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ORIGINAL PAPER

Gradual interval arithmetic and fuzzy interval arithmetic

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Abstract

This paper proposes an analysis of and a reflection on interval arithmetic (IA) and its extension to gradual interval arithmetic (GIA). Through this reflection, an overview of a part of IA that is directly related to fuzzy interval arithmetic (FIA) is analyzed, compared, and categorized according to two main families of IA: standard interval arithmetic (SIA) and instantiated interval arithmetic (IIA). Furthermore, SIA and IIA visions represent two viewpoints of computation that are different and they will cause modifications in interval interpretation and manipulation. This vision is essential in understanding the philosophy of IA and GIA computational mechanisms. The contribution of this paper is twofold. First, according to SIA and IIA visions, an analysis and a classification of a part of IAs are given. Equivalences and links between these IAs are analyzed and established. Second, an extension of IA to the gradual context is proposed. The GIA extension provides a new interpretation of FIA according to the gradual representation.

Keywords Standard interval arithmetic (SIA) · Instantiated interval arithmetic (IIA) · Gradual intervals · Gradual interval arithmetic (GIA) · Fuzzy interval arithmetic (FIA)

1 Introduction

Interval arithmetic (IA) represents an important tool for granular computing and more specifically for fuzzy arithmetic (FA) (Pedrycz et al. 2008). Dubois and Prade initiated the formalization of analytical fuzzy operations (Dubois and Prade 1980). They introduced the LR representation to allow for a better expression of arithmetic operations using fuzzy numbers (Dubois and Prade 1988). In the fuzzy literature, arithmetic operations are extended to the fuzzy context according to the extension principle. However, it is known that the computation based on Zadeh's extension principle is expensive, because of the necessity to solve a non-linear programming problem (Dong and Wong 1987; Oussalah and DeSchutter 2003). To overcome this difficulty, FA is often treated as IA using the α-cuts principle (Kaufmann and Gupta 1991; Giachetti and Young 1997; Bodjanova 2003; Moore and Lodwick 2003; Guerra and Stefanini 2005; Stefanini 2010).

Based on the fuzzy interval (FI) concept, several fuzzy interval arithmetic (FIA) approaches have been proposed (see Kaufmann and Gupta 1991; Giachetti and Young 1997; Bodjanova 2003; Guerra and Stefanini 2005; Stefanini 2010; Chalco-Cano et al. 2014; Lodwick and Dubois 2015). See (Dubois et al. 2000) for a survey on FIs.

IA distinguishes between syntax and semantics. While syntax focuses on variable expressions, semantics explores the meaning and sense given to the intervals used to represent these variables. When considering an interval expression, the computational result may sometimes be different depending on the meaning given to the intervals used. Generally, two main categories of IA methodologies are considered in the literature:

- Standard interval arithmetic (SIA), which was developed by Sunaga, Warmus, and Moore (S–W–M) (Warmus 1956; Sunaga 1958; Moore and Yang 1959; Moore 1962, 1966). SIA has several advantages and it allows rigorous enclosures for the ranges of operations and functions. Furthermore, SIA makes a qualitative difference in scientific computations, since the results are intervals in which the exact result must lie.

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However, the literature is unanimous about the problem of overestimation in SIA (Neumaier 1990; Markov 2001; Hanss 2005; Chalco-Cano et al. 2014). Moreover, it is well known that algebraic properties of SIA are often insufficient, if one wants to address inverse problems and solve interval equations: intervals do not have inverses with respect to the addition and multiplication operations (Markov 1995, 2001). Various IA extensions and hybridizations, which can be classified within the SIA category, have been proposed to make progress in SIA computing and allow, if they exist, exact resolutions of interval equations. For instance, we can mention the extended (generalized) IA of Kaucher (EIA) (Kaucher 1973, 1980), the inner IA of Markov (NIA) (Dimitrova et al. 2010; Markov 1977, 1979; Popova and Markov 1997), the generalized Hukuhara IA (GHIA) (Stefanini 2010) and the optimistic IA (OIA) (Boukezzoula and Galichet 2010; Boukezzoula et al. 2012, 2014).

- Instantiated interval arithmetic (IIA), which was proposed by Lodwick and Dubois (2015) for making the exact solving of interval equations possible. From a methodological point of view, IIA is different from SIA in the sense that IIA uses an instantiation of values inside the interval. In this context, the computation with these single instantiated values makes solving interval equations possible. Moreover, since the computations are based on instantiated single values, IIA possesses additive and multiplicative inverses. Furthermore, several promising arithmetics, such as constrained interval arithmetic (CIA) (Lodwick 1999, 2007; Lodwick and Jenkins 2013; Lodwick and Dubois 2015) and single constrained interval arithmetic (SCIA) (Chalco-Cano et al. 2014), can be classified in this IA category.

Although we use the same notation and basic operations on intervals both for SIA and IIA, the two visions are philosophically different and will cause modifications in the interpretation and manipulation of the interval. They do not give the same result, especially in the context of dependent variables and/or in the presence of multiple copies of the same variable. However, if the intervals involved are independent, the two visions produce equivalent results.

A conventional interval can be considered as a particular FI whose membership function takes the value 1, when a is in the interval and 0 elsewhere. When the bounds of an interval are flexible and represent a gradual transition over the interval, they can be represented by gradual numbers (GN) (Dubois and Prade 2008; Fortin et al. 2008; Boukezzoula et al. 2012, 2014). A GN is modeled by a function from (0, 1] to $\Re$. It is a real-valued number parameterized by a degree of relevance and/or likelihood $\lambda \in (0, 1]$. An interval $[a^-, a^+]$ becomes a gradual interval (GI) $[a^-(\lambda), a^+(\lambda)]$, when its boundaries are GN (Dubois and Prade 2008; Fortin et al. 2008; Boukezzoula et al. 2012, 2014). Analogous to a conventional interval, a GI is represented by the ordered pair of its two boundaries $a^-(\lambda)$ and $a^+(\lambda)$, called left and right profiles, respectively. The GI concept provides a new interpretation of FIA, called gradual interval arithmetic (GIA), based on the assignment function used to represent the gradualness of numbers. Conversely, the GI $[a^-(\lambda), a^+(\lambda)]$ can be interpreted as an FI, if its profiles $a^-(\lambda)$ and $a^+(\lambda)$ are injective and, respectively, non-decreasing and non-increasing. While an FI may be a particular GI, the opposite is false in so far as no monotonicity constraint is associated with the GI bounds (Fortin et al. 2008; Untiedt and Lodwick 2008; Boukezzoula et al. 2014). The concept of GI is much more general than that of FI. More details on GIs and their relationships with FIs are given in (Dubois and Prade 2008; Fortin et al. 2008; Boukezzoula et al. 2012). However, some FIAs can lead to non-monotonic GIs that are not fuzzy subsets and cannot be represented by FIs, since the interval boundaries are not monotonic (Fortin et al. 2008; Untiedt and Lodwick 2008; Boukezzoula et al. 2012). This finding is consistent with the criticisms detailed in notes published recently (see Allahviranloo et al. 2011; Gomes and Barros 2015). While based on GIA, these results are not controversial although they may be counterintuitive in FIA. Indeed, in the gradual context, no monotonicity constraint is imposed on the interval profiles. Furthermore, the gradual philosophy gives a new breath to these works and their scientific validity.

In our opinion, a thorough analysis and positioning of FIA and GIA in relation to SIA and IIA visions is beneficial. This analysis will improve the relevance and meaning of these arithmetic aspects, and thus avoid misinterpretations leading to impracticable considerations in practical applications. To provide an overview of the work that is presented here and to explain the reasoning behind our approach, we will begin by detailing the motivation for the proposed approach. The developments that are inherent to our method will be detailed in the next sections. This paper has two main motivations. The first one is to survey, analyze, and classify according to SIA and IIA visions, a part of IA that is directly related to FIA. Currently, it is not easy to choose an appropriate version of IA, because its numerous versions have been proposed. In this context, comparisons, equivalences, and links between these versions of IA are analyzed and established. The second motivation is to propose an extension of IA to the gradual context. To achieve this objective, a new alternative of FIA is developed. This representation is based on the use of GIs. Through the notion of GIs, a revision and new interpretation of FIA named GIA is made.

In this paper, we focus only on the elementary arithmetic operations $\{+, -, \times, \div\}$. These operations are of course a basis for more complicated problems of IA. Therefore, they are very important. If the elementary operations are formulated incorrectly, then using them for solving problems can sometimes lead to controversial results (Piegat and Landowski 2017). Moreover, special attention is paid to the inverse operators of addition and multiplication, i.e., subtraction and division. The paper is organized as follows. In Sect. 2, preliminaries about interval representation and notations are given. A quick overview of some SIA and IIA approaches is discussed in Sect. 3. Sections 4 and 5 are devoted to SIA and IIA methodologies and their extensions, respectively. The extension of IA to the gradual framework is detailed in Sect. 6. Illustrative examples are provided in Sect. 7. Concluding remarks are given in Sect. 8.

2 Preliminaries: interval representation and notations

A real interval, denoted $[a]$, is defined as a closed, compact, and bounded subset of $\Re$, such that:

$$[a] = [a^-, a^+] = \{a \in \Re | a^- \le a \le a^+\};\ \text{where } a^- \le a^+. \quad (1)$$

The real numbers $a^- = \inf([a])$ and $a^+ = \sup([a])$ are considered as the endpoints (the lower and upper bounds) of the interval $[a]$. In this notation, a refers to any element in the interval $[a]$.

The real interval $[a]$ can be characterized by its endpoints (EP) a^- and a^+, or by its midpoint M and radius R, i.e., $M([a]) = (a^- + a^+)/2$ and $R([a]) = (a^+ - a^-)/2$. The EP notation is the most used in the literature. Furthermore, the interval $[a]$ can also be represented by the pair $(M([a]), R([a]))$ in the MR space. Compared to the EP representation, the MR one highlights the central position of the interval and its width (radius). The MR and EP notations are strictly equivalent. The relation between them is obvious, i.e., $a^- = M([a]) - R([a])$ and $a^+ = M([a]) + R([a])$.

Throughout this paper, the set $\mathbb{I}_+ = \{[a^-, a^+] | a^- \le a^+; a^-, a^+ \in \Re\}$ denotes the set of proper intervals and $\mathbb{I}_- = \{[a^-, a^+] | a^- > a^+; a^-, a^+ \in \Re\}$ denotes the set of improper intervals. Proper intervals have a positive radius, whereas improper ones have a negative one.

In the sequel, two basic functions largely exploited in the literature are used to perform IA operations:

- The Chi (χ) function, initially introduced by Ratschek (Ratschek and Rokne 1995; Kulpa 2001), and defined by:

$$\chi([a]) = \begin{cases} -1 : \text{if } a^- = a^+ \\ a^+ \div a^- : \text{if } |a^-| \ge |a^+| \\ a^- \div a^+ : \text{if } |a^-| \le |a^+| \end{cases}. \quad (2)$$

If $a^- = 0$ or $a^+ = 0$, then $\chi([a]) = 0$. The Chi (χ) function is always defined, i.e., $-1 \le \chi([a]) \le 1$.

- The relative-extent function (rex) (Kulpa 2001), defined by:

$$\text{rex}([a]) = R([a])/M([a]). \quad (3)$$

If $M([a]) = 0$, it is assumed that the value of rex equals $\pm\infty$.

When considering two intervals $[a] = [a^-, a^+]$ and $[b] = [b^-, b^+]$, the four standard arithmetic operations both for SIA and IIA are defined by the following expression (Warmus 1956; Sunaga 1958; Moore and Yang 1959; Moore 1962, 1966; Lodwick 1999; Lodwick and Jenkins 2013; Lodwick and Dubois 2015; Piegat and Landowski 2017):

$$[a] \odot [b] = \left[\min_{a\in[a],\, b\in[b]} a \odot b,\ \max_{a\in[a],\, b\in[b]} a \odot b\right];\ \text{for } \odot \in \{+, -, \times, \div\}. \quad (4)$$

Furthermore, and as discussed in the paper introduction, if the handled intervals are considered to be strictly independent, SIA and IIA produce exactly the same results.

3 Quick overview of some SIA and IIA approaches

In the literature, many IA approaches exist. These are listed below with a brief analysis of their belonging and classification within SIA and IIA frameworks. Our aim here is not to compare the performance of these IA approaches, but to extract their most suitable essence and philosophy. We believe that each IA approach has its own interest, strength, and weakness. The essential question is not to know, in absolute terms, the best approach, but rather to determine which IA best suits a particular situation.

Historically, the first modern drafts of interval representation appeared in the 1920s and 1930s in England through works published by Burkill (1924) and Young (1931), in the 1950s in Japan through Sunaga's publications (Sunaga 1958), and in Poland thanks to the work of Warmus (1956). Furthermore, IA has been experiencing a real expansion of development following Moore's thesis in the USA (Moore and Yang 1959; Moore 1962) and the publication of his book, Interval analysis (Moore 1966).

As reported by Moore (1966), the initial philosophy behind SIA was to bind rounding errors (controlling errors) in mathematical computations, in which most real numbers cannot be represented by a finite precision floating-point number. Furthermore, SIA is an arithmetic defined on sets of real intervals, rather than sets of real numbers. SIA specifies a precise method for performing arithmetic operations on closed intervals (interval numbers). In SIA, each interval number represents some fixed real number between endpoints of the closed interval. Thus, an SIA operation produces two endpoints for each result. In this context, the true result certainly lies within these endpoints. According to this vision, SIA does not instantiate a single value in the interval, but considers all of its possible values. In fact, the SIA computations are only based on the intervals' endpoints, where the independence property between intervals is assumed. For instance, when considering a real interval, such as $[a] = [1, 2]$, then the basic operation $[a] - [a] = [-1, +1]$, which is not zero . The reason behind this assumption is that the two occurrences of the interval [1, 2] are not necessarily dependent, e.g., $[a] = [1, 2]$ and $[b] = [1, 2]$. Indeed, $[a]$ and $[b]$ can be two independent variables that just happen to have the same interval endpoints. Thus, in this case, SIA does not distinguish $[a] - [a]$ from $[a] - [b]$. Because SIA guarantees containing the set of all possible results, the pessimistic independence property between the intervals is implicitly assumed (Lodwick and Dubois 2015). More generally, while being based on SIA, some results such as $[a] - [a] \neq 0$ and $[a] \div [a] \neq 1$ are not controversial although counterintuitive. Furthermore, the $[x]$ "exact" solution of the linear interval equation $[a] + [x] = [b]$ is not, as we would expect, $[x] = [b] - [a]$. The same drawback appears when solving the interval equation $[a] \times [x] = [b]$, whose exact solution is not given by $[x] = [b] \div [a]$ as expected. Indeed, using SIA, it can be easily stated that substituting the solution $[x]$ gives a more imprecise result than the original $[b]$. At best, $[x] \subseteq [b]$, which means that the desired equality is generally not achieved. This problem is related to the lack of inverses in the calculus of interval quantities. In this context, it is well known that algebraic properties of SIA are not sufficient for addressing inverse problems, where the inverses of the addition and the multiplication operations do not exist (Markov 1995, 2001).

Since the pioneering works of S–W–M, research on IA has expanded considerably. Various IA extensions and hybridizations have been proposed for overcoming the SIA deficiencies, more particularly the absence of inverses, which is directly correlated to the interval equation solving problem. In this framework, SIA has been extended in the following two main directions:

- Extension of the set of proper intervals by improper intervals, which involves an extension of the definition of IA for extended (generalized) intervals (proper and improper intervals) (Kaucher 1973, 1980; Markov 1995, 1997; Costa et al. 2015). The corresponding extended interval arithmetic (EIA) structure was initially proposed by Ortolf (1969) and Kaucher (Kaucher 1973, 1980), and further investigated by Gardenes (Gardenes and Trepat 1980; Gardenes et al. 1986), Markov (1995, 1997) and others (Popova 2001). A rigorous and complete EIA algebraic study was made in Kaucher's work (1973, 1980). EIA coincides with SIA when only proper intervals are considered. Moreover, extended intervals based on EIA form a group, whereas SIA using proper intervals form a semigroup without invertibility (Markov 1995, 2001).
- Extension of the set of operations in IA on proper intervals by non-standard (inner) operations. The corresponding non-standard interval arithmetic (NIA) structure was initially proposed by Markov and investigated later by Markov, Dimitrova, Popova, and others (Markov 1977, 1979; Popova and Markov 1997; Dimitrova et al. 2010). Markov's first idea was to propose an alternative to IA in which addition and multiplication have inverse elements, while remaining in the set of proper intervals. The inner (non-standard) denomination is used as opposed to SIAs, considered as outer operations due to their overestimation property.

In the literature, other SIA alternatives have been proposed. They are often based on one or both of the above-mentioned directions. For instance, generalized Hukuhara IA (GHIA) (Stefanini 2010) and optimistic IA (OIA) (Boukezzoula and Galichet 2010; Boukezzoula et al. 2012, 2014) can be mentioned.

Recently, to make the exact resolution of interval equations possible, an alternative vision of SIA has been proposed (Lodwick and Dubois 2015). This new vision is based on the instantiation concept and gave birth to the instantiated IA (IIA) philosophy. From a conceptual point of view, IIA uses an instantiation of values inside the interval. In this context, the computation with these single instantiated values makes solving interval equations possible. Moreover, knowing that the computations are based on instantiated single values, IIA possesses additive and multiplicative inverses. Unlike SIA, IIA can capture the difference between dependent intervals (for example, multiple copies of the same intervals) and independent ones. Generally, IIA is an optimization problem, sometimes difficult to solve. One of the most pertinent methodologies that facilitates the implementation of IIA is undoubtedly the CIA proposed in the pioneering work of Lodwick et al. (Lodwick 1999; Lodwick and Jenkins 2013;

Lodwick and Dubois 2015). The set of constrained intervals belongs to a mathematical space that is richer in properties than the algebraic space of the intervals used in SIA.

The philosophy of CIA is to redefine an interval in such way that dependencies between variables are kept (Lodwick 1999; Lodwick and Jenkins 2013). This principle consists of transforming an interval $[a]$ into a constrained interval, which is viewed as a mapping from [0, 1] to polynomials of degree one (linear functions) with non-negative slopes. That is, a constrained interval merely transforms $a \in [a]$ into $a = f(\omega_a) = a^- + \omega_a (a^+ - a^-)$ for some $\omega_a \in [0, 1]$. In other words, the constrained interval is represented by the function $f(\omega_a)$ with $0 \leq \omega_a \leq 1$. If strong constraints are imposed on the intervals, i.e., the values are constrained by identical positions in their intervals, then CIA is transformed into single-level constrained interval arithmetic (SCIA), proposed by Chalco-Cano et al. (2014). The SCIA presupposes strong dependence between the manipulated variables; this condition is sometimes difficult to meet in real applications.

In a different register, Piegat and Landowski (2017) have developed another interesting vision of IIA, named multidimensional RDM-IA. However, while CIA and RDM-IA use similar instantiated notation, their philosophies and final results are different. The main difference resides in the fact that RDM-IA is based not on intervals, but on models of precise variable values (Piegat and Landowski 2017). In this case, the result obtained is also a model of precise variable values and not an interval. This approach, although very interesting, departs from the conventional IA context and is not considered in this paper.

Another vision of IIA was developed by Klir (Klir 1997; Klir and Pan 1998). His idea consists of taking into account relevant constraints among the operands involved. In this framework, the IA integrates not only the information contained in the operands, but also the additional information that may emanate from their meaning or from some external information about them. This additional information is considered to be a set of constraints. For instance, if an equality relation between two operands exists, it is viewed as an equality constraint. This arithmetic, defined for fuzzy intervals according to the α-cut principle, is known as requisite constraints interval arithmetic (RCIA) (Klir 1997; Klir and Pan 1998). To avoid overestimation due to the occurrence of interactive variables, the equality constraint is the most frequently applied constraint in RCIA. In addition, various inequality constraints can be used (Klir 1997; Klir and Pan 1998). Furthermore, if the equality constraint is used, RCIA and CIA are equivalent, although their formalisms are different.

Klir's arithmetic is generally insufficient since it depends on rules and does not encode correlation and/or interactivity between FIs into the representation. Another way to remedy this problem and to obtain arithmetic is from the "distributions" of membership functions of the FIs involved (similar to what is done for arithmetic of random variables) (Cabral and Barros 2015; Barros and Pedro 2017). This philosophy leads to interactive fuzzy arithmetic. In this framework, the notion of interactivity between FIs is described by a joint possibility distribution (according to chosen *t*-norms) (Fuller and Majlender 2004; Carlsson and Fuller 2005; Esmi et al. 2019). In interactive arithmetic, the operations are defined such that the interactivity relation between FIs (fuzzy numbers) is given by their joint possibility distribution (Esmi et al. 2019). For instance, based on interactive operations, differentiability and integrability are investigated and compared to Hukuhara differentiability and generalized Hukuhara differentiability (Barros and Pedro 2017; Cabral and Barros 2015).

The motivation of this paper is to approach operations for FIs and/or GIs from operations on CIs (extension of IA to FIA and/or GIA). The interactive arithmetic, which represents a very interesting and promising approach, is not based on this principle and explores another vision based on the extension principle with the use of joint possibility distributions of the manipulated FIs. This joint possibility distribution is often regarded as a measure of interactivity between the FIs. This difference of principles compared to the majority of the IA methods presented in this paper led us to exclude this approach in this work. In the paper sequel, IA methodologies are analyzed and classified. The IA alternatives relating to the SIA and IIA categories are thereby detailed in the two different forthcoming sections.

4 Standard interval arithmetic (SIA): extensions and hybridizations

In this section, an analysis of SIA and its extensions mentioned previously is given. To facilitate reading, the notations given in Table 1 are used for an operator $\odot \in \{+, -, \times, \div\}$.

4.1 Standard interval arithmetic (SIA)

The four elementary SIA operations between two intervals $[a]$ and $[b]$ are defined by Eq. (4). More specifically, Markov et al. (Markov 1995; Popova and Markov 1997) proposed the following elegant algebraic formulations of SIA:

- Standard addition: $\forall [a], [b] \in \mathbb{I}_+,\ [a] + [b] = [a^- + b^-, a^+ + b^+]$.
- Standard subtraction: $\forall [a], [b] \in \mathbb{I}_+,\ [a] - [b] = [a^- - b^+, a^+ - b^-]$.

Table 1 Notations for SIA and its extensions

SIA and its extensions denotations	Acronym	Index of the operator $\odot$
Standard interval arithmetic	SIA	No index
Extended (generalized) interval arithmetic	EIA	E
Non-standard (inner) interval arithmetic	NIA	N
Generalized Hukuhara interval arithmetic	GHIA	G
Optimistic interval arithmetic	OIA	O

- Standard multiplication: $\forall[a],\ [b] \in \mathbb{I}_+$:

$$[a]\times[b] = \begin{cases} [a^{-\sigma([b])}\times b^{-\sigma([a])},\ a^{\sigma([b])}\times b^{\sigma([a])}]\,;\ \text{for } [a],[b]\in\mathbb{I}_+\backslash\mathbb{Z}_+ \\ [a^{\sigma([a])}\times b^{-\sigma([a])},\ a^{\sigma([a])}\times b^{\sigma([a])}]\,;\ \text{for } [a]\in\mathbb{I}_+\backslash\mathbb{Z}_+, [b]\in\mathbb{Z}_+ \\ [a^{-\sigma([b])}\times b^{\sigma([b])},\ a^{\sigma([b])}\times b^{\sigma([b])}]\,;\ \text{for } [a]\in\mathbb{Z}_+, [b]\in\mathbb{I}_+\backslash\mathbb{Z}_+ \\ [\min\{a^-\times b^-, a^+\times b^-\}, \max(a^-\times b^-, a^+\times b^+)];\ \text{for } [a],[b]\in\mathbb{Z}_+ \end{cases} \tag{5}$$

In Eq. (5), $\mathbb{Z}_+$ represents the subset of intervals (in $\mathbb{I}_+$) containing zero in their interior and $\sigma([a])$ is the sign function defined by:

$$\sigma([a]) = +\ ,\ \text{if } a^- \geq 0 \text{ and } \sigma([a]) = -,\ \text{if } a^+ < 0.$$

- Standard division: $\forall[a] \in \mathbb{I}_+, \forall[b] \in \mathbb{I}_+\backslash\,\mathbb{Z}_+$:

$$[a]\div[b] = [a]\times(1\div[b]) = \begin{cases} [a^{-\sigma([b])}\div b^{\sigma([a])},\ a^{\sigma([b])}\div b^{-\sigma([a])}]\,;\ [a],[b]\in\mathbb{I}_+\backslash\,\mathbb{Z}_+ \\ [a^{-\sigma([b])}\div b^{-\sigma([b])},\ a^{\sigma([b])}\div b^{-\sigma([b])}]\,;\ [a]\in\mathbb{Z}_+, [b]\in\mathbb{I}_+\backslash\,\mathbb{Z}_+ \end{cases}. \tag{6}$$

According to SIA, it can be stated that $[a] + ([a] - [b]) \neq [a]$ and $[b] \times([a] \div [b]) \neq [a]$. Furthermore, and as discussed previously, $[a] - [a] \neq 0$ and $[a] \div [a] \neq 1$. Moreover, SIA cannot distinguish between the quantities $[a] - [b]$ (resp. $[a] \div [b]$) even when the intervals $[a]$ and $[b]$ are the same. This phenomenon is because when using SIA, all intervals are viewed as independent. Furthermore, algebraic properties of SIA in $_+$ are often insufficient for solving inverse problems (Markov 1995, 2001; Lodwick and Dubois 2015). The incompleteness of that algebraic structure stimulated attempts to create a more convenient IA extension based on $_+$.

4.2 Extended (generalized) interval arithmetic (EIA)

One of the most successful extensions of SIA is EIA. The latter is based on the concept of extended intervals (Kaucher 1973, 1980; Popova 2001). Extended intervals are intervals whose bounds are not constrained to be ordered. The main idea of EIA is to complete the set $\mathbb{I}_+$ of proper intervals by the set $\mathbb{I}_-$ of improper intervals, and the SIA is extended correspondingly. Indeed, knowing that the subsystems $(\mathbb{I}_+, +)$ and $(\mathbb{I}_+, \times)$ are only monoids (Markov 2001), the reciprocal operations for $+$ and $\times$ do not exist and the equation $[a] + [x] = [b]$ or $[a] \times [x] = [b]$ does not always have an "exact" solution in $\mathbb{I}_+$. The same problem appears when solving the equation $a + x = b$ in $\Re^+$, i.e., there is no solution in $\Re^+$ for $a > b$. To make $a + x = b$ always solvable for real numbers, it is primordial to extend the set $\Re^+$ by adding the set $\Re^-$. It amounts to solving the equation $a + x = b$ in $\Re = \Re^+ \cup \Re^-$. Considering the algebraic construction, the group $(\Re, +)$ is built by embedding the monoid $(\Re^+, +)$. By analogy, similar to $\Re^+$ being completed with $\Re^-$ to form $\Re$, the set of proper intervals $\mathbb{I}_+$ is completed with the set of improper intervals $\mathbb{I}_-$ to form the set of extended intervals $\mathbb{I}$,, i.e., $\mathbb{I} = \mathbb{I}_- \cup \mathbb{I}_+ = \{[a^-, a^+] \mid a^-, a^+ \in \Re\}$.

In the context of EIA, the operator pro (proper projection), which returns a proper interval, is defined by: $\forall a\in$, pro (a) = pro $([a^-, a^+]) = [\min\{a^-, a^+\}, \max\{a^-, a^+\}]$. The relationship between proper and improper interval is established by an operator dual defined by:

$$\forall[a] \in \mathbb{I}, \text{dual}([a]) = \text{dual}([a^-, a^+]) = [a^+, a^-]$$

EIA provides richer semantics than SIA, because it is possible to define the inverse operations of addition and multiplication (Markov 1995). However, though EIA is useful for solving inverse problems, the results can sometimes be improper intervals, which are not usable in practical applications. In such applications, it is important to find conditions that guarantee that the result belongs to $\mathbb{I}_+$. The SIA is extended to $\mathbb{I}$ in (Kaucher 1973, 1980; Markov 1995; Popova 2001); only extended subtraction and division operators are presented in this section (see Markov 1977, 1979; Popova and Markov 1997 for more details).

- Extended subtraction: this operator is defined as follows:

$$\forall[a] \in \mathbb{I}, \forall[b] \in \mathbb{I}, [a] -_{\mathrm{E}} [b] = [a^- - b^-, a^+ - b^+]. \tag{7}$$

From Eq. (7), we have $[a] -_E [a] = 0$ and $[a] -_{E} [b] \subseteq [a] - [b]$. It can also be proven (Markov 1979, 1997; Popova and Markov 1997) that:

$$[a] -_E [b] \in \begin{cases} \mathbb{I}_+ :\ \text{if } R([a]) \geq R([b]) \\ \mathbb{I}_- :\ \text{if } R([a]) \end{cases}.$$

- Extended division: the division of $[a]$ by $[b]$ is defined as follows:

$\forall[a] \in \mathbb{I},\ \forall[b] \in \mathbb{I}\backslash\mathbb{Z}:$

$$[a] \div_E [b] = \begin{cases} [a^{-\sigma([b])} \div b^{-\sigma([a])},\ a^{\sigma([b])} \div b^{\sigma([a])}];\ \text{for } [a],[b] \in \mathbb{I}\backslash\mathbb{Z} \\ [a^{-\sigma([b])} \div b^{\sigma([b])},\ a^{\sigma([b])} \div b^{\sigma([b])}];\ \text{for } [a] \in \mathbb{Z}, [b] \in \mathbb{I}\backslash\mathbb{Z} \end{cases} \tag{8}$$

In Eq. (8), $\mathbb{Z}$ represents the subset of intervals (in $\mathbb{I}$) containing zero in their interior. It can be seen from Eq. (8) that $[a] \div_E [a] = 1$. Furthermore, $[a] \div_E [b] \subseteq [a] \div [b]$. The quantity $[c] = [a] \div_E [b] \in \mathbb{I}_+$ if $R([c]) \geq 0$. It can be proved that $R([c]) \geq 0$, when $\chi([b]) \geq \chi([a])$ (Markov 1995); otherwise, $[a] \div_E [b] \in \mathbb{I}_-$, i.e.,

$$[a] \div_E],[b] \in \begin{cases} \mathbb{I}_+:\ \text{if } \chi([b]) \geq \chi([a]) \\ \mathbb{I}_-:\ \text{if } \chi([b]) < \chi([a]) \end{cases},$$

where χ is the Chi function defined by Eq. (2). More properties of extended the subtraction and division operators are given in (Markov 1995).

4.3 Non-standard (inner) interval arithmetic (NIA)

Markov's work, initially proposed in (Markov 1977), is one of the pertinent responses to the problem of the absence of inverses in SIA. He proposed an extension of SIA by introducing the so-called NIA. In this section, subtraction and division operators are discussed. Addition and multiplication operations are those used in SIA. More details concerning inner operations can be found in (Markov 1977, 1979; Dimitrova et al. 2010; Popova and Markov 1997).

- Non-standard subtraction: the subtraction operator of Markov, denoted $-_N$, is defined by:

$$\forall[a],[b] \in \mathbb{I}_+ : [a] -_N [b] = \begin{cases} [a^- - b^-, a^+ - b^+]\ ;\ \text{if } R([a]) \geq R([b]) \\ [a^+ - b^+, a^- - b^-]\ ;\ \text{if } R([a]) < R([b]) \end{cases}. \tag{9}$$

The operator $-_N$ can be rewritten as follows:

$$[a] -_N [b] = [\min\{a^- - b^-, a^+ - b^+\}, \max\{a^- - b^-, a^+ - b^+\}].$$

In general, $[a] -_N [b] \neq [a] - [b]$. Indeed, $[a] -_N [b] = [a] - [b]$ iff $R([a]).R([b]) = 0$. By analyzing Eq. (9), it leads to:

$$[a] -_N [b] = \begin{cases} [a^- - b^-, a^+ - b^+] = [a] -_E [b];\ \text{if } R([a]) \geq R([b]) \\ [a^+ - b^+, a^- - b^-] = \text{dual}([a] -_E [b])\ ;\ \text{if } R([a]) < R([b]) \end{cases} \tag{10}$$

According to Eq. (10), if $R([a]) \geq R([b])$ then $-_E$ and $-_N$ are equivalent and the resulting intervals in $\mathbb{I}_+$. In contrast, when $R([a]) < R([b])$, the operator $-_E$ gives an improper interval. To remedy this situation, the dual of this improper interval is taken in the definition of the operator $-_N$. Furthermore, the operator $-_N$ is a proper projection of $-_E$, i.e.,

$$\forall[a],[b] \in \mathbb{I}_+ : [a] -_N [b] = \text{pro}([a] -_E [b]) \subseteq [a] - [b]. \tag{11}$$

According to Eq. (11), $[a] -_N [a] = 0$ and $[a] -_N [b] = -_N ([b] -_N [a])$. Generally, $[a] + [b] = [c]$ implies that $[a] = [c] -_N [b]$ and $[b] = [c] -_N [a]$. Moreover, $([a] -_N [b]) + [b] = [a]$ only if $R([a]) \geq R([b])$. Otherwise, $([a] -_N [b]) + [b] \neq [a]$. Furthermore, $[a] -_N [b] \subseteq [a] - [b]$.

- Non-standard division: the inner division operator $\div_N$ is defined by:

$$[a] \div_N [b] = \begin{cases} \text{for } [a],[b] \in \mathbb{I}_+\backslash\mathbb{Z}_+ : \begin{cases} [a^{-\sigma([b])} \div b^{-\sigma([a])},\ a^{\sigma([b])} \div b^{\sigma([a])}];\ \text{if } \chi([b]) \geq \chi([a]) \\ [a^{\sigma([b])} \div b^{\sigma([a])},\ a^{-\sigma([b])} \div b^{-\sigma([a])}];\ \text{if } \chi([b]) < \chi([a]) \end{cases} \\ \text{for } [a] \in \mathbb{Z}_+, [b] \in \mathbb{I}_+\backslash\mathbb{Z}_+ : [a^{-\sigma([b])} \div b^{\sigma([b])},\ a^{\sigma([b])} \div b^{\sigma([b])}] \end{cases}. \tag{12}$$

If $\chi([b]) \geq \chi([a])$, then the operators $\div_E$ and $\div_N$ are equivalent and the interval obtained is in $\mathbb{I}_+$. In contrast, if $\chi([b]) < \chi([a])$, $\div_E$ gives an improper interval. In this case, the operator $\div_N$ transforms this improper interval into a proper one using the dual. Thus, $\div_N$ is a proper projection of $\div_E$, i.e.,

$$[a] \div_N [b] = \text{pro}([a] \div_E [b]) = \begin{cases} \text{for } [a],[b] \in \mathbb{I}_+\backslash\mathbb{Z}_+ : \begin{cases} [a] \div_E [b];\ \text{if } \chi([b]) \geq \chi([a]) \\ \text{dual}([a] \div_E [b]);\ \text{if } \chi([b]) < \chi([a]) \end{cases} \\ \text{for } [a] \in \mathbb{Z}_+, [b] \in \mathbb{I}_+\backslash\mathbb{Z}_+ : [a] \div_E [b]; \end{cases}. \tag{13}$$

According to Eq. (13), if $0 \notin [a]$, then $([a] \div_N [a]) = 1$. In general, $[a] \times [b] = [c]$ implies that $[a] = [c] \div_N [b]$ and $[b] = [c] \div_N [a]$. However, the equation $([a] \times [b]) \div_N [b] = [a]$ is valid only if $\chi([b]) \geq \chi([a])$. Otherwise, $([a] \times [b]) \div_N [b] \neq [a]$. Furthermore, in all circumstances $[a] \div_N [b] \subseteq [a] \div [b]$.

4.4 Generalized Hukuhara interval arithmetic: GHIA

Hukuhara proposed a difference between convex sets (Hukuhara 1967), known as the Hukuhara difference (HD) and denoted $-_H$. It can be translated into intervals and leads to the following expression (Dimitrova et al. 2010; Stefanini 2010):

$$\forall [a], [b] \in \mathbb{I}_+ : [a] -_H [b] = \begin{cases} [a^- - b^-, a^+ - b^+]; \text{if } R([a]) \geq R([b]) \\ \text{Not defined ; if } R([a]) < R([b]) \end{cases} \tag{14}$$

The existence condition of $-_H$ is imposed by the fact that $[a] -_H [b]$ must be proper ($\in \mathbb{I}_+$), i.e., $R([a]) \geq R([b])$ (Dimitrova et al. 2010). If this condition is not satisfied, then $[a] -_H [b]$ is not defined. This problem has found a solution thanks to Stefanini's works, in which a generalized HD is proposed (Stefanini 2010).

- Generalized Hukuhara subtraction: the generalized HD (Stefanini 2010), denoted $-_G$, is defined by:

$$\forall [a], [b] \in \mathbb{I}_+ : [a] -_G [b] = [c];\ \text{with} : \begin{cases} a^- = b^- + c^- \\ a^+ = b^+ + c^+ \end{cases} \text{(i)}\ or \begin{cases} b^- = a^- - c^+ \\ b^+ = a^+ - c^- \end{cases} \text{(ii)}. \tag{15}$$

The difference $[a] -_G [b]$ in Eq. (15) can be rewritten as:

$$[a] -_G [b] = \begin{cases} [a^- - b^-, a^+ - b^+];\ \text{if } R([a]) \geq R([b]) : \text{situation (i)} \\ [a^+ - b^+, a^- - b^-];\ \text{if } R([a]) < R([b]) : \text{situation (ii)} \end{cases}. \tag{16}$$

According to Eqs. (9) and (16), the equivalence between GHIA and NIA subtraction is established, i.e.,

$$[a] -_G [b] = [a] -_N [b] = \text{pro}([a] -_E [b]) \subseteq [a] - [b]. \tag{17}$$

- Generalized Hukuhara division: using the same philosophy as in the generalized HD, the following generalized division operator $\div_G$ is defined by Stefanini (2010):

$$\forall [a] \in \mathbb{I}_+, \forall [b] \in \mathbb{I}_+ \backslash \mathbb{Z}_+ : [a] \div_G [b] = [c];\ \text{with:} \begin{cases} \text{(i)} : [a] = [b] \times [c] \text{ or} \\ \text{(ii)} : [b] = [a] \times [c]^{-1} \end{cases}. \tag{18}$$

Six cases were analyzed by Stefanini (2010) with respect to Eq. (18) and the sign of the intervals $[a]$ and $[b]$. A new formulation is given here using the Chi function (see Table 2). It can be observed that situation (ii) consists of taking the dual of the interval in situation (i) when that interval is improper. The aggregation of the six cases in Table 2 gives the non-standard division operator.

In this case, the operator $\div_G$ is equivalent to the operator $\div_N$, i.e.,

$$[a] \div_G [b] = [a] \div_N [b] = \text{pro}([a] \div_E [b]) \subseteq [a] \div [b]. \tag{19}$$

The remarks given for NIA remain valid here for GHIA.

4.5 Optimistic interval arithmetic (OIA)

The philosophy of OIA (Boukezzoula and Galichet 2010; Boukezzoula et al. 2012, 2014) is based on the restriction of EIA to $\mathbb{I}_+$. EIA is exploited when it provides proper intervals; otherwise, OIA uses SIA. In this context, instead of using a proper projection as in the GHIA and NIA methodologies, an SIA vision is exploited to override the presence of improper intervals. OIA is developed in the midpoint-radius space (Kulpa 2001), but can be used naturally in the EP one. More details concerning SIA and OIA in the MR space are given in (Kulpa 2001; Boukezzoula and Galichet 2010; Boukezzoula et al. 2012, 2014). The optimistic subtraction and division operations are defined as follows:

- Optimistic subtraction: this operator is defined by:

$$\forall [a], [b] \in \mathbb{I}_+ : [a] -_O [b] = \begin{cases} [a] -_E [b];\ \text{if } R([a]) \geq R([b]) \\ [a] - [b];\ \text{if } R([a]) < R([b]) \end{cases} \tag{20}$$

As with EIA, the quantity $[a] -_O [b] \in \mathbb{I}_+$ iff $R([a]) \geq R([b])$. In this situation, the properties of EIA are preserved. In the optimistic vision, this condition is interpreted as follows: one cannot subtract from an imprecise quantity another more imprecise quantity without the risk of increasing the imprecision of the operation's result. If this principle is violated, the operation leads to an improper interval (in $\mathbb{I}_-$). In this situation, to avoid improper intervals, we must turn to SIA.

- Optimistic division: the division operator is defined as follows:

$$\forall [a] \in \mathbb{I}_+, \forall [b] \in \mathbb{I}_+ \backslash \mathbb{Z}_+ : [a] \div_O [b] = \begin{cases} \text{for } [a], [b] \in \mathbb{I}_+ \backslash \mathbb{Z}_+ : \begin{cases} [a] \div_E [b];\ \text{if } |\text{rex}([b])| \geq |\text{rex}([a])| \\ [a] \div [b];\ \text{if } |\text{rex}([b])| < |\text{rex}([a])| \end{cases}, \\ \text{for } [a] \in \mathbb{Z}_+, [b] \in \mathbb{I}_+ \backslash \mathbb{Z}_+ : [a] \div_E [b]; \end{cases} \tag{21}$$

In Eq. (21), rex denotes the relative-extent function defined by Eq. (3).

Table 2 Analysis of the generalized division operator

Case	Sign of $[a]$	Sign of $[b]$	Conditions	$[c] = [c^-, c^+]$	Situation
1	+	−	$a^- \times b^- \geq a^+ \times b^+ \Rightarrow \chi([b]) \geq \chi([a])$	$[a^+ \div b^-, a^- \div b^+]$	(i)
			$a^- \times b^- \leq a^+ \times b^+ \Rightarrow \chi([b]) < \chi([a])$	$[a^- \div b^+, a^+ \div b^-]$	(ii)
2	+	+	$a^- \times b^+ \leq a^+ \times b^- \Rightarrow \chi([b]) \geq \chi([a])$	$[a^- \div b^-, a^+ \div b^+]$	(i)
			$a^- \times b^+ \geq a^+ \times b^- \Rightarrow \chi([b]) < \chi([a])$	$[a^+ \div b^+, a^- \div b^-]$	(ii)
3	−	−	$a^+ \times b^- \leq a^- \times b^+ \Rightarrow \chi([b]) \geq \chi([a])$	$[a^+ \div b^+, a^- \div b^-]$	(i)
			$a^+ \times b^- \geq a^- \times b^+ \Rightarrow \chi([b]) < \chi([a])$	$[a^- \div b^-, a^+/b^+]$	(ii)
4	−	+	$a^- \times b^- \leq a^+ \times b^+ \Rightarrow \chi([b]) \geq \chi([a])$	$[a^- \div b^+, a^+ \div b^-]$	(i)
			$a^- \times b^- \geq a^+ \times b^+ \Rightarrow \chi([b]) < \chi([a])$	$[a^+/b^-, a^- \div b^+]$	(ii)
5	$0 \in [a]$	−	Does not depend on b^+: always true	$[a^+ \div b^-, a^- \div b^-]$	(i)
6	$0 \in [a]$	+	Does not depend on b^-: always true	$[a^- \div b^+, a^+ \div b^+]$	(i)

It was proven in (Boukezzoula and Galichet 2010; Boukezzoula et al. 2012, 2014) that $[a] \div_O [b] \in \mathbb{I}_+$ if $|\mathrm{rex}([b])| \geq |\mathrm{rex}([a])|$ and the EIA properties remain validated. This condition is strictly equivalent to the condition $\chi([b]) \geq \chi([a])$ used in EIA and NIA. In an optimistic context, this condition can be interpreted as follows: one cannot divide an imprecise quantity by another more imprecise one (in the sense of relative extent) without the risk of increasing the imprecision of the operation's result. If this principle is not respected, the operation leads to an improper interval (in $\mathbb{I}_-$). In this case, to avoid improper intervals, SIA division is used.

5 Instantiated interval arithmetic (IIA): extensions and hybridizations

An analysis of the IIA methodologies mentioned previously is detailed below. For ease of reading, the notations given in Table 3 are used for an operator $\odot \in \{+, -, \times, \div\}$.

5.1 Instantiated interval arithmetic (IIA)

As explained by Lodwick and Dubois (2015), the IIA approach is the initial vision proposed by the pioneers of IA (S–W–M) before its translation to the well-known SIA, where the independence property was assumed. IIA is defined on and restricted to $\mathbb{I}_+$. In this context, the four standard arithmetic operations for two intervals are the same as those given as follows, i.e.,

$$[a] \odot_I [b] = \left[\min_{a\in[a],\, b\in[b]} a \odot b,\ \max_{a\in[a],\, b\in[b]} a \odot b\right];\ \text{for } \odot \in \{+, -, \times, \div\}. \tag{22}$$

However, IIA uses the instantiated concept and does not impose independence. Unlike SIA, when the intervals are assumed to be dependent (copy of the same variable), IIA provides the following result:

$$\forall [a] \in \mathbb{I}_+, [a] -_I [a] = [a - a] = [0, 0] = 0 \neq [a] - [a] \text{ and}$$
$$\forall [a] \in \mathbb{I}_+ \backslash \mathbb{Z}_+, [a] \div_I [a] = [a \div a] = [1, 1] = 1 \neq [a] \div [a]$$

In contrast, where the handled intervals are assumed to be independent, it yields:

$$[a] \odot_I [b] = [a] \odot [b];\ \text{for } \odot \in \{+, -, \times, \div\}.$$

In other words, when the intervals $[a]$ and $[b]$ are considered to be independent, IIA is strictly equivalent to SIA (Lodwick and Dubois 2015). Contrasting to SIA, when the intervals $[a]$ and $[b]$ are the same, IIA provides different results $[a] -_I [b]$ (resp. $[a] \div_I [b]$) regardless of whether the intervals are associated with dependent (for example, multiple copies of the same interval) or independent variables. It can be seen that in IIA, $[b] +_I ([a] -_I [b]) = [a]$ and $[b] \times_I ([a] \div_I [b]) = [a]$. Moreover, $[a] -_I [a] = 0$ and $[a] \div_I [a] = 1$. Furthermore, unlike in SIA, IIA has

Table 3 Notations of IIA and its extensions

IIA and its extensions denotations	Acronym	Operator index
Instantiated interval arithmetic	IIA	I
Constrained interval arithmetic	CIA	C
Single-level constrained interval arithmetic	SCIA	S
Requisite constrained interval arithmetic	RCIA	R

additive and multiplicative inverses and enables the resolution of interval equations (Lodwick and Dubois 2015). It is clear that IIA implementation is a global optimization problem. In fact, a powerful and elegant implementation of IIA is undoubtedly CIA proposed by (Lodwick 1999; Lodwick and Jenkins 2013).

5.2 Constrained interval arithmetic (CIA)

CIA is based on the concept of a constrained interval. The interval $[a]$ is represented by a continuous and monotonic function $f(\omega_a)$: $[0, 1] \rightarrow [a^-, a^+]$ such that $f(0) = a^-$, $f(1) = a^+$ and f is non-decreasing. For simplicity, $f(\omega_a)$ is required to be linear and increasing, i.e.

$$f(\omega_a) = a^- + L_a\omega_a;\ L_a = (a^+ - a^-)\,;\ \text{with: } 0 \le \omega_a \le 1\,.$$

The concept of a constrained interval is used to express an ill-known value $a \in [a]$ as $a = f(\omega_a)$ for $\omega_a \in [0, 1]$. Thus, the choice of a unique value $a \in [a]$ is interpreted as the choice of a unique value of ω_a. For two intervals $[a]$ and $[b]$, the CIA operations are given by:

$$\begin{aligned}&\text{for } \odot \in \{+, -, \times, \div\}:\\ &[a] \odot_C [b] =\\ &\left[\min_{0 \le \omega_a \le 1; 0 \le \omega_b \le 1}\{f(\omega_a) \odot f(\omega_b)\}, \max_{0 \le \omega_a \le 1; 0 \le \omega_b \le 1}\{f(\omega_a) \odot f(\omega_b)\}\right].\end{aligned} \tag{23}$$

From Eq. (23), it can be seen that for independent variables, CIA is equivalent to IIA and SIA, i.e.,

$$[a] \odot_I [b] = [a] \odot_C [b] = [a] \odot [b]\,;\ \text{for } \odot \in \{+, -, \times, \div\}.$$

In the opposite case, the manipulation of the same variable leads to the following expressions for the subtraction and division operators:

- Subtraction operator: $\forall [a] \in \mathbb{I}_+$:

$$\begin{aligned}&[a] -_C [a]\\ &\quad = \left[\min_{0 \le \omega_a \le 1}\{f(\omega_a) - f(\omega_a)\}, \max_{0 \le \omega_a \le 1}\{f(\omega_a) - f(\omega_a)\}\right]\\ &= 0.\end{aligned}$$

- Division operator: $\forall [a] \in \mathbb{I}_+ \backslash \mathbb{Z}_+$:

$$\begin{aligned}&[a] \div_C [a]\\ &\quad = \left[\min_{0 \le \omega_a \le 1}\{f(\omega_a) \div f(\omega_a)\}, \max_{0 \le \omega_a \le 1}\{f(\omega_a) \div f(\omega_a)\}\right]\\ &= 1.\end{aligned}$$

The same analysis carried out for IIA remains valid here for CIA.

5.3 Single-level constrained interval arithmetic (SCIA)

The single-level constrained interval arithmetic (SCIA) is a special case of CIA proposed by Chalco-Cano et al. (2014). As in CIA, when considering an interval $[a]$, a function $f_a(\omega)$: $[0, 1] \rightarrow \Re$ is associated with $[a]$: $f_a(\omega) = a^- + L_a\omega\,;\ 0 \le \omega \le 1$. In SCIA, even when the intervals are independent, the instantiated variables are constrained by identical positions in their intervals, i.e., the same value of ω is fixed. For two intervals $[a]$ and $[b]$, the four standard SCIA operations are defined by:

$$\begin{aligned}&[a] \odot_S [b]\\ &\quad = \left[\min_{0 \le \omega \le 1}\{f_a(\omega) \odot f_b(\omega)\}, \max_{0 \le \omega \le 1}\{f_a(\omega) \odot f_b(\omega)\}\right.;\\ &\text{for } \odot \in \{+, -, \times, \div\}.\end{aligned} \tag{24}$$

For instance, this expression leads to the following subtraction and division operations:

- Subtraction operator:

$$\begin{aligned}&\forall [a], [b] \in \mathbb{I}_+:\\ &\begin{cases}[a] -_S [b] = \left[\min_{0 \le \omega \le 1}(f_a(\omega) - f_b(\omega)), \max_{0 \le \omega \le 1}(f_a(\omega) - f_b(\omega))\right]\\ \qquad = [\min\{a^- - b^-, a^+ - b^+\}, \max\{a^- - b^-, a^+ - b^+\}]\end{cases}.\end{aligned}$$

It can be stated that $[a] -_S [b] = [a] -_G [b] = [a] -_N [b] =$ pro$([a] -_E [b])$. This equivalence between the SCIA and GHIA subtraction operators was demonstrated in (Chalco-Cano et al. 2014). Although this equivalence deserves to be mentioned, the two representations are different, i.e., GHIA is an SIA extension and SCIA is inherent to IIA vision.

- Division operator:

$$\begin{aligned}&\forall [a] \in \mathbb{I}_+, \forall [b] \in \mathbb{I}_+ \backslash \mathbb{Z}_+:\ [a] \div_S [b]\\ &\quad = \left[\min_{0 \le \omega \le 1}\right.\\ &f_a(\omega) \div f_b(\omega), \max_{0 \le \omega \le 1} f_a(\omega) \div f_b(\omega)].\end{aligned}$$

It can be shown that this expression can be written as (Chalco-Cano et al. 2014):

$$\begin{aligned}&\forall [a] \in \mathbb{I}_+, \forall [b] \in \mathbb{I}_+ \backslash \mathbb{Z}_+:\\ &[a] \div_S [b] = [\min(a^- \div b^-, a^+ \div b^+), \max(a^- \div b^-, a^+ \div b^+)].\end{aligned} \tag{25}$$

As detailed in (Chalco-Cano et al. 2014), it can be proven that $[a] \div [b] \supseteq [a] \div_S [b] \supseteq [a] \div_G [b]$. Moreover, the equality $[a] \div_S [b] = [a] \div_G [b]$ is verified only for cases

2 and 3 in Table 2. Generally, the following property is verified (Chalco-Cano et al. 2014):

$$[a] \div [b] \supseteq [a] \div_S [b] \supseteq [a] \div_G [b] \\ = [a] \div_N [b] = \mathrm{pro}([a] \div_E [b]). \tag{26}$$

5.4 Requisite constrained interval arithmetic (RCIA)

To avoid counterintuitive results in the presence of multiple copies of the same variable, Klir proposes RCIA in which constraint relations on the quantities involved are introduced (Klir 1997; Klir and Pan 1998). Klir's idea consists of performing arithmetic operations with constraints dictated by the context of the problem. The constraints can be of various types: equality constraints, inequality constraints, and so on. However, the equality constraint is by far the most frequently used.

Let us consider a constraint relation Γ between the intervals $[a]$ and $[b]$. Each constraint Γ on $[a] \odot [b]$ is a relation on the Cartesian product $[a] \otimes [b]$.[1] In this context, RCIA is defined by (Klir 1997; Klir and Pan 1998):

$$([a] \odot_R [b])_\Gamma = \{[a] \odot [b] | (a, b) \in [a] \otimes [b] \cap \Gamma\}\text{; for } \odot \\ \in \{+, -, \times, \div\}. \tag{27}$$

When the constraint relation Γ is removed, definition in Eq. (27) turns into Eq. (4), i.e.,

$$[a] \odot_R [b] = \{[a] \odot [b] | (a, b) \in [a] \otimes [b]\} \\ = [a] \odot [b]\text{; for } \odot \in \{+, -, \times, \div\}, \tag{28}$$

which is equivalent to the definitions in SIA. For instance, if the equality constraint (i.e., $\Gamma = EQ$) is imposed (see Klir 1997; Klir and Pan 1998), then the following properties are obtained for subtraction and division:

- Subtraction operator: $\forall [a] \in \mathbb{I}_+ : ([a] -_R [a])_{\mathrm{EQ}} = \{a - a | a \in [a]\} = 0$.
- Division operator: $\forall [a] \in \mathbb{I}_+ \backslash \mathbb{Z}_+ : ([a] \div_R [a])_{\mathrm{EQ}} = \{a \div a | a \in [a]\} = 1$.

More details on the addition and multiplication operations are given in (Klir 1997; Klir and Pan 1998). In RCIA, the dependence between the manipulated variables can be captured by the equality constraint. This approach is efficient for avoiding overestimation due to the occurrence of interactive variables. However, the generalization of this approach to other types of constraints remains a difficult problem. When the equality constraint is used, the RCIA and CIA results are equivalent.

[1] Let us note that $\otimes$ is used to represent the cartesian product in place of the conventional symbol $\times$, which is used in this paper for the multiplication operator.

6 Gradual interval representation and arithmetic

6.1 Gradual interval arithmetic

From a theoretical point of view, all the SIA and IIA varieties of IA developed above are directly transposable to GI. However, from a practical point of view, there are some differences. Unlike a conventional interval, for which only a single horizontal dimension is considered, a GI is represented using two dimensions. In this context, particular attention will be paid to the GI profiles.

In this paper, we propose the use of the concept of gradual intervals (Dubois and Prade 2008; Fortin et al. 2008; Boukezzoula et al. 2012), which generalize conventional intervals, thereby making it possible to represent imprecision and uncertainty via the notion of gradualness (Dubois and Prade 2008; Fortin et al. 2008). A conventional interval $[a] = [a^-, a^+]$ simply becomes a GI: $[a(\lambda)] = [a^-(\lambda), a^+(\lambda)]$, when its boundaries are GN (Dubois and Prade 2008; Fortin et al. 2008). This GI can be interpreted as a conventional interval in a space of functions (i.e., the interval bounds are GNs) and it inherits the same algebraic properties as the CIs. In the gradual interval representation, two dimensions are considered. The first (the horizontal dimension) is similar to the one used in the representation of conventional intervals. The second (the vertical dimension) is related to the likelihood degrees or uncertainty/certainty and is limited to the unit interval [0, 1].

A fuzzy interval A is a normalized fuzzy subset of real numbers with a membership function μ_A. This fuzzy interval is a convex fuzzy set (its α-cuts are intervals). The upper semi-continuity of μ_A is equivalent to α-cuts being closed intervals. In this context, a fuzzy interval can be viewed as a stack of nested intervals that is defined by its α-cuts. However, in certain practical situations, gradual intervals may occur. They are generally characterized by ill-nested intervals in the vertical dimension. Conversely, a GI can be interpreted as an FI, if its profiles $a^-(\lambda)$ and $a^+(\lambda)$ are injective and, respectively, non-decreasing and non-increasing (Dubois and Prade 2008; Fortin et al. 2008; Boukezzoula et al. 2012, 2014). While an FI may be a particular GI, the opposite is false insofar as no monotonicity constraint is associated with GI. In the context of equivalence with FI, $a^-(\lambda)$ and $a^+(\lambda)$ are assumed to be continuous and their domains are extended to [0, 1], i.e., $a^-(0)$ and $a^+(0)$ are defined. For the remainder of this paper, a GI in which the profiles $a^-(\lambda)$ and $a^+(\lambda)$ are, respectively, non-decreasing and non-increasing is called a monotone (consonant) GI (or FI), while a non-monotone (non-consonant) GI that cannot be represented by FI is called a "pure GI". If $a^-(\lambda) \le a^+(\lambda)$, the GI is proper; otherwise, it is improper. The sets of proper and improper GIs are, respectively, denoted $\mathbb{GI}_+$ and $\mathbb{GI}_-$, and $\mathbb{GI} = \mathbb{GI}_- \cup \mathbb{GI}_+$ represents the set of generalized GIs.

The IA methodologies presented in sections IV and V can be directly extended to FIA. However, some FIAs, especially in the SIA context can lead to pure GIs, which cannot be represented by FI (Allahviranloo et al. 2011; Boukezzoula et al. 2012, 2014; Gomes and Barros 2015). These results are not controversial in the gradual context, because no constraint of monotonicity is imposed on the profiles of the GI. The gradual vision can provide a new interpretation for reinforcing the essence of FIA. GIA is elaborated upon by extending the IA expressions to the gradual case, where GIs replace the intervals in the equations. Furthermore, all the remarks and analysis given in sections IV and V remain valid, where the conventional intervals are replaced by GIs. For more details on basic GIA operations, see (Boukezzoula and Galichet 2010; Boukezzoula et al. 2012, 2014). For instance, the gradual SIA, EIA, and CIA subtraction operators between two intervals are defined by:

- Gradual SIA subtraction:

$$\forall\, [a(\lambda)], [b(\lambda)] \in \mathbb{GI}_+,\ [a(\lambda)] - [b(\lambda)] = [a^-(\lambda) - b^+(\lambda), a^+(\lambda) - b^-(\lambda)].$$

- Gradual EIA subtraction:

$$\forall [a(\lambda)], [b(\lambda)] \in \mathbb{GI}, [a(\lambda)] -_E [b(\lambda)] = [a^-(\lambda) - b^-(\lambda), a^+(\lambda) - b^+(\lambda)].$$ Thus, $[a(\lambda)] -_E [b(\lambda)] \in \begin{cases} \mathbb{GI}_+ : \text{ if } R([a(\lambda)]) \ge R([b(\lambda)]) \\ \mathbb{GI}_- : \text{ if } R([a(\lambda)]) < R([b(\lambda)]) \end{cases}.$

- Gradual CIA subtraction:

$$\forall [a(\lambda)],\ [b(\lambda)] \in \mathbb{GI}_+ : [a(\lambda)] -_C [b(\lambda)] = \left[\min_{0 \le \omega_a \le 1; 0 \le \omega_b \le 1} \{f_\lambda(\omega_a) - f_\lambda(\omega_b)\}, \max_{0 \le \omega_a \le 1; 0 \le \omega_b \le 1} \{f_\lambda(\omega_a) - f_\lambda(\omega_b)\}\right].$$

As mentioned previously, for simplicity and without loss of generality, the function $f_\lambda(\omega_a)$ is required to be linear and increasing. For instance:

$$f_\lambda(\omega_a) = a^-(\lambda) + L_a(\lambda).\omega_a;\ L_a(\lambda) = (a^+(\lambda) - a^-(\lambda))\,;\ \text{with: } 0 \le \omega_a \le 1\,.$$

6.2 Remarks and discussions

- It is important to be able to compare different arithmetics. For instance, we provide a comparison here of the different SIA alternatives. Of course, this comparison can be applied in the IIA context or between SIA and IIA approaches. Since SIA is defined only in $\mathbb{GI}_+$, the comparison can only be carried out in this context.

  The first comparison is related to the inclusion property. The computations based on EIA, NIA, GHIA, and OIA are always at least as precise as those based on SIA, i.e.,$\forall \blacklozenge \in \{E, N, G, O\}$:

$$\begin{cases} \forall\, [a(\lambda)], [b(\lambda)] \in \mathbb{GI}_+, [a(\lambda)] -_\blacklozenge [b(\lambda)] \subseteq [a(\lambda)] - [b(\lambda)] \\ \forall [a(\lambda)] \in \mathbb{GI}_+, \forall [b(\lambda)] \in \mathbb{GI}_+ \backslash \mathbb{Z}_+,\ [a(\lambda)] \div_\blacklozenge [b(\lambda)] \subseteq [a(\lambda)] \div [b(\lambda)] \end{cases}. \tag{29}$$

- The second comparison consists of evaluating the gain in precision. To evaluate the precision gain between gradual SIA and the other gradual IA, the following indicator is proposed:

$$\Xi_{\odot}^{\blacklozenge}(\lambda) = \frac{R([a(\lambda)] \odot_\blacklozenge [b(\lambda)])}{R([a(\lambda)] \odot [b(\lambda)])};\ \text{for: } \odot \in \{-, \div\}\,;\ \text{and } \blacklozenge \in \{E, N, G, O\}. \tag{30}$$

This precision indicator is not limited to basic operators, but can be applied to more complicated mathematical expressions. In Eq. (30), the special case of 0/0 is interpreted as 1. According to the inclusion property of Eq. (29), it can be deduced that the precision gain indicator is ≤ 1. More specifically, this indicator is interpreted as follows (see Table 4).

- In a fuzzy framework, when considering independent consonant GI (or FI), the results produced by SIA are equivalent to the one given by Zadeh's extension principle. In this context, IIA and SIA approaches are generally equivalent. However, in the presence of multiple copies of the same variable (dependent variables), the extension principle implements an IIA view.
- GIA is useful for solving interval equations. Since addition and subtraction (resp. multiplication and division) are not inverse operations in SIA, it is not possible to accurately solve the interval equations $[a(\lambda)] = [b(\lambda)] + [x(\lambda)]$ and $[a(\lambda)] = [b(\lambda)] \times [x(\lambda)]$. Furthermore, the gradual interval equation $[a(\lambda)] = [b(\lambda)] + [x(\lambda)]$ has an exact solution in $\mathbb{GI}_+$ according to EIA, NIA, GHIA, and OIA, iff $R([a(\lambda)]) \ge R([b(\lambda)])$. This solution is unique and is given by $[x(\lambda)] = [a(\lambda)] -_\blacklozenge [b(\lambda)]$, for $\blacklozenge \in \{E, N, G, O\}$. In the same way, $[b(\lambda)] \times [x(\lambda)] = [a(\lambda)]$ has an exact solution in $\mathbb{GI}_+$ using EIA, NIA, GIA, and OIA if $\chi([b(\lambda)]) \ge \chi([a(\lambda)])$. Moreover, the solution is unique and given by: $[x(\lambda)] = [a(\lambda)] \div_\blacklozenge [b(\lambda)]$. In an IIA context, the inverse operators always exist and it is possible to solve these equations accurately.

Table 4 Interpretation of the indicator $\Xi_{\odot}^{\blacklozenge}(\lambda)$

Indicator value	Indicator value interpretation
$\Xi_{\odot}^{\blacklozenge}(\lambda) < 0$	The operation $\odot_{\blacklozenge}$ gives an improper gradual interval
$\Xi_{\odot}^{\blacklozenge}(\lambda) = 0$	The operation $\odot_{\blacklozenge}$ gives a crisp gradual number
$0 < \Xi_{\odot}^{\blacklozenge}(\lambda) < 1$	The imprecision between $\odot_{\blacklozenge}$ and $\odot$ is reduced by $1/\ \Xi_{\odot}^{\blacklozenge}(\lambda)$
$\Xi_{\odot}^{\blacklozenge}(\lambda) = 1$	The operators $\odot_{\blacklozenge}$ and $\odot$ give the same result

7 Illustrative examples

In this section, simulation results using the different gradual IAs are presented and five examples are considered. Moreover, to be able to compare the results, the proposed method is implemented using examples extracted from (Stefanini 2010; Liu et al. 2012; Chalco-Cano et al. 2014; Lodwick and Dubois 2015). The first example illustrates the behavior of SIA and IIA operators in the situation of independent variables. The second example shows gradual computing in evaluating an analytic expression with dependent variables. The third example clarifies the possible passage through improper intermediate GI during gradual computing. The fourth example shows the potential of IA alternatives in solving fuzzy linear equations. The fifth example is a three-term fuzzy weighted average in which the variables are dependent. For reasons of conciseness, only the subtraction and division operators are illustrated.

7.1 Example 1

This example is taken from the paper of Stefanini (2010), in which the two triangular consonant GIs (FIs) $[a(\lambda)] = [1 + 0.5\lambda,\ 5 - 3.5\lambda]$ and $[b(\lambda)] = [-4 + 2\lambda,\ -1 - \lambda]$ are considered.

A. SIA vision and its extensions:

The SIA results are given by:

$$[a(\lambda)] - [b(\lambda)] = [2 + 1.5\lambda,\ 9 - 5.5\lambda] \text{and} [a(\lambda)] \div [b(\lambda)] = [(5 - 3.5\lambda) \div (-1 - \lambda),\ (1 + 0.5\lambda) \div (-4 + 2\lambda)].$$

Since $\forall\ \lambda \in [0,\ 1]$, $R_a(\lambda) = 2 - 2\lambda \geq R_b(\lambda) = 1.5 - 1.5\lambda$, we have:

$$[a(\lambda)] -_E [b(\lambda)] = [a(\lambda)] -_N [b(\lambda)] = [a(\lambda)] -_G [b(\lambda)] = [a(\lambda)] -_O [b(\lambda)] = [5 - 1.5\lambda,\ 6 - 2.5\lambda] \in \mathbb{GI}_+.$$

In the same way, since $\forall\lambda \in [0,\ 1]$: $\chi([b(\lambda)]) = (-1 - \lambda) \div (-4 + 2\lambda) \geq \chi([a(\lambda)]) = (1 + 0.5\lambda) \div (5 - 3.5\lambda)$, we have:

$$[a(\lambda)] \div_E [b(\lambda)] = [a(\lambda)] \div_N [b(\lambda)] = [a(\lambda)] \div_G [b(\lambda)] = [a(\lambda)] \div_O [b(\lambda)] = [(5 - 3.5\lambda) \div (-4 + 2\lambda),\ (1 + 0.5\lambda) \div (-1 - \lambda)] \in \mathbb{GI}_+.$$

It can be seen that the GHIA of Stefanini (2010) is strictly equivalent to EIA, NIA, and OIA and leads to the same proper GI (see Figs. 1, 2). According to these arithmetics, although the operands $[a(\lambda)]$and $[b(\lambda)]$ are FIs, the subtraction and division results are purely GIs and cannot be represented by FIs. This result is in accordance with the criticisms given in (Allahviranloo et al. 2011; Gomes and Barros 2015). For instance and as stated in (Gomes and Barros 2015), the assertion that the generalized difference between two fuzzy numbers is always a fuzzy number is incorrect. If these results are controversial in an FIA context, they are accepted in a gradual framework, where no constraint of monotonicity is imposed on the interval profiles, thereby giving the gradual concept its full meaning. Moreover, the EIA, NIA, GHIA, and OIA results are less imprecise than the SIA ones, i.e., $[a(\lambda)] \odot_{\blacklozenge} [b(\lambda)] \subseteq [a(\lambda)] \odot [b(\lambda)]$, for $\odot \in \{-, \div\}$ and $\blacklozenge \in \{E, N, G, O\}$.

Compared to methods published in the literature and cited above, the proposed approach allows the quantification of the gain in precision between the different IAs through the overestimation indicator. For instance, the overestimation indicator (see (30)) between NIA and SIA is given by:

$$\Xi_{-}^{N}(\lambda) = \frac{R([a(\lambda)] -_N [b(\lambda)])}{R([a(\lambda)] - [b(\lambda)])} = \frac{0.5 - 0.5\lambda}{3.5 - 3.5\lambda} \text{ and}$$
$$\Xi_{\div}^{N}(\lambda) = \frac{R([a(\lambda)] \div_N [b(\lambda)])}{R([a(\lambda)] \div [b(\lambda)])} = \frac{2 + 3\lambda - 5\lambda^2}{38 - 51\lambda + 13\lambda^2}.$$

In this case, for $\lambda = 0$, the NIA subtraction reduces the imprecision of the SIA subtraction by a factor of 7. For $\lambda = 1$, the two arithmetics give the same result, i.e., $\Xi_{-}^{N}(\lambda) = 0/0 = 1$. The same remarks can be made about the division operator. Indeed, for $\lambda = 0$, the NIA division reduces the imprecision of the SIA division by a factor of 19. The same remarks can be made on the other arithmetics, which lead to the same results.

B. IIA vision and its extensions:

IIA is implemented in the same way as CIA. The intervals $[a(\lambda)]$ and $[b(\lambda)]$ are expressed as follows:

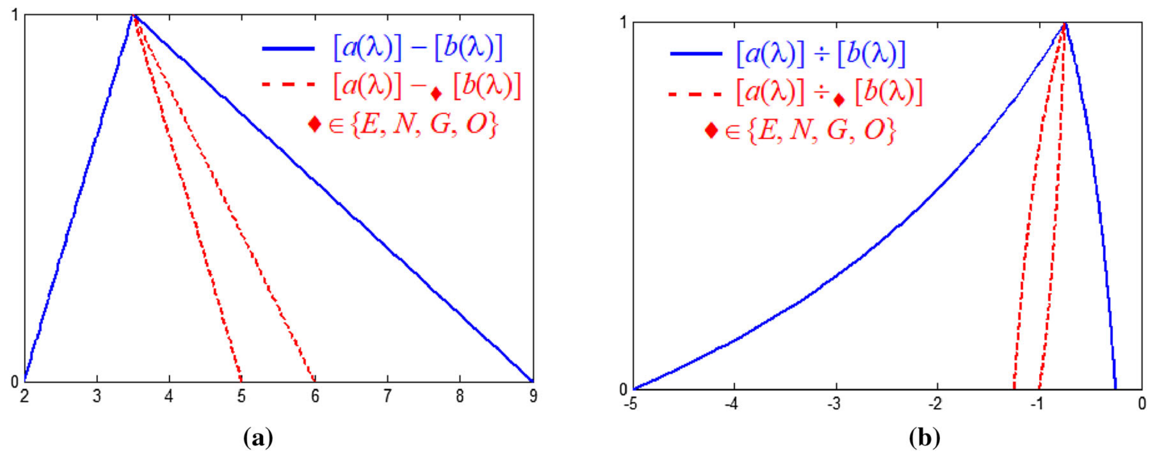

Fig. 1 SIA vision of division and subtraction operators: **a** subtraction operators, **b** division operators

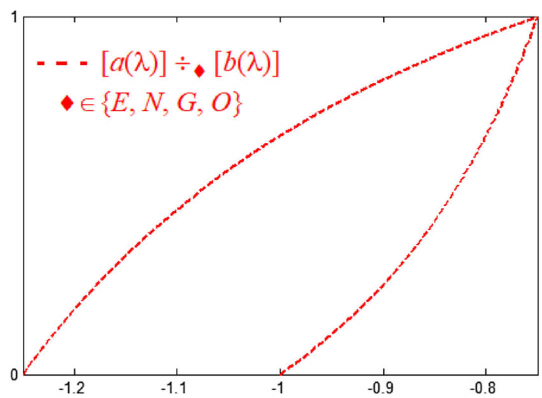

Fig. 2 Zoom-in view of the division operator

$$f_\lambda(\omega_a) = a^-(\lambda) + L_a(\lambda)\omega_a$$
$$= 1 + 0.5\lambda + (4 - 4\lambda)\omega_a;\;\; 0 \le \omega_a \le 1$$
$$f_\lambda(\omega_b) = b^-(\lambda) + L_b(\lambda)\omega_b||$$

The CIA subtraction and division are given by:

$$[a(\lambda)] -_C [b(\lambda)] =$$
$$\left[\min_{0\le\omega_a\le1;0\le\omega_b\le1}\{f_\lambda(\omega_a) - f_\lambda(\omega_b)\}, \max_{0\le\omega_a\le1;0\le\omega_b\le1} cr\{f_\lambda(\omega_a) - f_\lambda(\omega_b)\}\right];$$
$$= [2 + 1.5\lambda,\, 9 - 5.5\lambda]$$

$$[a(\lambda)] \div_C [b(\lambda)]$$
$$= [\min_{0\le\omega_a\le1;0\le\omega_b\le1}\{f_\lambda(\omega_a) \div f_\lambda(\omega_b)\}, \max_{0\le\omega_a\le1;0\le\omega_b\le1}\{f_\lambda(\omega_a) \div f_\lambda(\omega_b)\}]$$
$$= [(5 - 3.5\lambda) \div (-1 - \lambda), (1 + 0.5\lambda) \div (-4 + 2\lambda)].$$

As discussed in Sect. 5, since the operands $[a(\lambda)]$ and $[b(\lambda)]$ are assumed to be independent, SIA and CIA give the same result. In the same way, it can be verified that CIA and RCIA (under the equality constraint) are equivalent, i.e.,

$$[a(\lambda)] \odot [b(\lambda)] = [a(\lambda)] \odot_C [b(\lambda)]$$
$$= ([a(\lambda)] \odot_R [b(\lambda)])_{\mathrm{EQ}}\;;\;\text{for } \odot \in \{-, \div\}.$$

In SCIA, the GI $[a(\lambda)]$ and $[b(\lambda)]$ are transformed into the following functions:

$$f_\lambda^a(\omega) = a^-(\lambda) + L_a(\lambda)\omega = 1 + 0.5\lambda + (4 - 4\lambda)\omega;\;\; 0 \le \omega \le 1$$
$$f_\lambda^b(\omega) = b^-(\lambda) + L_b(\lambda)\omega = -4 + 2\lambda + (3 - 3\lambda)\omega;\;\; 0 \le \omega \le 1\,.$$

The SCIA subtraction and division are given by:

$$[a(\lambda)] -_S [b(\lambda)]$$
$$= \left[\min_{0\le\omega\le1}\{f_\lambda^a(\omega) - f_\lambda^b(\omega)\}, \max_{0\le\omega\le1}\{f_\lambda^a(\omega) - f_\lambda^b(\omega)\}\right]$$
$$= [5 - 1.5\lambda,\, 6 - 2.5\lambda]$$

and

$$[a(\lambda)] \div_S [b(\lambda)]$$
$$= \left[\min_{0\le\omega\le1}\{f_\lambda^a(\omega) \div f_\lambda^b(\omega)\}, \max_{0\le\omega\le1}\{f_\lambda^a(\omega) \div f_\lambda^b(\omega)\}\right]$$
$$= [(5 - 3.5\lambda) \div (-1 - \lambda),\, (1 + 0.5\lambda) \div (-4 + 2\lambda)].$$

These results lead to:

$$[a(\lambda)] -_S [b(\lambda)] = [a(\lambda)] -_E [b(\lambda)]\,;\;\text{and } [a(\lambda)] \div_S [b(\lambda)]$$
$$= [a(\lambda)] \div [b(\lambda)].$$

As discussed previously, although their philosophy and context of application are different, the SCIA subtraction is always equivalent to the EIA one. The SCIA division, in this special case, is equivalent to the SIA one. This observation is not always true, as will be illustrated in the following example. However, in all cases , property of Eq. (29) holds.

7.2 Example 2

This example is taken from the paper of Chalco-Cano et al. (2014). The objective is to compute the expression $E(a, b) = (a - b)/b$ when the variables a and b are viewed as triangular FIs (consonant GIs) given by $[a(\lambda)] = [\lambda, 2-\lambda]$ and $[b(\lambda)] = [2+\lambda, 5-2\lambda]$. The SIA and IIA approaches are considered below. To simplify the mathematical notation, the expression $E(a, b)$ will be denoted as E.

According to SIA, the evaluation of the expression E leads to:

$$[E(\lambda)]_{\mathrm{SIA}} = ([a(\lambda)] - [b(\lambda)]) \div [b(\lambda)] = [(-5+3\lambda) \div (2+\lambda), (-2\lambda) \div (2+\lambda)] \in \mathbb{GI}_{+}.$$

The computation of the expression E according to EIA is given by:

Since $\forall\ \lambda \in [0, 1]$, $R([a(\lambda)]) < R([b(\lambda)])$, which leads to:

$$[a(\lambda)] -_{E} [b(\lambda)] = [-2, -3+\lambda] \in \mathbb{GI}_{-}.$$

At the same time, since $\chi([b(\lambda)]) < \chi([a(\lambda)] -_{E} [b(\lambda)])$, it follows that:

$$[E(\lambda)]_{\mathrm{EIA}} = ([a(\lambda)] -_{E} [b(\lambda)]) \div_{E} [b(\lambda)] = [(-3+\lambda) \div (5-2\lambda), (-2) \div (2+\lambda)] \in \mathbb{GI}_{-}.$$

It can be observed from these results that EIA results in improper GIs (see Fig. 3b), i.e., the left and right profiles are permuted. In this situation, to obtain proper intervals, NIA and GHIA use a proper projection of the EIA results (see Fig. 3a):

$$[a(\lambda)] -_{N} [b(\lambda)] = [a(\lambda)] -_{G} [b(\lambda)] = \mathrm{pro}([a(\lambda)] -_{E} [b(\lambda)]) = [-3+\lambda, -2] \in \mathbb{GI}_{+} \text{ and}$$
$$[E(\lambda)]_{\mathrm{NIA}} = [E(\lambda)]_{\mathrm{GHIA}} = \mathrm{pro}([E(\lambda)]_{\mathrm{EIA}}) = [(-2) \div (2+\lambda), (-3+\lambda) \div (5-2\lambda)] \in \mathbb{GI}_{+};$$

where

$$[E(\lambda)]_{\mathrm{NIA}} = ([a(\lambda)] -_{N} [b(\lambda)]) \div_{N} [b(\lambda)]; \text{ and: } [E(\lambda)]_{GHIA} = ([a(\lambda)] -_{G} [b(\lambda)]) \div_{G} [b(\lambda)];$$

Since the EIA results are improper, to keep an interpretable and realistic result, OIA uses SIA. In this case, we have:

$$[a(\lambda)] -_{O} [b(\lambda)] = [a(\lambda)] - [b(\lambda)]; \text{ and: } [E(\lambda)]_{\mathrm{OIA}} = [E(\lambda)]_{\mathrm{SIA}}.$$

In this application, SIA, EIA, NIA, GHIA, and OIA are not equivalent. As the condition for obtaining proper intervals is not guaranteed, the EIA produces improper results. However, the subtraction and division operators are exact inverse operations of addition and multiplication. In this context, NIA and GHIA operate an interval proper projection of the EIA results. NIA and GHIA are equivalent, but their subtraction and division are not exact inverse operations of the addition and multiplication operations. Furthermore, OIA turns to SIA and they are equivalent.

- IIA vision and its extension

In the implementation of CIA, the intervals are transformed as follows:

$f_{\lambda}(\omega_a) = \lambda + (2-2\lambda)\omega_a;\ 0 \le \omega_a \le 1$ and $f_{\lambda}(\omega_b) = 2 + \lambda + (3-3\lambda)\omega_b;\ 0 \le \omega_b \le 1$.

The expression of E in CIA is given by:

$$[E(\lambda)]_{CIA} = \left[\min_{0 \le \omega_a \le 1; 0 \le \omega_b \le 1} (f_{\lambda}(\omega_a) - f_{\lambda}(\omega_b)) \div f_{\lambda}(\omega_b), \max_{0 \le \omega_a \le 1; 0 \le \omega_b \le 1} (f_{\lambda}(\omega_a) - f_{\lambda}(\omega_b)) \div f_{\lambda}(\omega_b)\right] = [(-5+3\lambda) \div (5-2\lambda), (-2\lambda) \div (2+\lambda)]$$

For SCIA, the intervals are expressed as follows:

$f^{a}_{\lambda}(\omega) = \lambda + (2-2\lambda)\omega;\ f^{b}_{\lambda}(\omega) = 2 + \lambda + (3-3\lambda)\omega;\ 0 \le \omega \le 1$ and

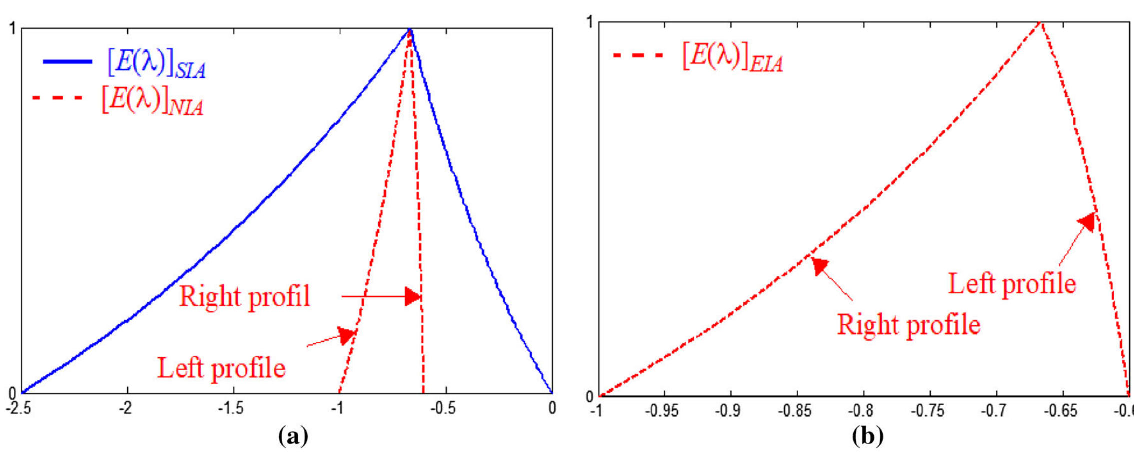

Fig. 3 The expression $E(\lambda)$ according to SIA, EIA, and NIA: **a** SIA and NIA results, **b** EIA result

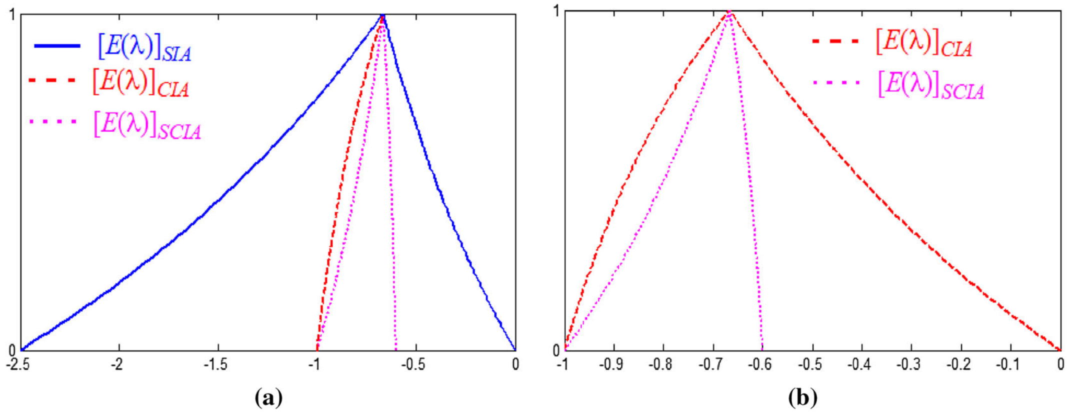

Fig. 4 The expression of E according to SIA, CIA, and SCIA: **a** SIA, CIA, and SCIA results, **b** CIA and SCIA results

$$[E(\lambda)]_{SCIA} = \left[\min_{0\le\omega\le 1}(f^a_\lambda(\omega) - f^b_\lambda(\omega)) \div f^b_\lambda(\omega), \max_{0\le\omega\le 1}(f^a_\lambda(\omega) - f^b_\lambda(\omega)) \div f^b_\lambda(\omega)\right] = [(-2)\div(\lambda+2), (-3+\lambda)\div(5-2\lambda)].$$

The expression of E according to CIA and SCIA and its comparison with SIA are given in Fig. 4. In this situation, it can be seen that CIA and SCIA give different results (see Fig. 4b) and they are not equivalent.

From a methodological perspective, SIA and CIA are different. The differentiation between them is based on the meaning and interpretation attributed to the considered intervals. Furthermore, if the handled intervals $[a]$ and $[b]$ are considered to be strictly independent, the SIA and CIA produce exactly the same results. In the opposite case, when the intervals are dependent (for example, multiple copies of the same interval as in the case of the expression E), the results provided by SIA and CIA may be different. In this case, CIA is always less imprecise than SIA. Moreover, the following inclusion is obtained $[E(\lambda)]_{\mathrm{SCIA}} \subseteq [E(\lambda)]_{\mathrm{CIA}} \subseteq [E(\lambda)]_{\mathrm{SIA}}$. As illustrated in Fig. 4a, the right profiles of the CIA and SIA results coincide. In addition, as detailed in Sect. 7.1, the indicator of Eq. (30) can be implemented to specify the gain in precision between the IA.

7.3 Example 3

The possible passage through improper intermediate GIs during gradual computing is illustrated in this example. Let us consider the interval expression $[f(\lambda)] = [f_1(\lambda)] - [f_2(\lambda)]$; $[f_1(\lambda)] = [b(\lambda)] \div [a(\lambda)]$ and $[f_2(\lambda)] = [b(\lambda)] - [a(\lambda)]$, where $[a(\lambda)]$ and $[b(\lambda)]$ are trapezoidal GIs given by $[a(\lambda)] = [1 + 2\lambda, 9 - 4\lambda]$ and $[b(\lambda)] = [1 + \lambda, 5 - 2\lambda]$. The computation of $[f(\lambda)]$ using EIA gives the following result:

$$[f_1(\lambda)]_{\mathrm{EIA}} = [b(\lambda)] \div_E [a(\lambda)] = [(1+\lambda)\div(1+2\lambda), (5-2\lambda)\div(9-4\lambda)] \in \mathbb{GI}_-;$$

$$[f_2(\lambda)]_{\mathrm{EIA}} = [b(\lambda)] -_E [a(\lambda)] = [-\lambda, -4+2\lambda] \in \mathbb{GI}_-; \text{ and}$$

$$[f(\lambda)]_{\mathrm{EIA}} = [f_1(\lambda)]_{\mathrm{EIA}} -_E [f_2(\lambda)]_{\mathrm{EIA}} = [(1+2\lambda+2\lambda^2)\div(1+2\lambda), (41-36\lambda+8\lambda^2) \div (9-4\lambda)] \in \mathbb{GI}_+.$$

Although the two quantities $[f_1(\lambda)]_{\mathrm{EIA}}$ and $[f_2(\lambda)]_{\mathrm{EIA}}$ are improper, the result of the expression $[f(\lambda)]_{\mathrm{EIA}}$ is proper (see Fig. 5).

For comparison purposes, the computation of $[f(\lambda)]$ using SIA and EIA is illustrated in Fig. 6, where

$$[f(\lambda)]_{\mathrm{SIA}} = [(35-53\lambda+16\lambda^2)\div(-9+4\lambda), (13+9\lambda - 10\lambda^2)\div(1+2\lambda)].$$

Unlike the FIA approach, the originality of the proposed GIA method lies in its ability to compute with pure and/or improper GIs. A philosophical debate can be made on the meaning of improper gradual intervals. We believe that this concept does not conflict with a viable mathematical reasoning. These improper intervals (issued from EIA) have no physical meaning, but they can be used in intermediate computing. An analogy can be made with complex numbers when they are used for computation, but real numbers are required for the result. For instance, this example is used for highlighting the possible passage through improper intermediate GIs during gradual computing.

Clearly, the set of generalized intervals (proper and improper) is useful for solving inverse problems. However, the final computational results can sometimes be unrealistic for real applications, i.e., when improper intervals are obtained. An important issue is to be able to find conditions that guarantee that the result belongs to the set of proper intervals. In this paper, these conditions are given for the subtraction and division operators, but can be obtained for any given mathematical interval expression. More generally, in the situation when the final computational result is

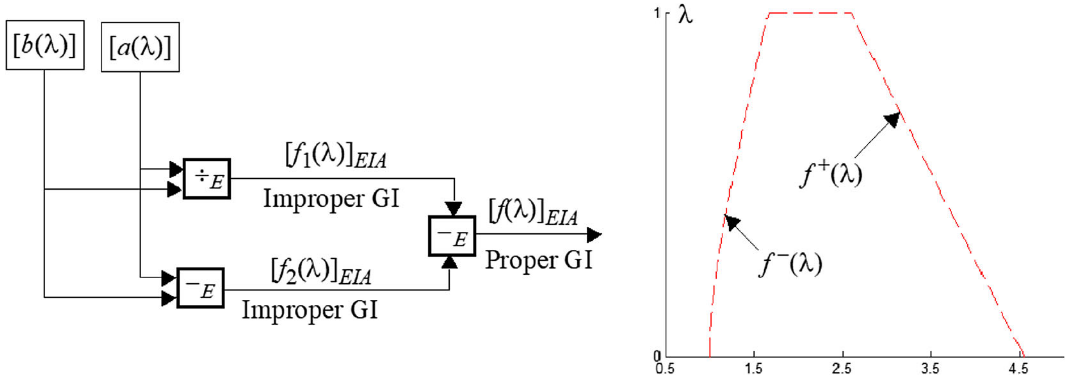

Fig. 5 Illustration of possible passages through improper gradual intervals

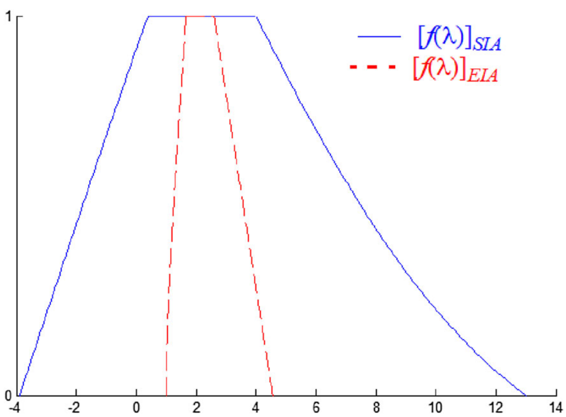

Fig. 6 $[f(\lambda)]$ results using SIA and EIA

improper, one must turn to another IA. This is clearly the philosophy of NIA, GHIA, and OIA

7.4 Example 4

This example gives an illustration of the potential applicability of the SIA alternatives in solving fuzzy linear equations. Consider the resolution of the fuzzy equation $[a(\lambda)] \times x \subseteq [b(\lambda)]$ given in (Lodwick and Dubois 2015), where $[a(\lambda)]$ and $[b(\lambda)]$ are triangular fuzzy intervals, $[a(\lambda)]$ with a support [1, 3] and a core $\{2\}$; $[b(\lambda)]$ with a support [3, 5] and a core $\{4\}$. If the set of solutions of this equation is denoted $[x(\lambda)]$, it can be expressed as follows:

$$[a(\lambda)] \times [x(\lambda)] = [b(\lambda)].$$

The solution $[x(\lambda)]$ of this equation can be obtained through the EIA, i.e.,

$$\begin{aligned}[x(\lambda)] &= [x^-(\lambda), x^+(\lambda)] = [b(\lambda)] \div_E [a(\lambda)] \\ &= [(\lambda+3) \div (\lambda+1), (\lambda-3) \div (\lambda-1)].\end{aligned}$$

The result of this operation is an improper gradual interval, because $\chi([b(\lambda)]) < \chi([a(\lambda)])$. Moreover, it can be seen that $R([x(\lambda)]) < 0$ and the bounds of the gradual interval are reversed (see Fig. 7a, situation 1). In this case, the solution set $\Omega_{\forall\exists}$, such that $x^-(\lambda) \le x \le x^+(\lambda)$ is reduced to $x = [x(1)]$. This result is in concordance with the solution given in (Lodwick and Dubois 2015). Moreover, the proposed approach allows us to determine a priori whether the result will be a proper or improper GI. In the same way, we consider the previous equation in the case, where $[a(\lambda)]$ and $[b(\lambda)]$ are triangular and trapezoidal fuzzy intervals given by $[a(\lambda)] = [2 + 5\lambda, 12 - 5\lambda]$ and $[b(\lambda)] = [2 + 3\lambda, 12 - 3\lambda]$. The set of solutions is given by:

$$\begin{aligned}[x(\lambda)] &= [x^-(\lambda), x^+(\lambda)] = [b(\lambda)] \div_E [a(\lambda)] \\ &= [(3\lambda+2) \div (5\lambda+2), (3\lambda-12) \div (5\lambda-12)].\end{aligned}$$

The solution of this equation is a proper gradual interval, because $\chi([b(\lambda)]) \ge \chi([a(\lambda)])$. The set of solutions that is equivalent to that obtained in (Lodwick and Dubois 2015), is expressed as follows:

$$\begin{aligned}\Omega_{\forall\exists} = \{x | (3\lambda+2) \div (5\lambda+2) \le x \le (3\lambda-12) \div (5\lambda \\ -12)\}.\end{aligned}$$

However, it can be seen that $[x(\lambda)]$ is a pure gradual interval (see Fig. 7b, situation 2). In this case and as discussed in (Lodwick and Dubois 2015), the only plausible solution of this equation is the singleton $x = [x(0)] = 1$.

This example shows that GIA can be used to accurately solve fuzzy equations. The results obtained are equivalent to those obtained in (Lodwick and Dubois 2015). However, the proposed approach makes it possible to verify a priori whether the desired results are acceptable or not (proper or improper GIs).

7.5 Example 5

In this example, the three-term FWA example given in (Liu et al. 2012) is used. The scores and weights are summarized in Table 5. These fuzzy intervals are special cases of consonant gradual intervals.

According to the meaning given to the GI, SIA, and IIA weighted average versions can be considered. This vision

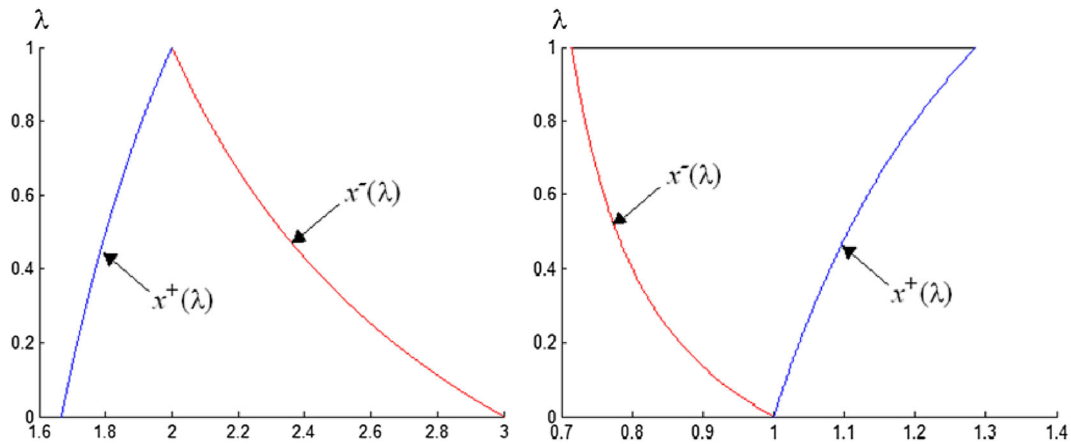

Fig. 7 Solutions of the fuzzy equation $[a(\lambda)] \times x \subseteq [b(\lambda)]$: **a** situation 1, **b** situation 2

Table 5 Scores and weights for gradual intervals

Scores $[x_i] = [x_i^-(\lambda), x_i^+(\lambda)]$	Weights $w_i = [w_i^-(\lambda), w_i^+(\lambda)]$
$[x_1] = [\lambda, 2 - \lambda]$	$[w_1] = [0.3\lambda, 0.9 - 0.6\lambda]$
$[x_2] = [2 + \lambda, 4 - \lambda]$	$[w_2] = [0.4 + 0.3\lambda, 1 - 0.3\lambda]$
$[x_3] = [4 + \lambda, 6 - \lambda]$	$[w_3] = [0.6 + 0.2\lambda, 1 - 0.2\lambda]$

provides a new interpretation of FWA through the notion of GI.

A. SIA vision of gradual weighted average (GWA)

The computation of GWA using SIA leads to the following expression:

$$\begin{aligned}&[\mathrm{GWA}(\lambda)]_{\mathrm{SIA}}\\ &= [\mathrm{num}(\lambda)] \div [\mathrm{den}(\lambda)] = ([x_1(\lambda)] \times [w_1(\lambda)] + \cdots + [x_5(\lambda)] \times [w_5(\lambda)])\\ &\quad \div ([w_1(\lambda)] + \cdots + [w_5(\lambda)])\\ &= [(8\lambda^2 + 24\lambda + 32) \div (29 - 11\lambda),\ (5.5\lambda^2 - 32.5\lambda + 59) \div (5 + 4\lambda)].\end{aligned}$$

Since $\chi([\mathrm{den}(\lambda)]) \geq \chi([\mathrm{num}(\lambda)])$, computing the GWA using EIA, NIA, GHIA, and OIA gives the same results, interpreted by the following expression:

$$\begin{aligned}&[\mathrm{GWA}(\lambda)]_{\mathrm{EIA}} = [\mathrm{GWA}(\lambda)]_{\mathrm{INIA}} = [\mathrm{GWA}(\lambda)]_{\mathrm{GHIA}}\\ &= [\mathrm{GWA}(\lambda)]_{\mathrm{OIA}}\\ &= [(4\lambda^2 + 12\lambda + 16) \div (5 + 4\lambda), (11\lambda^2 - 65\lambda + 118)\\ &\div (29 - 11\lambda)] \in \mathbb{GI}_+.\end{aligned}$$

The GWA results are illustrated in Fig. 8.

It can be observed from Fig. 8 that the GWA computed using EIA, NIA, GHIA, and OIA is a purely gradual interval and cannot be represented by an FI.

B. IIA vision of gradual weighted average (GWA)

The computation of GWA using CIA leads to the following expression:

$$\begin{aligned}&[GWA(\lambda)]_{\mathrm{CIA}} = [\mathrm{num}(\lambda)] \div_C [\mathrm{den}(\lambda)]\\ &= ([x_1(\lambda)] \times [w_1(\lambda)] + \cdots + [x_5(\lambda)] \times [w_5(\lambda)])\\ &\div ([w_1(\lambda)] + \cdots + [w_5(\lambda)]), \text{thus}\end{aligned}$$

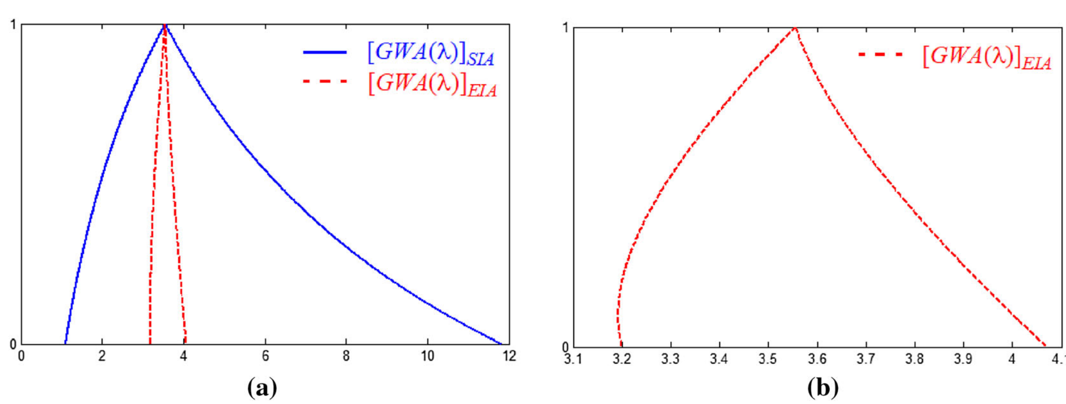

Fig. 8 GWA using SIA and EIA: **a** GWA using SIA and EIA, **b** GWA using EIA

$$[\mathrm{GWA}(\lambda)]_{\mathrm{CIA}} = \begin{cases} [(-\lambda^2 + 33\lambda + 32) \div (19 - \lambda),\ (-2\lambda^2 - 4\lambda + 38) \div (7 + 2\lambda)]\,;\ \text{if } 0 \le \lambda < 0.375 \\ [(-7\lambda^2 + 27\lambda + 44) \div (25 - 7\lambda),\ (-2\lambda^2 - 4\lambda + 38) \div (7 + 2\lambda)]\,;\ \text{if } 0.375 \le \lambda < 1 \end{cases}.$$

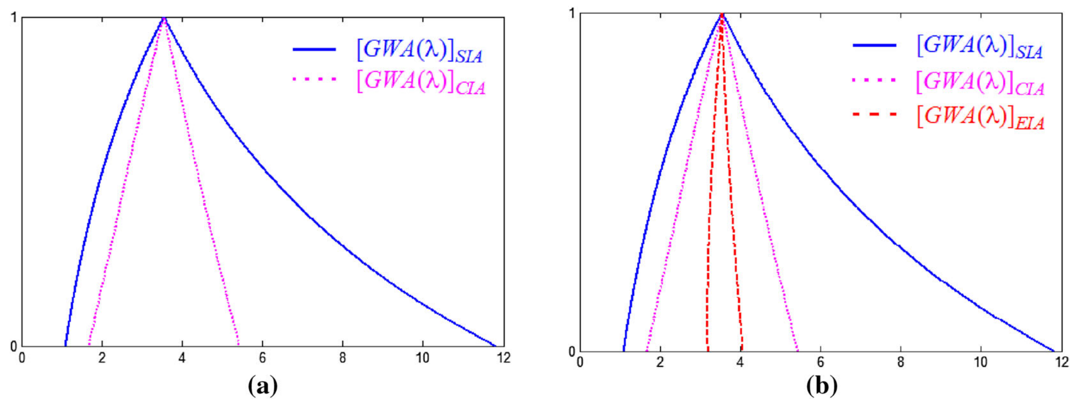

Fig. 9 GWA results using SIA, EIA, and CIA: **a** GWA using SIA and CIA, **b** GWA using SIA, CIA, and EIA

Table 6 Overestimation errors between GWA visions

Indicator	$\lambda = 0$	$\lambda = 0.5$	$\lambda = 1$
Between $[\mathrm{GWA}]_{\mathrm{EIA}}$ and $[\mathrm{GWA}]_{\mathrm{SIA}}$	0.149	0.16	1
Between $[\mathrm{GWA}]_{\mathrm{EIA}}$ and $[\mathrm{GWA}]_{\mathrm{CIA}}$	0.508	0.495	1

For comparison, Fig. 9 regroups the results of GWA using SIA, EIA, and CIA.

When $\lambda = 1$, all approaches give the same result, i.e., the weighted average for precise weights and numbers. At the other levels, the following inclusion is obtained: $[\mathrm{GWA}]_{\mathrm{EIA}} \subseteq [\mathrm{GWA}]_{\mathrm{CIA}} \subseteq [\mathrm{GWA}]_{\mathrm{SIA}}$. In this case, $[\mathrm{GWA}]_{\mathrm{EIA}}$ gives the less imprecise result. Furthermore, CIA is less imprecise than SIA. Moreover, the precision gain indicator can be implemented. For instance, the indicator Eq. (30) for three values of λ is given in Table 6.

It can be seen from these results that a value of 0.508 between the $[\mathrm{GWA}]_{\mathrm{EIA}}$ and $[\mathrm{EGWA}]_{\mathrm{CIA}}$ means that the latter reduces the imprecision by a factor of 2. More generally, the indicator is a non-linear function with regard to λ and can be evaluated for any given value of λ. The result obtained by $[\mathrm{EGWA}]_{\mathrm{CIA}}$ is equivalent to the one obtained using the fuzzy weighted average according to Zadeh's extension principle by applying the Karnik–Mendel algorithm (Liu et al. 2012). By analyzing the behavior of the KM, it is easy to show that the latter implements a direct extension of the weighted average operator to fuzzy intervals according to IIA. It is important to emphasize here that the given comparisons in this example are for illustrative purposes only. In reality, a comparison between SIA and IIA approaches is not relevant here, because these two visions have two different philosophies.

8 Conclusion

In this paper, using the concept of GIs, a contribution and a reflection on IA and its extension to GIA through SIA and IIA visions have been proposed. This GIA extension provides a revision and a new interpretation of FIA according to the concept of GIs. The results obtained illustrate the existing equivalences between the IAs and their interest. For instance, in an SIA framework, the extensions EIA, NIA, GHIA, and OIA are less imprecise than the original SIA. Moreover, it has been demonstrated that under some conditions, these IAs are equivalent. Furthermore, if EIA results are proper intervals, they are equivalent to those obtained by NIA, GIA, and OIA. In the opposite case, when EIA produces improper intervals, NIA and GHIA operate an interval proper projection, while OIA turns to SIA. In the IIA context, improper intervals cannot occur. Furthermore, if the variables are independent, SIA, CIA, and RCIA are equivalent. SCIA sometimes gives different results from CIA, but in any case, the SCIA results are included in SIA. More generally, if the manipulated intervals are dependent, the SIA and IIA approaches do not give the same results. Future work will be focused on the extension of this methodology for computing and solving more complicated interval expressions and equations.

In the paper, for simplicity and without loss of generality, illustrative examples are carried out using linear FIs and GIs. However, the proposed concepts remain transposable, regardless of the shape of the considered FIs and/or GIs. Furthermore, our approach can be applied to any analytical form of the considered FIs (Dutta and Saikia 2019; Dutta and Doley 2019; Fahmi et al. 2019). In the near future, a methodological reflection about the extension of FI arithmetic to interval valued intuitionistic fuzzy sets (Atanassov and Gargov 1989; Vidhya and Irene Hepzibah 2017) and interval type-2 fuzzy sets (Mendel et al. 2006) will be conducted. This extension could be inspired by the concepts of thick intervals and thick gradual intervals for its implementation (Boukezzoula et al. 2019). From practical perspectives, this extension could help in the implementation of multicriteria and multiple attribute decision making strategies, when the handled information is represented by interval valued intuitionistic fuzzy sets (Chen et al. 2012a, b; Wang and Chen 2017) and/or interval type-2 fuzzy sets (Qin and Liu 2015; Qin et al. 2017; Runkler et al. 2017). More generally, on the basis of ontic and epistemic interpretations of intervals (Cuso and Dubois 2014; Dubois 2011, 2014; Lodwick and Dubois 2015), another dichotomy for the categorization of IA and its extensions will be presented in our future paper.

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