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ORIGINAL PAPER

Novel distance measures for Pythagorean fuzzy sets with applications to pattern recognition problems

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Abstract

Pythagorean fuzzy set (PFS) is a concept that generalizes intuitionistic fuzzy sets. The notion of PFSs is very much applicable in decision science because of its unique nature of indeterminacy. The main feature of PFSs is that it is characterized by membership degree, non-membership degree, and indeterminate degree in such a way that the sum of the square of each of the parameters is one. In this paper, we propose some novel distance measures for PFSs by incorporating the conventional parameters that describe PFSs. We provide a numerical example to illustrate the validity and applicability of the distance measures for PFSs. While analyzing the reliability of the proposed distance measures in comparison with similar distance measures for PFSs in the literature, we discover that the proposed distance measures, especially, d_5 yields the most reasonable measure. Finally, some applications of d_5 to pattern recognition problems are explicated. These novel distance measures for Pythagorean fuzzy sets could be applied in decision making of real-life problems embedded with uncertainty.

Keywords Distance measure · Fuzzy set · Intuitionistic fuzzy set · Pattern recognition · Pythagorean fuzzy set

1 Introduction

Zadeh (1965) proposed the concept of fuzzy sets to cope uncertainty in real-life problems. Fuzzy set theory has achieved a great success in several fields due to its ability to cope uncertainty. Fuzzy set is characterized by a membership function, μ which takes value from a crisp set to a unit interval, $I = [0, 1]$. Many application of fuzzy sets have been carried out (see Chen et al. 2001; Chen and Tanuwijaya 2011; Chen and Chang 2011; Chen et al. 2012; Chen and Huang 2003; Lee and Chen 2008; Cheng et al. 2016; Chen and Wang 1995; Wang and Chen 2008). Out of several generalizations of fuzzy set theory for various objectives, the notion of intuitionistic fuzzy sets (IFSs) introduced by Atanassov (1983, 1986) is interesting and useful.

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A lot of attentions were drawn to the development of distance measures between IFSs in a quest to apply IFSs to solve many real-life problems. As such, several measures were proposed (see Hatzimichailidis et al. 2012; Szmidt and Kacprzyk 2000; Szmidt 2014; Wang and Xin 2005). Some applications of IFSs in real-life problems have been extensively researched by Davvaz and Sadrabadi (2016), Chen and Chang (2015) , Chen et al. (2016a, b), Liu and Chen (2017, 2018), Liu et al. (2017), De et al. (2001), Ejegwa et al. (2014), Ejegwa (2015), Ejegwa and Modom (2015), Ejegwa and Onasanya (2019), Szmidt and Kacprzyk (2001, 2004).

In a pursuit to reasonably cope uncertainty in real-life problems, Yager (2013a, b) proposed a concept called Pythagorean fuzzy sets (PFSs). The theory of PFSs is a new approach to deal with vagueness more precisely in comparison with IFSs. Albeit, the origin of PFSs emanated from IFSs of second type (IFSST) introduced by Atanassov (1989) as generalized IFSs. Some theoretical aspects of PFSs have been extensively studied (see Dick et al. 2016; Gou et al. 2016; He et al. 2016; Peng and Yang 2015). Pythagorean fuzzy set theory has attracted attentions of many scholars, and the concept has been applied to several application areas. For some applications of PFSs, see Ejegwa 2019a, b; Gao and Wei 2018; Rahman et al. 2018; Khan et al. 2018; Garg 2017, 2018; Du et al. 2017; Hadi-Venchen and Mirjaberi 2014; Yager 2016; Yager and Abbasov 2013.

The notion of distance measure for PFSs is of immense important, especially in terms of applications. Several authors have worked on distance measures for PFSs from different perspectives. Zhang and Xu (2014) first proposed distance measure for PFSs by incorporating the three traditional parameters of PFSs. Li and Zeng (2018) introduced PFS that is characterized by four parameters and consequently proposed a variety of distance measures for PFSs, which take into account the four proposed parameters. In this same vein, Peng (2018) proposed a new distance measure for PFSs by incorporating four parameters more than the three traditional components of PFSs. Zeng et al. (2018) extended the distance measures for PFSs studied by Li and Zeng (2018) by incorporating five parameters. Howbeit, the four or five parameters captured in (Li and Zeng 2018; Peng 2018; Zeng et al. 2018) are not the traditional components of PFSs. Ejegwa (2018) proposed some distance measures for PFSs which satisfied the metric conditions by incorporating the three parameters of PFSs.

Sequel to the exploration of some distance measures for PFSs (see Zhang and Xu 2014; Li and Zeng 2018; Peng 2018; Zeng et al. 2018; Ejegwa 2018); especially, those distance measures (Zhang and Xu 2014; Ejegwa 2018) that incorporated the three conventional parameters of PFSs, the need to propose new distance measures for PFSs with more reasonable, reliable, and efficient output, are undeniable. Thus, the motivation of this work. In this paper, we explore some novel distance measures for PFSs. By taking into account the three parameters characterization of PFSs (viz, membership degree, non-membership degree, and indeterminate degree), we propose some new distance measures for PFSs with application to pattern recognition problems. Before applying the proposed distance measures to some cases of pattern recognition, we provide a numerical example to illustrate the validity and applicability of the proposed distance measures for PFSs in comparison with the distance measures in (Zhang and Xu 2014; Ejegwa 2018), and find that the proposed distance measures, especially, d_5 yield the most reasonable measure. Hence, we apply the most reasonable of the distance measures for PFSs, that is, d_5 to pattern recognition problems.

This paper is organized by presenting some mathematical preliminaries of fuzzy sets, IFSs, and PFSs in Sect. 2. In Sect. 3, we reiterate some distance measures for PFSs studied in (Zhang and Xu 2014; Ejegwa 2018) with a numerical example. In addition, in Sect. 3, some novel distance measures for PFSs are proposed with a numerical example. Section 4 discusses the application of d_5 to pattern recognition problems. Finally, Sect. 5 summarises the resulted outcomes of the paper with future direction of research.

2 Preliminaries

We recall some basic notions of fuzzy sets and IFSs as background to PFSs.

Definition 2.1 (Zadeh 1965). Let X be a nonempty set. A fuzzy set A of X is characterized by a membership function:

$$\mu_A : X \to [0, 1].$$

That is

$$\mu_A(x) = \begin{cases} 1, & \text{if } x \in X \\ 0, & \text{if } x \notin X \\ (0,1) & \text{if } x \text{ispartlyin} X \end{cases}$$

Alternatively, a fuzzy set A of X is an object having the form:

$$A = \{\langle x, \mu_A(x)\rangle \mid x \in X\} \text{ or } A = \left\{\left\langle \frac{\mu_A(x)}{x} \right\rangle \mid x \in X\right\},$$

where the function

$$\mu_A(x) : X \to [0, 1]$$

defines the degree of membership of the element, $x \in X$.

Definition 2.2 (Atanassov 1983, 1986). Let a nonempty set X be fixed. An IFS A of X is an object having the form:

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in X\}$$

or

$$A = \left\{\left\langle \frac{\mu_A(x), \nu_A(x)}{x} \right\rangle \mid x \in X\right\},$$

where the functions

$$\mu_A(x) : X \to [0, 1] \text{ and } \nu_A(x) : X \to [0, 1]$$

define the degree of membership and the degree of non-membership, respectively, of the element $x \in X$ to A, which is a subset of X, and for every $x \in X$

$$0 \le \mu_A(x) + \nu_A(x) \le 1.$$

For each A in X

$$\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$$

is the intuitionistic fuzzy set index or hesitation margin of x in X. The hesitation margin $\pi_A(x)$ is the degree of non-determinacy of $x \in X$, to the set A and $\pi_A(x) \in [0, 1]$. The hesitation margin is the function that expresses lack of knowledge of whether $x \in X$ or $x \notin X$. Thus

$$\mu_A(x) + \nu_A(x) + \pi_A(x) = 1.$$

Definition 2.3 (Yager 2013a, b). Let X be a universal set. Then, a Pythagorean fuzzy set A which is a set of ordered pairs over X, is defined by

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in X\}$$

or

$$A = \left\{\left\langle \frac{\mu_A(x), \nu_A(x)}{x} \right\rangle \mid x \in X\right\},$$

where the functions

$$\mu_A(x) : X \to [0,1] \text{ and } \nu_A(x) : X \to [0,1]$$

define the degree of membership and the degree of non-membership, respectively, of the element $x \in X$ to A, which is a subset of X, and for every $x \in X$:

$$0 \leq (\mu_A(x))^2 + (\nu_A(x))^2 \leq 1.$$

Supposing $(\mu_A(x))^2 + (\nu_A(x))^2 \leq 1$, then there is a degree of indeterminacy of $x \in X$ to A defined by $\pi_A(x) = \sqrt{1 - [(\mu_A(x))^2 + (\nu_A(x))^2]}$ and $\pi_A(x) \in [0,1]$. In what follows, $(\mu_A(x))^2 + (\nu_A(x))^2 + (\pi_A(x))^2 = 1$. Otherwise, $\pi_A(x) = 0$ whenever $(\mu_A(x))^2 + (\nu_A(x))^2 = 1$.

We denote the set of all *PFSs* over X by *PFS*(X).

Example 2.4 Let $A \in PFS(X)$. Suppose $\mu_A(x) = 0.70$ and $\nu_A(x) = 0.50$ for $X = \{x\}$. Clearly, $0.70 + 0.50 \nleq 1$, but $0.70^2 + 0.50^2 \leq 1$. Thus, $\pi_A(x) = 0.5099$, and hence, $(\mu_A(x))^2 + (\nu_A(x))^2 + (\pi_A(x))^2 = 1$.

Table 1 explains the difference between Pythagorean fuzzy sets and intuitionistic fuzzy sets (Ejegwa 2018).

Definition 2.5 (Yager 2013a). Let $A, B \in PFS(X)$. Then A and B are equal iff $\mu_A(x) = \mu_B(x)$ and $\nu_A(x) = \nu_B(x)$ $\forall x \in X$.

Definition 2.6 (Yager 2013a, b). Let $A, B \in PFS(X)$. Then, we define the following:

Table 1 Pythagorean fuzzy sets and intuitionistic fuzzy sets

Intuitionistic fuzzy sets	Pythagorean fuzzy sets
$\mu + \nu \leq 1$	$\mu + \nu \leq 1$ or $\mu + \nu \geq 1$
$0 \leq \mu + \nu \leq 1$	$0 \leq \mu^2 + \nu^2 \leq 1$
$\pi = 1 - (\mu + \nu)$	$\pi = \sqrt{1 - [\mu^2 + \nu^2]}$
$\mu + \nu + \pi = 1$	$\mu^2 + \nu^2 + \pi^2 = 1$

(i) $A^c = \{\langle x, \nu_A(x), \mu_A(x)\rangle | x \in X\}$.
(ii) $A \cup B = \{\langle x, max(\mu_A(x), \mu_B(x)), min(\nu_A(x), \nu_B(x))\rangle | x \in X\}$.
(iii) $A \cap B = \{\langle x, min(\mu_A(x), \mu_B(x)), max(\nu_A(x), \nu_B(x))\rangle | x \in X\}$.

Remark 2.7 (Ejegwa 2018). Let $A, B, C \in PFS(X)$. By Definition 2.6, the following properties hold:

$$\begin{aligned}
(A^c)^c &= A \\
A \cap A &= A \\
A \cup A &= A \\
A \cap B &= B \cap A \\
A \cup B &= B \cup A \\
A \cap (B \cap C) &= (A \cap B) \cap C \\
A \cup (B \cup C) &= (A \cup B) \cup C \\
A \cap (B \cup C) &= (A \cap B) \cup (A \cap C) \\
A \cup (B \cap C) &= (A \cup B) \cap (A \cup C) \\
(A \cap B)^c &= A^c \cup B^c \\
(A \cup B)^c &= A^c \cap B^c
\end{aligned}$$

Definition 2.8 (Ejegwa and Onasanya, 2019). Let $A \in PFS(X)$. Then, the level/ground set of A is defined by

$$A_* = \{x \in X | \mu_A(x) > 0,\ \nu_A(x) < 1\}.$$

Certainly, A_* is a subset of X.

3 Some distance measures for Pythagorean fuzzy sets

Here, we present some distance measures (DM) for PFSs. First, let us consider the definition of distance measure for PFSs. Distance measure for PFSs is a term that describes the difference between Pythagorean fuzzy sets.

Definition 3.1 (Ejegwa 2018). Let X be nonempty set and $A, B, C \in PFS(X)$. The distance measure d between A and B is a function $d : PFS \times PFS \to [0,1]$ that satisfies

(i) $0 \leq d(A,B) \leq 1$ (boundedness).
(ii) $d(A,B) = 0$ iff $A = B$ (separability).
(iii) $d(A,B) = d(B,A)$ (symmetric).
(iv) $d(A,C) + d(B,C) \geq d(A,B)$ (triangle inequality).

Now, we recall a proposition from Ejegwa (2018), and state a new result.

Proposition 3.2 (Ejegwa 2018). *Let $A, B, C \in PFS(X)$. Suppose $A \subseteq B \subseteq C$, then $d(A,C) \geq d(A,B)$ and $d(A,C) \geq d(B,C)$.*

Proposition 3.3 *If $A, B, C \in PFS(X)$, such that $A \subseteq B \subseteq C$, then*

$d(A,C) \geq max[d(A,B), d(B,C)]$.

Proof Let $A, B, C \in PFS(X)$. Assume that $A \subseteq B \subseteq C$, then by Proposition 3.2, we have $d(A,B) \leq d(A,C)$ and $d(B,C) \leq d(A,C)$. Hence

$d(A,C) \geq max[d(A,B), d(B,C)]$.

□

Let A and B be PFSs of $X = \{x_1, \ldots, x_n\}$, and by incorporating the three parameters of PFSs, the following distance measures have been proposed in the literature:

$$d_1(A,B) = \frac{1}{2n}\sum_{i=1}^{n}[|\mu_A(x_i) - \mu_B(x_i)| + |\nu_A(x_i) - \nu_B(x_i)| + |\pi_A(x_i) - \pi_B(x_i)|]$$

$$d_2(A,B) = \left(\frac{1}{2n}\sum_{i=1}^{n}[(\mu_A(x_i) - \mu_B(x_i))^2 + (\nu_A(x_i) - \nu_B(x_i))^2 + (\pi_A(x_i) - \pi_B(x_i))^2]\right)^{\frac{1}{2}}$$

$$d_3(A,B) = \frac{1}{2n}\sum_{i=1}^{n}[|(\mu_A(x_i))^2 - (\mu_B(x_i))^2| + |(\nu_A(x_i))^2 - (\nu_B(x_i))^2| + |(\pi_A(x_i))^2 - (\pi_B(x_i))^2|]$$

$$d_4(A,B) = \frac{1}{2}\sum_{i=1}^{n}[|(\mu_A(x_i))^2 - (\mu_B(x_i))^2| + |(\nu_A(x_i))^2 - (\nu_B(x_i))^2| + |(\pi_A(x_i))^2 - (\pi_B(x_i))^2|],$$

where

$$\pi_A(x_i) = \sqrt{1 - [(\mu_A(x_i))^2 + (\nu_A(x_i))^2]}$$

and

$$\pi_B(x_i) = \sqrt{1 - [(\mu_B(x_i))^2 + (\nu_B(x_i))^2]}.$$

These are the distance measures studied in Pythagorean fuzzy set setting that take account of the three conventional parameters of PFSs. Note that $d_1(A,B)$–$d_3(A,B)$ were introduced by Ejegwa (2018); Zhang and Xu (2014) proposed $d_4(A,B)$. Certainly, $d_3(A,B)$ normalizes $d_4(A,B)$.

3.1 New distance measures for Pythagorean fuzzy sets

We propose some new distance measures for Pythagorean fuzzy sets, and exemplify the measures to ascertain their compliant to Definition 3.1.

Let $A, B \in PFS(X)$, such that $X = \{x_1, \ldots, x_n\}$. By incorporating the three parameters of PFSs, we propose the following new distance measures for PFSs:

$$d_5(A,B) = \frac{1}{4n}\sum_{i=1}^{n}[|\mu_A(x_i) - \mu_B(x_i)| + ||\mu_A(x_i) - \nu_A(x_i)| - |\mu_B(x_i) - \nu_B(x_i)|| + ||\mu_A(x_i) - \pi_A(x_i)| - |\mu_B(x_i) - \pi_B(x_i)|||]$$

$$d_6(A,B) = \frac{1}{4n}\sum_{i=1}^{n}[|\mu_A(x_i) - \mu_B(x_i)| + |\nu_A(x_i) - \nu_B(x_i)| + |\pi_A(x_i) - \pi_B(x_i)| + 2\,max\{|\mu_A(x_i) - \mu_B(x_i)|,\ |\nu_A(x_i) - \nu_B(x_i)|,\ |\pi_A(x_i) - \pi_B(x_i)|\}]$$

$$d_7(A,B) = \left(\frac{1}{4n}\sum_{i=1}^{n}[(\mu_A(x_i) - \mu_B(x_i))^2 + (\nu_A(x_i) - \nu_B(x_i))^2 + (\pi_A(x_i) - \pi_B(x_i))^2 + 2\,max\{(\mu_A(x_i) - \mu_B(x_i))^2,\ (\nu_A(x_i) - \nu_B(x_i))^2,\ (\pi_A(x_i) - \pi_B(x_i))^2\}]\right)^{\frac{1}{2}},$$

where

$$\pi_A(x_i) = \sqrt{1 - [(\mu_A(x_i))^2 + (\nu_A(x_i))^2]}$$

and

$$\pi_B(x_i) = \sqrt{1 - [(\mu_B(x_i))^2 + (\nu_B(x_i))^2]}.$$

3.2 Numerical verification

Now, we verify whether the proposed distance measures for PFSs satisfy the conditions in Definition 3.1.

For example, let $A, B, C \in PFS(X)$ for $X = \{x_1, x_2, x_3\}$. Suppose

$$A = \left\{\left\langle\frac{0.6, 0.2}{x_1}\right\rangle, \left\langle\frac{0.4, 0.6}{x_2}\right\rangle, \left\langle\frac{0.5, 0.3}{x_3}\right\rangle\right\},$$

$$B = \left\{\left\langle\frac{0.8, 0.1}{x_1}\right\rangle, \left\langle\frac{0.7, 0.3}{x_2}\right\rangle, \left\langle\frac{0.6, 0.1}{x_3}\right\rangle\right\}$$

and

$$C = \left\{\left\langle\frac{0.9, 0.2}{x_1}\right\rangle, \left\langle\frac{0.8, 0.2}{x_2}\right\rangle, \left\langle\frac{0.7, 0.3}{x_3}\right\rangle\right\}.$$

Calculating the distance using the proposed distance measures above, we have

$$
\begin{aligned}
d_5(A,B) =& \frac{1}{12}\sum_{i=1}^{3}[||0.6-0.8| + ||0.6-0.2| - |0.8-0.1||\\
&+ ||0.6-0.7746| - |0.8-0.5916||\\
&+ |0.4-0.7|\\
&+ ||0.4-0.6| - |0.7-0.3|| + ||0.4-0.6928|\\
&- |0.7-0.6481||\\
&+ |0.5-0.6| + ||0.5-0.3| - |0.6-0.1||\\
&+ ||0.5-0.8124| - |0.6-0.7937||]\\
=& 0.1495
\end{aligned}
$$

$$
\begin{aligned}
d_6(A,B) =& \frac{1}{12}\sum_{i=1}^{3}[|0.6-0.8| + |0.2-0.1|\\
&+ |0.7746-0.5916|\\
&+ 2\,max\{|0.6-0.8|,\ |0.2-0.1|,\\
&\ |0.7746-0.5916|\}\\
&+ |0.4-0.7| + |0.6-0.3| + |0.6928-0.6481|\\
&+ 2\,max\{|0.4-0.7|,\ |0.6-0.3|,\\
&\ |0.6928-0.6481|\}\\
&+ |0.5-0.6| + |0.3-0.1| + |0.8124-0.7937|\\
&+ 2\,max\{|0.5-0.6|,\ |0.3-0.1|,\\
&\ |0.8124-0.7937|\}]\\
=& 0.2372
\end{aligned}
$$

$$
\begin{aligned}
d_7(A,B) =& \Bigg(\frac{1}{12}\sum_{i=1}^{3}[(0.6-0.8)^2 + (0.2-0.)^2\\
&+ (0.7746-0.5916)^2\\
&+ 2\,max\{(0.6-0.8)^2,\ (0.2-0.1)^2,\\
&\ (0.7746-0.5916)^2\}\\
&+ (0.4-0.7)^2 + (0.6-0.3)^2\\
&+ (0.6928-0.6481)^2\\
&+ 2\,max\{(0.4-0.7)^2,\ (0.6-0.3)^2,\\
&\ (0.6928-0.6481)^2\}\\
&+ (0.5-0.6)^2 + (0.3-0.1)^2\\
&\ + (0.8124-0.7937)^2\\
&+ 2\,max\{(0.5-0.6)^2,\ (0.3-0.1)^2,\\
&\ (0.8124-0.7937)^2\}]\Bigg)^{\frac{1}{2}}\\
=& 0.2338
\end{aligned}
$$

Similarly, we obtain

$$
\begin{aligned}
d_5(A,C) =& 0.2048,\ d_6(A,C) = 0.3294,\ d_7(A,C) = 0.3346,\\
d_5(B,C) =& 0.1024,\ d_6(B,C) = 0.1784,\ d_7(B,C) = 0.1691.
\end{aligned}
$$

3.2.1 Comments

The following are observed from the above computations:

(i) It follows that, $d_i(A,B), d_i(A,C), d_i(B,C) \in [0,1]$, $\forall d_i$, where $i = 5,6,7$,

(ii) $d_i(A,B) = 0$, $d_i(A,C) = 0$ and $d_i(B,C) = 0$ if and only if $A = B$, $A = C$ and $B = C$ $\forall d_i$, where $i = 5,6,7$,

(iii) $\forall d_i$, where $i = 5,6,7$, it follows that

$$
d_i(A,B) = d_i(B,A),\ d_i(A,C) = d_i(C,A) \text{ and } d_i(B,C) = d_i(C,B)
$$

because of the use of square and absolute value,

(iv) $d_i(A,C) + d_i(B,C) \geq d_i(A,B)$ holds $\forall d_i$, where $i = 5,6,7$.

Clearly, Conditions (i)–(iv) of Definition 3.1 hold for all the distance measures.

Applying the distance measures in (Zhang and Xu 2014; Ejegwa 2018), that is, d_1, d_2, d_3, and d_4 to calculate the distances between PFSs A, B, and C, we get the following results in Table 2.

Table 3 contains all the values of distance measures (that is, d_1, d_2, d_3, d_4, d_5, d_6, and d_7) between the PFSs A, B, and C defined over $X = \{x_1, x_2, x_3\}$.

3.2.2 Discussion

From Table 3, we observe that the distance measure proposed by Zhang and Xu (2014), that is, d_4 does not completely satisfies the conditions of distance measures for PFSs as seen in Definition 3.1, since $d_4(A,C) \notin [0,1]$. However, the distance measures in (Ejegwa 2018), that is, d_1, d_2, and d_3 completely satisfy the conditions of distance measures for PFSs in Definition 3.1. Similarly, the proposed distance measures, say d_5, d_6, and d_7, completely satisfy the conditions of distance measures for PFSs. Thus, d_1, d_2, d_3, d_5, d_6, and d_7 are appropriate distance measures for PFSs.

Notwithstanding, d_5 is the most reasonable/efficient of the distance measures discussed, since

Table 2 Numerical outputs

DM	d_1	d_2	d_3	d_4
$d(A, B)$	0.2411	0.2294	0.2400	0.7200
$d(A, C)$	0.3298	0.3274	0.3900	1.1700
$d(B, C)$	0.1887	0.1632	0.1867	0.5600

Table 3 Numerical outputs

DM	d_1	d_2	d_3	d_4	d_5	d_6	d_7
$d(A, B)$	0.2411	0.2294	0.2400	0.7200	0.1495	0.2372	0.2338
$d(A, C)$	0.3298	0.3274	0.3900	1.1700	0.2048	0.3294	0.3346
$d(B, C)$	0.1887	0.1632	0.1867	0.5600	0.1024	0.1784	0.1691

$d_5(A, B) < d_i(A, B)$,
$d_5(A, C) < d_i(A, C)$ and
$d_5(B, C) < d_i(B, C)\ \forall i = 1, 2, 3, 6, 7.$

Hence, we adopt d_5 for application to pattern recognition problems.

4 Application to pattern recognition problems

In this section, we apply d_5 to deal with some pattern recognition problems experience in real life.

Example 4.1 Suppose we have a pattern recognition problem about the classification of building materials. Given three classes of building materials represented by PFSs $\tilde{A_1}$, $\tilde{A_2}$, and $\tilde{A_3}$ in the feature space $X = \{x_1, x_2, x_3\}$, respectively, shown as follows:

$$\tilde{A_1} = \left\{ \frac{\langle 0.34, 0.34 \rangle}{x_1}, \frac{\langle 0.19, 0.48 \rangle}{x_2}, \frac{\langle 0.02, 0.12 \rangle}{x_3} \right\}$$
$$\tilde{A_2} = \left\{ \frac{\langle 0.35, 0.33 \rangle}{x_1}, \frac{\langle 0.20, 0.47 \rangle}{x_2}, \frac{\langle 0.02, 0.5 \rangle}{x_3} \right\}$$
$$\tilde{A_3} = \left\{ \frac{\langle 0.33, 0.35 \rangle}{x_1}, \frac{\langle 0.21, 0.46 \rangle}{x_2}, \frac{\langle 0.01, 0.13 \rangle}{x_3} \right\}.$$

Given another kind of unknown building material represented by an PFS $\tilde{B}$ in the feature space $X = \{x_1, x_2, x_3\}$. The goal is to classify the unknown pattern $\tilde{B}$ into one of the pattern $\tilde{A_1}$, $\tilde{A_2}$ or $\tilde{A_3}$, where

$$\tilde{B} = \left\{ \frac{\langle 0.37, 0.31 \rangle}{x_1}, \frac{\langle 0.23, 0.44 \rangle}{x_2}, \frac{\langle 0.04, 0.10 \rangle}{x_3} \right\}.$$

Calculating the distances between the known building materials and the unknown building material using d_5, we get

$$\begin{aligned} d_5(\tilde{A_1}, \tilde{B}) = \frac{1}{12} \sum_{i=1}^{3} [|\mu_{\tilde{A_1}}(x_i) - \mu_{\tilde{B}}(x_i)| \\ + ||\mu_{\tilde{A_1}}(x_i) - \nu_{\tilde{A_1}}(x_i)| - |\mu_{\tilde{B}}(x_i) - \nu_{\tilde{B}}(x_i)|| \\ + ||\mu_{\tilde{A_1}}(x_i) - \pi_{\tilde{A_1}}(x_i)| - |\mu_{\tilde{B}}(x_i) - \pi_{\tilde{B}}(x_i)||] \\ = 0.0273, \end{aligned}$$

$$d_5(\tilde{A_2}, \tilde{B}) = 0.0618,\ d_5(\tilde{A_3}, \tilde{B}) = 0.0261.$$

From the results, we can see that the distance between $\tilde{A_3}$ and $\tilde{B}$ is the smallest and the distance between $\tilde{A_2}$ and $\tilde{B}$ is the greatest. Hence, $\tilde{B}$ belongs to $\tilde{A_3}$.

Example 4.2 Given three kinds of mineral fields, each is featured by the content of three minerals and has one kind of typical hybrid mineral. We can express the three kinds of typical hybrid mineral by three PFSs $\tilde{C_1}$, $\tilde{C_2}$, and $\tilde{C_3}$ in the feature $X = \{x_1, x_2, x_3\}$, respectively, shown as follows:

$$\tilde{C_1} = \left\{ \frac{\langle 0.5, 0.4 \rangle}{x_1}, \frac{\langle 0.8, 0.0 \rangle}{x_2}, \frac{\langle 0.3, 0.7 \rangle}{x_3} \right\}$$
$$\tilde{C_2} = \left\{ \frac{\langle 0.6, 0.3 \rangle}{x_1}, \frac{\langle 0.9, 0.1 \rangle}{x_2}, \frac{\langle 0.6, 0.4 \rangle}{x_3} \right\}$$
$$\tilde{C_3} = \left\{ \frac{\langle 0.6, 0.3 \rangle}{x_1}, \frac{\langle 0.9, 0.1 \rangle}{x_2}, \frac{\langle 0.5, 0.5 \rangle}{x_3} \right\}.$$

Given another kind of hybrid mineral represented by an PFS $\tilde{D}$ in the feature $X = \{x_1, x_2, x_3\}$. The aim is to find which field should mineral $\tilde{D}$ belongs, where

$$\tilde{D} = \left\{ \frac{\langle 0.4, 0.2 \rangle}{x_2}, \frac{\langle 0.9, 0.05 \rangle}{x_3} \right\}.$$

Using d_5, we calculate the distances between the three kinds of mineral fields and the unknown as thus:

$$\begin{aligned} d_5(\tilde{C_1}, \tilde{D}) = \frac{1}{12} \sum_{i=1}^{3} [|\mu_{\tilde{C_1}}(x_i) - \mu_{\tilde{D}}(x_i)| \\ + ||\mu_{\tilde{C_1}}(x_i) - \nu_{\tilde{C_1}}(x_i)| - |\mu_{\tilde{D}}(x_i) - \nu_{\tilde{D}}(x_i)|| \\ + ||\mu_{\tilde{C_1}}(x_i) - \pi_{\tilde{C_1}}(x_i)| - |\mu_{\tilde{D}}(x_i) - \pi_{\tilde{D}}(x_i)||] \\ = 0.3443, \end{aligned}$$

$$d_5(\tilde{C_2}, \tilde{D}) = 0.3237,\ d_5(\tilde{C_3}, \tilde{D}) = 0.3392.$$

From the results, we can say that mineral field $\tilde{D}$ belongs to $\tilde{C_2}$, since the distance between $\tilde{C_2}$ and $\tilde{D}$ is the shortest.

Example 4.3 Consider three known patterns of building materials represented by PFSs $\tilde{E_1}$, $\tilde{E_2}$, and $\tilde{E_3}$ of the feature space $X = \{x_1, x_2, x_3, x_4\}$, respectively, shown as follows:

$$\tilde{E}_1 = \left\{ \frac{\langle 0.5, 0.3\rangle}{x_1}, \frac{\langle 0.7, 0.0\rangle}{x_2}, \frac{\langle 0.4, 0.5\rangle}{x_3}, \frac{\langle 0.7, 0.3\rangle}{x_4} \right\}$$
$$\tilde{E}_2 = \left\{ \frac{\langle 0.5, 0.2\rangle}{x_1}, \frac{\langle 0.6, 0.1\rangle}{x_2}, \frac{\langle 0.2, 0.7\rangle}{x_3}, \frac{\langle 0.7, 0.3\rangle}{x_4} \right\}$$
$$\tilde{E}_3 = \left\{ \frac{\langle 0.5, 0.4\rangle}{x_1}, \frac{\langle 0.7, 0.1\rangle}{x_2}, \frac{\langle 0.4, 0.6\rangle}{x_3}, \frac{\langle 0.7, 0.2\rangle}{x_4} \right\}.$$

The aim is to classify an unknown pattern of building material represented by an PFS $\tilde{F}$ in the feature space $X = \{x_1, x_2, x_3, x_4\}$ into one of the pattern $\tilde{E}_1$, $\tilde{E}_2$ or $\tilde{E}_3$, where

$$\tilde{F} = \left\{ \frac{\langle 0.4, 0.3\rangle}{x_1}, \frac{\langle 0.7, 0.1\rangle}{x_2}, \frac{\langle 0.3, 0.6\rangle}{x_3}, \frac{\langle 0.7, 0.3\rangle}{x_4} \right\}.$$

Calculating the distances between the known pattern of building materials and the unknown pattern using d_5, we get

$$\begin{aligned} d_5(\tilde{E}_1, \tilde{F}) &= \frac{1}{16}\sum_{i=1}^{4}[|\mu_{\tilde{E}_1}(x_i) - \mu_{\tilde{F}}(x_i)| \\ &\quad + ||\mu_{\tilde{E}_1}(x_i) - \nu_{\tilde{E}_1}(x_i)| - |\mu_{\tilde{F}}(x_i) - \nu_{\tilde{F}}(x_i)|| \\ &\quad + ||\mu_{\tilde{E}_1}(x_i) - \pi_{\tilde{E}_1}(x_i)| - |\mu_{\tilde{F}}(x_i) - \pi_{\tilde{F}}(x_i)||] \\ &= 0.0834, \end{aligned}$$
$$d_5(\tilde{E}_2, \tilde{F}) = 0.0721,\ d_5(\tilde{E}_3, \tilde{F}) = 0.0490.$$

From the results, we can say that the distance between $\tilde{E}_3$ and $\tilde{F}$ is the shortest, and the distance between $\tilde{E}_1$ and $\tilde{F}$ is the longest. Thus, it follows that $\tilde{F}$ belongs to $\tilde{E}_3$.

Example 4.4 Given three kinds of mineral fields, each is featured by the content of five minerals and has one kind of typical hybrid mineral. We can express the three kinds of mineral fields by three PFSs $\tilde{G}_1$, $\tilde{G}_2$, and $\tilde{G}_3$ in the feature $X = \{x_1, x_2, x_3, x_4, x_5\}$, respectively, shown as follows:

$$\tilde{G}_1 = \left\{ \frac{\langle 0.9, 0.1\rangle}{x_1}, \frac{\langle 0.8, 0.0\rangle}{x_2}, \frac{\langle 0.7, 0.1\rangle}{x_3}, \frac{\langle 0.7, 0.4\rangle}{x_5} \right\}$$
$$\tilde{G}_2 = \left\{ \frac{\langle 0.8, 0.1\rangle}{x_1}, \frac{\langle 0.8, 0.2\rangle}{x_2}, \frac{\langle 0.9, 0.0\rangle}{x_4} \right\}$$
$$\tilde{G}_3 = \left\{ \frac{\langle 0.6, 0.2\rangle}{x_2}, \frac{\langle 0.8, 0.0\rangle}{x_3}, \frac{\langle 0.6, 0.1\rangle}{x_5} \right\}.$$

Suppose we have an unknown mineral field represented by an PFS $\tilde{H}$ in the feature space $X = \{x_1, x_2, x_3, x_4, x_5\}$, where

$$\tilde{H} = \left\{ \frac{\langle 0.5, 0.3\rangle}{x_1}, \frac{\langle 0.6, 0.2\rangle}{x_2}, \frac{\langle 0.8, 0.1\rangle}{x_3}, \frac{\langle 0.9, 0.0\rangle}{x_4}, \frac{\langle 0.7, 0.4\rangle}{x_5} \right\}.$$

Our task is to justify which mineral field the unknown mineral field $\tilde{H}$ belong to.

Calculating the distances between each of $\tilde{G}_1$, $\tilde{G}_2$ and $\tilde{G}_3$, and the unknown mineral field $\tilde{H}$ using d_5, we have

$$\begin{aligned} d_5(\tilde{G}_1, \tilde{H}) &= \frac{1}{20}\sum_{i=1}^{5}[|\mu_{\tilde{G}_1}(x_i) - \mu_{\tilde{H}}(x_i)| \\ &\quad + ||\mu_{\tilde{G}_1}(x_i) - \nu_{\tilde{G}_1}(x_i)| - |\mu_{\tilde{H}}(x_i) - \nu_{\tilde{H}}(x_i)|| \\ &\quad + ||\mu_{\tilde{G}_1}(x_i) - \pi_{\tilde{G}_1}(x_i)| - |\mu_{\tilde{H}}(x_i) - \pi_{\tilde{H}}(x_i)||] \\ &= 0.1827, \end{aligned}$$
$$d_5(\tilde{G}_2, \tilde{H}) = 0.2040,\ d_5(\tilde{C}_3, \tilde{D}) = 0.1785.$$

From the results, we can say that the unknown mineral field $\tilde{H}$ belongs to $\tilde{G}_3$, since the distance between $\tilde{G}_3$ and $\tilde{H}$ is the shortest.

5 Conclusion

The concept of PFSs is of immense importance in real-life problems because of its ability to cope with embedded imprecision more effective than IFSs. Some of the applications of PFSs have been explored (see Ejegwa 2019a, b; Perez-Dominguez et al. 2018; Yager 2013b, 2014, 2016). So far, we have proposed some new distance measures for PFSs that satisfied the properties of distance measure, by taking into account the conventional parameters of PFSs. We verified the authenticity of the proposed distance measures in comparison with some distance measures for PFSs that also used the conventional parameters (see Ejegwa 2018; Zhang and Xu 2014), and found that the proposed distance measures, especially, d_5 yields better output. To test the applicability of the proposed distance measures in real-life problems, some pattern recognition problems were considered via d_5, for reliable output. The novel distance measures for Pythagorean fuzzy sets proposed in this work could be applied in decision making of real-life problems embedded with uncertainty.

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Compliance with ethical standards

Conflict of interest The authors declare that there is no conflict of interest toward the publication of this manuscript.

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