\tilde{\mathcal{M}}_{1\mu}(u_i)(a) = \tilde{\mathcal{M}}_{2\mu}(u_i)(Q(a))$, $\tilde{\mathcal{M}}_{1\nu}(u_i)(a) = \tilde{\mathcal{M}}_{2\nu}(u_i)(Q(a))$.
2. $\tilde{\mathcal{N}}_{1\mu}(u_i)(ab) = \tilde{\mathcal{N}}_{2\mu}(u_i)(Q(a)Q(b))$, $\tilde{\mathcal{N}}_{1\nu}(u_i)(ab) = \tilde{\mathcal{N}}_{2\nu}(u_i)(Q(a)Q(b)), \forall\, a, b \in \mathcal{X},\ ab \in \mathcal{E}_1,\ u_i \in \mathcal{U}$.

Definition 2.8 A weak isomorphism $Q : P_{D_1} \to P_{D_2}$ of two PDFSGs $P_{D_1} = (\tilde{\mathcal{M}}_1, \tilde{\mathcal{N}}_1, \mathcal{U})$ and $P_{D_2} = (\tilde{\mathcal{M}}_2, \tilde{\mathcal{N}}_2, \mathcal{U})$ is a bijective mapping $Q : \mathcal{X}_1 \to \mathcal{X}_2$ satisfying

1. Q is a homomorphism.
2. $\tilde{\mathcal{M}}_{1\mu}(u_i)(a) = \tilde{\mathcal{M}}_{2\mu}(u_i)(Q(a))$, $\tilde{\mathcal{M}}_{1\nu}(u_i)(a) = \tilde{\mathcal{M}}_{2\nu}(u_i)(Q(a))$, $\forall\, a, b \in \mathcal{X},\ u_i \in \mathcal{U}$.

Definition 2.9 A co-weak isomorphism $Q : P_{D_1} \to P_{D_2}$ of two PDFSGs $P_{D_1} = (\tilde{\mathcal{M}}_1, \tilde{\mathcal{N}}_1, \mathcal{U})$ and $P_{D_2} = (\tilde{\mathcal{M}}_2, \tilde{\mathcal{N}}_2, \mathcal{U})$ is a bijective mapping $Q : \mathcal{X}_1 \to \mathcal{X}_2$ satisfying

1. Q is a homomorphism.
2. $\tilde{\mathcal{N}}_{1\mu}(u_i)(ab) = \tilde{\mathcal{N}}_{2\mu}(u_i)(Q(a)Q(b))$, $\tilde{\mathcal{N}}_{1\nu}(u_i)(ab) = \tilde{\mathcal{N}}_{2\nu}(u_i)(Q(a)Q(b)), \forall\, ab \in \mathcal{E}_1,\ u_i \in \mathcal{U}$.

Definition 2.10 A PDFSG $P_D = (\tilde{\mathcal{M}}, \tilde{\mathcal{N}}, \mathcal{U})$ is called self-complementary if $\overline{P}_D \cong P_D$.

Proposition 2.11 *If $P_D = (\tilde{\mathcal{M}}, \tilde{\mathcal{N}}, \mathcal{U})$ is a self-complementary PDFSG, then*

$$\sum_{a \neq b} \tilde{\mathcal{N}}_{\mu}(u_i)(ab) = \frac{1}{2} \sum_{a \neq b} \frac{(\tilde{\mathcal{M}}_{\mu}(u_i)(a))(\tilde{\mathcal{M}}_{\mu}(u_i)(b))}{\tilde{\mathcal{M}}_{\mu}(u_i)(a) + \tilde{\mathcal{M}}_{\mu}(u_i)(b) - \tilde{\mathcal{M}}_{\mu}(u_i)(a)\tilde{\mathcal{M}}_{\mu}(u_i)(b)}, \tag{1}$$

$$\sum_{a \neq b} \tilde{\mathcal{N}}_{\nu}(u_i)(ab) = \frac{1}{2} \sum_{a \neq b} \frac{\tilde{\mathcal{M}}_{\nu}(u_i)(a) + \tilde{\mathcal{M}}_{\nu}(u_i)(b) - 2\tilde{\mathcal{M}}_{\nu}(u_i)(a)\tilde{\mathcal{M}}_{\nu}(u_i)(b)}{1 - \tilde{\mathcal{M}}_{\nu}(u_i)(a)\tilde{\mathcal{M}}_{\nu}(u_i)(b)}. \tag{2}$$

Proof Suppose that P_D is a self-complementary PDFSG; then there occurs an isomorphism $Q : \mathcal{X} \to \mathcal{X}$ such that $\overline{\tilde{\mathcal{M}}_{\mu}(u_i)(Q(a))} = \tilde{\mathcal{M}}_{\mu}(u_i)(a)$, $\overline{\tilde{\mathcal{M}}_{\nu}(u_i)(Q(a))} = \tilde{\mathcal{M}}_{\nu}(u_i)(a), \forall\, u_i \in \mathcal{U}, a, b \in \mathcal{X}$.

$$\overline{\tilde{\mathcal{N}}_{\mu}(u_i)(Q(a)Q(b))} = \tilde{\mathcal{N}}_{\mu}(u_i)(ab), \overline{\tilde{\mathcal{N}}_{\nu}(u_i)(Q(a)Q(b))} = \tilde{\mathcal{N}}_{\nu}(u_i)(ab), \forall\, u_i \in \mathcal{U}, ab \in \mathcal{E}.$$

By complement of P_D, we have

$$\overline{\tilde{\mathcal{N}}_{\mu}(u_i)(Q(a)Q(b))} = \frac{(\overline{\tilde{\mathcal{M}}_{\mu}(u_i)(Q(a))})(\overline{\tilde{\mathcal{M}}_{\mu}(u_i)(Q(b))})}{\overline{\tilde{\mathcal{M}}_{\mu}(u_i)(Q(a))} + \overline{\tilde{\mathcal{M}}_{\mu}(u_i)(Q(b))} - (\overline{\tilde{\mathcal{M}}_{\mu}(u_i)(Q(a))})(\overline{\tilde{\mathcal{M}}_{\mu}(u_i)(Q(b))})} - \tilde{\mathcal{N}}_{\mu}(u_i)(Q(a)Q(b))$$

$$\tilde{\mathcal{N}}_{\mu}(u_i)(ab) = \frac{(\tilde{\mathcal{M}}_{\mu}(u_i)(a))(\tilde{\mathcal{M}}_{\mu}(u_i)(b))}{\tilde{\mathcal{M}}_{\mu}(u_i)(a) + \tilde{\mathcal{M}}_{\mu}(u_i)(b) - (\tilde{\mathcal{M}}_{\mu}(u_i)(a))(\tilde{\mathcal{M}}_{\mu}(u_i)(b))} - \tilde{\mathcal{N}}_{\mu}(u_i)(Q(a)Q(b))$$

$$\sum_{a \neq b} \tilde{\mathcal{N}}_{\mu}(u_i)(ab) + \sum_{a \neq b} \tilde{\mathcal{N}}_{\mu}(u_i)(Q(a)Q(b)) = \sum_{a \neq b} \frac{(\tilde{\mathcal{M}}_{\mu}(u_i)(a))(\tilde{\mathcal{M}}_{\mu}(u_i)(b))}{\tilde{\mathcal{M}}_{\mu}(u_i)(a) + \tilde{\mathcal{M}}_{\mu}(u_i)(b) - (\tilde{\mathcal{M}}_{\mu}(u_i)(a))(\tilde{\mathcal{M}}_{\mu}(u_i)(b))}$$

$$2\sum_{a \neq b} \tilde{\mathcal{N}}_{\mu}(u_i)(ab) = \sum_{a \neq b} \frac{(\tilde{\mathcal{M}}_{\mu}(u_i)(a))(\tilde{\mathcal{M}}_{\mu}(u_i)(b))}{\tilde{\mathcal{M}}_{\mu}(u_i)(a) + \tilde{\mathcal{M}}_{\mu}(u_i)(b) - (\tilde{\mathcal{M}}_{\mu}(u_i)(a))(\tilde{\mathcal{M}}_{\mu}(u_i)(b))}$$

$$\sum_{a \neq b} \tilde{\mathcal{N}}_{\mu}(u_i)(ab) = \frac{1}{2}\sum_{a \neq b} \frac{(\tilde{\mathcal{M}}_{\mu}(u_i)(a))(\tilde{\mathcal{M}}_{\mu}(u_i)(b))}{\tilde{\mathcal{M}}_{\mu}(u_i)(a) + \tilde{\mathcal{M}}_{\mu}(u_i)(b) - (\tilde{\mathcal{M}}_{\mu}(u_i)(a))(\tilde{\mathcal{M}}_{\mu}(u_i)(b))}.$$

Similarly,

$$\overline{\tilde{\mathcal{N}}_{\nu}(u_i)(Q(a)Q(b))} = \frac{\overline{\tilde{\mathcal{M}}_{\nu}(u_i)(Q(a))} + \overline{\tilde{\mathcal{M}}_{\nu}(u_i)(Q(b))} - 2(\overline{\tilde{\mathcal{M}}_{\nu}(u_i)(Q(a))})(\overline{\tilde{\mathcal{M}}_{\nu}(u_i)(Q(b))})}{1 - (\overline{\tilde{\mathcal{M}}_{\nu}(u_i)(Q(a))})(\overline{\tilde{\mathcal{M}}_{\nu}(u_i)(Q(b))})} - \tilde{\mathcal{N}}_{\nu}(u_i)(Q(a)Q(b))$$

$$\tilde{\mathcal{N}}_{\nu}(u_i)(ab) = \frac{\tilde{\mathcal{M}}_{\nu}(u_i)(a) + \tilde{\mathcal{M}}_{\nu}(u_i)(b) - 2(\tilde{\mathcal{M}}_{\nu}(u_i)(a))(\tilde{\mathcal{M}}_{\nu}(u_i)(b))}{1 - (\tilde{\mathcal{M}}_{\nu}(u_i)(a))(\tilde{\mathcal{M}}_{\nu}(u_i)(b))} - \tilde{\mathcal{N}}_{\nu}(u_i)(Q(a)Q(b))$$

$$\sum_{a \neq b} \tilde{\mathcal{N}}_{\nu}(u_i)(ab) + \sum_{a \neq b} \tilde{\mathcal{N}}_{\nu}(u_i)(Q(a)Q(b)) = \sum_{a \neq b} \frac{\tilde{\mathcal{M}}_{\nu}(u_i)(a) + \tilde{\mathcal{M}}_{\nu}(u_i)(b) - 2(\tilde{\mathcal{M}}_{\nu}(u_i)(a))(\tilde{\mathcal{M}}_{\nu}(u_i)(b))}{1 - (\tilde{\mathcal{M}}_{\nu}(u_i)(a))(\tilde{\mathcal{M}}_{\nu}(u_i)(b))}$$

$$2\sum_{a \neq b} \tilde{\mathcal{N}}_{\nu}(u_i)(ab) = \sum_{a \neq b} \frac{\tilde{\mathcal{M}}_{\nu}(u_i)(a) + \tilde{\mathcal{M}}_{\nu}(u_i)(b) - 2(\tilde{\mathcal{M}}_{\nu}(u_i)(a))(\tilde{\mathcal{M}}_{\nu}(u_i)(b))}{1 - (\tilde{\mathcal{M}}_{\nu}(u_i)(a))(\tilde{\mathcal{M}}_{\nu}(u_i)(b))}$$

$$\sum_{a \neq b} \tilde{\mathcal{N}}_{\nu}(u_i)(ab) = \frac{1}{2}\sum_{a \neq b} \frac{\tilde{\mathcal{M}}_{\nu}(u_i)(a) + \tilde{\mathcal{M}}_{\nu}(u_i)(b) - 2(\tilde{\mathcal{M}}_{\nu}(u_i)(a))(\tilde{\mathcal{M}}_{\nu}(u_i)(b))}{1 - (\tilde{\mathcal{M}}_{\nu}(u_i)(a))(\tilde{\mathcal{M}}_{\nu}(u_i)(b))}.$$

□

Proposition 2.12 *Let $P_D = (\tilde{\mathcal{M}}, \tilde{\mathcal{N}}, \mathcal{U})$ be the PDFSG. If*

$$\tilde{\mathcal{N}}_\mu(u_i)(ab) = \frac{1}{2}\left(\frac{(\tilde{\mathcal{M}}_\mu(u_i)(a))(\tilde{\mathcal{M}}_\mu(u_i)(b))}{\tilde{\mathcal{M}}_\mu(u_i)(a) + \tilde{\mathcal{M}}_\mu(u_i)(b) - (\tilde{\mathcal{M}}_\mu(u_i)(a))(\tilde{\mathcal{M}}_\mu(u_i)(b))}\right) \tag{3}$$

$$\tilde{\mathcal{N}}_v(u_i)(ab) = \frac{1}{2}\left(\frac{\tilde{\mathcal{M}}_v(u_i)(a) + \tilde{\mathcal{M}}_v(u_i)(b) - 2(\tilde{\mathcal{M}}_v(u_i)(a))(\tilde{\mathcal{M}}_v(u_i)(b))}{1 - (\tilde{\mathcal{M}}_v(u_i)(a))(\tilde{\mathcal{M}}_v(u_i)(b))}\right) \tag{4}$$

$\forall a, b \in \mathcal{X}$, then P_D is self-complementary.

Proof Suppose that $P_D = (\tilde{\mathcal{M}}, \tilde{\mathcal{N}}, \mathcal{U})$ is the PDFSG that satisfies

$$\tilde{\mathcal{N}}_\mu(u_i)(ab) = \frac{1}{2}\left(\frac{(\tilde{\mathcal{M}}_\mu(u_i)(a))(\tilde{\mathcal{M}}_\mu(u_i)(b))}{\tilde{\mathcal{M}}_\mu(u_i)(a) + \tilde{\mathcal{M}}_\mu(u_i)(b) - (\tilde{\mathcal{M}}_\mu(u_i)(a))(\tilde{\mathcal{M}}_\mu(u_i)(b))}\right)$$

$$\tilde{\mathcal{N}}_v(u_i)(ab) = \frac{1}{2}\left(\frac{\tilde{\mathcal{M}}_v(u_i)(a) + \tilde{\mathcal{M}}_v(u_i)(b) - 2(\tilde{\mathcal{M}}_v(u_i)(a))(\tilde{\mathcal{M}}_v(u_i)(b))}{1 - (\tilde{\mathcal{M}}_v(u_i)(a))(\tilde{\mathcal{M}}_v(u_i)(b))}\right)$$

$\forall a, b \in \mathcal{X}$, then the identity mapping $I : \mathcal{X} \to \mathcal{X}$ is an isomorphism from P_D to $\overline{P}_D$ that fulfilled the following conditions :

$\tilde{\mathcal{M}}_\mu(u_i)(a) = \overline{\tilde{\mathcal{M}}_\mu(u_i)(I(a))}$, $\tilde{\mathcal{M}}_v(u_i)(a) = \overline{\tilde{\mathcal{M}}_v(u_i)(I(a))}$, $\forall a \in \mathcal{X}$. Since

$$\tilde{\mathcal{N}}_\mu(u_i)(ab) = \frac{1}{2}\left(\frac{(\tilde{\mathcal{M}}_\mu(u_i)(a))(\tilde{\mathcal{M}}_\mu(u_i)(b))}{\tilde{\mathcal{M}}_\mu(u_i)(a) + \tilde{\mathcal{M}}_\mu(u_i)(b) - (\tilde{\mathcal{M}}_\mu(u_i)(a))(\tilde{\mathcal{M}}_\mu(u_i)(b))}\right), \forall a, b \in \mathcal{X}.$$

we have, $\overline{\tilde{\mathcal{N}}_\mu(u_i)(I(a)I(b))}$

$$= \overline{\tilde{\mathcal{N}}_\mu(u_i)(ab)}$$

$$= \frac{(\tilde{\mathcal{M}}_\mu(u_i)(a))(\tilde{\mathcal{M}}_\mu(u_i)(b))}{\tilde{\mathcal{M}}_\mu(u_i)(a) + \tilde{\mathcal{M}}_\mu(u_i)(b) - (\tilde{\mathcal{M}}_\mu(u_i)(a))(\tilde{\mathcal{M}}_\mu(u_i)(b))} - \tilde{\mathcal{N}}_\mu(u_i)(ab)$$

$$= \frac{(\tilde{\mathcal{M}}_\mu(u_i)(a))(\tilde{\mathcal{M}}_\mu(u_i)(b))}{\tilde{\mathcal{M}}_\mu(u_i)(a) + \tilde{\mathcal{M}}_\mu(u_i)(b) - (\tilde{\mathcal{M}}_\mu(u_i)(a))(\tilde{\mathcal{M}}_\mu(u_i)(b))} - \frac{1}{2}\left(\frac{(\tilde{\mathcal{M}}_\mu(u_i)(a))(\tilde{\mathcal{M}}_\mu(u_i)(b))}{\tilde{\mathcal{M}}_\mu(u_i)(a) + \tilde{\mathcal{M}}_\mu(u_i)(b) - (\tilde{\mathcal{M}}_\mu(u_i)(a))(\tilde{\mathcal{M}}_\mu(u_i)(b))}\right)$$

$$= \frac{1}{2}\left(\frac{(\tilde{\mathcal{M}}_\mu(u_i)(a))(\tilde{\mathcal{M}}_\mu(u_i)(b))}{\tilde{\mathcal{M}}_\mu(u_i)(a) + \tilde{\mathcal{M}}_\mu(u_i)(b) - (\tilde{\mathcal{M}}_\mu(u_i)(a))(\tilde{\mathcal{M}}_\mu(u_i)(b))}\right)$$

$$= \tilde{\mathcal{N}}_\mu(u_i)(ab).$$

Similarly , $\overline{\tilde{\mathcal{N}}_v(u_i)(I(a)I(b))}$

$$= \overline{\tilde{\mathcal{N}}_v(u_i)(ab)}$$

$$= \frac{\tilde{\mathcal{M}}_v(u_i)(a) + \tilde{\mathcal{M}}_v(u_i)(b) - 2(\tilde{\mathcal{M}}_v(u_i)(a))(\tilde{\mathcal{M}}_v(u_i)(b))}{1 - (\tilde{\mathcal{M}}_v(u_i)(a))(\tilde{\mathcal{M}}_v(u_i)(b))} - \tilde{\mathcal{N}}_v(u_i)(ab)$$

$$= \frac{\tilde{\mathcal{M}}_v(u_i)(a) + \tilde{\mathcal{M}}_v(u_i)(b) - 2(\tilde{\mathcal{M}}_v(u_i)(a))(\tilde{\mathcal{M}}_v(u_i)(b))}{1 - (\tilde{\mathcal{M}}_v(u_i)(a))(\tilde{\mathcal{M}}_v(u_i)(b))} - \frac{1}{2}\left(\frac{\tilde{\mathcal{M}}_v(u_i)(a) + \tilde{\mathcal{M}}_v(u_i)(b) - 2(\tilde{\mathcal{M}}_v(u_i)(a))(\tilde{\mathcal{M}}_v(u_i)(b))}{1 - (\tilde{\mathcal{M}}_v(u_i)(a))(\tilde{\mathcal{M}}_v(u_i)(b))}\right)$$

$$= \frac{1}{2}\left(\frac{\tilde{\mathcal{M}}_v(u_i)(a) + \tilde{\mathcal{M}}_v(u_i)(b) - 2(\tilde{\mathcal{M}}_v(u_i)(a))(\tilde{\mathcal{M}}_v(u_i)(b))}{1 - (\tilde{\mathcal{M}}_v(u_i)(a))(\tilde{\mathcal{M}}_v(u_i)(b))}\right)$$

$$= \tilde{\mathcal{N}}_v(u_i)(ab)$$

Since the conditions of isomorphism $\overline{\tilde{\mathcal{N}}_\mu(u_i)(I(a)I(b))} = \tilde{\mathcal{N}}_\mu(u_i)(ab)$ and $\overline{\tilde{\mathcal{N}}_v(u_i)(I(a)I(b))} = \tilde{\mathcal{N}}_v(u_i)(ab)$ are satisfied by I, $P_D = (\tilde{\mathcal{M}}, \tilde{\mathcal{N}}, \mathcal{U})$ is self-complementary. □

Proposition 2.13 *If $P_{D_1} = (\tilde{\mathcal{M}}_1, \tilde{\mathcal{N}}_1, \mathcal{U})$ and $P_{D_2} = (\tilde{\mathcal{M}}_2, \tilde{\mathcal{N}}_2, \mathcal{U})$ are two isomorphic PDFSGs, then $\overline{P}_{D_1} \cong \overline{P}_{D_2}$ and conversely.*

Proof Suppose that $P_{D_1} = (\tilde{\mathcal{M}}_1, \tilde{\mathcal{N}}_1, \mathcal{U})$ and $P_{D_2} = (\tilde{\mathcal{M}}_2, \tilde{\mathcal{N}}_2, \mathcal{U})$ are two isomorphic PDFSGs. Then, there exists a bijective mapping $Q : \mathcal{X}_1 \to \mathcal{X}_2$ that satisfies

$$\tilde{\mathcal{M}}_{1\mu}(u_i)(a) = \tilde{\mathcal{M}}_{2\mu}(u_i)(Q(a)), \tilde{\mathcal{M}}_{1v}(u_i)(a) = \tilde{\mathcal{M}}_{2v}(u_i)(Q(a)), \forall a \in \mathcal{X}_1, u_i \in \mathcal{U}.$$

$$\tilde{\mathcal{N}}_{1\mu}(u_i)(ab) = \tilde{\mathcal{N}}_{2\mu}(u_i)(Q(a)Q(b)), \tilde{\mathcal{N}}_{1v}(u_i)(ab) = \tilde{\mathcal{N}}_{2v}(u_i)(Q(a)Q(b)), \forall ab \in \mathcal{E}_1,\ u_i \in \mathcal{U}.$$

By the complement of PDFSG, the membership value of an edge ab is

$$\overline{\tilde{\mathcal{N}}_{1\mu}(u_i)(ab)} = \frac{(\tilde{\mathcal{M}}_{1\mu}(u_i)(a))(\tilde{\mathcal{M}}_{1\mu}(u_i)(b))}{\tilde{\mathcal{M}}_{1\mu}(u_i)(a) + \tilde{\mathcal{M}}_{1\mu}(u_i)(b) - (\tilde{\mathcal{M}}_{1\mu}(u_i)(a))(\tilde{\mathcal{M}}_{1\mu}(u_i)(b))} - \tilde{\mathcal{N}}_{1\mu}(u_i)(ab)$$

$$= \frac{(\tilde{\mathcal{M}}_{2\mu}(u_i)(Q(a)))(\tilde{\mathcal{M}}_{2\mu}(u_i)(Q(b)))}{\tilde{\mathcal{M}}_{2\mu}(u_i)(Q(a)) + \tilde{\mathcal{M}}_{2\mu}(u_i)(Q(b)) - (\tilde{\mathcal{M}}_{2\mu}(u_i)(Q(a)))(\tilde{\mathcal{M}}_{2\mu}(u_i)(Q(b)))} - \tilde{\mathcal{N}}_{2\mu}(u_i)(Q(a)Q(b))$$

$$= \overline{\tilde{\mathcal{N}}_{2\mu}(u_i)(Q(a)Q(b))}$$

The nonmembership value of an edge ab is

$$\begin{aligned}&\overline{\tilde{\mathcal{N}}_{1v}(u_i)(ab)}\\&=\frac{\tilde{\mathcal{M}}_{1v}(u_i)(a)+\tilde{\mathcal{M}}_{1v}(u_i)(b)-2(\tilde{\mathcal{M}}_{1v}(u_i)(a))(\tilde{\mathcal{M}}_{1v}(u_i)(b))}{1-(\tilde{\mathcal{M}}_{1v}(u_i)(a))(\tilde{\mathcal{M}}_{1v}(u_i)(b))}\\&\quad-\tilde{\mathcal{N}}_{1v}(u_i)(ab)\\&=\frac{\tilde{\mathcal{M}}_{2v}(u_i)(Q(a))+\tilde{\mathcal{M}}_{2v}(u_i)(Q(b))-2(\tilde{\mathcal{M}}_{2v}(u_i)(Q(a)))(\tilde{\mathcal{M}}_{2v}(u_i)(Q(b)))}{1-(\tilde{\mathcal{M}}_{2v}(u_i)(Q(a)))(\tilde{\mathcal{M}}_{2v}(u_i)(Q(b)))}\\&\quad-\tilde{\mathcal{N}}_{2v}(u_i)(Q(a)Q(b))\\&=\overline{\tilde{\mathcal{N}}_{2v}(u_i)(Q(a)Q(b))}\end{aligned}$$

Hence, $\overline{P}_{D_1}\cong\overline{P}_{D_2}$. Similarly, the converse part can be proved. □

Proposition 2.14 *If $P_{D_1}=(\tilde{\mathcal{M}}_1,\tilde{\mathcal{N}}_1,\mathcal{U})$ and $P_{D_2}=(\tilde{\mathcal{M}}_2,\tilde{\mathcal{N}}_2,\mathcal{U})$ are two weak isomorphic PDFSGs, then the complements of P_{D_1} and P_{D_2} are also weak isomorphic.*

Proof Suppose that $P_{D_1}=(\tilde{\mathcal{M}}_1,\tilde{\mathcal{N}}_1,\mathcal{U})$ and $P_{D_2}=(\tilde{\mathcal{M}}_2,\tilde{\mathcal{N}}_2,\mathcal{U})$ are two weak isomorphic PDFSGs. Then by definition of weak isomorphism, there exists a bijective mapping $Q:\mathcal{X}_1\rightarrow\mathcal{X}_2$ that satisfies

$$\begin{aligned}&\tilde{\mathcal{M}}_{1\mu}(u_i)(a)=\tilde{\mathcal{M}}_{2\mu}(u_i)(Q(a)),\tilde{\mathcal{M}}_{1v}(u_i)(a)=\\&\quad\tilde{\mathcal{M}}_{2v}(u_i)(Q(a)),\forall a\in\mathcal{X}_1,u_i\in\mathcal{U}.\\&\tilde{\mathcal{N}}_{1\mu}(u_i)(ab)\le\tilde{\mathcal{N}}_{2\mu}(u_i)(Q(a)Q(b)),\tilde{\mathcal{N}}_{1v}(u_i)(ab)\le\\&\quad\tilde{\mathcal{N}}_{2v}(u_i)(Q(a)Q(b)),\forall ab\in\mathcal{E}_1,\ u_i\in\mathcal{U}.\end{aligned}$$

As

$$\begin{aligned}&\tilde{\mathcal{N}}_{1\mu}(u_i)(ab)\le\tilde{\mathcal{N}}_{2\mu}(u_i)(Q(a)Q(b))\\&\quad-\tilde{\mathcal{N}}_{1\mu}(u_i)(ab)\ge-\tilde{\mathcal{N}}_{2\mu}(u_i)(Q(a)Q(b))\\&T(\tilde{\mathcal{M}}_{1\mu}(u_i)(a),\tilde{\mathcal{M}}_{1\mu}(u_i)(b))-\tilde{\mathcal{N}}_{1\mu}(u_i)(ab)\\&\quad\ge T(\tilde{\mathcal{M}}_{1\mu}(u_i)(a),\tilde{\mathcal{M}}_{1\mu}(u_i)(b))-\tilde{\mathcal{N}}_{2\mu}(u_i)(Q(a)Q(b))\\&T(\tilde{\mathcal{M}}_{1\mu}(u_i)(a),\tilde{\mathcal{M}}_{1\mu}(u_i)(b))-\tilde{\mathcal{N}}_{1\mu}(u_i)(ab)\\&\quad\ge T(\tilde{\mathcal{M}}_{2\mu}(u_i)(Q(a)),\tilde{\mathcal{M}}_{2\mu}(u_i)(Q(b)))\\&\quad\quad-\tilde{\mathcal{N}}_{2\mu}(u_i)(Q(a)Q(b))\\&\overline{\tilde{\mathcal{N}}_{1\mu}(u_i)(ab)}\ge\overline{\tilde{\mathcal{N}}_{2\mu}(u_i)(Q(a)Q(b))}\end{aligned}$$

Similarly, as

$$\begin{aligned}&\tilde{\mathcal{N}}_{1v}(u_i)(ab)\le\tilde{\mathcal{N}}_{2v}(u_i)(Q(a)Q(b))\\&-\tilde{\mathcal{N}}_{1v}(u_i)(ab)\ge-\tilde{\mathcal{N}}_{2v}(u_i)(Q(a)Q(b))\\&T(\tilde{\mathcal{M}}_{1v}(u_i)(a),\tilde{\mathcal{M}}_{1v}(u_i)(b))-\tilde{\mathcal{N}}_{1v}(u_i)(ab)\\&\quad\ge T(\tilde{\mathcal{M}}_{1v}(u_i)(a),\tilde{\mathcal{M}}_{1v}(u_i)(b))-\tilde{\mathcal{N}}_{2v}(u_i)(Q(a)Q(b))\\&T(\tilde{\mathcal{M}}_{1v}(u_i)(a),\tilde{\mathcal{M}}_{1v}(u_i)(b))-\tilde{\mathcal{N}}_{1v}(u_i)(ab)\\&\quad\ge T(\tilde{\mathcal{M}}_{2v}(u_i)(Q(a)),\tilde{\mathcal{M}}_{2v}(u_i)(Q(b)))\\&\quad\quad-\tilde{\mathcal{N}}_{2v}(u_i)(Q(a)Q(b))\\&\overline{\tilde{\mathcal{N}}_{1v}(u_i)(ab)}\ge\overline{\tilde{\mathcal{N}}_{2v}(u_i)(Q(a)Q(b))}\end{aligned}$$

Hence, $\overline{P}_{D_1}$ and $\overline{P}_{D_2}$ are weak isomorphic. □

Definition 2.15 Let P_D be a PDFSG on $\mathcal{X}$. Then P_D is regular PDFSG if $\tilde{\mathbf{H}}(u_i)$ is a regular PDFG, $\forall\, u_i\in\mathcal{U}$, $i=1,2,\ldots,m$.

Definition 2.16 Let P_D be a PDFSG on $\mathcal{X}$. Then P_D is totally regular PDFSG if $\tilde{\mathbf{H}}(u_i)$ is a totally regular PDFG, $\forall\, u_i\in\mathcal{U}$.

Theorem 2.17 *Let $P_D=(\tilde{\mathcal{M}},\tilde{\mathcal{N}},\mathcal{U})$ be a regular PDFSG. Then the size of PDFGs $S(\tilde{\mathbf{H}}(u_i))=(\frac{nR_i}{2},\frac{nR_i'}{2})$, where $|\mathcal{X}|=n$ and (R_i,R_i') is the degree of a vertex in $\tilde{\mathbf{H}}(u_i),\forall\, u_i\in\mathcal{U}$.*

Theorem 2.18 *Let $P_D=(\tilde{\mathcal{M}},\tilde{\mathcal{N}},\mathcal{U})$ be a totally regular PDFSG. Then $2S(\tilde{\mathbf{H}}(u_i))+O(\tilde{\mathbf{H}}(u_i))=(nf_i,nf_i')$, where $|\mathcal{X}|=n$ and (f_i,f_i') is the total degree of a vertex in $\tilde{\mathbf{H}}(u_i),\forall\, u_i\in\mathcal{U}$.*

Proof Suppose that P_D is a PDFSG, i.e., $\tilde{\mathbf{H}}(u_i)$ is PDFG; the total degree of a vertex is

$$\begin{aligned}((\mathcal{TD})_\mu(a),(\mathcal{TD})_v(a))=\Bigg(&\sum_{a,b\neq a\in\mathcal{X}}\tilde{\mathcal{N}}_\mu(u_i)(ab)\\&+\tilde{\mathcal{M}}_\mu(u_i)(a),\sum_{a,b\neq a\in\mathcal{X}}\tilde{\mathcal{N}}_v(u_i)(ab)\\&+\tilde{\mathcal{M}}_v(u_i)(a)\Bigg),\forall\, u_i\in\mathcal{U}.\end{aligned}$$

Since P_D is totally regular PDFSG, i.e., $\tilde{\mathbf{H}}(u_i)$ is (f_i,f_i') totally regular PDFG, so

$$\begin{aligned}(f_i,f_i')=\Bigg(&\sum_{a,b\neq a\in\mathcal{X}}\tilde{\mathcal{N}}_\mu(u_i)(ab)\\&+\tilde{\mathcal{M}}_\mu(u_i)(a),\sum_{a,b\neq a\in\mathcal{X}}\tilde{\mathcal{N}}_v(u_i)(ab)\\&+\tilde{\mathcal{M}}_v(u_i)(a)\Bigg)\\(\sum_{a\in\mathcal{X}}f_i,\sum_{a\in\mathcal{X}}f_i')=\Bigg(&\sum_{a\in\mathcal{X}}\sum_{a,b\neq a\in\mathcal{X}}\tilde{\mathcal{N}}_\mu(u_i)(ab)\\&+\sum_{a\in\mathcal{X}}\tilde{\mathcal{M}}_\mu(u_i)(a),\sum_{a\in\mathcal{X}}\sum_{a,b\neq a\in\mathcal{X}}\tilde{\mathcal{N}}_v(u_i)(ab)\\&+\sum_{a\in\mathcal{X}}\tilde{\mathcal{M}}_v(u_i)(a)\Bigg)\end{aligned}$$

Since each edge is double time counted, one time for vertex a and one time for vertex b, so

$$(nf_i, nf_i') = \Bigg(2 \sum_{a,b \neq a \in \mathcal{X}} \tilde{\mathcal{N}}_\mu(u_i)(ab) + \sum_{a \in \mathcal{X}} \tilde{\mathcal{M}}_\mu(u_i)(a), 2 \sum_{a,b \neq a \in \mathcal{X}} \tilde{\mathcal{N}}_v(u_i)(ab) + \sum_{a \in \mathcal{X}} \tilde{\mathcal{M}}_v(u_i)(a)\Bigg)$$

$$(nf_i, nf_i') = 2\Bigg(\sum_{a,b \neq a \in \mathcal{X}} \tilde{\mathcal{N}}_\mu(u_i)(ab), \sum_{a,b \neq a \in \mathcal{X}} \tilde{\mathcal{N}}_v(u_i)(ab)\Bigg) + \Bigg(\sum_{a \in \mathcal{X}} \tilde{\mathcal{M}}_\mu(u_i)(a), \sum_{a \in \mathcal{X}} \tilde{\mathcal{M}}_v(u_i)(a)\Bigg)$$

$$(nf_i, nf_i') = 2S(\tilde{\mathbf{H}}(u_i)) + O(\tilde{\mathbf{H}}(u_i)) \Rightarrow 2S(\tilde{\mathbf{H}}(u_i)) + O(\tilde{\mathbf{H}}(u_i))$$

□

Theorem 2.19 *Let $P_D = (\tilde{\mathcal{M}}, \tilde{\mathcal{N}}, \mathcal{U})$ be a regular and totally regular PDFSG. Then $O(\tilde{\mathbf{H}}(u_i)) = n\{(f_i - R_i), (f_i' - R_i')\}$, where $|\mathcal{X}| = n$ and (R_i, R_i') is the degree and (f_i, f_i') is the total degree of a vertex in $\tilde{\mathbf{H}}(u_i), \forall u_i \in \mathcal{U}$.*

Proof Suppose that P_D is a regular PDFSG, i.e., $\tilde{\mathbf{H}}(u_i)$ is a regular PDFG. Then size of $\tilde{\mathbf{H}}(u_i)$ is

$$S(\tilde{\mathbf{H}}(u_i)) = (\frac{nR_i}{2}, \frac{nR_i'}{2})$$
$$2S(\tilde{\mathbf{H}}(u_i)) = (nR_i, nR_i')$$

Also, P_D is a totally regular PDFSG, i.e., $\tilde{\mathbf{H}}(e_i)$ is a totally regular PDFG, from Theorem 2.18

$$\begin{aligned} 2S(\tilde{\mathbf{H}}(u_i)) + O(\tilde{\mathbf{H}}(u_i)) &= (nf_i, nf_i') \\ O(\tilde{\mathbf{H}}(u_i)) &= (nf_i, nf_i') - 2S(\tilde{\mathbf{H}}(u_i)) \\ O(\tilde{\mathbf{H}}(u_i)) &= (nf_i, nf_i') - (nR_i, nR_i') \\ O(\tilde{\mathbf{H}}(u_i)) &= n\{(f_i, f_i') - (R_i, R_i')\} \\ O(\tilde{\mathbf{H}}(u_i)) &= n\{(f_i - R_i), (f_i' - R_i')\}. \end{aligned}$$

□

Theorem 2.20 *Consider $P_{D_1} = (\tilde{\mathcal{M}_1}, \tilde{\mathcal{N}_1}, \mathcal{U})$ is isomorphic to $P_{D_2} = (\tilde{\mathcal{M}_2}, \tilde{\mathcal{N}_2}, \mathcal{U})$:*

1. *If P_{D_1} is regular PDFSG, then P_{D_2} is also regular PDFSG.*
2. *If P_{D_1} is totally regular PDFSG, then P_{D_2} is also totally regular PDFSG.*

Proof Suppose that P_{D_1} is isomorphic to P_{D_2}, i.e., each $\tilde{\mathbf{H}}_1(u_i)$ is isomorphic to $\tilde{\mathbf{H}}_2(u_i)$ and P_{D_1} is regular PDFSG; then the degree of each vertex in $\tilde{\mathbf{H}}_1(u_i)$ is given by

$$\begin{aligned} &(\mathcal{D}_{1\mu}(a), \mathcal{D}_{1v}(a)) \\ &= \Bigg(\sum_{ab \in \mathcal{E}} \tilde{\mathcal{N}}_{1\mu}(u_i)(ab), \sum_{ab \in \mathcal{E}} \tilde{\mathcal{N}}_{1v}(u_i)(ab)\Bigg), \forall u_i \in \mathcal{U}. \\ &= (R_i, R_i'). \end{aligned}$$

Since $P_{D_1} \cong P_{D_2}$, i.e., $\tilde{\mathbf{H}}_1(u_i) \cong \tilde{\mathbf{H}}_2(u_i)$, we have

$$\begin{aligned} (R_i, R_i') &= (\mathcal{D}_{1\mu}(a), \mathcal{D}_{1v}(a)) \\ &= \Bigg(\sum_{ab \in \mathcal{E}} \tilde{\mathcal{N}}_{1\mu}(u_i)(ab), \sum_{ab \in \mathcal{E}} \tilde{\mathcal{N}}_{1v}(u_i)(ab)\Bigg) \\ &= \Bigg(\sum_{ab \in \mathcal{E}} \tilde{\mathcal{N}}_{2\mu}(u_i)(Q(a)Q(b)), \sum_{ab \in \mathcal{E}} \tilde{\mathcal{N}}_{2v}(u_i)(Q(a)Q(b))\Bigg) \\ &= (\mathcal{D}_{2\mu}(Q(a)), \mathcal{D}_{2v}(Q(a))). \end{aligned}$$

Therefore, $\tilde{\mathbf{H}}_2(u_i)$ is (R_i, R_i')-regular PDFG. Hence, P_{D_2} is regular PDFSG.

Assume that P_{D_1} is totally regular PDFSG; then the total degree of each vertex in $\tilde{\mathbf{H}}_1(u_i)$ is

$$\begin{aligned} &\big((\mathcal{TD})_{1\mu}(a), (\mathcal{TD})_{1v}(a)\big) \\ &= \Bigg(\sum_{ab \in \mathcal{E}} \tilde{\mathcal{N}}_{1\mu}(u_i)(ab) + \tilde{\mathcal{M}}_{1\mu}(u_i)(a), \sum_{ab \in \mathcal{E}} \tilde{\mathcal{N}}_{1v}(u_i)(ab) + \tilde{\mathcal{M}}_{1v}(u_i)(a)\Bigg), \forall u_i \in \mathcal{U}. \\ &= (f_i, f_i') \end{aligned}$$

Since $P_{D_1} \cong P_{D_2}$, i.e., $\tilde{\mathbf{H}}_1(u_i) \cong \tilde{\mathbf{H}}_2(u_i)$, we have

$$\begin{aligned} (f_i, f_i') &= \big((\mathcal{TD})_{1\mu}(a), (\mathcal{TD})_{1v}(a)\big) \\ &= \Bigg(\sum_{ab \in \mathcal{E}} \tilde{\mathcal{N}}_{1\mu}(u_i)(ab) + \tilde{\mathcal{M}}_{1\mu}(u_i)(a), \sum_{ab \in \mathcal{E}} \tilde{\mathcal{N}}_{1v}(u_i)(ab) + \tilde{\mathcal{M}}_{1v}(u_i)(a)\Bigg) \\ &= \Bigg(\sum_{ab \in \mathcal{E}} \tilde{\mathcal{N}}_{2\mu}(u_i)(Q(a)Q(b)) + \tilde{\mathcal{M}}_{2\mu}(u_i)(Q(a)), \sum_{ab \in \mathcal{E}} \tilde{\mathcal{N}}_{2v}(u_i)(Q(a)Q(b)) + \tilde{\mathcal{M}}_{2v}(u_i)(Q(a))) \\ &= \big((\mathcal{TD})_{2\mu}(Q(a)), (\mathcal{TD})_{2v}(Q(a))\big). \end{aligned}$$

Therefore, $\tilde{\mathbf{H}}_2(u_i)$ is (f_i, f_i')-totally regular PDFG. Hence, P_{D_2} is totally regular PDFSG. □

Theorem 2.21 *Let $P_D = (\tilde{\mathcal{M}}, \tilde{\mathcal{N}}, \mathcal{U})$ be a PDFSG. If $\tilde{\mathcal{M}}$ is a constant function in $\tilde{\mathbf{H}}(u_i), \forall u_i \in \mathcal{U}$, then following statements are equivalent:*

1. P_D *is regular PDFSG.*
2. P_D *is totally regular PDFSG.*

Proof Suppose that $\tilde{\mathcal{M}}$ is a constant function in $\tilde{\mathbf{H}}(u_i)$, i.e., $\tilde{\mathcal{M}}_\mu(u_i)(a) = c_i$ and $\tilde{\mathcal{M}}_v(u_i)(a) = c_i'$. Also, suppose that P_D is regular PDFSG, i.e., $\tilde{\mathbf{H}}(u_i)$ is (R_i, R_i')-regular PDFG, then

$$\mathcal{D}(a) = \left(\sum_{a,b\neq a\in\mathcal{X}} \tilde{\mathcal{N}}_\mu(u_i)(ab), \sum_{a,b\neq a\in\mathcal{X}} \tilde{\mathcal{N}}_v(u_i)(ab)\right) = (R_i, R_i').$$

The total degree of a vertex is

$$\begin{aligned}\mathcal{TD}(a) &= \left(\sum_{a,b\neq a\in\mathcal{X}} \tilde{\mathcal{N}}_\mu(u_i)(ab) + \tilde{\mathcal{M}}_\mu(u_i)(a), \sum_{a,b\neq a\in\mathcal{X}} \tilde{\mathcal{N}}_v(u_i)(ab) + \tilde{\mathcal{M}}_v(u_i)(a)\right)\\ &= (R_i + c_i, R_i' + c_i')\\ &= (f_i, f_i').\end{aligned}$$

Therefore, $\tilde{\mathbf{H}}(u_i)$ is (f_i, f_i')-totally regular PDFG. Hence, P_D is totally regular PDFSG. Therefore, $(1) \Rightarrow (2)$ is proved.

Now suppose that P_D is totally regular PDFSG, i.e., $\tilde{\mathbf{H}}(u_i)$ is (f_i, f_i')-totally regular PDFG, then

$$\mathcal{TD}(a) = \left((\mathcal{TD})_\mu(a), (\mathcal{TD})_v(a)\right)$$

$$(f_i, f_i') = \left(\sum_{a,b\neq a\in\mathcal{X}} \tilde{\mathcal{N}}_\mu(u_i)(ab) + \tilde{\mathcal{M}}_\mu(u_i)(a), \sum_{a,b\neq a\in\mathcal{X}} \tilde{\mathcal{N}}_v(u_i)(ab) + \tilde{\mathcal{M}}_v(u_i)(a)\right)$$

$$(f_i, f_i') = \left(\sum_{a,b\neq a\in\mathcal{X}} \tilde{\mathcal{N}}_\mu(u_i)(ab) + \tilde{\mathcal{M}}_\mu(u_i)(a), \sum_{a,b\neq a\in\mathcal{X}} \tilde{\mathcal{N}}_v(u_i)(ab) + \tilde{\mathcal{M}}_v(u_i)(a)\right)$$

$$(f_i, f_i') = \left(\sum_{a,b\neq a\in\mathcal{X}} \tilde{\mathcal{N}}_\mu(u_i)(ab) + c_i, \sum_{a,b\neq a\in\mathcal{X}} \tilde{\mathcal{N}}_v(u_i)(ab) + c_i'\right)$$

$$(f_i - c_i, f_i' - c_i') = \left(\sum_{a,b\neq a\in\mathcal{X}} \tilde{\mathcal{N}}_\mu(u_i)(ab), \sum_{a,b\neq a\in\mathcal{X}} \tilde{\mathcal{N}}_v(u_i)(ab)\right)$$

$$\begin{aligned}\left(\mathcal{D}_\mu(a), \mathcal{D}_v(a)\right) &= (f_i - c_i, f_i' - c_i')\\ &= (R_i, R_i').\end{aligned}$$

Therefore, $\tilde{\mathbf{H}}(u_i)$ is (R_i, R_i')-regular PDFG. Hence, P_D is regular PDFSG. Therefore, $(2) \Rightarrow (1)$ is proved. □

Theorem 2.22 *Let P_D be a PDFSG. If P_D is regular and totally regular PDFSG, then $\tilde{\mathcal{M}}$ is a constant function, $\forall\, u_i \in \mathcal{U}$.*

Proof Suppose that P_D is regular and totally regular PDFSG, i.e., $\tilde{\mathbf{H}}(u_i)$ is regular and totally regular PDFG. Then the degree of vertex is

$$\begin{aligned}\left(\mathcal{D}_\mu(a), \mathcal{D}_v(a)\right) &= \left(\sum_{a,b\neq a\in\mathcal{X}} \tilde{\mathcal{N}}_\mu(u_i)(ab), \sum_{a,b\neq a\in\mathcal{X}} \tilde{\mathcal{N}}_v(u_i)(ab)\right)\\ &= (R_i, R_i')\end{aligned}$$

and the total degree of vertex is

$$\begin{aligned}\left((\mathcal{TD})_\mu(a), (\mathcal{TD})_v(a)\right) &= \left(\sum_{a,b\neq a\in\mathcal{X}} \tilde{\mathcal{N}}_\mu(u_i)(ab) + \tilde{\mathcal{M}}_\mu(u_i)(a), \sum_{a,b\neq a\in\mathcal{X}} \tilde{\mathcal{N}}_v(u_i)(ab) + \tilde{\mathcal{M}}_v(u_i)(a)\right)\\ &= (f_i, f_i').\end{aligned}$$

Now, $\left((\mathcal{TD})_\mu(a), (\mathcal{TD})_v(a)\right) = (f_i, f_i')$

$$\left(R_i + \tilde{\mathcal{M}}_\mu(u_i)(a), R_i' + \tilde{\mathcal{M}}_v(u_i)(a)\right) = (f_i, f_i')$$

$$\begin{aligned}\left(\tilde{\mathcal{M}}_\mu(u_i)(a), \tilde{\mathcal{M}}_v(u_i)(a)\right) &= (f_i - R_i, f_i' - R_i')\\ &= (c_i, c_i')\end{aligned}$$

$\Rightarrow \tilde{\mathcal{M}}$ is a constant function, $\forall\, u_i \in \mathcal{U}$. □

Remark Converse of Theorem 2.22 may not be true in general:

Consider a PDFSG $P_G = \{\tilde{\mathbf{H}}(u_1)\}$ as shown in Fig. 4. Since $\mathcal{D}(a_1) = (1.6, 1.8) \neq \mathcal{D}(a_3) = (1.4, 1.8)$. Also, $\mathcal{TD}(a_1) = (2.2, 2.5) \neq \mathcal{TD}(a_3) = (2.0, 2.5)$. Therefore, $\tilde{\mathbf{H}}(u_1)$ is neither regular nor totally regular PDFG. Hence, P_D is neither regular nor totally regular PDFSG.

Definition 2.23 A PDFSG P_D on n vertices is said to be strongly regular if the following properties hold:

1. P_D is regular PDFSG, i.e., $\tilde{\mathbf{H}}(e_i)$ is regular PDFG of degree (R_i, R_i'),
2. the sum of the membership and nonmembership values of the common neighboring vertices of any pair of adjoining vertices of $\tilde{\mathbf{H}}(u_i)$ is equal and represented by $M_i^* = (M_i, M_i')$,
3. the sum of the membership and nonmembership values of the common neighboring vertices of any pair of nonadjoining vertices of $\tilde{\mathbf{H}}(u_i)$ is equal and represented by $N_i^* = (N_i, N_i'), \forall\, u_i \in \mathcal{U}$.

Example 2.24 Consider a PDFSG $P_D = \{\tilde{\mathbf{H}}(u_1)\}$ as shown in Fig. 5.

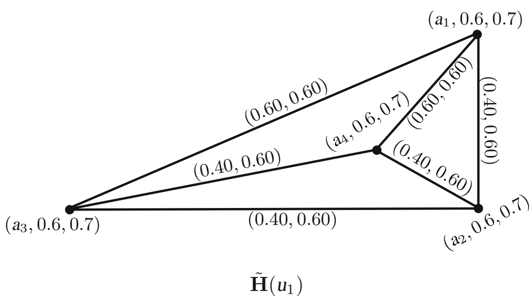


Fig. 4 PDFSG $P_D = \{\tilde{\mathbf{H}}(u_1)\}$

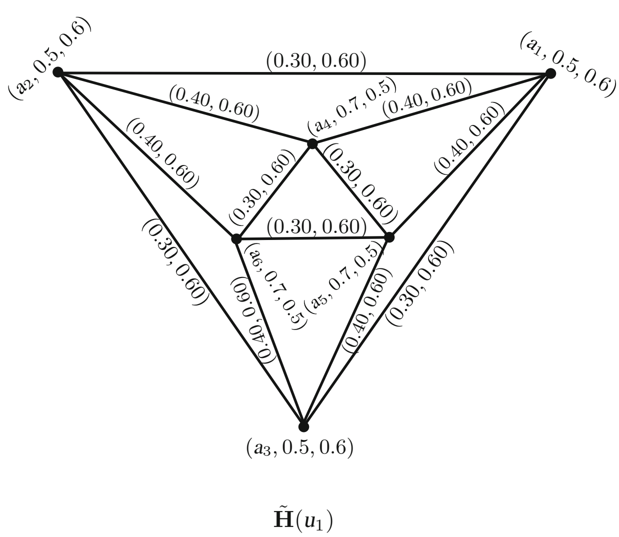


Fig. 5 Strongly regular PDFSG $P_D = \{\tilde{\mathbf{H}}(u_1)\}$

Clearly, in $\tilde{\mathbf{H}}(u_1)$, $\mathcal{D}(a_1) = (1.4, 2.4) = \mathcal{D}(a_2) = \mathcal{D}(a_3) = D(a_4) = D(a_5) = D(a_6)$ and $M_1^* = (1.2, 1.1)$ and $N_1^* = (2.4, 2.2)$. This implies that $\tilde{\mathbf{H}}(u_1)$ is a strongly regular PDFG. So, $P_D = \{\tilde{\mathbf{H}}(u_1)\}$ is a strongly regular PDFSG.

Definition 2.25 A PDFSG P_D is known as bipartite if $\mathcal{X} = \{a_1, a_2, \ldots, a_n\}$ can be partitioned into two nonempty disjoint sets $\mathcal{X}_1$ and $\mathcal{X}_2$ such that $\tilde{\mathcal{N}}_\mu(u_i)(a_j a_k) = 0$ and $\tilde{\mathcal{N}}_\nu(u_i)(a_j a_k) = 0$ if $a_j, a_k \in \mathcal{X}_1$ or $a_j, a_k \in \mathcal{X}_2$. Furthermore, if

$$\tilde{\mathcal{N}}_\mu(u_i)(a_j a_k) = \frac{(\tilde{\mathcal{M}}_\mu(u_i)(a_j))(\tilde{\mathcal{M}}_\mu(u_i)(a_k))}{\tilde{\mathcal{M}}_\mu(u_i)(a_j) + \tilde{\mathcal{M}}_\mu(u_i)(a_k) - (\tilde{\mathcal{M}}_\mu(u_i)(a_j))(\tilde{\mathcal{M}}_\mu(u_i)(a_k))}, \tag{5}$$

$$\tilde{\mathcal{N}}_\nu(u_i)(a_j a_k) = \frac{\tilde{\mathcal{M}}_\nu(u_i)(a_j) + \tilde{\mathcal{M}}_\nu(u_i)(a_k) - 2(\tilde{\mathcal{M}}_\nu(u_i)(a_j))(\tilde{\mathcal{M}}_\nu(u_i)(a_k))}{1 - (\tilde{\mathcal{M}}_\nu(u_i)(a_j))(\tilde{\mathcal{M}}_\nu(u_i)(a_k))}, \tag{6}$$

$\forall\, a_j \in \mathcal{X}_1$ and $a_k \in \mathcal{X}_2, u_i \in \mathcal{U}$, then P_D is complete bipartite PDFSG.

Definition 2.26 A bipartite PDFSG $P_D = \{\tilde{\mathcal{M}}, \tilde{\mathcal{N}}, \mathcal{U}\}$ is known as biregular if each vertex in $\mathcal{X}_1$ and $\mathcal{X}_2$ has equal degree $\alpha_i^* = (\alpha_i, \alpha_i')$ and $\beta_i^* = (\beta_i, \beta_i')$, respectively, where α_i^*, β_i^* are constants, $\forall\, u_i \in \mathcal{U}$.

Example 2.27 Consider a nonempty set $\mathcal{X}$ which is partitioned into two nonempty sets $\mathcal{X}_1 = \{a_1, a_2, a_3\}$ and $\mathcal{X}_2 = \{a_4, a_5, a_6\}$ such that $\mathcal{E} = \{a_1a_5, a_1a_6, a_2a_4, a_2a_5, a_3a_4, a_3a_6\}$. In $\mathcal{X}_1$ and $\mathcal{X}_2$, each vertex has equal degree. So, $\tilde{\mathbf{H}}(u_1)$ is biregular PDFG. Hence, P_D is a biregular PDFSG as shown in Fig. 6.

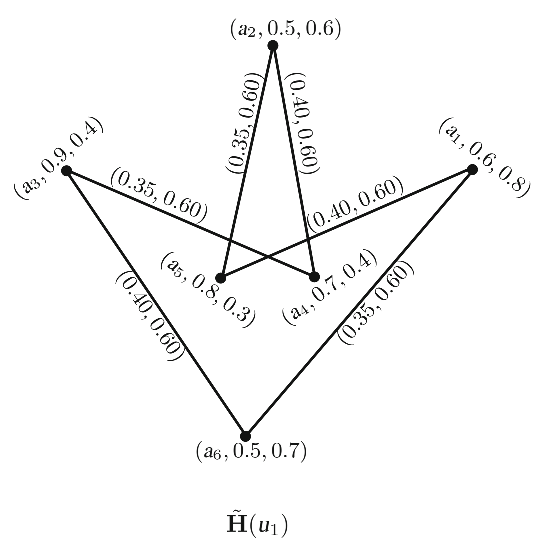


Fig. 6 Biregular PDFSG $P_D = \{\tilde{\mathbf{H}}(u_1)\}$

Theorem 2.28 *Let $P_D = (\tilde{\mathcal{M}}, \tilde{\mathcal{N}}, \mathcal{U})$ be a PDFSG. If P_D is complete and $\tilde{\mathcal{M}}$, $\tilde{\mathcal{N}}$ are constant functions, then $P_D = (\tilde{\mathcal{M}}, \tilde{\mathcal{N}}, \mathcal{U})$ is strongly regular PDFSG.*

Proof Suppose that P_D is a complete PDFSG with $\mathcal{X} = \{a_1, a_2, \ldots, a_n\}$. As $\tilde{\mathcal{M}}$ and $\tilde{\mathcal{N}}$ are constant functions, $\tilde{\mathcal{M}}_\mu(u_i)(a) = c_i, \tilde{\mathcal{M}}_v(u_i)(a_j) = c'_i, \forall\, u_i \in \mathcal{U}, \quad a_j \in \mathcal{X}$ and $\tilde{\mathcal{N}}_\mu(u_i)(a_ja_k) = d_i, \tilde{\mathcal{N}}_v(u_i)(a_ja_k) = d'_i, \forall\, u_i \in \mathcal{U}, \quad a_ja_k \in \mathcal{E}$. To prove that $P_D = (\tilde{\mathcal{M}}, \tilde{\mathcal{N}}, \mathcal{U})$ is a strongly regular PDFSG, we show that P_D is regular PDFSG, i.e., $\tilde{\mathbf{H}}(u_i)$ is (R_i, R'_i)-regular PDFG, $\forall\, u_i \in \mathcal{U}$. Furthermore, the adjoining and nonadjoining vertices have equal common neighborhood $M^*_i = (M_i, M'_i)$ and $N^*_i = (N_i, N'_i)$, respectively.

As P_D is complete PDFSG, i.e., $\tilde{\mathbf{H}}(u_i)$ is complete PDFG, $\forall\, u_i \in \mathcal{U}$. So

$$
\begin{aligned}
&\Big((\mathcal{TD})_\mu(a_j), (\mathcal{TD})_v(a_j)\Big)\\
&\quad = \Bigg(\sum_{a_j, a_k \neq a_j \in \mathcal{X}} \tilde{\mathcal{N}}_\mu(u_i)(a_ja_k)\\
&\qquad + \tilde{\mathcal{M}}_\mu(u_i)(a_j), \sum_{a_j, a_k \neq a_j \in \mathcal{X}} \tilde{\mathcal{N}}_v(u_i)(a_ja_k)\\
&\qquad + \tilde{\mathcal{M}}_v(u_i)(a_j)\Big)\\
&\quad = \big((n-1)d_i, (n-1)d'_i\Bigg)
\end{aligned}
$$

$\Rightarrow \tilde{\mathbf{H}}(u_i)$ is $\big((n-1)d_i, (n-1)d'_i\big)$-regular PDFG. Hence, P_D is regular PDFSG. Furthermore, the sum of the membership and nonmembership values of common neighboring vertices of any pair of adjoining vertices of any pair of adjoining vertices $M^*_i = (M_i, M'_i) = \big((n-1)c_i, (n-1)c'_i\big)$ is equal. As $\tilde{\mathbf{H}}(u_i)$ is complete, the sum of the membership and nonmembership values of common neighboring vertices of any pair of nonadjoining vertices of any pair of adjoining vertices $N^*_i = (N_i, N'_i) = (0, 0)$ is equal. As all conditions are fulfilled, so $\tilde{\mathbf{H}}(u_i)$ is strongly regular PDFG, $\forall\, u_i \in \mathcal{U}$. Hence, P_D is a strongly PDFSG. □

Theorem 2.29 *Let $P_D = (\tilde{\mathcal{M}}, \tilde{\mathcal{N}}, \mathcal{U})$ be a PDFSG. If P_D is strongly regular and strong, then $\overline{P}_D$ is a regular PDFSG.*

Proof Suppose that P_D is strongly regular PDFSG; then by definition P_D is regular, i.e., $\tilde{\mathbf{H}}(u_i)$ is regular, $\forall\, u_i \in \mathcal{U}$. Also, P_D is strong, and then $\overline{P}_D$ is also strong. Therefore,

$$
\begin{aligned}
&\overline{\tilde{\mathcal{N}}_\mu(u_i)(ab)}\\
&= \begin{cases} \dfrac{(\tilde{\mathcal{M}}_\mu(u_i)(a))(\tilde{\mathcal{M}}_\mu(u_i)(b))}{\tilde{\mathcal{M}}_\mu(u_i)(a) + \tilde{\mathcal{M}}_\mu(u_i)(b) - (\tilde{\mathcal{M}}_\mu(u_i)(a))(\tilde{\mathcal{M}}_\mu(u_i)(b))}, & \text{if } \tilde{\mathcal{N}}_\mu(u_i)(ab) = 0\\ 0, & \text{if } 0 \leq \tilde{\mathcal{N}}_\mu(u_i)(ab) \leq 1. \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
&\overline{\tilde{\mathcal{N}}_v(u_i)(ab)}\\
&= \begin{cases} \dfrac{\tilde{\mathcal{M}}_v(u_i)(a) + \tilde{\mathcal{M}}_v(u_i)(b) - 2(\tilde{\mathcal{M}}_v(u_i)(a))(\tilde{\mathcal{M}}_v(u_i)(b))}{1 - (\tilde{\mathcal{M}}_v(u_i)(a))(\tilde{\mathcal{M}}_v(u_i)(b))}, & \text{if } \tilde{\mathcal{N}}_v(u_i)(ab) = 0\\ 0, & \text{if } 0 \leq \tilde{\mathcal{N}}_v(u_i)(ab) \leq 1. \end{cases}
\end{aligned}
$$

In $\overline{\tilde{\mathbf{H}}(u_i)}$, the degree of vertex a is

$$
\begin{aligned}
&(\overline{\mathcal{D}}_\mu(a), \overline{\mathcal{D}}_v(a))\\
&\quad = \left(\sum_{a, b \neq a \in \mathcal{X}} \overline{\tilde{\mathcal{N}}_\mu(u_i)(ab)}, \sum_{a, b \neq a \in \mathcal{X}} \overline{\tilde{\mathcal{N}}_v(u_i)(ab)}\right)\\
&\quad = \left(\sum_{a, b \neq a \in \mathcal{X}} \frac{(\tilde{\mathcal{M}}_\mu(u_i)(a))(\tilde{\mathcal{M}}_\mu(u_i)(b))}{\tilde{\mathcal{M}}_\mu(u_i)(a) + \tilde{\mathcal{M}}_\mu(u_i)(b) - (\tilde{\mathcal{M}}_\mu(u_i)(a))(\tilde{\mathcal{M}}_\mu(u_i)(b))},\right.\\
&\qquad \left.\sum_{a, b \neq a \in \mathcal{X}} \frac{\tilde{\mathcal{M}}_v(u_i)(a) + \tilde{\mathcal{M}}_v(u_i)(b) - 2(\tilde{\mathcal{M}}_v(u_i)(a))(\tilde{\mathcal{M}}_v(u_i)(b))}{1 - (\tilde{\mathcal{M}}_v(u_i)(a))(\tilde{\mathcal{M}}_v(u_i)(b))}\right)\\
&\quad = (R_i, R'_i).
\end{aligned}
$$

$\Rightarrow \overline{\tilde{\mathbf{H}}(u_i)}$ is (R_i, R'_i)-regular PDFG. Hence, $\overline{P}_D$ is regular PDFSG. □

Corollary 2.30 *Let $P_D = (\tilde{\mathcal{M}}, \tilde{\mathcal{N}}, \mathcal{U})$ be a PDFSG. If $\overline{P}_D$ is strongly regular and strong, then P_D is a regular PDFSG.*

Theorem 2.31 *Consider $P_D = (\tilde{\mathcal{M}}, \tilde{\mathcal{N}}, \mathcal{U})$ be a PDFSG; then P_D is strongly regular PDFSG if and only if $\overline{P}_D$ is a strongly regular PDFSG.*

Proof Suppose that P_D is strongly regular PDFSG; then, P_D is regular PDFSG. Also, the adjoining and nonadjoining vertices of $\tilde{\mathbf{H}}(u_i)$ have equal common neighborhood $M^*_i = (M_i, M'_i)$ and $N^*_i = (N_i, N'_i)$,, respectively. To prove that $\overline{P}_D$ is a strongly regular PDFSG, we show that $\overline{P}_D$ is regular PDFSG. Since P_D is strongly regular and strong PDFSG, then by Theorem 2.29, $\overline{P}_D$ is regular PDFSG. Moreover, suppose that $S_i = \{(a_j, a_k) \mid (a_j, a_k) \in \mathcal{E}\}$ and $S'_i = \{(a_j, a_k) \mid (a_j, a_k) \notin \mathcal{E}\}$ be the sets of all adjoining and nonadjoining vertices of $\tilde{\mathbf{H}}(u_i)$, where a_j and a_k have equal common neighborhood $M^*_i = (M_i, M'_i)$ and $N^*_i = (N_i, N'_i)$, respectively. Then $\overline{S_i} = \{(a_j, a_k) \mid (a_j, a_k) \in \overline{\mathcal{E}}\}$ and $\overline{S'_i} = \{(a_j, a_k) \mid (a_j, a_k) \notin \overline{\mathcal{E}}\}$ be the sets of all adjoining and nonadjoining vertices of $\overline{\tilde{\mathbf{H}}(u_i)}$, where a_j and a_k have equal common neighborhood $N^*_i = (N_i, N'_i)$ and $M^*_i = (M_i, M'_i)$, respectively. Hence, $\overline{P_D}$ is strongly regular PDFSG.

Conversely, $\overline{P}_D$ is strongly regular PDFSG; then, $\overline{P}_D$ is regular PDFSG. Also, the adjoining and nonadjoining vertices of $\overline{\tilde{\mathbf{H}}(u_i)}$ have equal common neighborhood $N^*_i = (N_i, N'_i)$ and $M^*_i = (M_i, M'_i)$, respectively. To prove that P_D

is a strongly regular PDFSG, we must show that P_D is regular PDFSG. Since $\overline{P}_D$ is strongly regular and strong PDFSG, then by Corollary 2.30, P_D is regular PDFSG. Moreover, suppose that $\overline{S_i} = \{(a_j, a_k) \mid (a_j, a_k) \in \overline{\mathcal{E}}\}$ and $\overline{S_i'} = \{(a_j, a_k) \mid (a_j, a_k) \notin \overline{\mathcal{E}}\}$ be the sets of all adjoining and nonadjoining vertices of $\overline{\tilde{\mathbf{H}}(u_i)}$, where a_j and a_k have equal common neighborhood $N_i^* = (N_i, N_i')$ and $M_i^* = (M_i, M_i')$, respectively. Then $S_i = \{(a_j, a_k) \mid (a_j, a_k) \in \mathcal{E}\}$ and $S_i' = \{(a_j, a_k) \mid (a_j, a_k) \notin \mathcal{E}\}$ be the sets of all adjoining and nonadjoining vertices of vertices of $\tilde{\mathbf{H}}(u_i)$, where a_j and a_k have equal common neighborhood $M_i^* = (M_i, M_i')$ and $N_i^* = (N_i, N_i')$, respectively. Hence, P_D is strongly regular PDFSG. □

Definition 2.32 Let $P_D = (\tilde{\mathcal{M}}, \tilde{\mathcal{N}}, \mathcal{U})$ be a PDFSG and $(\mathcal{D})_{i1}, (\mathcal{D})_{i2}, \ldots, (\mathcal{D})_{ir}$ be the degree of vertices in $\tilde{\mathbf{H}}(e_i)$ of P_D. Then the degree sequence is expressed by $((\mathcal{D})_{i1}, (\mathcal{D})_{i2}, \ldots, (\mathcal{D})_{ir}) = ((\mathcal{D})_{\mu_{i1}}, (\mathcal{D})_{\mu_{i2}}, \ldots, (\mathcal{D})_{\mu_{ir}}; (\mathcal{D})_{\nu_{i1}}, (\mathcal{D})_{\nu_{i2}}, \ldots, (\mathcal{D})_{\nu_{ir}})$, where $(\mathcal{D})_{\mu_{i1}} \geq (\mathcal{D})_{\mu_{i2}} \geq (\mathcal{D})_{\mu_{i3}} \geq \cdots \geq (\mathcal{D})_{\mu_{ir}}$ and $(\mathcal{D})_{\nu_{i1}} \geq (\mathcal{D})_{\nu_{i2}} \geq (\mathcal{D})_{\nu_{i3}} \geq \cdots \geq (\mathcal{D})_{\nu_{ir}}$.

Definition 2.33 The set of different positive real values arising in the degree sequence of $\tilde{\mathbf{H}}(u_i)$ of a PDFSG P_D is known as the degree set of $\tilde{\mathbf{H}}(u_i)$.

Example 2.34 Consider a PDFSG $P_D = \{\tilde{\mathbf{H}}(u_1)\}$ as shown in Fig. 7.

In $\tilde{\mathbf{H}}(u_1)$, the degree of vertices is $\mathcal{D}(a_1) = (1.4, 2.0)$, $\mathcal{D}(a_2) = (1, 1.1), \mathcal{D}(a_3) = (2.1, 1.5)$, $\mathcal{D}(a_4) = (1.6, 1.9)$, $\mathcal{D}(a_5) = (0.85, 1.4), \mathcal{D}(a_6) = (1.75, 1.9)$. Hence, the degree sequence of the membership values and nonmembership values in Fig. 7 is (2.1, 1.75, 1.6, 1.4, 1, 0.85) and (2.0, 1.9, 1.9, 1.5, 1.4, 1.1), whereas the corresponding degree sets are $\{2.1, 1.75, 1.6, 1.4, 1, 0.85\}$ and $\{2.0, 1.9, 1.5, 1.4, 1.1\}$.

Theorem 2.35 *Let P_D be a strongly regular PDFSG; then the degree sequence of n elements of $\tilde{\mathbf{H}}(u_i)$ is a constant sequence $(R_i, R_i, \ldots, R_i; R_i', R_i', \ldots, R_i')$.*

Proof Suppose that P_D is a strongly regular PDFSG; then, P_D is a regular PDFSG, i.e., $\tilde{\mathbf{H}}(u_i)$ is a (R_i, R_i')-regular PDFG. Thus the degree of all vertices in $\tilde{\mathbf{H}}(u_i)$ is $\mathcal{D}(a) = (R_i, R_i'), \forall\, a \in \mathcal{X}$. Hence, the degree sequence of $\tilde{\mathbf{H}}(u_i)$ is a constant sequence $(R_i, R_i, \ldots, R_i; R_i', R_i', \ldots, R_i')$. □

Theorem 2.36 *Let P_D be a strongly regular PDFSG; then the degree set of the membership and nonmembership values of $\tilde{\mathbf{H}}(u_i)$ is a singleton set $\{R_i\}$ and $\{R_i'\}$, respectively.*

Proof Suppose that P_D is a strongly regular PDFSG; then by definition, P_D is a regular PDFSG, i.e., $\tilde{\mathbf{H}}(u_i)$ is a (R_i, R_i')-regular PDFG. Thus the degree of all vertices in $\tilde{\mathbf{H}}(u_i)$ is $\mathcal{D}(a) = (R_i, R_i'), \forall\, a \in \mathcal{X}$. As the degree sequence of $\tilde{\mathbf{H}}(u_i)$ is a constant sequence $(R_i, R_i, \ldots, R_i; R_i', R_i', \ldots, R_i')$, then the corresponding membership and nonmembership degree set is $\{R_i\}$ and $\{R_i'\}$, respectively. □

Remark Converse of Theorems 2.35 and 2.36 may not be true in general.

Consider a PDFSG $P_D = \{\tilde{\mathbf{H}}(u_1)\}$ as shown in Fig. 8.

It can be seen from Fig. 8 that $\tilde{\mathbf{H}}(u_1)$ has constant membership and nonmembership degree sequence (1.2, 1.2, 1.2, 1.2, 1.2, 1.2) and (1.5, 1.5, 1.5, 1.5, 1.5, 1.5), respectively, whereas the corresponding membership and nonmembership degree set is $\{1.2\}$ and $\{1.5\}$, respectively. But the sum of membership and nonmembership values of the common neighboring vertices of any pair of adjoining vertices of $\tilde{\mathbf{H}}(u_1)$ is not equal. Hence, P_D is not strongly regular PDFSG.

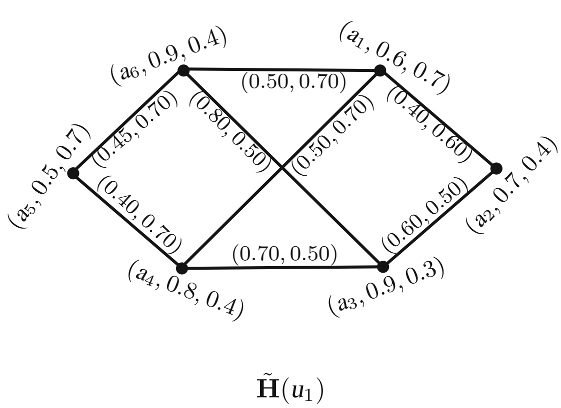


Fig. 7 PDFSG $P_D = \{\tilde{\mathbf{H}}(u_1)\}$

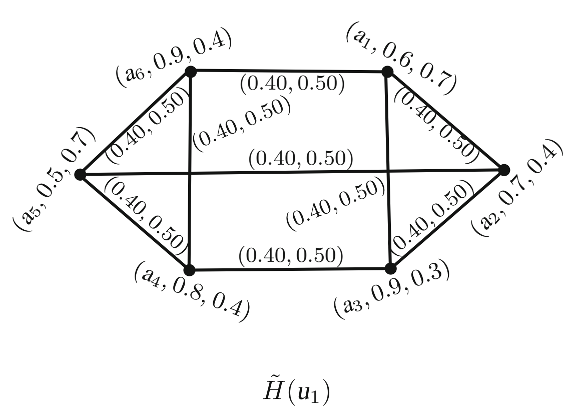


Fig. 8 PDFSG $P_D = \{\tilde{\mathbf{H}}(u_1)\}$

Definition 2.37 Let $\tilde{\mathcal{N}} = \{(ab, \tilde{\mathcal{N}}_\mu(u_i)(ab), \tilde{\mathcal{N}}_v(u_i)(ab)) \mid ab \in \mathcal{E}\}$ be a PDFS edge set in PDSFG P_D; then

- The degree of edge $ab \in \mathcal{E}$ is denoted by $\mathcal{D}(ab)$ and defined by $\mathcal{D}(ab) = ((\mathcal{D})_\mu(ab), (\mathcal{D})_v(ab))$, where

$$(\mathcal{D})_\mu(ab) = \mathcal{D}_\mu(a) + \mathcal{D}_\mu(b) - 2\tilde{\mathcal{N}}_\mu(ab), \forall u_i \in \mathcal{U} \tag{7}$$

$$(\mathcal{D})_v(ab) = \mathcal{D}_v(a) + \mathcal{D}_v(b) - 2\tilde{\mathcal{N}}_v(ab), \forall u_i \in \mathcal{U}. \tag{8}$$

- The total degree of edge $ab \in \mathcal{E}$ is denoted by $\mathcal{TD}(ab)$ and defined by $\mathcal{TD}(ab) = ((\mathcal{TD})_\mu(ab), (\mathcal{TD})_v(ab))$, where

$$(\mathcal{TD})_\mu(ab) = \mathcal{D}_\mu(a) + \mathcal{D}_\mu(b) - \tilde{\mathcal{N}}_\mu(ab), \forall u_i \in \mathcal{U}. \tag{9}$$

$$(\mathcal{TD})_v(ab) = \mathcal{D}_v(a) + \mathcal{D}_v(b) - \tilde{\mathcal{N}}_v(ab), \forall u_i \in \mathcal{U}. \tag{10}$$

Example 2.38 Consider a PDFSG $P_D = \{\tilde{\mathbf{H}}(u_1)\}$ as shown in Fig. 9.

In $\tilde{\mathbf{H}}(u_1)$, the degree of an edge a_1a_2 is $(\mathcal{D})(a_1a_2) = \Big((\mathcal{D}_\mu(a_1) + \mathcal{D}_\mu(a_2) - 2\tilde{\mathcal{N}}_\mu(a_1a_2)), \quad (\mathcal{D}_v(a_1) + \mathcal{D}_v(a_2) - 2\tilde{\mathcal{N}}_v(a_1a_2))\Big) = (0.8, 1.45)$ and the total degree of an edge a_1a_2 is $(\mathcal{TD})(a_1a_2) = \Big(((\mathcal{D})_\mu(a_1) + (\mathcal{D})_\mu(a_2) - \tilde{\mathcal{N}}_\mu(a_1a_2)), \quad ((\mathcal{D})_v(a_1) + (\mathcal{D})_v(a_2) - \tilde{\mathcal{N}}_v(a_1a_2))\Big) = (1.4, 2.15)$.

Remark If P_D is a strongly regular PDFSG, then the membership and nonmembership degree sequence of edge in $\tilde{\mathbf{H}}(u_i)$ need not to be constant sequence.

Consider a PDFSG $P_D = \{\tilde{\mathbf{H}}(u_i)\}$ as shown in Fig. 10. It can be seen from Fig. 10 that P_D is strongly regular PDFSG because $\tilde{\mathbf{H}}(u_1)$ is strongly regular PDFG with $R_1 = (2.3, 1.4)$, $M_1 = (1.8, 0.8)$ and $N_1 = (0, 0)$. The edge degree sequence of the membership values and nonmembership values is (3.2, 3.2, 3, 3, 3, 3) and (2, 2, 1.8, 1.8, 1.8, 1.8), respectively, whereas the corresponding edge degree sets $\{3.2, 3\}$ and $\{2, 1.8\}$ are not constant sequence.

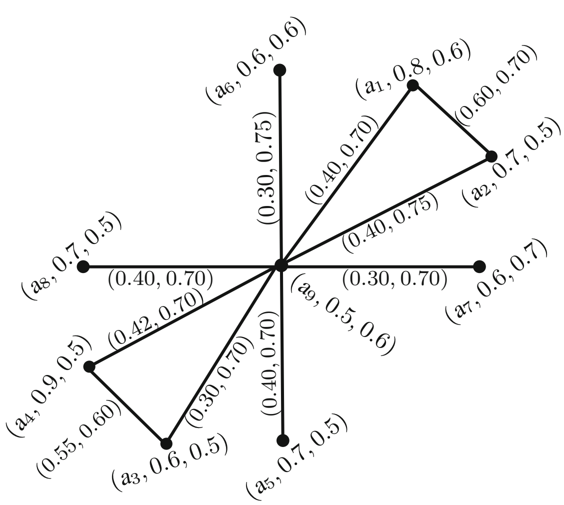
Theorem 2.39 *Let P_D be a strongly regular PDFSG with $\tilde{\mathcal{N}}$ is a constant function; then the edge degree sequence and edge degree set are constant sequence and singleton set, respectively.*

Proof Suppose that $\tilde{\mathcal{N}}_\mu(u_i)(ab) = c_i, \tilde{\mathcal{N}}_v(u_i)(ab) = c'_i, \forall u_i \in \mathcal{U}, ab \in \mathcal{E}$ and P_D is a strongly regular PDFSG; then P_D is regular PDFSG, i.e., $\tilde{\mathbf{H}}(u_i)$ is (R_i, R'_i)-regular PDFG such that

$$\mathcal{D}_\mu(a) = \sum_{a,b \neq a \in \mathcal{X}} \tilde{\mathcal{N}}_\mu(u_i)(ab) = R_i,$$

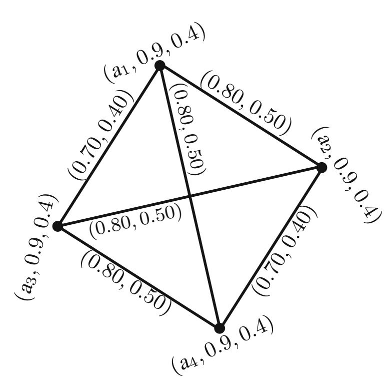
$$\mathcal{D}_v(a) = \sum_{a,b \neq a \in \mathcal{X}} \tilde{\mathcal{N}}_v(u_i)(ab) = R'_i, \forall u_i \in \mathcal{U}, a \in \mathcal{X}.$$

Therefore, the degree of an edge is



$\tilde{\mathbf{H}}(u_1)$

Fig. 9 PDFSG $P_D = \{\tilde{\mathbf{H}}(u_1)\}$



$\tilde{\mathbf{H}}(u_1)$

Fig. 10 $P_D = \{\tilde{\mathbf{H}}(u_1)\}$

$$
\begin{aligned}
(\mathcal{D})_\mu(ab) &= \mathcal{D}_\mu(a) + \mathcal{D}_\mu(b) - 2\tilde{\mathcal{N}}_\mu(u_i)(ab)\\
&= R_i + R_i - 2c_i\\
&= 2(R_i - c_i).\\
(\mathcal{D})_\nu(ab) &= \mathcal{D}_\nu(a) + \mathcal{D}_\nu(b) - 2\tilde{\mathcal{N}}_\nu(u_i)(ab)\\
&= R'_i + R'_i - 2c'_i\\
&= 2(R'_i - c'_i), \forall\, u_i \in \mathcal{U}, ab \in E.
\end{aligned}
$$

Hence, the edge degree sequence of membership and nonmembership values is constant sequence and its corresponding edge degree sets $\{2(R_i - c_i)\}, \{2(R'_i - c'_i)\}$ are singleton sets. □

Definition 2.40 A PDFSG P_D is edge regular PDFSG if $\tilde{\mathbf{H}}(u_i)$ is edge regular PDFG of degree $(q_i, q'_i), \forall\, u_i \in \mathcal{U}$.

Example 2.41 Consider a PDFSG $P_D = \{\tilde{\mathbf{H}}(u_1)\}$ as shown in Fig. 11.

The degree of each edge in $\tilde{\mathbf{H}}(u_1)$ is (1.4, 1.2). So, P_D is a edge regular PDFSG.

Definition 2.42 A PDFSG P_D is totally edge regular PDFSG if $\tilde{\mathbf{H}}(u_i)$ is totally edge regular PDFG of degree $(w_i, w'_i), \forall\, u_i \in \mathcal{U}$.

Example 2.43 Consider a PDFSG P_D as shown in Fig. 12.

The total degree of each edge is (1.05, 1.8) in $\tilde{\mathbf{H}}(u_1)$. Hence, $P_D = \{\tilde{\mathbf{H}}(u_1)\}$ is a totally edge regular PDFSG.

Theorem 2.44 *Let P_D be a regular PDFSG such that $\tilde{\mathcal{N}}$ is a constant function; then P_D is edge regular PDFSG.*

Proof Suppose that P_D is a regular PDFSG, i.e., $\tilde{\mathbf{H}}(u_i)$ is (R_i, R'_i) regular PDFG such that

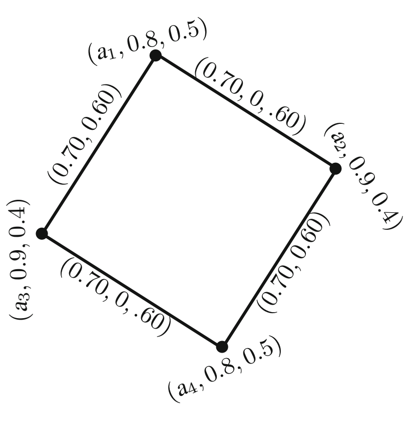


Fig. 11 Edge regular PDFSG $P_D = \{\tilde{\mathbf{H}}(u_1)\}$

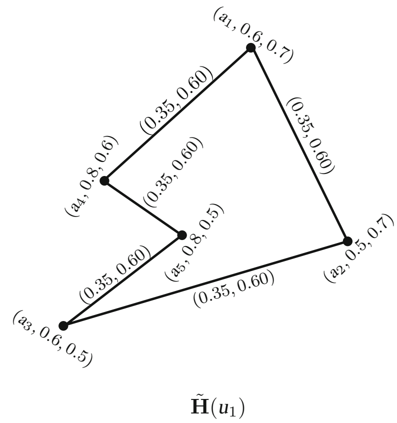


Fig. 12 Totally edge regular PDFSG $P_D = \{\tilde{\mathbf{H}}(u_1)\}$

$$
\begin{aligned}
&\mathcal{D}_\mu(a) = R_i,\\
&\mathcal{D}_\nu(a) = R'_i, \forall\, u_i \in \mathcal{U}, a \in \mathcal{X}.
\end{aligned}
$$

Let $\tilde{\mathcal{N}}_\mu(u_i)(ab) = c_i$ and $\tilde{\mathcal{N}}_\nu(u_i)(ab) = c'_i, \forall\, u_i \in \mathcal{U}, ab \in \mathcal{E}$. The degree of edge is

$$
\begin{aligned}
(\mathcal{D})_\mu(ab) &= \mathcal{D}_\mu(a) + \mathcal{D}_\mu(b) - 2\tilde{\mathcal{N}}_\mu(u_i)(ab)\\
&= R_i + R_i - 2c_i\\
&= 2(R_i - c_i)\\
&= q_i.\\
(\mathcal{D})_\nu(ab) &= \mathcal{D}_\nu(a) + \mathcal{D}_\nu(b) - 2\tilde{\mathcal{N}}_\nu(u_i)(ab)\\
&= R'_i + R'_i - 2c'_i\\
&= 2(R'_i - c'_i)\\
&= q'_i, \forall\, u_i \in \mathcal{U}, ab \in \mathcal{E}.
\end{aligned}
$$

Hence, P_D is an edge regular PDFSG. v

Theorem 2.45 *Let P_D be both edge regular and totally edge regular PDFSG; then $\tilde{\mathcal{N}}$ is a constant function.*

Proof Suppose that P_D is an edge regular and totally edge regular PDFSG. Then the degree of an edge in $\tilde{\mathbf{H}}(u_i)$ is

$$
\begin{aligned}
(\mathcal{D})_\mu(ab) &= \mathcal{D}_\mu(a) + \mathcal{D}_\mu(b) - 2\tilde{\mathcal{N}}_\mu(u_i)(ab)\\
&= q_i.\\
(\mathcal{D})_\nu(ab) &= \mathcal{D}_\nu(a) + \mathcal{D}_\nu(b) - 2\tilde{\mathcal{N}}_\nu(u_i)(ab)\\
&= q'_i, \forall\, u_i \in \mathcal{U}, ab \in \mathcal{E}.
\end{aligned}
$$

The total degree of an edge is

$$
\begin{aligned}
(\mathcal{TD})_\mu(ab) &= (\mathcal{D})_\mu(a) + (\mathcal{D})_\mu(b) - \tilde{\mathcal{N}}_\mu(u_i)(ab)\\
&= w_i.\\
(\mathcal{TD})_\nu(ab) &= (\mathcal{D})_\nu(a) + (\mathcal{D})_\nu(b) - \tilde{\mathcal{N}}_\nu(u_i)(ab)\\
&= w'_i, \forall\, u_i \in \mathcal{U}, ab \in \mathcal{E}.
\end{aligned}
$$

Further, it follows that

$$(\mathcal{TD})_{\mu}(ab) = w_i$$
$$(\mathcal{D})_{\mu}(a) + (\mathcal{D})_{\mu}(b) - \tilde{\mathcal{N}}_{\mu}(u_i)(ab) = w_i$$
$$(D)_{\mu}(ab) + \tilde{\mathcal{N}}_{\mu}(u_i)(ab) = w_i$$
$$\Rightarrow \tilde{\mathcal{N}}_{\mu}(u_i)(ab) = w_i - R_i$$
$$(\mathcal{TD})_{\nu}(ab) = w'_i$$
$$(\mathcal{D})_{\nu}(a) + (\mathcal{D})_{\nu}(b) - \tilde{\mathcal{N}}_{\nu}(u_i)(ab) = w'_i$$
$$(\mathcal{D})_{\nu}(ab) + \tilde{\mathcal{N}}_{\nu}(u_i)(ab) = w'_i$$
$$\Rightarrow \tilde{\mathcal{N}}_{\nu}(u_i)(ab) = w'_i - R'_i.$$

Hence, $\tilde{\mathcal{N}}$ is a constant function. □

3 Application to decision-making problem

In this section, we propose the decision-making problems using PDFSG environment. To handle the decision-making problems, we adopt some steps given in the following algorithm.

Algorithm

1. **Input**:
   Possible alternatives,
   Possible parameters.
2. Construction of the PFPR $O = (o_{il})$ corresponding to given parameters.
3. Use PDFAA operator to calculate the combined PFE corresponding to each parameter

$$o_j = \text{PDFAA}(o_{jl}, o_{j2}, \ldots, o_{jn}) = \left(\sqrt{1 - \frac{1}{1 + \left[\sum_{k=1}^{n} \frac{1}{n} \left(\frac{\mu_{jk}^2}{1-\mu_{jk}^2}\right)^{\gamma}\right]^{\frac{1}{\gamma}}}}, \frac{1}{1 + \left[\sum_{k=1}^{n} \frac{1}{n} \left(\frac{1-\nu_{jk}}{\nu_{jk}}\right)^{\gamma}\right]^{\frac{1}{\gamma}}} \right), j = 1, 2, \ldots, n. \tag{11}$$

4. To find the combined overall preference value $o_j (j = 1, 2, \ldots, 4)$, compute the score functions Ⓢ(o_j) given by (Zhang and Xu 2014).

$$Ⓢ(o_j) = (\mu_j)^2 - (\nu_j)^2 \tag{12}$$

5. **Output:** The decision is $\max\{\min Ⓢ(o_k)(u_i), i = 1, 2, \ldots, m, k = 1, 2, \ldots, n\}$.

3.1 Selection of suitable ETL software for a business intelligence project

Business intelligence (BI) helps to convert raw data into meaningful information for informed business decisions. BI helps to gain insights into consumer behavior. It is necessary for a business to understand the demands of customers so that resources can be invested into beneficial products. The central part of BI is established on data warehouses powered by ETL (Extract, Transform and Load). With the continuous development of BI usage, ETL, the initial point of the project, has become a key factor that affects the failure or success of the BI project. The main work of BI project is the selection of most suitable ETL software which maximizes the profits, limits the costs and is flexible to accommodate future advancements in the project. Mr. X wants to select a ETL software for BI (adopted from Akram et al. 2019, 2020). Let $\mathcal{X} = \{a_1, a_2, a_3, a_4, a_5\}$ be the set of five ETL software as the universal set. Mr. X compares the five ETL software $a_i (i = 1, 2, \ldots, 5)$ pairwise for the selection and provides its preference information in the form of PFPR $O = (o_{il})_{5\times5}$, where $o_{il} = (\mu_{il}, \nu_{il})$ is the Pythagorean fuzzy element assigned by the Mr. X (an expert) with μ_{il} as the degree to which the ETL software a_i is preferred over the other ETL software a_l and ν_{il} as the degree to which the ETL software a_i is not preferred over the other ETL software a_l. The PFPR $O = (o_{il})_{5\times5}$ is expressed in the following tabular form in Table 1.

The PF directed network corresponding to PFPR O is given in Table 1, presented in Fig. 13.

In order to compute the combined PFE $o_{il} = (\mu_{il}, \nu_{il}), (i, l = 1, 2, \ldots, 5)$ of the ETL software a_1, over all the other ETL software, we use Pythagorean Dombi fuzzy arithmetic averaging (PDFAA) operator given by (Akram et al. 2019, 2020) given in Eq. 13.

$$o_j = \text{PDFAA}(o_{jl}, o_{j2}, \ldots, o_{jn}) = \left(\sqrt{1 - \frac{1}{1 + \left[\sum_{k=1}^{n} \frac{1}{n} \left(\frac{\mu_{jk}^2}{1-\mu_{jk}^2}\right)^{\gamma}\right]^{\frac{1}{\gamma}}}}, \frac{1}{1 + \left[\sum_{k=1}^{n} \frac{1}{n} \left(\frac{1-\nu_{jk}}{\nu_{jk}}\right)^{\gamma}\right]^{\frac{1}{\gamma}}} \right). \tag{13}$$

Take $\gamma = 1$ in Eq. 13 to obtain the combined overall preference value $o_j (j = 1, 2, \ldots, 4)$.

Table 1 PFPR

O	a_1	a_2	a_3	a_4	a_5
a_1	(0.5, 0.5)	(0.7, 0.5)	(0.5, 0.6)	(0.3, 0.8)	(0.4, 0.3)
a_2	(0.5, 0.7)	(0.5, 0.5)	(0.2, 0.9)	(0.8, 0.4)	(0.1, 0.8)
a_3	(0.6, 0.5)	(0.9, 0.2)	(0.5, 0.5)	(0.6, 0.6)	(0.5, 0.4)
a_4	(0.8, 0.3)	(0.4, 0.8)	(0.6, 0.6)	(0.5, 0.5)	(0.6, 0.7)
a_5	(0.3, 0.4)	(0.8, 0.1)	(0.4, 0.5)	(0.7, 0.6)	(0.5, 0.5)

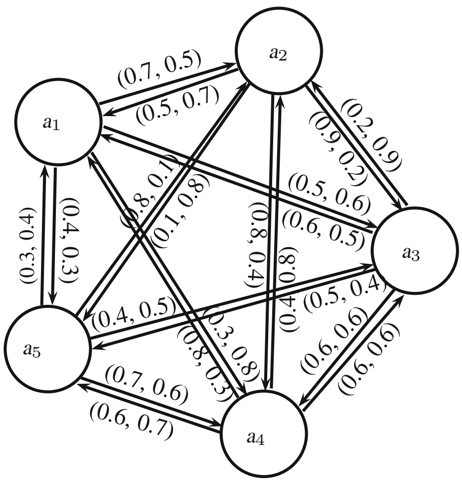


Fig. 13 Directed network of PFPR

$$o_1 = (0.5264, 0.4878),$$
$$o_2 = (0.5770, 0.6032),$$
$$o_3 = (0.7401, 0.3798),$$
$$o_4 = (0.6377, 0.5166),$$
$$o_5 = (0.6341, 0.2752).$$

The score functions $Ⓢ(o_j)$ of the combined overall preference value $o_j (j = 1, 2, \ldots, 5)$ are calculated by using score function given by Zhang and Xu (2014).

$$\begin{aligned} &Ⓢ(o_j) = (\mu_j)^2 - (\nu_j)^2 \\ &Ⓢ(o_1) = 0.0386, Ⓢ(o_2) = -0.0262, Ⓢ(o_3) = 0.3603, \\ &Ⓢ(o_4) = 0.1211, Ⓢ(o_5) = 0.3589. \end{aligned} \tag{14}$$

On the basis of score functions, ranking is

$$a_3 > a_5 > a_4 > a_1 > a_2$$

Hence, a_3 is the most suitable ETL software.

3.2 Evaluation of electronics companies

In this modern area of life, electronics has too much importance in different aspects of our life. It is difficult to find an electrical item in our home that does not have electronics partnered with it, in some way. Electronics or electronic components can be found everywhere from music to cooking in some way. There are many electronic devices which have made our daily life too much easy, for example, television, camera, laptop, fridge, oven, etc.

Mr. X wants to purchase some electronic devices for his home like oven, fridge and television. There are different electronics companies which supply these things. But he wants to select that company for purchasing of things which is most "affordable" and having "best quality." Let $\mathcal{X} = \{a_1, a_2, a_3, a_4\}$ be the set of four electronics companies as the universal set and $\mathcal{U} = \{u_1, u_2\}$ be the set of parameters that particularize the electronics companies, the parameters u_1 and u_2 stand for "affordable," "best quality," respectively. Mr. X compares the four companies $a_i (i = 1, 2, \ldots, 4)$ pairwise for the selection on the basis of the parameters "affordable" and "best quality" and provides its preference information in the form of PFPR $O = (o_{il})_{4\times4}$, where $o_{il} = (\mu_{il}, \nu_{il})$ is the Pythagorean fuzzy element assigned by the Mr. X expert with μ_{il} as the degree to which the company a_i is preferred over the company a_l with respect to the given parameter and ν_{il} as the degree to which the company a_i is not preferred over the company a_l with respect to the given parameter. The PFPR $O = (o_{il})_{4\times4}$ for the given parameters is expressed in the following tabular form in Tables 2 and 3, respectively.

The PF directed network corresponding to PFPR O is given in Tables 2 and 3, presented in Figs. 14 and 15.

In order to compute the combined PFE $o_{il} = (\mu_{il}, \nu_{il}), (i, l = 1, 2, \ldots, 4)$ of the company a_1, over all the other companies, we use Pythagorean Dombi fuzzy arithmetic averaging (PDFAA) operator given by Akram et al. (2019, 2020) given in Eq. 13. Take $\gamma = 1$ in Eq. 13 to obtain the combined overall preference value $o_j (j = 1, 2, \ldots, 4)$.

For parameter u_1,

Table 2 PFPR for parameter u_1

O	a_1	a_2	a_3	a_4
a_1	(0.5, 0.5)	(0.8, 0.6)	(0.8, 0.3)	(0.4, 0.6)
a_2	(0.6, 0.8)	(0.5, 0.5)	(0.4, 0.9)	(0.5, 0.7)
a_3	(0.3, 0.8)	(0.9, 0.4)	(0.5, 0.5)	(0.7, 0.4)
a_4	(0.6, 0.4)	(0.7, 0.5)	(0.4, 0.7)	(0.5, 0.5)

Table 3 PFPR for parameter u_2

O	a_1	a_2	a_3	a_4
a_1	(0.5, 0.5)	(0.2, 0.9)	(0.4, 0.8)	(0.9, 0.4)
a_2	(0.9, 0.2)	(0.5, 0.5)	(0.5, 0.6)	(0.5, 0.7)
a_3	(0.8, 0.4)	(0.6, 0.5)	(0.5, 0.5)	(0.7, 0.4)
a_4	(0.4, 0.9)	(0.7, 0.5)	(0.4, 0.7)	(0.5, 0.5)

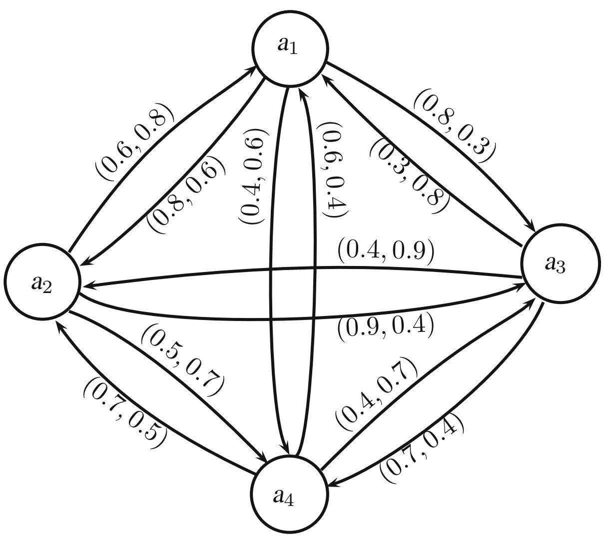


Fig. 14 Directed network of PFPR corresponding to parameter u_1

$o_1(u_1) = (0.710, 0.461),$
$o_2(u_1) = (0.512, 0.691),$
$o_3(u_1) = (0.765, 0.485),$
$o_4(u_1) = (0.582, 0.505).$

For parameter u_2,

$o_1(u_2) = (0.710, 0.461),$
$o_2(u_2) = (0.512, 0.691),$
$o_3(u_2) = (0.765, 0.485),$
$o_4(u_2) = (0.582, 0.505).$

The score functions $\circledS(o_j)$ of the combined overall preference value $o_j(j = 1, 2, \ldots, 4)$ are calculated by using score function given in Eq. 14.

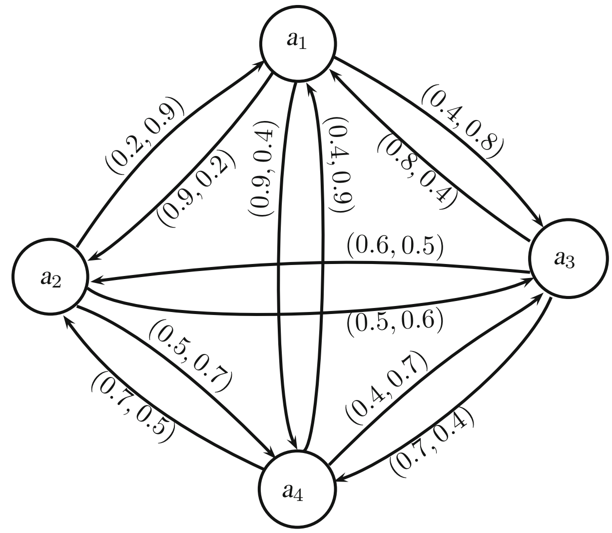


Fig. 15 Directed network of PFPR corresponding to parameter u_2

For parameter u_1,

$\circledS(o_1)(u_1) = 0.290,$
$\circledS(o_2)(u_1) = -0.215,$
$\circledS(o_3)(u_1) = 0.350,$
$\circledS(o_4)(u_1) = 0.084.$

For parameter u_2,

$\circledS(o_1)(u_2) = 0.208,$
$\circledS(o_2)(u_2) = 0.412,$
$\circledS(o_3)(u_2) = 0.279,$
$\circledS(o_4)(u_2) = -0.082.$

The decision is $\max\{\min \circledS(o_k)(u_i), i = 1, 2, k = 1, 2, 3, 4\} = \max\{0.208, -0.215, 0.279, -0.082\} = 0.279$. So, Mr. X will select the a_3 company to purchase the electronics.

3.3 Comparison analysis

In this section, we discuss the importance and logic behind the development of our proposed model. The example shown in Sect. 3.1 taken from the existing model (Akram et al. 2019, 2020) shows that we can get a best alternative by comparing pairwise these alternatives using preference values and after ranking by score function, we get the best alternative. But this model does not handle the situations when we have a list of parameters for the selection of any alternative. It is clear from application Sect. 3.1, we cannot take a set of attributes for the judgment of any alternative. Then we need our proposed model. The logics behind this model are:

1. A Pythagorean fuzzy soft graph, as an extension of intuitionistic fuzzy soft graph, is useful in representing the parametric relationships between objects where relationship is ambiguous, while Dombi operators with operational parameters have creditable flexibility.
2. PDFSG model provides us information, about various parameters for the selection of any attribute.
3. Proposed model reduces to PDFG model, when we consider only one parameter.
4. The application in Sect. 3.2, describes the importance of proposed model.

4 Conclusion

Soft set is considered useful tool for the parameterized point of view, whereas the Pythagorean fuzzy set is taken as the more general concept as compared to the intuitionistic fuzzy set, because the space of Pythagorean fuzzy values is greater than the space of intuitionistic fuzzy values. A Pythagorean fuzzy soft graph, an extension of intuitionistic fuzzy soft graph, is a powerful tool to handle the pairwise relationships between objects corresponding to different parameters, while Dombi operators are more helpful to handle decision-making problems. Using these two concepts, we have introduced the idea of Pythagorean Dombi fuzzy soft graph (PDFSG) in this paper. We have described certain properties of PDFSGs. Further, we have studied the idea of edge regular PDFSG with consequential properties. We have also solved the decision-making problems using the Pythagorean Dombi fuzzy arithmetic averaging (PDFAA) operators. We aim to extend our studies to: (1) complex Pythagorean Dombi fuzzy soft graphs; (2) complex Pythagorean Dombi fuzzy soft hypergraphs; (3) a study on Dijkstra algorithm for a network with complex picture trapezoidal fuzzy numbers.

Compliance with ethical standards

Conflict of interest: The authors declare that they have no conflict of interest regarding the publication of this research article.

References

Akram M, Ali G (2019) Group decision making approach under multi (Q, N)-soft multi granulation rough model. Granul Comput. https://doi.org/10.1007/s41066-019-00190-6

Akram M, Ali G (2020) Hybrid models for decision making based on rough Pythagorean fuzzy bipolar soft information. Granul Comput 5(1):1–15

Akram M, Dudek WA, Dar JM (2019) Pythagorean Dombi fuzzy aggregation operators with application in multicriteria decision making. Int J Intell Syst 34(11):3000–3019

Akram M, Dar JM, Naz S (2020) Pythagorean Dombi fuzzy graphs. Complex Intell Syst 6:29–54

Akram M, Davvaz B (2012) Strong intuitionistic fuzzy graphs. Filomat 26(1):177–196

Akram M, Nawaz S (2016) Fuzzy soft graphs with applications. J Intell Fuzzy Syst 30(6):3619–3632

Ali MI (2011) A note on soft sets, rough soft sets and fuzzy soft sets. Appl Soft Comput 11(4):3329–3332

Ali MI, Feng F, Liu X, Min WK, Shabir M (2009) On some new operations in soft set theory. Comput Math Appl 57(9):1547–1553

Ashraf S, Naz S, Kerre EE (2018) Dombi fuzzy graphs. Fuzzy Inf Eng 10(1):58–79

Atanassov KT (1986) Intuitionistic fuzzy sets. Fuzzy Sets Syst 20(1):87–96

Bai SM, Chen SM (2008a) Automatically constructing grade membership functions of fuzzy rules for students evaluation. Expert Syst Appl 35(3):1408–1414

Bai SM, Chen SM (2008b) Automatically constructing concept maps based on fuzzy rules for adapting learning systems. Expert Syst Appl 35(1–2):41–49

Chen SM (1996) A fuzzy reasoning approach for rule-based systems based on fuzzy logics. IEEE Trans Syst Man Cybern B (Cybern) 26(5):769–778

Chen SM, Cheng SH (2016) Fuzzy multi-attribute decision making based on transformation techniques of intuitionistic fuzzy values and intuitionistic fuzzy geometric averaging operators. Inf Sci 352:133–149

Chen SM, Cheng SH, Lan TC (2016a) A novel similarity measure between intuitionistic fuzzy sets based on the centroid points of transformed fuzzy numbers with applications to pattern recognition. Inf Sci 343:15–40

Chen SM, Cheng SH, Lan TC (2016b) Multi-criteria decision making based on the TOPSIS method and similarity measures between intuitionistic fuzzy values. Inf Sci 367:279–295

Chen SM, Manalu GMT, Pan JS, Liu HC (2013) Fuzzy forecasting based on two-factors second-order fuzzy-trend logical relationship groups and particle swarm optimization techniques. IEEE Trans Cybern 43(3):1102–1117

Chen J, Ye J (2017) Some single-valued neutrosophic Dombi weighted aggregation operators for multiple attribute decision making. Symmetry 9(6):82

Dombi J (1982) A general class of fuzzy operators, the De Morgan class of fuzzy operators and fuzziness measures induced by fuzzy operators. Fuzzy Sets Syst 8(2):149–163

Dubois D, Ostasiewicz W, Prade H (2000) Fuzzy sets: history and basic notions. Handbook of fuzzy sets and possibility theory. Springer, New York, pp 121–124

Feng F, Li C, Davvaz B, Ali MI (2011a) Soft sets combined with fuzzy sets and rough sets: a tentative approach. Soft Comput 14(9):899–911

Feng F, Liu X, Fotea VL, Jun YB (2011b) Soft sets and soft rough sets. Inf Sci 181(6):1125–1137

Hamacher H (1978) On logical aggregations of non-binar explicit decision criteria. Rita G. Fischer Verlag, Frankfurt

Jana C, Pal M, Wang JQ (2019) Bipolar fuzzy Dombi aggregation operators and its application in multiple-attribute decision making process. J Ambient Intell Human Comput 10(9):3533–3549

Klement EP, Mesiar R, Pap E (2013) Triangular norms, vol 8. Springer Science and Business Media, Berlin

Kuwagaki A (1952) On the rational functional equation of function unknown of two variables. Mem Coll Sci 28(2)

Liu P, Liu J, Chen SM (2018) Some intuitionistic fuzzy Dombi Bonferroni mean operators and their application to multi-attribute group decision making. J Oper Res Soc 69(1):1–24

Liu P, Chen SM, Liu J (2017) Multiple-attribute group decision making based on intuitionistic fuzzy interaction partitioned Bonferroni mean operators. Inf Sci 411:98–121

Liu X, Wang L (2020) An extension approach of TOPSIS method with OWAD operator for multiple criteria decision making. Granul Comput 5(1):135–148

Maji PK, Biswas R, Roy AR (2001a) Fuzzy soft sets. J Fuzzy Math 9(3):589–602

Maji PK, Biswas R, Roy AR (2001b) Intuitionistic fuzzy soft sets. J Fuzzy Math 9(3):677–692

Menger K (1942) Statistical metrics. Proc Natl Acad Sci U S A 28(12):535–537

Mishra AR, Chandel A, Motwani D (2020) Extended MABAC method based on divergence measures for multi-criteria assessment of programming language with interval-valued intuitionistic fuzzy sets. Granul Comput 5(1):97–117

Molodtsov DA (1999) Soft set theory-first results. Comput Math Appl 37:19–31

Naz S, Ashraf S, Akram M (2018) A novel approach to decision making with Pythagorean fuzzy information. Mathematics 6:1–28

Parvathi R, Karunambigai MG (2006) Intuitionistic fuzzy graphs. Computational intelligence, theory and applications. Springer, Berlin, pp 139–150

Peng X, Yang Y, Song J, Jiang Y (2015) Pythagorean fuzzy soft set and its application. Comput Eng 41(7):224–229

Rosenfeld A (1975) Fuzzy graphs, fuzzy sets and their applications. Academic Press, New York, pp 77–95

Roy AR, Maji PK (2007) A fuzzy soft set theoretic approach to decision making problems. J Comput Appl Math 203(2):461–472

Schweizer B, Sklar S (1983) Probabilistic metric spaces. Probab Appl Math

Shahzadi S, Akram M (2017) Intuitionistic fuzzy soft graphs with applications. J Appl Math Comput 55(12):369–392

Shi L, Ye J (2018) Dombi Aggregation operators of neutrosophic cubic sets for multiple attribute decision making. Algorithms. https://doi.org/10.3390/a11030029

Som T (2006) On the theory of soft sets, soft relations and fuzzy soft relation. In: Proceedings of the national conference on uncertainty: a mathematical approach, UAMA-06, Burdwan, pp 1–9

Thumbakara RK, George B (2014) Soft graphs. Gen Math Notes 21(2):75–86

Yager RR (2013) Pythagorean fuzzy subsets. In: 2013 Joint IFSA world congress and NAFIPS annual meeting (IFSA/NAFIPS). IEEE, pp 57–61

Zadeh LA (1965) Fuzzy sets. Inf Control 8(3):338–353

Zhang X, Xu Z (2014) Extension of TOPSIS to multiple criteria decision making with Pythagorean fuzzy sets. Int J Intell Syst 29(12):1061–1078