ORIGINAL PAPER

A novel equilibrium solution concept for intuitionistic fuzzy bi-matrix games considering proportion mix of possibility and necessity expectations

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Abstract

This paper considers a class of bi-matrix games involving two players with their payoffs matrices having entries from the set of trapezoidal intuitionistic fuzzy numbers. We generalize the notion of Nash equilibrium for such a class of games by modeling the variations in the proportion of the actual realization of the expected values from the game by the two players in terms of the two matrices $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ depending upon the subjectivity of the respective players. Using the intuitionistic fuzzy measure approach, a convex combination of the possibility and the necessity measures, we introduce the notion of $(\boldsymbol{\alpha}, \boldsymbol{\beta})$-intuitionistic fuzzy measure equilibrium solutions for games in this class. A methodology is developed to extract the proposed equilibrium solution of an intuitionistic fuzzy bi-matrix game by solving an equivalent quadratic programming problem with linear constraints. The viability of the proposed concept is depicted through two real-life examples of decision-making on marketing strategies to magnetize the preference degrees of the customers. Conclusively, a comparison is drawn between the proposed solution concept with a few of the proximate equilibrium solutions concepts existing for such a class of games.

1 Introduction

A bi-matrix game is a two-person non-cooperative non-zero sum matrix game in which each player has a finite strategies sets to choose from to play the game. Two matrices, called their respective payoffs matrices, are used to explain the payoffs of players. The aim is to obtain a Nash-equilibrium solution of a game which is defined typically in the sense of optimizing a utility function involving the payoffs matrices of two players.

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The matrix and the bi-matrix games have been extensively studied and successfully applied in a variety of areas. The pioneer work of Campos (1989) laid the foundation of modeling fuzzy matrix games with fuzzy payoffs applying the linear programming models. Several new models describing matrix games with fuzzy payoffs have emerged since then. Bector et al. (2004) use a suitable defuzzification function to develop duality results for linear programming with fuzzy parameters and apply them to solve the matrix games with fuzzy payoffs. Based on fuzzy 'max' order, Maeda (2003) defines three types of min–max equilibrium strategies and utilizes its properties to design a solution procedure for the fuzzy matrix games. Li (1999) presents a multi-objective linear programming model to solve fuzzy matrix games when entries in the payoffs matrix are triangular fuzzy numbers. Li (2012) develops a method for solving matrix games with triangular fuzzy numbers which assure triangular fuzzy values for such games. Using the fuzzy relational approach, Vijay et al. (2007) extend the duality results of Inuiguchi et al. (2003)

and Ramík (2005, 2006) to study a generalized model of fuzzy matrix games with fuzzy goals and fuzzy payoffs. Recently, Xu et al. (2017) introduce the possibility and the necessity measures for matrix game with fuzzy payoffs and define an (α, β)-PN equilibrium strategy for two players playing the game.

In contrast to the enormous literature available on fuzzy matrix games, few studies have reported on fuzzy bi-matrix games. In Mangasarian and Stone (1964), show that every equilibrium point of a two-person non-zero sum game can always be obtained by solving a quadratic programming problem. Maeda (2000) considers a fuzzy bi-matrix game with fuzzy payoffs and study the existence conditions for a Nash-equilibrium of such a game. Vidyottama et al. (2004) establish an equivalence of a fuzzy bi-matrix game with the fuzzy goal to a crisp nonlinear programming problem and extend this study to the class of fuzzy bi-matrix games with fuzzy payoffs and fuzzy goals. Larbani (2009) establishes the existence results for a Nash-equilibrium of a bi-matrix fuzzy game where Nature is also a participating player. To deal with vagueness and imprecision information in real-life problems which are expressed by upper and lower approximation rough set is used. Ammar and Brikaa (2018) proposed a useful technique for solving constraint matrix games with rough interval payoffs and obtain the α-trust equilibrium strategies and the expected equilibrium strategies.

Atanassov (1986, 1989) extends the notion of the fuzzy set by adding to it another component called the non-membership degree of an element to belong to the given set. The membership and the non-membership degrees are more-or-less independent of each other except that their sum is less than or equal to one. This newly defined set is named an intuitionistic fuzzy set. Hence, an intuitionistic fuzzy set is characterized by two functions describing the membership degree and the non-membership degree. An intuitionistic fuzzy set helps to model situations requiring affirmation, negation, and hesitation, simultaneously. For instance, assuming one wishes to present the test statistics depicting the level of difficulty of a question paper in a competitive exam. It can be represented by the membership degree describing the proportion of questions attempted but incorrectly answered by the majority of examinees (assuming that incorrect answering is a consequence of a question being difficult), the non-membership degree explaining the proportion of questions attempted and correctly answered by a majority, and the hesitation degree depicting a portion of questions not attempted by almost anyone (assuming that such questions are either wrongly framed or carry ambiguity in their comprehension). An intuitionistic fuzzy set thus embeds more information than the standard fuzzy set. Atanassov (1994, 1999), and many other researchers define various mathematical operators on intuitionistic fuzzy sets to enrich this class.

Since its inception, intuitionistic fuzzy sets are applied widely to model real-life decision-making problems, more specifically the multi-attribute decision problems. The literature on applications of intuitionistic fuzzy sets in multi-attribute decision problems is enormous to quote all. Li (2005, 2008, 2010a, b, c), Li et al. (2010), Liu and Wang (2007), Liu and Chen (2017), Chen et al. (2016), Wang et al. (2009), and many more, made significant contributions in context of multi-attribute decision problems under intuitionistic fuzzy set-up and its other generalizations. However, research in intuitionistic fuzzy sets is not limited to such a class of decision-making problems. Among other applications, Li and Chuntian (2002), Vlachos and Sergiadis (2007), and more recently Chen and Chang (2015) apply various concepts of intuitionistic fuzzy sets to the pattern recognition problems. Garg and Kaur (2018) describe a series of distance measures for interval-valued intuitionistic fuzzy sets based on different metrics and utilize them to illustrate applications in medical diagnosis and pattern recognition problems. De et al. (2001) also explore intuitionistic fuzzy sets in problems of medical diagnosis. Vlachos and Sergiadis (2009) model the inde-terminacy that originates from quantization noise in the pixel values of digital images using intuitionistic fuzzy sets. Garai et al. (2018) extend the possibility and necessity theory to the intuitionistic fuzzy sets and apply the same to the problems of manufacturing systems. Cheng et al. (2016) formulate a fuzzy time series forecasting model, Chen and Tanuwijaya (2011) use high-order fuzzy logical relationships and automatic clustering techniques in fuzzy forecasting, while Kumar and Gangwar (2016) extend to built on intuitionistic fuzzy time series to handle the non-determinism in time series forecasting. Xu et al. (2008) study clustering method for intuitionistic fuzzy sets.

The initial years of intuitionistic fuzzy set witnessed a controversy surrounding its nomenclature (Dubois et al. 2005; Grzegorzewski and Mrówka 2005). A very similar name has also been used to describe intuitionistic logic, though the two concepts differ in their mathematical structure and treatment. To avoid confusion in the terminology, Dubois et al. (2005) and Grzegorzewski and Mrówka (2005) advise calling the newly defined set by 'Atanassov's intuitionistic fuzzy set' or simply an 'intuitionistic fuzzy set'. Following their suggestion, hence-forth, in this paper, we shall stick to using *intuitionistic fuzzy set* only.

Aggarwal et al. (2012a, b) develop duality results for the class of intuitionistic fuzzy linear programming problems involving linear membership and non-membership functions and apply these results to study the two-person intuitionistic fuzzy matrix games with intuitionistic fuzzy goals. Khan et al. (2017) analyze the two-person zero-sum matrix games with intuitionistic fuzzy goals by resolving

indeterminacy or hesitancy factor from each goal, and therefore, converting games into two-person fuzzy matrix games. In a series of papers, Seikh et al. (2013, 2015b, 2016) study intuitionistic fuzzy matrix games with intuitionistic fuzzy payoffs under different set-ups and conditions and obtain the optimal strategies for players along with optimal values of the games. Although a few more good studies Bandyopadhyay et al. (2013) and Nan et al. (2010) highlight applications of intuitionistic fuzzy sets in matrix games, yet there does not exist much in the literature to report on the use of intuitionistic fuzzy sets in the bi-matrix games. A systematic search through the literature reveals the work of Nayak and Pal (2010) which comes closest to study intuitionistic fuzzy bi-matrix games. They define the notion of Nash-equilibrium solution for a class of bi-matrix games with intuitionistic fuzzy goals on the lines of Nishizaki and Sakawa (2013). Seikh et al. (2015a) propose the solution methodology for bi-matrix games with intuitionistic fuzzy goals by applying the aspiration level approach and formulate the equivalent crisp bi-matrix game under certain conditions. In another paper, Seikh et al. (2015c) define a new ranking function for triangular intuitionistic fuzzy numbers and use it to solve bi-matrix games. Khan et al. (2016) define a kind of equilibrium solution for intuitionistic fuzzy bi-matrix game and show that finding such a solution is equivalent to solving a fuzzy bi-matrix game with piecewise linear S-shaped membership functions. They pointed out a wrong way in defining the membership and non-membership functions of intuitionistic fuzzy goals in the work of Nayak and Pal (2010). Recently, Nan et al. (2017) present a nonlinear programming model to solve bi-matrix games in which the values of the game for two players are regarded as intuitionistic fuzzy sets, and the payoffs are expressed using triangular intuitionistic fuzzy numbers.

Taking motivation from the work of Xu et al. (2017) and Garai et al. (2018), and employing the intuitionistic fuzzy measure approach, in this paper, we study a class of bi-matrix games having trapezoidal intuitionistic fuzzy numbers entries in the payoffs matrices of two players. We define an (α, β)-intuitionistic fuzzy measure equilibrium solution for such games. Our primary contributions are twofold. First, we apply the convex combination of the possibility and necessity expectations to describe the expectation of the intuitionistic fuzzy measure and use it to introduce a new notion of intuitionistic fuzzy measure equilibrium solution of the bi-matrix games with intuitionistic payoffs. Second, we bring in two diagonal matrices α and β to represent the proportion of the actual realizations of the expected values to the two players. Using the illustrative examples, we perform a comparative analysis of the proposed method with some proximity researches. Our noting depicts the usefulness of the proposed solution concept as it incorporates an optimistic and pessimistic attitude of the players in the form of a parameter and also model uncertainty in the actual realizations of the expected values to the players in the form of two diagonal matrices.

1.1 Comparison of the present paper vis-a-vis other relevant works

We provide some highlighting differences with other relevant works from the literature to justify what we believe to be the contribution of the present work.

(i) If compared to the work of Xu et al. (2017), where only the membership degree is used to define the concept of the possibility and necessity equilibrium strategies for the fuzzy matrix games, our proposed approach analyzes the Intuitionistic fuzzy information including the non-membership degree for evaluating Nash-equilibrium strategies in the Intuitionistic fuzzy bi-matrix games. Consequently, the work in Xu et al. (2017) falls as a particular case of our proposed work.

(ii) If compared to the work of Maeda (2000), who studied bi-matrix games in the fuzzy framework, this paper utilizes the trapezoidal intuitionistic fuzzy numbers in the payoffs. Since the trapezoidal intuitionistic fuzzy numbers add one more layer of granularity in the subjective assessments of strategies by players, they are better off in describing the problems comprehensively and accurately than the fuzzy numbers. Moreover, unlike the work in Maeda (2000) which requires to pre-specify the real parameters u and v to describe the aspiration levels of two players in the (u, v)-possible Nash equilibrium, or parameters α and β in [0, 1] to define (α, β)-Nash equilibrium of fuzzy bi-matrix games, our approach is different. We applied the intuitionistic fuzzy measure which is a convex combination of possibility and necessity measures, and hence it is encompassing the features of both these measures. In addition, our proposed equilibrium solution also involves certain parameters, but these can be considered as weights attached to the possibility and necessity measures (through $\lambda \in [0, 1]$), and the strategies of two players (through the diagonal matrices α and β) in an intuitionistic fuzzy bi-matrix game.

(iii)   If compared to the work of Fan et al. (2014), who put forward two kinds of nonlinear programming algorithms to compute the Nash equilibrium of bi-matrix games with payoffs of intuitionistic fuzzy values, our proposed work differ significantly. First, we considered trapezoidal intuitionistic fuzzy numbers in payoffs than simply intuitionistic fuzzy values, second, we used intuitionistic fuzzy measure to model the viewpoints of two players through a parameter $\lambda \in [0, 1]$, which can be adjusted accordingly to represent a wider spectrum of situations.

(iv)   If compared to the recent works of An et al. (2017) and Yang et al. (2016), where different ranking functions [weighted mean-area ranking method in An et al. (2017) and value-index and ambiguity-index in Yang et al. (2016)] are used to formulate nonlinear optimization models for solving the bi-matrix games with intuitionistic fuzzy payoffs, our proposed approach uses the Intuitionistic fuzzy measure to defuzzify the trapezoidal intuitionistic fuzzy numbers and seeks to solve a quadratic optimization problem with linear constraints. Since the intuitionistic fuzzy measure includes the possibility and necessity measures, it imparts more flexibility to players to express and incorporate their opinions into the model.

(v)   If compared to another recent work of Nan et al. (2017), where the proposed solution methodology for the intuitionistic fuzzy bi-matrix games with intuitionistic fuzzy goals requires a complete prior description of the intuitionistic fuzzy goals of two players along with their tolerances in the membership and non-membership values, the present paper does not ask for any such information about the goals or values of the game for two players. In fact, our proposed model computes the optimal expected values of the game for the two players corresponding to the Nash-equilibrium solution. Moreover, the optimization model proposed in Nan et al. (2017) is a nonlinear program with linear objective function and quadratic constraints, whereas the model proposed in this work has a computational advantage in optimizing quadratic objective function with linear constraints.

(vi)   If compared to the work of Verma et al. (2015), who studied equilibrium solutions of the intuitionistic fuzzy matrix games using α, β-cuts of intuitionistic fuzzy numbers and the ranking order as described for comparing intervals (i. e. $[a, b] \leq [c, d] \Leftrightarrow a \leq c$ and $b \leq d$), in this paper we study the bi-matrix games and use the intuitionistic fuzzy measure approach to describe Nash-equilibrium solutions. Unlike the work in Verma et al. (2015), our proposed Nash-equilibrium solution does not depend on the selection of cuts of intuitionistic fuzzy numbers. Moreover it encompasses flexibility by incorporating the optimistic–pessimistic attitude of the players.

(vii)   Compared to the numerous studies on fuzzy and Intuitionistic fuzzy matrix games Aggarwal et al. (2012a, b), Maeda (2003), Nan et al. (2010, 2016) and Seikh et al. (2015c), and many more, this paper broadens the field by advancing the study to the class of intuitionistic fuzzy bi-matrix games. Furthermore, the proposed methodology presents an evaluation of Nash-equilibrium of such games by solving the computationally tractable quadratic optimization models.

(viii)   If compared to the work of Seikh et al. (2015a), where the aspiration levels for the intuitionistic fuzzy goals of the two players with their respective tolerances in the intuitionistic fuzzy bi-matrix game are specified, our proposed approach does not require the players to define any of these parameters, thereby reducing the task of the players. Seikh et al. (2015a) apply Angelov (1997) approach to formulate a crisp equivalent linear program to compute Nash equilibrium. Our methodology proposes a quadratic program which calculates the expected optimal values of the two players. Our scheme reduces the task of the players in supplying additional numbers in the form of aspiration levels and tolerance parameters which in the absence of any well laid out procedure may otherwise be hard to inform. On the other hand, it is more reasonable to articulate the outlook of players on pessimism to optimism scale.

The paper unfolds as follows. Section 2 revisits the basic definitions and preliminaries on intuitionistic fuzzy sets and intuitionistic fuzzy measures. Section 3 introduces the concept of an $(\boldsymbol{\alpha}, \boldsymbol{\beta})$-intuitionistic fuzzy measure equilibrium solution for intuitionistic fuzzy bi-matrix games with Intuitionistic fuzzy goals. The section also presents a formulation of an equivalent quadratic programming problem to extract such a solution. Section 4 demonstrates the

proposed solution scheme through real-life examples. Section 5 displays a comparison of an (α, β)-intuitionistic fuzzy measure equilibrium solution with some of the existing solution concepts for the intuitionistic fuzzy bimatrix games from the literature. Section 6 summarizes our findings with an outlook on future research.

2 Preliminaries

We briefly recollect certain preliminaries on intuitionistic fuzzy sets, intuitionistic fuzzy numbers, and intuitionistic fuzzy measures, needed in the sequel.

Definition 1 (*Intuitionistic fuzzy set*) (Atanassov 1986) Let X be a universal set. An intuitionistic fuzzy set $\tilde{a}$ in X is described by

$$\tilde{a} = \{\langle x, \mu_{\tilde{a}}(x), v_{\tilde{a}}(x) \rangle \mid x \in X, \ \mu_{\tilde{a}}(x) + v_{\tilde{a}}(x) \leq 1\},$$

where $\mu_{\tilde{a}} : X \to [0,1]$ and $v_{\tilde{a}} : X \to [0,1]$ define respectively the membership function and the non-membership function of an element $x \in X$ to belong to the set $\tilde{a}$.

If $\mu_{\tilde{a}}(x) + v_{\tilde{a}}(x) = 1$, for all $x \in X$, then $\tilde{a}$ degenerates to the conventional fuzzy set Zadeh (1965), where the non-belongingness of x to the set $\tilde{a}$ is taken as the complement of its belongingness to the set.

In addition, the function $\pi_{\tilde{a}}(x)$ defined by $\pi_{\tilde{a}}(x) = 1 - \mu_{\tilde{a}}(x) - v_{\tilde{a}}(x)$, $x \in X$, measures the hesitation degree or indeterminacy degree of the membership of x in $\tilde{a}$, and it usually has a more intuitive interpretation than the non-membership degree De et al. (2001) and Vlachos and Sergiadis (2007). It provides another layer of granularity in decision-making by giving more resilience to the decision maker in representing information in terms of the membership, non-membership, and hesitation degrees.

For $X = \mathbb{R}$, the set of real numbers, we recall an intuitionistic fuzzy number as follows.

Definition 2 (*Intuitionistic fuzzy number*) (Nehi 2010) An intuitionistic fuzzy number $\tilde{a}$ is an intuitionistic fuzzy set over $\mathbb{R}$ whose membership function $\mu_{\tilde{a}} : \mathbb{R} \to [0,1]$ and non-membership function $v_{\tilde{a}} : \mathbb{R} \to [0,1]$ satisfy the following conditions:

(i) there exist $c \in \mathbb{R}$ such that $\mu_{\tilde{a}}(c) = 1$ (hence, $v_{\tilde{a}}(c) = 0$);

(ii) $\mu_{\tilde{a}}$ is quasi-concave and $v_{\tilde{a}}$ is quasi-convex on $\mathbb{R}$;

(iii) $\mu_{\tilde{a}}$ is upper semi-continuous and $v_{\tilde{a}}$ is lower semi-continuous on $\mathbb{R}$;

(iv) the two support sets, $\{x \in \mathbb{R} \mid \mu_{\tilde{a}}(x) > 0\}$ and $\{x \in \mathbb{R} \mid v_{\tilde{a}}(x) < 1\}$, are bounded in $\mathbb{R}$.

We shall be denoting the set of all intuitionistic fuzzy numbers on $\mathbb{R}$ by $\mathrm{IFN}(\mathbb{R})$.

From the above definition, we get at once that for any intuitionistic fuzzy number $\tilde{a}$ there exists eight real numbers $a_i, c_i, i = 1, \ldots, 4$, such that $c_1 \leq a_1 \leq c_2 \leq a_2 \leq a_3 \leq c_3 \leq a_4 \leq c_4$, and four functions $f_i : \mathbb{R} \to [0,1]$, $i = 1, \ldots, 4$, called sides of the intuitionistic fuzzy number, where f_1 and f_4 are non-decreasing and f_2 and f_3 are non-increasing functions on $\mathbb{R}$. The membership function $\mu_{\tilde{a}}$ and the non-membership function $v_{\tilde{a}}$ of an intuitionistic fuzzy number $\tilde{a}$ are respectively described as follows:

$$\mu_{\tilde{a}}(x) = \begin{cases} 0 & x < a_1 \\ f_1(x) & a_1 \leq x < a_2 \\ 1 & a_2 \leq x \leq a_3 \\ f_2(x) & a_3 < x \leq a_4 \\ 0 & x > a_4, \end{cases}$$

$$v_{\tilde{a}}(x) = \begin{cases} 1 & x < c_1 \\ f_3(x) & c_1 \leq x < c_2 \\ 0 & c_2 \leq x \leq c_3 \\ f_4(x) & c_3 < x \leq c_4 \\ 1 & x > c_4. \end{cases}$$

It is worth noting that an intuitionistic fuzzy number $\tilde{a}$ is a conjunction of two fuzzy numbers, one with membership function $\mu_{\tilde{a}}$ and the other one having membership function $1 - v_{\tilde{a}}$.

In particular, if functions f_i, $i = 1, \ldots, 4$, are linear and $a_2 = c_2$, $a_3 = c_3$, then the intuitionistic fuzzy number is a trapezoidal intuitionistic fuzzy number (TrIFN). The membership and the non-membership functions of the trapezoidal intuitionistic fuzzy number are respectively defined as follows:

$$\mu_{\tilde{a}}(x) = \begin{cases} 0 & x < a_1 \\ \dfrac{x - a_1}{a_2 - a_1} & a_1 \leq x < a_2 \\ 1 & a_2 \leq x \leq a_3 \\ \dfrac{a_3 - x}{a_4 - a_3} & a_3 < x \leq a_4 \\ 0 & x > a_4, \end{cases}$$

$$v_{\tilde{a}}(x) = \begin{cases} 1 & x < c_1 \\ \dfrac{x - c_2}{c_1 - c_2} & c_1 \leq x < c_2 \\ 0 & c_2 \leq x \leq c_3 \\ \dfrac{x - c_3}{c_4 - c_3} & c_3 < x \leq c_4 \\ 1 & x > c_4. \end{cases}$$

A TrIFN is represented by $\tilde{a} = \langle [b, c, l, r], [b, c, p, q] \rangle$ with $a_2 = c_2 = b$, $a_3 = c_3 = c$, $a_1 = b - l$, $a_4 = c + r$, $c_1 = b - p$, $c_4 = c + q$, and therefore, $p \geq l$ and $q \geq r$. Figure 1 describes one such instance.

The set of all trapezoidal intuitionistic fuzzy numbers on $\mathbb{R}$ shall be denoted by $\mathrm{TrIFN}(\mathbb{R})$.
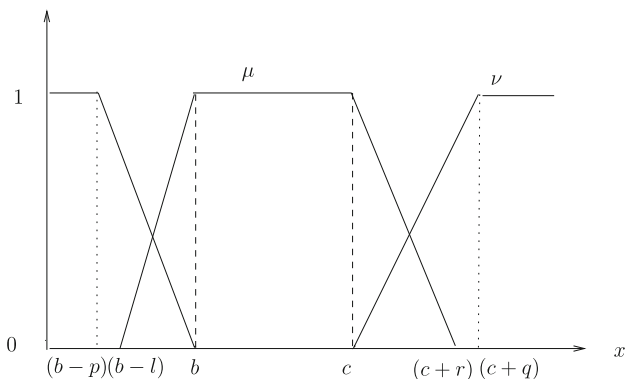
Fig. 1 Membership and non-membership functions of a TrIFN $\tilde{a} = \langle[b, c, l, r], [b, c, p, q]\rangle$

Definition 3 (Chakraborty et al. 2014) Let $\tilde{a} = \langle[a_1, a_2, \ell, r], [a_1, a_2, p, q]\rangle$, $\tilde{b} = \langle[b_1, b_2, m, n], [b_1, b_2, s, t]\rangle$ $\in$ TrIFN($\mathbb{R}$), and $k \in \mathbb{R}$. The addition, subtraction, and scalar multiplication are defined as follows:

(i) $\tilde{a} + \tilde{b} = \langle[a_1 + b_1, a_2 + b_2, \ell + m, r + n],$
    $[a_1 + b_1, a_2 + b_2, p + s, q + t]\rangle$;

(ii) $\tilde{a} - \tilde{b} = \langle[a_1 - b_2, a_2 - b_1, \ell + n, r + m],$
     $[a_1 - b_2, a_2 - b_1, p + t, q + s]\rangle$;

(iii) $k\tilde{a} = \langle[ka_1, ka_2, k\ell, kr],$
      $[ka_1, ka_2, kp, kq]\rangle,$ if $k \geq 0$;

(iv) $k\tilde{a} = \langle[ka_2, ka_1, -kr, -k\ell],$
     $[ka_2, ka_1, -kq, -kp]\rangle,$ if $k < 0$.

Some authors use $\langle[b, c, l, r], [b, c, p, q], w, u\rangle$ to denote a generalized trapezoidal intuitionistic fuzzy numbers with height of membership function w (instead of 1) and height of one minus the non-membership function being $1 - u$ (instead of 1). However, in the present study, we are working with $w = 1$ and $u = 0$ case.

Remark 1 If $b = c$ in a TrIFN $\langle[b, c, l, r], [b, c, p, q]\rangle$, then the number is called triangular intuitionistic fuzzy number (TIFN), expressed as $\tilde{a} = \langle[b, l, r], [b, p, q]\rangle$.

The parameters involved in the trapezoidal intuitionistic fuzzy number are a part of the information coming from the decision maker in the form of subjective perception about some number. Generally, it is easy for the decision maker to describe a range for the most possible value of the outcome and the extremes beyond which the acceptability of errors in the outcome is zero. Adding to it one more layer of granularity is the expression for the negation of such an outcome resulting in the four parameters each for the membership and the non-membership of the outcome. The values of these parameters along with their linear membership and

non-membership functions can be extracted using fuzzy interpolative reasoning based rules (Chen and Chang 2011), statistical techniques (Falsafain et al. 2008) or intuitionistic fuzzy cognitive maps (Papageorgiou and Iakovidis 2013).

2.1 Possibility and necessity measures for a trapezoidal intuitionistic fuzzy number

Let $\tilde{a}$, $\tilde{b} \in$ IFN($\mathbb{R}$). The possibility and necessity measures are defined as follows (see, Chakraborty et al. 2014 and Garai et al. 2018).

$$\text{Poss}_\mu(\tilde{a} \leq \tilde{b}) = \sup_{x \leq y} \min\{\mu_{\tilde{a}}(x), \mu_{\tilde{b}}(y)\},$$

$$\text{Poss}_v(\tilde{a} \leq \tilde{b}) = \sup_{x \leq y} \min\{v_{\tilde{a}}(x), v_{\tilde{b}}(y)\};$$

$$\text{Nec}_\mu(\tilde{a} < \tilde{b}) = \inf_{x \geq y} \max\{\mu_{\tilde{a}}(x), \mu_{\tilde{b}}(y)\}$$
$$= 1 - \sup_{x \geq y} \max\{\mu_{\tilde{a}}(x), \mu_{\tilde{b}}(y)\},$$

$$\text{Nec}_v(\tilde{a} < \tilde{b}) = \inf_{x \geq y} \max\{v_{\tilde{a}}(x), v_{\tilde{b}}(y)\}$$
$$= 1 - \sup_{x \geq y} \max\{v_{\tilde{a}}(x), v_{\tilde{b}}(y)\}.$$

Let $\tilde{a} = \langle[b, c, l, r], [b, c, p, q]\rangle \in$ TrIFN($\mathbb{R}$) and $x \in \mathbb{R}$. The possibility measures for the membership function of an intuitionistic fuzzy number are shown in Xu et al. (2017) to be as follows:

$$\text{Poss}_\mu(\tilde{a} \geq x) = \begin{cases} 1 & x \leq c \\ \dfrac{c + r - x}{r} & c \leq x \leq c + r \\ 0 & x \geq c + r, \end{cases}$$

and $\text{Poss}_\mu(\tilde{a} \leq x) = \begin{cases} 0 & x \leq b - l \\ \dfrac{x - b + l}{l} & b - l \leq x \leq b \\ 1 & x \geq b. \end{cases}$

Similarly, the possibility measure for the non-membership function of an intuitionistic fuzzy number are defined as follows.

$$\text{Poss}_v(\tilde{a} \geq x) = \begin{cases} 0 & x \leq c \\ \dfrac{x - c}{q} & c \leq x \leq c + q \\ 1 & x \geq c + q, \end{cases}$$

and $\text{Poss}_v(\tilde{a} \leq x) = \begin{cases} 1 & x \leq b - p \\ \dfrac{b - x}{p} & b - p \leq x \leq b \\ 0 & x \geq b. \end{cases}$

The necessity measures for the membership function of an intuitionistic fuzzy number are given as follows (Xu et al. 2017).

$$\mathrm{Nec}_\mu(\tilde{a} < x) = \begin{cases} 0 & x \le c \\ \dfrac{x-c}{r} & c \le x \le c+r \\ 1 & x \ge c+r, \end{cases}$$

$$\text{and} \quad \mathrm{Nec}_\mu(\tilde{a} > x) = \begin{cases} 1 & x \le b-l \\ \dfrac{b-x}{l} & b-l \le x \le b \\ 0 & x \ge b. \end{cases}$$

And the necessity measures for the non-membership function of an intuitionistic fuzzy number are defined as follows.

$$\mathrm{Nec}_v(\tilde{a} < x) = \begin{cases} 1 & x \le c \\ \dfrac{c+q-x}{q} & c \le x \le c+q \\ 0 & x \ge c+q, \end{cases}$$

$$\text{and} \quad \mathrm{Nec}_v(\tilde{a} > x) = \begin{cases} 0 & x \le b-p \\ \dfrac{p-b+x}{p} & b-p \le x \le b \\ 1 & x \ge b. \end{cases}$$

Figure 2 summarizes the above concepts.

We introduce the notion of intuitionistic fuzzy measure as follows. Recently, we came across the work of Garai et al. (2018) who also formulated the same concept.

Definition 4 (*Intuitionistic fuzzy measure*) Let $\tilde{a} \in \mathrm{IFN}(\mathbb{R})$. The intuitionistic fuzzy measure $\mathrm{Me}_\mu(\cdot)$ for the membership function and the intuitionistic fuzzy measure $\mathrm{Me}_v(\cdot)$ for the non-membership function of $\tilde{a}$ are defined as follows.

$$\mathrm{Me}_\mu(\tilde{a} \ge x) = \lambda \mathrm{Poss}_\mu(\tilde{a} \ge x) + (1-\lambda)\mathrm{Nec}_\mu(\tilde{a} \ge x),$$
$$\mathrm{Me}_v(\tilde{a} \ge x) = \lambda \mathrm{Poss}_v(\tilde{a} \ge x) + (1-\lambda)\mathrm{Nec}_v(\tilde{a} \ge x),$$
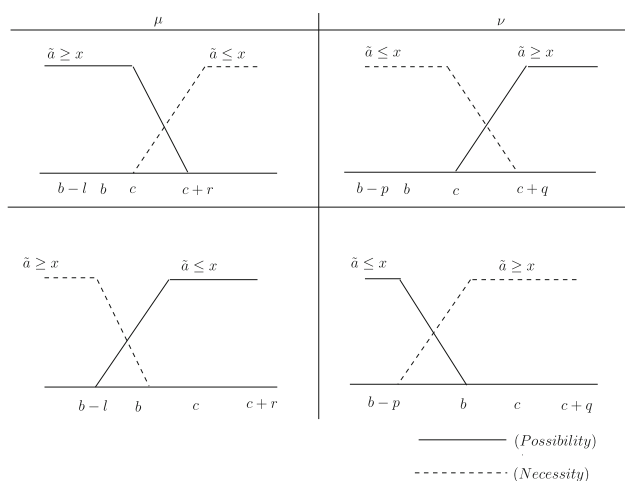

Fig. 2 Membership and non-membership functions for possibility and necessity measures of a TrIFN $\tilde{a} = \langle [b, c, l, r], [b, c, p, q] \rangle$, where the possibility measure functions are depicted by solid line and the necessity measure functions are depicted by dotted line

where $\lambda \in (0,1)$, is a parameter to capture an outlook or perception of the decision maker. While $\lambda = 0$ describes a complete pessimistic behavior, $\lambda = 1$ explains precisely the opposite optimistic behavior of the decision maker.

One can analogously define the notions of $\mathrm{Me}_\mu(\tilde{a} \le x)$ and $\mathrm{Me}_v(\tilde{a} \le x)$.

Definition 5 (*Credibility measure*) If $\lambda = \frac{1}{2}$ in Definition 4, then the intuitionistic fuzzy measures of $\tilde{a}$ are equivalent to the credibility measures of $\tilde{a}$ defined in Chakraborty et al. (2014) as follows.

$$\mathrm{Cr}_\mu(\tilde{a} \ge x) = \frac{1}{2}(\mathrm{Poss}_\mu(\tilde{a} \ge x) + \mathrm{Nec}_\mu(\tilde{a} > x)),$$
$$\mathrm{Cr}_v(\tilde{a} \ge x) = \frac{1}{2}(\mathrm{Poss}_v(\tilde{a} \ge x) + \mathrm{Nec}_v(\tilde{a} > x)).$$

On the similar lines, we can explain $\mathrm{Cr}_\mu(\tilde{a} \le x)$ and $\mathrm{Cr}_v(\tilde{a} \le x)$. The creditability measure for fuzzy numbers is widely studied and applied in many domains. For a comprehensive details on creditability measure, one can refer to an excellent article (Liu 2006).

2.2 Possibility and necessity expectations

We define the expectation with respect to the possibility, necessity, and intuitionistic fuzzy measures.

Let $\tilde{a} \in \mathrm{IFN}(\mathbb{R})$. Define

$$E_P^\mu(\tilde{a}) = \int_0^{+\infty} \mathrm{Poss}_\mu(\tilde{a} \ge x)\, \mathrm{d}x$$
$$- \int_{-\infty}^0 (1 - \mathrm{Poss}_\mu(\tilde{a} \ge x))\, \mathrm{d}x,$$
$$E_P^v(\tilde{a}) = \int_0^{+\infty} (1 - \mathrm{Poss}_v(\tilde{a} \ge x))\, \mathrm{d}x$$
$$- \int_{-\infty}^0 \mathrm{Poss}_v(\tilde{a} \ge x)\, \mathrm{d}x,$$

provided all involved improper integrals exist. Then, $E_P^\mu(\tilde{a})$ and $E_P^v(\tilde{a})$ are called the possibility expectations (or risk proneness expectations) according to the membership function and the non-membership function of the intuitionistic fuzzy number $\tilde{a}$, respectively.

For $\tilde{a} \in \mathrm{TrIFN}(\mathbb{R})$, it is easy to work out that $E_P^\mu(\tilde{a}) = c + \frac{r}{2}$ and $E_P^v(\tilde{a}) = c + \frac{q}{2}$ (see, Xu et al. 2017).

Similarly, the necessity expectations to the membership and the non-membership functions of $\tilde{a} \in \mathrm{IFN}(\mathbb{R})$ are defined as follows.

$$E_N^\mu(\tilde{a}) = \int_0^{+\infty} \mathrm{Nec}_\mu(\tilde{a} > x)\, dx$$
$$- \int_{-\infty}^0 (1 - \mathrm{Nec}_\mu(\tilde{a} > x))\, dx,$$
$$E_N^v(\tilde{a}) = \int_0^{+\infty} (1 - \mathrm{Nec}_v(\tilde{a} > x))\, dx$$
$$- \int_{-\infty}^0 \mathrm{Nec}_v(\tilde{a} > x)\, dx,$$

provided all involved improper integrals exist.

For $\tilde{a} \in \mathrm{TrIFN}(\mathbb{R})$, we have, $E_N^\mu(\tilde{a}) = b - \frac{l}{2}$ and $E_N^v(\tilde{a}) = b - \frac{p}{2}$. One can refer to Xu et al. (2017) for details of integrals evaluation for TrIFN.

Definition 6 Let $\tilde{a} \in \mathrm{IFN}(\mathbb{R})$. The expectation of the intuitionistic fuzzy measure of $\tilde{a}$ is defined by

$$E_{\mathrm{Me}_\mu}(\tilde{a}) = \int_0^{+\infty} \mathrm{Me}_\mu(\tilde{a} \geq x)\, dx$$
$$- \int_{-\infty}^0 (1 - \mathrm{Me}_\mu(\tilde{a} \geq x))\, dx,$$
$$E_{\mathrm{Me}_v}(\tilde{a}) = \int_0^{+\infty} (1 - \mathrm{Me}_v(\tilde{a} \geq x))\, dx$$
$$- \int_{-\infty}^0 \mathrm{Me}_v(\tilde{a} \geq x)\, dx,$$

provided all four improper integrals exist.

Proposition 1 For $\lambda \in [0, 1]$, we have,

$$E_{\mathrm{Me}_\mu}(\tilde{a}) = \lambda E_P^\mu(\tilde{a}) + (1 - \lambda) E_N^\mu(\tilde{a}),$$
and $\quad E_{\mathrm{Me}_v}(\tilde{a}) = \lambda E_P^v(\tilde{a}) + (1 - \lambda) E_N^v(\tilde{a}).$

Proof Since all the integrals are assumed to exist, we have,

$$E_{\mathrm{Me}_\mu}(\tilde{a}) = \int_0^{+\infty} \mathrm{Me}_\mu(\tilde{a} \geq x)\, dx - \int_{-\infty}^0 (1 - \mathrm{Me}_\mu(\tilde{a} \geq x))\, dx$$
$$= \int_0^{+\infty} \left(\lambda \mathrm{Poss}_\mu(\tilde{a} \geq x) + (1 - \lambda)\mathrm{Nec}_\mu(\tilde{a} \geq x)\right) dx$$
$$- \int_{-\infty}^0 1 - \left(\lambda \mathrm{Poss}_\mu(\tilde{a} \geq x) + (1 - \lambda)\mathrm{Nec}_\mu(\tilde{a} \geq x)\right) dx$$
$$= \lambda \int_0^{+\infty} \mathrm{Poss}_\mu(\tilde{a} \geq x)\, dx - \lambda \int_{-\infty}^0 \left(1 - \mathrm{Poss}_\mu(\tilde{a} \geq x)\right) dx$$
$$+ (1 - \lambda) \int_0^{+\infty} \mathrm{Nec}_\mu(\tilde{a} \geq x)\, dx$$
$$- (1 - \lambda) \int_{-\infty}^0 \left(1 - \mathrm{Nec}_\mu(\tilde{a} \geq x)\right) dx$$
$$= \lambda E_P^\mu(\tilde{a}) + (1 - \lambda) E_N^\mu(\tilde{a}).$$

The other relation for the non-membership case can analogously be worked out.

Remark 2 For $\tilde{a} = \langle [b, c, l, r], [b, c, p, q] \rangle \in \mathrm{TrIFN}(\mathbb{R})$,

$$E_{\mathrm{Me}_\mu}(\tilde{a}) = \lambda\left(c + \frac{r}{2}\right) + (1 - \lambda)\left(b - \frac{l}{2}\right),$$
$$E_{\mathrm{Me}_v}(\tilde{a}) = \lambda\left(c + \frac{q}{2}\right) + (1 - \lambda)\left(b - \frac{p}{2}\right).$$

Remark 3 We use the scalar λ for representing the outlook of the player on the scale of pessimism to optimism; $\lambda = 0$ is used to depict the total pessimistic viewpoint while $\lambda = 1$ describes the contrast fully optimistic view. We carry the same interpretation for λ throughout the remaining discussion in the paper.

3 Intuitionistic fuzzy measure equilibrium solution of an intuitionistic fuzzy bi-matrix game

Let $\mathbb{R}^n$ be n-dimensional Euclidean space and $\mathbb{R}_+^n$ be its non-negative orthant. Let $\tilde{A}, \tilde{B} \in \mathrm{TrIFN}(\mathbb{R}^{m \times n})$ be $m \times n$ matrices having each entry as a trapezoidal intuitionistic fuzzy number, that is, $\tilde{A} = [\langle [a_{ij}, c_{ij}, l_{ij}, r_{ij}], [a_{ij}, c_{ij}, p_{ij}, q_{ij}] \rangle]_{m \times n}$ and $\tilde{B} = [\langle [b_{ij}, d_{ij}, m_{ij}, n_{ij}], [b_{ij}, d_{ij}, t_{ij}, s_{ij}] \rangle]_{m \times n}$.

We associate six matrices each with $\tilde{A}$ and $\tilde{B}$, respectively, as follows.

$$A = [a_{ij}]_{m \times n}, \quad C = [c_{ij}]_{m \times n}, \quad L = [l_{ij}]_{m \times n}$$
$$R = [r_{ij}]_{m \times n}, \quad P = [p_{ij}]_{m \times n}, \quad Q = [q_{ij}]_{m \times n};$$
$$B = [b_{ij}]_{m \times n}, \quad D = [d_{ij}]_{m \times n}, \quad M = [m_{ij}]_{m \times n},$$
$$N = [n_{ij}]_{m \times n}, \quad T = [t_{ij}]_{m \times n}, \quad S = [s_{ij}]_{m \times n}.$$

Let $S^m = \{x \in \mathbb{R}_+^m \mid e'x = 1\}$ and $S^n = \{y \in \mathbb{R}_+^n \mid e'y = 1\}$ be the strategy spaces for player I and player II, respectively, where $e' = (1, \ldots, 1)$ is a vector of ones whose dimension is taken specific to context. Thus, a two person non-zero sum bi-matrix game with intuitionistic fuzzy payoffs, denoted by BMGIFP, is defined by a quadruple, $\mathrm{BMGIFP} = (S^m, S^n, \tilde{A}, \tilde{B})$.

Let $\alpha_i, \beta_j \in [0, 1]$, $i = 1, \ldots, m$, $j = 1, \ldots, n$. Define two diagonal matrices

$$\boldsymbol{\alpha} = \mathrm{diag}\,(\alpha_1, \alpha_2 \ldots, \alpha_m)_{m \times m}, \quad \boldsymbol{\beta} = \mathrm{diag}\,(\beta_1, \beta_2, \ldots, \beta_n)_{n \times n}.$$

Remark 4 For given matrices $\boldsymbol{\alpha}$ and $\tilde{A}$, and scalar $\lambda \in [0, 1]$, we shall be using the notations

$$F_\mu(\tilde{A}, \boldsymbol{\alpha}, \mathbf{I} - \boldsymbol{\alpha}, \lambda) = \lambda \boldsymbol{\alpha} E_P^\mu(\tilde{A})(1 - \lambda)(\mathbf{I} - \boldsymbol{\alpha}) E_N^\mu(\tilde{A}),$$
$$G_v(\tilde{A}, \boldsymbol{\alpha}, \mathbf{I} - \boldsymbol{\alpha}, \lambda) = \lambda \boldsymbol{\alpha} E_P^v(\tilde{A}) + (1 - \lambda)(\mathbf{I} - \boldsymbol{\alpha}) E_N^v(\tilde{A}),$$

where $\mathbf{I}$ is the identity matrix of an appropriate order relative to the context. Also, $E_P^\mu(\tilde{A}) = [E_P^\mu(\tilde{a}_{ij})]$ and $E_N^\mu(\tilde{A}) = [E_N^\mu(\tilde{a}_{ij})]$ are the matrices of possibility and necessity expectations to the membership respectively of the

intuitionistic fuzzy matrix $\tilde{A}$ taken in the spirit of expectation to the membership of each TrIFN entry $\tilde{a}_{ij}$ of the matrix $\tilde{A}$. The other two notations $E_P^v(\tilde{A})$ and $E_N^v(\tilde{A})$ have similar interpretation with the possibility and necessity expectations to the non-membership of matrices defined by the respective expectations to the non-membership of each TrIFN entry of the matrix.

Similarly, for given matrices $\boldsymbol{\beta}$ and $\tilde{B}$, and scalar $\lambda \in [0, 1]$, we shall be using the notations

$$\widehat{F}_\mu(\tilde{B}, \boldsymbol{\beta}, \mathbf{I} - \boldsymbol{\beta}, \lambda) = \lambda E_P^\mu(\tilde{B}) \boldsymbol{\beta} + (1 - \lambda) E_N^\mu(\tilde{B}) (\mathbf{I} - \boldsymbol{\beta}),$$
$$\widehat{G}_v(\tilde{B}, \boldsymbol{\beta}, \mathbf{I} - \boldsymbol{\beta}, \lambda) = \lambda E_P^v(\tilde{B}) \boldsymbol{\beta} + (1 - \lambda) E_N^v(\tilde{B}) (\mathbf{I} - \boldsymbol{\beta}).$$

Remark 5 The scalar $\boldsymbol{\alpha}_i \in [0, 1]$ is the proportion of the actual realization of the expected payoff to player I concerning possibility measure on playing his i-th strategy. Similarly, the scalar $\boldsymbol{\beta}_j \in [0, 1]$ is the proportion of the actual realization of the expected payoff to player II concerning the possibility measure on playing his j-th strategy. Thus, if player I plays his i-th strategy and player II plays his j-th strategy then the actual realizations of the expected payoffs with respect to the possibility measure are $\boldsymbol{\alpha}_i E_P(\tilde{a}_{ij})$ for player I and $\boldsymbol{\beta}_j E_P(\tilde{b}_{ij})$ for player II. Note that if $\boldsymbol{\alpha}_i = \boldsymbol{\beta}_j = 1$, then the two players will receive the payoffs that they have been expecting initially for the possibility measure. In case $0 < \boldsymbol{\alpha}_i = \boldsymbol{\beta}_j < 1$, the expected payoffs received by the two players are the fractions of their original expected payoffs for the possibility measure. Moreover, if $\boldsymbol{\alpha}_i$ and $\boldsymbol{\beta}_j$ are proportions of the expected payoffs to the players associated with the possibility measure then, $1 - \boldsymbol{\alpha}_i$ and $1 - \boldsymbol{\beta}_j$ are taken to be the corresponding proportions of expected payoffs from the necessity measure. To sum up, in practice, the uncertain market forces or some unknown external factors, the expected payoffs of the two players on playing their respective strategies may not be realized truly to their full values, but some fraction of it is what the players receive. The matrices $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ attach a kind of proportionality with the actual realization of the expected payoffs of the two players.

Since an intuitionistic fuzzy number is a conjunction of two fuzzy numbers, we define the notion of *equilibrium solution* of the intuitionistic fuzzy matrix game BMGIFP as follows.

Definition 7 An element $(x^*, y^*) \in S^m \times S^n$ is called the $(\boldsymbol{\alpha}, \boldsymbol{\beta})$-intuitionistic fuzzy measure equilibrium solution of the intuitionistic fuzzy bi-matrix game BMGIFP if the following inequalities are satisfied:

$$x^\mathrm{T} F_\mu(\tilde{A}, \boldsymbol{\alpha}, \mathbf{I} - \boldsymbol{\alpha}, \lambda_1) y^* \leq x^{*\mathrm{T}} F_\mu(\tilde{A}, \boldsymbol{\alpha}, \mathbf{I} - \boldsymbol{\alpha}, \lambda_1) y^*, \quad \forall x \in S^m,$$
$$x^\mathrm{T} G_v(\tilde{A}, \boldsymbol{\alpha}, \mathbf{I} - \boldsymbol{\alpha}, \lambda_1) y^* \geq x^{*\mathrm{T}} G_v(\tilde{A}, \boldsymbol{\alpha}, \mathbf{I} - \boldsymbol{\alpha}, \lambda_1) y^*, \quad \forall x \in S^m,$$
$$x^{*\mathrm{T}} \widehat{F}_\mu(\tilde{B}, \boldsymbol{\beta}, \mathbf{I} - \boldsymbol{\beta}, \lambda_2) y \leq x^{*\mathrm{T}} \widehat{F}_\mu(\tilde{B}, \boldsymbol{\beta}, \mathbf{I} - \boldsymbol{\beta}, \lambda_2) y^*, \quad \forall y \in S^n,$$
$$x^{*\mathrm{T}} \widehat{G}_v(\tilde{B}, \boldsymbol{\beta}, \mathbf{I} - \boldsymbol{\beta}, \lambda_2) y \geq x^{*\mathrm{T}} \widehat{G}_v(\tilde{B}, \boldsymbol{\beta}, \mathbf{I} - \boldsymbol{\beta}, \lambda_2) y^*, \quad \forall y \in S^n,$$

for given scalars $\lambda_1 \in [0, 1]$ for player I and $\lambda_2 \in [0, 1]$ for player II.

The above definition, in view of Remark 4, immediately leads to the following inequalities to be satisfied by an $(\boldsymbol{\alpha}, \boldsymbol{\beta})$-intuitionistic fuzzy measure equilibrium solution (x^*, y^*). For $\lambda_1, \lambda_2 \in (0, 1)$,

$$\lambda_1 x^\mathrm{T} \boldsymbol{\alpha} E_P^\mu(\tilde{A}) y^* + (1 - \lambda_1) x^\mathrm{T} (\mathbf{I} - \boldsymbol{\alpha}) E_N^\mu(\tilde{A}) y^*$$
$$\leq \lambda_1 x^{*\mathrm{T}} \boldsymbol{\alpha} E_P^\mu(\tilde{A}) y^* + (1 - \lambda_1) x^{*\mathrm{T}} (\mathbf{I} - \boldsymbol{\alpha}) E_N^\mu(\tilde{A}) y^*, \qquad (1)$$
$$\forall x \in S^m,$$

$$\lambda_1 x^\mathrm{T} \boldsymbol{\alpha} E_P^v(\tilde{A}) y^* + (1 - \lambda_1) x^\mathrm{T} (\mathbf{I} - \boldsymbol{\alpha}) E_N^v(\tilde{A}) y^*$$
$$\geq \lambda_1 x^{*\mathrm{T}} \boldsymbol{\alpha} E_P^v(\tilde{A}) y^* + (1 - \lambda_1) x^{*\mathrm{T}} (\mathbf{I} - \boldsymbol{\alpha}) E_N^v(\tilde{A}) y^*, \qquad (2)$$
$$\forall x \in S^m,$$

$$\lambda_2 x^{*\mathrm{T}} E_P^\mu(\tilde{B}) \boldsymbol{\beta} y + (1 - \lambda_2) x^{*\mathrm{T}} E_N^\mu(\tilde{B}) (\mathbf{I} - \boldsymbol{\beta}) y$$
$$\leq \lambda_2 x^{*\mathrm{T}} E_P^\mu(\tilde{B}) \boldsymbol{\beta} y^* + (1 - \lambda_2) x^{*\mathrm{T}} E_N^\mu(\tilde{B}) (\mathbf{I} - \boldsymbol{\beta}) y^*, \qquad (3)$$
$$\forall y \in S^n,$$

$$\lambda_2 x^{*\mathrm{T}} E_P^v(\tilde{B}) \boldsymbol{\beta} y + (1 - \lambda_2) x^{*\mathrm{T}} E_N^\mu(\tilde{B}) (\mathbf{I} - \boldsymbol{\beta}) y$$
$$\geq \lambda_2 x^{*\mathrm{T}} E_P^v(\tilde{B}) \boldsymbol{\beta} y^* + (1 - \lambda_2) x^{*\mathrm{T}} E_N^\mu(\tilde{B}) (\mathbf{I} - \boldsymbol{\beta}) y^*, \qquad (4)$$
$$\forall y \in S^n.$$

Remark 6

(i) If $\lambda_1 = \lambda_2 = 1/2$, in (1)–(4), then (x^*, y^*) is called an $(\boldsymbol{\alpha}, \boldsymbol{\beta})$-creditability equilibrium solution for the intuitionistic fuzzy bi-matrix game BMGIFP.

(ii) If $\lambda_1 = \lambda_2 = 1$, (means optimistic attitude adopted by both the players), then (x^*, y^*) satisfying (1)–(4) is the $(\boldsymbol{\alpha}, \boldsymbol{\beta})$-possibility equilibrium solution for the intuitionistic fuzzy bi-matrix game BMGIFP. In addition, if $\boldsymbol{\alpha} = I_m$ and $\boldsymbol{\beta} = I_n$, then (x^*, y^*) is the Nash equilibrium solution of the game with respect to the possibility measure.

(iii) If $\lambda_1 = \lambda_2 = 0$, (means pessimistic attitude adopted by both the players), then (x^*, y^*) satisfying (1)–(4) is the $(\boldsymbol{\alpha}, \boldsymbol{\beta})$-necessity equilibrium for the intuitionistic fuzzy bi-matrix game BMGIFP. Moreover, if $\boldsymbol{\alpha} = O_m$ and $\boldsymbol{\beta} = O_n$ (zero matrices), then (x^*, y^*) is the Nash equilibrium solution of the game with respect to the necessity measure.

(iv) If we consider $\tilde{B} = -\tilde{A}$, and the non-membership function $v(x) = 1 - \mu(x)$, then (x^*, y^*) satisfying (1)–(4) reduces to the $(\boldsymbol{\alpha}, \boldsymbol{\beta})$-PN equilibrium solution for fuzzy matrix game defined by Xu

et al. (2017). In this spirit, Definition 7 can be taken as the generalization of the work of Xu et al. (2017) in the intuitionistic fuzzy set-up.

Taking

$$x^{\mathrm{T}}\boldsymbol{\alpha} E_\mu(\tilde{A})y = \left(\sum_{i=1}^{m}\sum_{j=1}^{n}\alpha_i x_i a_{ij} y_j, \quad \sum_{i=1}^{m}\sum_{j=1}^{n}\alpha_i x_i c_{ij} y_j, \right.$$
$$\left. \sum_{i=1}^{m}\sum_{j=1}^{n}\alpha_i x_i l_{ij} y_j, \quad \sum_{i=1}^{m}\sum_{j=1}^{n}\alpha_i x_i r_{ij} y_j \right)$$
$$= (v_1, v_2, v_3, v_4),$$

$$x^{\mathrm{T}}(\mathbf{I}-\boldsymbol{\alpha})E_v(\tilde{A})y = \left(\sum_{i=1}^{m}\sum_{j=1}^{n}(1-\alpha_i)x_i a_{ij} y_j, \right.$$
$$\sum_{i=1}^{m}\sum_{j=1}^{n}(1-\alpha_i)x_i c_{ij} y_j,$$
$$\sum_{i=1}^{m}\sum_{j=1}^{n}(1-\alpha_i)x_i p_{ij} y_j,$$
$$\left. \sum_{i=1}^{m}\sum_{j=1}^{n}(1-\alpha_i)x_i q_{ij} y_j \right)$$
$$= (v'_1, v'_2, v'_3, v'_4),$$

we can express,

$$x^{*\mathrm{T}} F_\mu(\tilde{A}, \boldsymbol{\alpha}, \mathbf{I}-\boldsymbol{\alpha}, \lambda_1)y^*$$
$$= \lambda_1\left(v_4 + \frac{v_2}{2}\right) + (1-\lambda_1)\left(v_3 - \frac{v_1}{2}\right) = V.$$

Similarly one can define

$$V' = x^{*\mathrm{T}} G_v(\tilde{A}, \boldsymbol{\alpha}, \mathbf{I}-\boldsymbol{\alpha}, \lambda_1)y^*,$$
$$W = x^{*\mathrm{T}} \widehat{F}_\mu(\tilde{B}, \boldsymbol{\beta}, \mathbf{I}-\boldsymbol{\beta}, \lambda_2)y^*,$$
$$W' = x^{*\mathrm{T}} \widehat{G}_v(\tilde{B}, \boldsymbol{\beta}, \mathbf{I}-\boldsymbol{\beta}, \lambda_2)y^*.$$

The value of the intuitionistic fuzzy bi-matrix game BMGIFP for player I is explained by a trapezoidal intuitionistic fuzzy number

$$\widetilde{V} = \langle [v_1, v_2, v_3, v_4], [v'_1, v'_2, v'_3, v'_4]\rangle.$$

The expectations of the intuitionistic fuzzy measure of $\widetilde{V}$ according to the membership and the non-membership functions are respectively V and V'. Similar interpretation can be imparted for the trapezoidal intuitionistic fuzzy value of the game for Player II to receive

$$\widetilde{W} = \langle [w_1, w_2, w_3, w_4], [w'_1, w'_2, w'_3, w'_4]\rangle,$$

and W and W' are the intuitionistic fuzzy measure expectations from the game for player II with respect to the membership and the non-membership functions, respectively.

Note that V and V' depend on the parameter λ_1 and the matrix $\boldsymbol{\alpha}$, while W and W' depend on the parameter λ_2 and the matrix $\boldsymbol{\beta}$. Thus, for the given set of parameters $\lambda_1, \lambda_2 \in [0, 1]$, and matrices $\boldsymbol{\alpha}_{m\times m}$ and $\boldsymbol{\beta}_{n\times n}$, the four expected values, V, V' for player I, and W, W' for player II, can be determined.

3.1 Equivalent optimization models

An element $(x^*, y^*) \in S^m \times S^n$ is an $(\boldsymbol{\alpha}, \boldsymbol{\beta})$-intuitionistic fuzzy measure equilibrium solution for the game BMGIFP if and only if (x^*, y^*, V, W, V', W') is the solution of following system of linear inequalities:

$$\sum_{j=1}^{n}\left(\lambda_1\alpha_i\left(c_{ij} + \frac{r_{ij}}{2}\right) + (1-\lambda_1)(1-\alpha_i)\left(a_{ij} - \frac{l_{ij}}{2}\right)\right)y_j$$
$$\leq V, \quad i = 1, \ldots, m,$$
$$\sum_{j=1}^{n}\left(\lambda_1\alpha_i\left(c_{ij} + \frac{q_{ij}}{2}\right) + (1-\lambda_1)(1-\alpha_i)\left(a_{ij} - \frac{p_{ij}}{2}\right)\right)y_j$$
$$\geq V', \quad i = 1, \ldots, m,$$
$$\sum_{i=1}^{m}x_i\left(\lambda_2\beta_j\left(d_{ij} + \frac{n_{ij}}{2}\right) + (1-\lambda_2)(1-\beta_j)\left(b_{ij} - \frac{m_{ij}}{2}\right)\right)$$
$$\leq W, \quad j = 1, \ldots, n,$$
$$\sum_{i=1}^{m}x_i\left(\lambda_2\beta_j\left(d_{ij} + \frac{s_{ij}}{2}\right) + (1-\lambda_2)(1-\beta_j)\left(b_{ij} - \frac{t_{ij}}{2}\right)\right)$$
$$\geq W', \quad j = 1, \ldots, n,$$
$$\sum_{i=1}^{m}x_i = \sum_{j=1}^{n}y_j = 1$$
$$x_i \geq 0, \ y_j \geq 0, \quad i = 1, \ldots, m, j = 1, \ldots, n.$$

The next theorem is an immediate consequence of the above discussion.

Theorem 1 *An element (x^*, y^*) is an $(\boldsymbol{\alpha}, \boldsymbol{\beta})$-intuitionistic fuzzy measure equilibrium solution of the game* BMGIFP *if and only if (x^*, y^*, V, W, V', W') is an optimal solution of the following equivalent quadratic programming problem* (QPP):

(QPP) $\max f(x, y, v, w, v', w')$

subject to

$$\sum_{j=1}^{n} \left(\lambda_1 \alpha_i \left(c_{ij} + \frac{r_{ij}}{2} \right) + (1 - \lambda_1)(1 - \alpha_i) \left(a_{ij} - \frac{l_{ij}}{2} \right) \right) y_j$$
$$\leq v, \quad i = 1, \ldots, m,$$

$$\sum_{j=1}^{n} \left(\lambda_1 \alpha_i \left(c_{ij} + \frac{q_{ij}}{2} \right) + (1 - \lambda_1)(1 - \alpha_i) \left(a_{ij} - \frac{p_{ij}}{2} \right) \right) y_j$$
$$\geq v', \quad i = 1, \ldots, m,$$

$$\sum_{i=1}^{m} \left(\lambda_2 \beta_j \left(d_{ij} + \frac{n_{ij}}{2} \right) + (1 - \lambda_2)(1 - \beta_j) \left(b_{ij} - \frac{m_{ij}}{2} \right) \right) x_i$$
$$\leq w, \quad j = 1, \ldots, n,$$

$$\sum_{i=1}^{m} \left(\lambda_2 \beta_j \left(d_{ij} + \frac{s_{ij}}{2} \right) + (1 - \lambda_2)(1 - \beta_j) \left(b_{ij} - \frac{t_{ij}}{2} \right) \right) x_i$$
$$\geq w', \quad j = 1, \ldots, n,$$

$$\sum_{i=1}^{m} x_i = \sum_{j=1}^{n} y_j = 1,$$

$$x_i \geq 0, \ y_j \geq 0, \quad i = 1, \ldots, m, \ j = 1, \ldots, n,$$

where $f(x, y, v, w, v', w') = \frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{n} x_i (\lambda_1 \alpha_i (r_{ij} - q_{ij}) + \lambda_2 \beta_j (n_{ij} - s_{ij}) + (1 - \lambda_1)(1 - \alpha_i)(p_{ij} - l_{ij}) + (1 - \lambda_2)(1 - \beta_j)(t_{ij} - m_{ij})) y_j - v - w + v' + w'$.

For any other feasible solution (x, y, v, w, v', w') of (QPP), we have, $f(x, y, v, w, v', w') \leq 0$, and at the optimal (x^*, y^*, V, W, V', W') of (QPP), we have,

$$f(x^*, y^*, V, W, V', W') = 0.$$

Throughout the discussion in the sequel, the optimal solution of (QPP) is denoted by (x^*, y^*, V, W, V', W'), where V, V' and W, W' are used to denote the optimal expected payoffs to the membership and the non-membership of player I and player II, respectively.

3.2 Special cases

(1) When $\lambda_1 = \lambda_2 = 0$, then model (QPP) becomes

(QPP-1) $\max f(x, y, v, w, v', w')$

subject to

$$(1 - \alpha_i) \sum_{j=1}^{n} \left(a_{ij} - \frac{l_{ij}}{2} \right) y_j \leq v, \quad i = 1, \ldots, m,$$

$$(1 - \alpha_i) \sum_{j=1}^{n} \left(a_{ij} - \frac{p_{ij}}{2} \right) y_j \geq v', \quad i = 1, \ldots, m,$$

$$(1 - \beta_j) \sum_{i=1}^{m} \left(b_{ij} - \frac{m_{ij}}{2} \right) x_i \leq w, \quad j = 1, \ldots, n,$$

$$(1 - \beta_j) \sum_{i=1}^{m} \left(b_{ij} - \frac{t_{ij}}{2} \right) x_i \geq w', \quad j = 1, \ldots, n,$$

$$\sum_{i=1}^{m} x_i = \sum_{j=1}^{n} y_j = 1,$$

$$x_i \geq 0, \ y_j \geq 0, \quad i = 1, \ldots, m, \ j = 1, \ldots, n.$$

where $f(x, y, v, w, v', w') = \frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{n} x_i ((1 - \alpha_i)(p_{ij} - l_{ij}) + (1 - \beta_j)(t_{ij} - m_{ij})) y_j - v - w + v' + w'$.

In such a case, (x^*, y^*) is called the $(\boldsymbol{\alpha}, \boldsymbol{\beta})$-necessity equilibrium solution of BMGIFP.

Let $\boldsymbol{\alpha}_i \neq 1$, $\boldsymbol{\beta}_j \neq 1$, $\forall i, j$. Then, (QPP-1) can be expressed as follows.

(EQPP-1) $\max f(x, y, v, w, v', w')$

subject to

$$\sum_{j=1}^{n} \left(a_{ij} - \frac{l_{ij}}{2} \right) y_j \leq v_i, \quad i = 1, \ldots, m,$$

$$\sum_{j=1}^{n} \left(a_{ij} - \frac{p_{ij}}{2} \right) y_j \geq v'_i, \quad i = 1, \ldots, m,$$

$$\sum_{i=1}^{m} \left(b_{ij} - \frac{m_{ij}}{2} \right) x_i \leq w_j, \quad j = 1, \ldots, n,$$

$$\sum_{i=1}^{m} \left(b_{ij} - \frac{t_{ij}}{2} \right) x_i \geq w'_j, \quad j = 1, \ldots, n,$$

$$v = (1 - \boldsymbol{\alpha}_i) v_i, \quad i = 1, \ldots, m,$$

$$v' = (1 - \boldsymbol{\alpha}_i) v'_i, \quad i = 1, \ldots, m,$$

$$w = (1 - \boldsymbol{\beta}_j) w_j, \quad j = 1, \ldots, n,$$

$$w' = (1 - \boldsymbol{\beta}_j) w'_j, \quad j = 1, \ldots, n,$$

$$\sum_{i=1}^{m} x_i = \sum_{j=1}^{n} y_j = 1,$$

$$x_i \geq 0, \ y_j \geq 0, \quad i = 1, \ldots, m, \ j = 1, \ldots, n,$$

where $f(x, y, v, w, v', w') = \frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{n} x_i ((1 - \alpha_i)(p_{ij} - l_{ij}) + (1 - \beta_j)(t_{ij} - m_{ij})) y_j - v - w + v' + w'$.

Remark 7 On carefully examining the two problems (QPP-1) and (EQPP-1), we note that if both players adopt the pessimistic outlook towards the game (depicted by taking $\lambda_1 = \lambda_2 = 0$), then the necessity equilibrium solution (x^*, y^*) of problem (QPP-1) does not change with the change in the values of $\boldsymbol{\alpha}_i$ and $\boldsymbol{\beta}_j$. However, the optimal expected payoffs V, V', W, and W', from (QPP-1) would see changes.

(2) When $\lambda_1 = \lambda_2 = 1$, then the model (QPP) reduces to the following problem:

(QPP-2) $\max f(x, y, v, w, v', w')$

subject to

$$\alpha_i \sum_{j=1}^{n} \left(c_{ij} + \frac{r_{ij}}{2}\right) y_j \leq v, \quad i = 1, \ldots, m,$$

$$\alpha_i \sum_{j=1}^{n} \left(c_{ij} + \frac{q_{ij}}{2}\right) y_j \geq v', \quad i = 1, \ldots, m,$$

$$\beta_j \sum_{i=1}^{m} \left(d_{ij} + \frac{n_{ij}}{2}\right) x_i \leq w, \quad j = 1, \ldots, n,$$

$$\beta_j \sum_{i=1}^{m} \left(d_{ij} + \frac{s_{ij}}{2}\right) x_i \geq w', \quad j = 1, \ldots, n,$$

$$\sum_{i=1}^{m} x_i = \sum_{j=1}^{n} y_j = 1,$$

$$x_i \geq 0, \ y_j \geq 0 \quad i = 1, \ldots, m, \ j = 1, \ldots, n,$$

where $f(x, y, v, w, v', w') = \frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{n} x_i(\alpha_i(r_{ij} - q_{ij}) + \beta_j(n_{ij} - s_{ij}))y_j - v - w + v' + w'$.

In this case, (x^*, y^*) is called the $(\boldsymbol{\alpha}, \boldsymbol{\beta})$-possibility equilibrium solution of game BMGIFP.

Remark 8 An observation similar to Remark 7 can be noted for problem (QPP-2). If $\boldsymbol{\alpha}_i \neq 0$, $\boldsymbol{\beta}_j \neq 0$, $\forall i, j$, then, the possibility equilibrium solution (x^*, y^*) of (QPP-2) does not depend on the changing values of $\boldsymbol{\alpha}_i$ and $\boldsymbol{\beta}_j$, although the optimal expected payoffs V, V', W, W' from (QPP-2) would see changes.

(3) When $\lambda_1 = \lambda_2 = 1/2$, then problem (QPP) becomes the following problem to solve:

(QPP-3) $\max f(x, y, v, w, v', w')$

subject to

$$\sum_{j=1}^{n} \left(\alpha_i \left(c_{ij} + \frac{r_{ij}}{2}\right) + (1 - \alpha_i)\left(a_{ij} - \frac{l_{ij}}{2}\right)\right) y_j$$
$$\leq 2v, \quad i = 1, \ldots, m,$$

$$\sum_{j=1}^{n} \left(\alpha_i \left(c_{ij} + \frac{q_{ij}}{2}\right) + (1 - \alpha_i)\left(a_{ij} - \frac{p_{ij}}{2}\right)\right) y_j$$
$$\geq 2v', \quad i = 1, \ldots, m,$$

$$\sum_{i=1}^{m} \left(\beta_j \left(d_{ij} + \frac{n_{ij}}{2}\right) + (1 - \beta_j)\left(b_{ij} - \frac{m_{ij}}{2}\right)\right) x_i$$
$$\leq 2w, \quad j = 1, \ldots, n,$$

$$\sum_{i=1}^{m} \left(\beta_j \left(d_{ij} + \frac{s_{ij}}{2}\right) + (1 - \beta_j)\left(b_{ij} - \frac{t_{ij}}{2}\right)\right) x_i$$
$$\geq 2w', \quad j = 1, \ldots, n,$$

$$\sum_{i=1}^{m} x_i = \sum_{j=1}^{n} y_j = 1,$$

$$x_i \geq 0, \ y_j \geq 0 \quad i = 1, \ldots, m, \ j = 1, \ldots, n,$$

where $f(x, y, v, w, v', w') = \frac{1}{4} \sum_{i=1}^{m} \sum_{j=1}^{n} x_i(\alpha_i(r_{ij} - q_{ij}) + \beta_j(n_{ij} - s_{ij}) + (1 - \alpha_i)(p_{ij} - l_{ij}) + \beta_j(t_{ij} - m_{ij}))y_j - v - w + v' + w'$.

Then, (x^*, y^*) is called the $(\boldsymbol{\alpha}, \boldsymbol{\beta})$-credibility equilibrium solution of BMGIFP representing an equal compromise between the possibility and necessity equilibrium solutions.

(4) When $\alpha = O_m$ and $\beta = O_n$ (zero matrices), then problem (QPP) reduces to the following problem:

(QPP-4) $\max f(x, y, v, w, v', w')$

subject to

$$(1 - \lambda_1) \sum_{j=1}^{n} \left(a_{ij} - \frac{l_{ij}}{2}\right) y_j \leq v, \quad i = 1, \ldots, m,$$

$$(1 - \lambda_1) \sum_{j=1}^{n} \left(a_{ij} - \frac{p_{ij}}{2}\right) y_j \geq v', \quad i = 1, \ldots, m,$$

$$(1 - \lambda_2) \sum_{i=1}^{m} \left(b_{ij} - \frac{m_{ij}}{2}\right) x_i \leq w, \quad j = 1, \ldots, n,$$

$$(1 - \lambda_2) \sum_{i=1}^{m} \left(b_{ij} - \frac{t_{ij}}{2}\right) x_i \geq w', \quad j = 1, \ldots, n,$$

$$\sum_{i=1}^{m} x_i = \sum_{j=1}^{n} y_j = 1,$$

$$x_i \geq 0, \ y_j \geq 0 \quad i = 1, \ldots, m, \ j = 1, \ldots, n,$$

where $f(x, y, v, w, v', w') = \frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{n} x_i((1 - \lambda_1)(p_{ij} - l_{ij}) + (1 - \lambda_2)(t_{ij} - m_{ij}))y_j - v - w + v' + w'$.

Remark 9 Note that (QPP-4) is identical to (QPP-1) except that the factors $(1 - \boldsymbol{\alpha}_i)$ and $(1 - \boldsymbol{\beta}_j)$ in constraints and objective functions are now replaced by the factors $(1 - \lambda_1)$, $\forall i$, and $(1 - \lambda_2)$, $\forall j$, respectively. And hence, Remark 7 holds for problem (QPP-4), that is, equilibrium solution would not change with change in λ_1 and λ_2 values although optimal values of the expected payoffs V, V', W, W' would change.

(5) When $\alpha = I_m$ and $\beta = I_n$, then the problem (QPP) reduces to the following problem:

(QPP-5) $\max f(x, y, v, w, v', w')$

subject to

$$\lambda_1 \sum_{j=1}^{n} \left(c_{ij} + \frac{r_{ij}}{2}\right) y_j \leq v, \quad i = 1, \ldots, m,$$

$$\lambda_1 \sum_{j=1}^{n} \left(c_{ij} + \frac{q_{ij}}{2}\right) y_j \geq v', \quad i = 1, \ldots, m,$$

$$\lambda_2 \sum_{i=1}^{m} \left(d_{ij} + \frac{n_{ij}}{2}\right) x_i \leq w, \quad j = 1, \ldots, n,$$

$$\lambda_2 \sum_{i=1}^{m} \left(d_{ij} + \frac{s_{ij}}{2}\right) x_i \geq w', \quad j = 1, \ldots, n,$$

$$\sum_{i=1}^{m} x_i = \sum_{j=1}^{n} y_j = 1,$$

$$x_i \geq 0, \ y_j \geq 0, \quad i = 1, \ldots, m, \ j = 1, \ldots, n,$$

where $f(x, y, v, w, v', w') = \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n} x_i\big(\lambda_1(r_{ij} - q_{ij}) + \lambda_2(n_{ij} - s_{ij})\big)y_j - v - w + v' + w'$.

Remark 10 Again, we notice that (QPP-5) is identical to (QPP-2) except replacing α_i and β_j in the latter problem by λ_1 and λ_2, $\forall\, i, j$. Consequently, Remark 8 holds for problem (QPP-5) also.

(6) When $\alpha = 0.5\,I_m$ and $\beta = 0.5\,I_n$, then the problem (QPP) becomes the following problem to solve:

(QPP-6) $\max f(x, y, v, w, v', w')$

subject to

$$\sum_{j=1}^{n}\left(\lambda_1\left(c_{ij} + \frac{r_{ij}}{2}\right) + (1 - \lambda_1)\left(a_{ij} - \frac{l_{ij}}{2}\right)\right)y_j$$
$$\leq 2v, \quad i = 1, \ldots, m,$$

$$\sum_{j=1}^{n}\left(\lambda_1\left(c_{ij} + \frac{q_{ij}}{2}\right) + (1 - \lambda_1)\left(a_{ij} - \frac{p_{ij}}{2}\right)\right)y_j$$
$$\geq 2v', \quad i = 1, \ldots, m,$$

$$\sum_{i=1}^{m}\left(\lambda_2\left(d_{ij} + \frac{n_{ij}}{2}\right) + (1 - \lambda_2)\left(b_{ij} - \frac{m_{ij}}{2}\right)\right)x_i$$
$$\leq 2w, \quad j = 1, \ldots, n,$$

$$\sum_{i=1}^{m}\left(\lambda_2\left(d_{ij} + \frac{s_{ij}}{2}\right) + (1 - \lambda_2)\left(b_{ij} - \frac{t_{ij}}{2}\right)\right)x_i$$
$$\geq 2w', \quad j = 1, \ldots, n,$$

$$\sum_{i=1}^{m} x_i = \sum_{j=1}^{n} y_j = 1,$$
$$x_i \geq 0, \ y_j \geq 0, \quad i = 1, \ldots, m, \ j = 1, \ldots, n,$$

where $f(x, y, v, w, v', w') = \frac{1}{4}\sum_{i=1}^{m}\sum_{j=1}^{n} x_i(\lambda_1(r_{ij} - q_{ij}) + \lambda_2(n_{ij} - s_{ij}) + (1 - \lambda_1)(p_{ij} - l_{ij}) + (1 - \lambda_2)(t_{ij} - m_{ij}))y_j - v - w + v' + w'$.

Solving problem (QPP-6) is the same as to solving (QPP-3) on replacing all α_i and β_j in (QPP-3) by λ_1 and λ_2, respectively.

(7) Zero-sum matrix game with intuitionistic fuzzy payoffs

We consider $\tilde{B} = -\tilde{A}$ in the intuitionistic fuzzy bi-matrix game BMGIFP, that is, a case of zero-sum intuitionistic fuzzy game, then

$$\langle[b_{ij}, d_{ij}, m_{ij}, n_{ij}], [b_{ij}, d_{ij}, t_{ij}, s_{ij}]\rangle$$
$$= \langle[-c_{ij}, -a_{ij}, r_{ij}, l_{ij}], [-c_{ij}, -a_{ij}, q_{ij}, p_{ij}]\rangle.$$

To compute an (α, β)-intuitionistic fuzzy measure equilibrium solution of this game, we have to solve the following quadratic programming problem:

(QPP-7) $\max f(x, y, v, v', w, w')$

subject to

$$\sum_{j=1}^{n}\left(\lambda_1\alpha_i\left(c_{ij} + \frac{r_{ij}}{2}\right) + (1 - \lambda_1)(1 - \alpha_i)\left(a_{ij} - \frac{l_{ij}}{2}\right)\right)y_j$$
$$\leq v, \quad i = 1, \ldots, m,$$

$$\sum_{j=1}^{n}\left(\lambda_1\alpha_i\left(c_{ij} + \frac{q_{ij}}{2}\right) + (1 - \lambda_1)(1 - \alpha_i)\left(a_{ij} - \frac{p_{ij}}{2}\right)\right)y_j$$
$$\geq v', \quad i = 1, \ldots, m,$$

$$\sum_{i=1}^{m}\left(\lambda_2\beta_j\left(a_{ij} - \frac{l_{ij}}{2}\right) + (1 - \lambda_2)(1 - \beta_j)\left(c_{ij} + \frac{r_{ij}}{2}\right)\right)x_i$$
$$\geq -w, \quad j = 1, \ldots, n,$$

$$\sum_{i=1}^{m}\left(\lambda_2\beta_j\left(a_{ij} - \frac{p_{ij}}{2}\right) + (1 - \lambda_2)(1 - \beta_j)\left(c_{ij} + \frac{q_{ij}}{2}\right)\right)x_i$$
$$\leq -w', \quad j = 1, \ldots, n,$$

$$\sum_{i=1}^{m} x_i = \sum_{j=1}^{n} y_j = 1,$$
$$x_i \geq 0, \ y_j \geq 0, \quad i = 1, \ldots, m, \ j = 1, \ldots, n,$$

where $f(x, y, v, v', w, w') = \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n} x_i\,((\lambda_1\alpha_i - (1 - \lambda_2)(1 - \beta_j))(r_{ij} - q_{ij}) + (\lambda_2\beta_j - (1 - \lambda_1)(1 - \alpha_i))(l_{ij} - p_{ij}))y_j - v - w + v' + w'$.

Furthermore, taking $\lambda_1 = \lambda_2 = 1/2$ and $2\alpha = \mathbf{I}_m$, $2\beta = \mathbf{I}_n$, we get the following problem to solve:

$$\max f(x, y, v, w, v', w') = -v - w + v' + w'$$

subject to

$$\sum_{j=1}^{n}\left(\frac{2c_{ij} + r_{ij} + 2a_{ij} - l_{ij}}{8}\right)y_j \leq v, \quad i = 1, \ldots, m,$$

$$\sum_{j=1}^{n}\left(\frac{2c_{ij} + q_{ij} + 2a_{ij} - p_{ij}}{8}\right)y_j \geq v', \quad i = 1, \ldots, m,$$

$$\sum_{i=1}^{m}\left(\frac{2c_{ij} + r_{ij} + 2a_{ij} - l_{ij}}{8}\right)x_i \geq -w, \quad j = 1, \ldots, n,$$

$$\sum_{i=1}^{m}\left(\frac{2c_{ij} + q_{ij} + 2a_{ij} - p_{ij}}{8}\right)x_i \leq -w', \quad j = 1, \ldots, n,$$

$$\sum_{i=1}^{m} x_i = \sum_{j=1}^{n} y_j = 1,$$
$$x_i \geq 0, \ y_j \geq 0, \quad i = 1, \ldots, m, \ j = 1, \ldots, n.$$

Remark 11 If $\tilde{B} = -\tilde{A}$, $\lambda_1 = \lambda_2 = 1/2$, $2\alpha = \mathbf{I}_m$, $2\beta = \mathbf{I}_n$, and $v = 1 - \mu$, then the game BMGIFP reduces to the zero-sum fuzzy matrix game and the solution for it is the credibility expectation equilibrium solution determined by solving the following problem:

(QPP-8) $\max f(x, y, v, w) = -v - w$

   subject to

$$\sum_{j=1}^{n} \left(\frac{2c_{ij} + r_{ij} + 2a_{ij} - l_{ij}}{8} \right) y_j \le v, \quad i = 1, \ldots, m,$$

$$\sum_{i=1}^{m} \left(\frac{2c_{ij} + r_{ij} + 2a_{ij} - l_{ij}}{8} \right) x_i \ge -w, \quad j = 1, \ldots, n,$$

$$\sum_{i=1}^{m} x_i = \sum_{j=1}^{n} y_j = 1,$$

$$x_i \ge 0, \ y_j \ge 0, \quad i = 1, \ldots, m, \ j = 1, \ldots, n.$$

The above set of constraint inequalities are the same (except for the multiplicative factor (1/2)) as described by (Xu et al. 2017, Section 4, Theorem 11) to find the credibility expectation equilibrium solution (x^*, y^*) of the zero-sum fuzzy matrix game. The value of this fuzzy matrix game is

$$v^* = \sum_{j=1}^{n} \sum_{i=1}^{m} x_i^* \left(\frac{2c_{ij} + r_{ij} + 2a_{ij} - l_{ij}}{8} \right) y_j^*.$$

(8) When $v = 1 - \mu$, amounting to the fuzzy case of the I-fuzzy situation, the problem (QPP) reduces to the following problem.

(QPP-9) $\max f(x, y, v, w)$

   subject to

$$\sum_{j=1}^{n} \left(\lambda_1 \alpha_i \left(c_{ij} + \frac{r_{ij}}{2} \right) + (1 - \lambda_1)(1 - \alpha_i) \left(a_{ij} - \frac{l_{ij}}{2} \right) \right) y_j$$
$$\le v, \quad i = 1, \ldots, m,$$

$$\sum_{i=1}^{m} \left(\lambda_2 \beta_j \left(d_{ij} + \frac{n_{ij}}{2} \right) + (1 - \lambda_2)(1 - \beta_j) \left(b_{ij} - \frac{m_{ij}}{2} \right) \right) x_i$$
$$\le w, \quad j = 1, \ldots, n,$$

$$\sum_{i=1}^{m} x_i = \sum_{j=1}^{n} y_j = 1,$$

$$x_i \ge 0, \ y_j \ge 0, \quad i = 1, \ldots, m, \ j = 1, \ldots, n,$$

where $f(x, y, v, w) = \frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{n} x_i \big(\lambda_1 \alpha_i r_{ij} + \lambda_2 \beta_j n_{ij} - (1 - \lambda_1)(1 - \alpha_i) l_{ij} - (1 - \lambda_2)(1 - \beta_j) m_{ij} \big) y_j - v - w.$

An element (x^*, y^*) is then called an $(\boldsymbol{\alpha}, \boldsymbol{\beta})$-fuzzy measure equilibrium solution of the bi-matrix game with fuzzy payoffs.

4 Examples of marketing strategies for illustration of the proposed equilibrium solution

To demonstrate the efficacy and validity of the proposed solution concepts, we present two examples.

The following example is taken from Seikh et al. (2015c).

Example 1 Consider two rival television media companies T_1 and T_2 aiming to raise their respective television rating point (TRP) by increasing their viewer-ship base. Suppose the management of both companies are rational to choose optimal strategies to maximize their own TRPs. To accomplish this, let T_1 and T_2 have to choose decision on the types of programs to be broadcasted daily during the peak hours 8:00 p.m.–11:00 p.m. Each of them suppose have two strategies to choose from—TV serials (strategy S_1) and reality show (strategy S_2). The above problem can be casted as a bi-matrix game having companies T_1 and T_2 as two players. Moreover, due to insufficient information or precise data, the managements of the two companies hold indecisive views on exact number of viewer-ship on implementation of these strategies. They can provide some estimated payoffs values with a certain degree of confidence mixed with some hesitation degree. The trapezoidal intuitionistic fuzzy numbers seem perfect to model such a scenario. Assuming that the marketing research departments of the two companies analyzed some data (through survey, or past experience) and supplied the following payoffs matrices on the number (all numbers can be treated with a common multiple of 100 or higher power of 10) of viewer-ship.

$$\tilde{A} = \begin{pmatrix} \langle [8, 9, 2, 4], [8, 9, 4, 6] \rangle & \langle [12, 13, 4, 6], [12, 13, 6, 8] \rangle \\ \langle [10, 12, 4, 6], [10, 12, 6, 6] \rangle & \langle [6, 7, 2, 4], [6, 7, 4, 6] \rangle \end{pmatrix},$$

$$\tilde{B} = \begin{pmatrix} \langle [8, 9, 2, 4], [8, 9, 4, 6] \rangle & \langle [6, 7, 2, 4], [6, 7, 4, 6] \rangle \\ \langle [6, 7, 2, 4], [6, 7, 4, 6] \rangle & \langle [10, 12, 4, 4], [10, 12, 6, 6] \rangle \end{pmatrix}.$$

Considering $\boldsymbol{\alpha} = 0.6\,\mathbf{I}_2$, $\lambda_1 = 0.3$, for player I, and $\boldsymbol{\beta} = 0.5\,\mathbf{I}_2$, $\lambda_2 = 0.6$, for player II. We have to solve the following quadratic programming problem to find the $(\boldsymbol{\alpha}, \boldsymbol{\beta})$-intuitionistic fuzzy measure equilibrium solution of this BMGIFP:

$$\max -2.6\,x_1y_1 + 0.28\,x_2y_1 + 0.18\,x_1y_2 - 0\,x_2y_2$$
$$\quad - v - w + v' + w'$$

subject to

$$3.94y_1 + 5.96y_2 \le v,$$
$$4.94y_1 + 3.02y_2 \le v,$$
$$6.64y_1 + 5.58y_2 \ge v',$$
$$4.66y_1 + 2.92y_2 \ge v',$$
$$4.9x_1 + 3.7x_2 \le w,$$
$$3.7x_1 + 3.4x_2 \le w,$$
$$4.8x_1 + 3.8x_2 \ge w',$$
$$3.8x_1 + 5.9x_2 \ge w',$$
$$x_1 + x_2 = 1,$$
$$y_1 + y_2 = 1,$$
$$x_1, x_2, y_1, y_2 \ge 0.$$

$$\tilde{B} = \begin{pmatrix} \langle[-9,-8,4,2],[-9,-8,6,4]\rangle & \langle[-13,-12,6,4],[13,-12,8,6]\rangle \\ \langle[-12,-10,6,4],[-12,-10,6,6]\rangle & \langle[-7,-6,4,2],[-7,-6,6,4]\rangle \end{pmatrix}.$$

The optimal solution is $(x^* = (0,1),\ y^* = (1,0))$, indicating T_1 should show reality shows while T_2 must choose to telecast TV serials. The corresponding optimal values depicting the intuitionistic fuzzy measure expected membership and non-membership (expressed as tuple) in viewer-ship are $(V,\ V') = (4.94, 4.66)$ and $(W,\ W') = (3.70, 3.80)$.

For $\alpha = \beta = \mathbf{I}_2$, the (α, β)-intuitionistic fuzzy measure equilibrium solutions of this bi-matrix game $(S^2, S^2, \tilde{A}, \tilde{B})$, obtained by solving (QPP) for certain specific values of parameters λ_1, λ_2, are listed in Table 1.

In Table 1, note that (x_1^*, x_2^*) and (y_1^*, y_2^*) do not change with the changing values of λ_1 and λ_2. This observation is in line with Remark 10. It is worth noting that the optimal values V, V' and W, W' are increasing with increase in λ_1 and λ_2, respectively. This pattern in optimal expected values can be explained in the sense that $\alpha = I_m$ and $\beta = I_n$ lead to only the possibility measure term in calculation of intuitionistic fuzzy measure expectation.

For $\alpha = \beta = \mathbf{O}_2$ (zero matrices), and for different values of λ_1, λ_2, the necessity equilibrium solutions of the game are depicted in Table 2.

Again, (x_1^*, x_2^*) and (y_1^*, y_2^*) in Table 2 do not change with the changing values of λ_1 and λ_2, but the optimal values V, V' and W, W' do change. This observation is in line with Remark 9. Moreover, values of V, V' and W, W' are decreasing with increase in λ_1 and λ_2, respectively. This

phenomena is reasonable as the case $\alpha = O$ and $\beta = O$ correspond to the expected values of game concerning necessity measure only.

For $\alpha = \beta = 0.5\,\mathbf{I}_2$, and for different values of λ_1, λ_2, the (α, β)-intuitionistic fuzzy measure equilibrium solutions of the given game are depicted in Table 3.

Note that in Table 3, the credibility equilibrium solutions as well as the corresponding optimal expected values of the game are changing with change in values of λ_1 and λ_2.

We next demonstrate the solution methodology for the special case when the bi-matrix game reduces to a zero-sum game. In other words, we consider the case when the payoff matrix of media company T_2 is negative of the payoff matrix of the media company T_1, that is, $\tilde{B} = -\tilde{A}$, therefore, the 2×2 matrix

By considering model (QPP-7) with $\alpha = \beta = \mathbf{I}_2$, and for different values of λ_1, λ_2, the $(\mathbf{I}, \mathbf{I})$-intuitionistic fuzzy measure equilibrium solutions of the zero-sum intuitionistic fuzzy matrix game are reported in Table 4.

Similarly, Table 5 is constructed by solving model (QPP-7) for $\alpha = \beta = \mathbf{O}_2$. Table 6 reports the (α, β)-intuitionistic fuzzy measure equilibrium solutions of the zero-sum matrix game for $\alpha = \beta = 0.5\,\mathbf{I}_2$.

In Tables 4 and 5, observe that equilibrium solutions (x_1^*, x_2^*) and (y_1^*, y_2^*) of two players remain the same with changing values of λ_1 and λ_2. This is in conformity with Remarks 9 and 10, respectively. Moreover, V, V' increase with increasing λ_1 and similar pattern for W, W' with respect to λ_2, in Table 4. And, V, V' decrease with increasing λ_1 and similar noting for W, W' with respect to λ_2, in Table 5.

We next present an example from Nan et al. (2017) suitably changed for illustrating our proposed methodology.

Example 2 Consider the case of two commerce retailers P_1 and P_2 (i.e., player I and player II) making a decision aiming to enhance the satisfaction degrees of the customers. The judgments of players on satisfaction degrees of customers, including preferences and experiences, are vague. Assume that P_1 and P_2 are non-cooperative and rational in the sense that they choose optimal strategies to maximize their own profit. Suppose P_1 has two pure strategies: establishing a scientific and rational service

systems and providing customers with satisfaction products, while P_2 also possesses same pure strategies.

Let the 2×2 payoffs matrices of P_1 and P_2 be described by triangular intuitionistic fuzzy numbers be as follows:

$$\tilde{A} = \begin{pmatrix} \langle[0.8,0.1,0.1],[0.8,0.2,0.2]\rangle & \langle[0.4,0.1,0.1],[0.4,0.3,0.3]\rangle \\ \langle[0.4,0.1,0.1],[0.4,0.3,0.3]\rangle & \langle[0.5,0.1,0.1],[0.5,0.2,0.2]\rangle \end{pmatrix},$$

$$\tilde{B} = \begin{pmatrix} \langle[0.36,0.06,0.04],[0.36,0.08,0.06]\rangle & \langle[0.285,0.085,0.015],[0.285,0.09,0.02]\rangle \\ \langle[0.25,0.05,0.05],[0.25,0.1,0.1]\rangle & \langle[0.40,0.05,0.05],[0.25,0.07,0.07]\rangle \end{pmatrix}.$$

We need to solve the following quadratic programming problem to find an (α, β)-intuitionistic fuzzy measure equilibrium solution of this problem:

$$\max -0.2\lambda_1 - 0.04\lambda_2 + 0.12)x_1y_1$$
$$+ (-0.4\lambda_1 - 0.10\lambda_2 + 0.25)x_2y_1$$
$$+ (-0.4\lambda_1 - 0.01\lambda_2 + 0.205)x_1y_2$$
$$- (-0.2\lambda_1 - 0.04\lambda_2 + 0.12)x_2y_2$$
$$- v - w + v' + w'$$

subject to

$$\lambda_1(0.85y_1 + 0.45y_2) + (1 - \lambda_1)(0.75y_1 + 0.35y_2)$$
$$\leq 2v,$$
$$\lambda_1(0.45y_1 + 0.55y_2) + (1 - \lambda_1)(0.35y_1 + 0.45y_2)$$
$$\leq 2v,$$
$$\lambda_1(0.90y_1 + 0.55y_2) + (1 - \lambda_1)(0.70y_1 + 0.25y_2)$$
$$\geq 2v',$$
$$\lambda_1(0.55y_1 + 0.60y_2) + (1 - \lambda_1)(0.25y_1 + 0.40y_2)$$
$$\geq 2v',$$
$$\lambda_2(0.38x_1 + 0.275x_2) + (1 - \lambda_2)(0.33x_1 + 0.225x_2)$$
$$\leq 2w,$$
$$\lambda_2(0.2925x_1 + 0.425x_2) + (1 - \lambda_2)(0.2425x_1 + 0.375x_2)$$
$$\leq 2w,$$
$$\lambda_2(0.39x_1 + 0.30x_2) + (1 - \lambda_2)(0.32x_1 + 0.20x_2)$$
$$\geq 2w',$$
$$\lambda_2(0.29x_1 + 0.435x_2) + (1 - \lambda_2)(0.24x_1 + 0.36x_2)$$
$$\geq 2w',$$
$$x_1 + x_2 = 1,$$
$$y_1 + y_2 = 1,$$
$$x_1, x_2, y_1, y_2 \geq 0.$$

For $\alpha = \beta = 0.5\,\mathbf{I}_2$, and for different values of λ_1, λ_2, the (α, β)-intuitionistic fuzzy measure equilibrium solutions of the given game are depicted in Table 6.

It is worth noting, in Table 7, that the $(0.5I, 0.5I)-$intuitionistic fuzzy equilibrium solution as well as the

optimal expected payoffs values get change with change in values of λ_1 and λ_2.

Table 1 For $\alpha = \beta = \mathbf{I}_2$, the (α, β)-intuitionistic fuzzy measure equilibrium solutions for the intuitionistic fuzzy matrix game between the two television media companies

λ_1	λ_2	(x_1^*, x_2^*)	(y_1^*, y_2^*)
0	1	(0.714, 0.285)	(0, 1)
0.25	0.75	(0.714, 0.285)	(0.636, 0.363)
0.50	0.50	(0.714, 0.285)	(0.636, 0.363)
0.75	0.25	(0.714, 0.285)	(0.636, 0.363)
1	0	(1, 0)	(0.636, 0.363)

V	V'	W	W'
0	0	10.42	11.42
3.20	3.29	7.82	8.57
6.40	6.90	5.21	5.71
9.61	9.81	2.60	2.85
12.81	13.81	0	0

Table 2 For $\alpha = \beta = O_2$, the (α, β)-intuitionistic fuzzy measure equilibrium solutions for the intuitionistic fuzzy matrix game between television media companies

λ_1	λ_2	(x_1^*, x_2^*)	(y_1^*, y_2^*)
0	1	(0, 1)	(0.833, 0.167)
0.25	0.75	(0.60, 0.40)	(0.833, 0.167)
0.50	0.50	(0.60, 0.40)	(0.833, 0.167)
0.75	0.25	(0.60, 0.40)	(0.833, 0.167)
1	0	(0.60, 0.40)	(0, 1)

V	V'	W	W'
7.5	6.5	0	0
5.625	4.875	1.55	1.30
3.75	3.25	3.10	2.60
1.87	1.62	4.65	3.90
0	0	6.20	5.20

We can analogously construct $(\boldsymbol{\alpha}, \boldsymbol{\beta})$-I-fuzzy measure solutions for other cases. However, the equilibrium solutions thus obtained cannot be compared with the one in Nan et al. (2017) simply because the approach in the latter work is different than the one proposed in this paper. In Nan et al. (2017), the authors assume the intuitionistic fuzzy aspiration levels and tolerances in them for P_1 and P_2.

5 A comparative study with existing approaches

Since the study by Maeda (2000) is most proximate to the present study, we first perform a peer comparison with the approach in Maeda (2000).

Table 3 For $\boldsymbol{\alpha} = \boldsymbol{\beta} = 0.5\,\mathbf{I}_2$, the $(\boldsymbol{\alpha}, \boldsymbol{\beta})$-intuitionistic fuzzy measure equilibrium solutions for the intuitionistic fuzzy matrix game between television media companies

λ_1	λ_2	(x_1^*, x_2^*)	(y_1^*, y_2^*)
0	1	(0.715, 0.285)	(0.833, 0.166)
0.25	0.75	(0.692, 0.307)	(0.517, 0.482)
0.50	0.50	(0.666, 0.334)	(0.705, 0.295)
0.75	0.25	(0.636, 0.364)	(0.666, 0.333)
1	0	(0.60, 0.40)	(0.636, 0.364)

V	V'	W	W'
3.75	3.25	5.21	5.71
4.42	3.65	4.69	4.94
5.08	4.91	4.16	4.16
5.75	5.75	3.63	3.38
6.40	6.59	3.10	2.60

Table 4 The $(\mathbf{I}, \mathbf{I})$-intuitionistic fuzzy measure equilibrium solutions of the zero-sum matrix game with $\tilde{B} = -\tilde{A}$

λ_1	λ_2	(x_1^*, x_2^*)	(y_1^*, y_2^*)
0	1	(0.50, 0.50)	(0, 1)
0.25	0.75	(0.50, 0.50)	(0.636, 0.363)
0.50	0.50	(0.50, 0.50)	(0.667, 0.333)
0.75	0.25	(0.50, 0.50)	(0.727, 0.272)
1	0	(0, 1)	(0.636, 0.364)

V	V'	W	W'
0	0	7.5	6.5
3.20	3.29	5.62	4.87
6.40	6.59	3.75	3.25
10.02	10.02	1.87	1.62
12.81	13.81	0	0

Table 5 The $(\mathbf{O}, \mathbf{O})$-intuitionistic fuzzy measure equilibrium solutions of the zero-sum matrix game with $\tilde{B} = -\tilde{A}$

λ_1	λ_2	(x_1^*, x_2^*)	(y_1^*, y_2^*)
0	1	(0, 1)	(0.833, 0.167)
0.25	0.75	(0.545, 0.454)	(0.833, 0.167)
0.50	0.50	(0.545, 0.454)	(0.833, 0.167)
0.75	0.25	(0.545, 0.454)	(0.833, 0.167)
1	0	(0.50, 0.50)	(0, 1)

V	V'	W	W'
7.5	6.5	0	0
5.62	4.87	3.20	3.45
3.76	3.25	6.40	6.90
1.87	1.62	9.61	10.36
0	0	12.5	13.5

Table 6 The $(0.5\,\mathbf{I}, 0.5\,\mathbf{I})$-intuitionistic fuzzy measure equilibrium solutions of the zero-sum matrix game with $\tilde{B} = -\tilde{A}$

λ_1	λ_2	(x_1^*, x_2^*)	(y_1^*, y_2^*)
0	1	(0.50, 0.50)	(0.833, 0.167)
0.25	0.75	(0.517, 0.482)	(0.758, 0.241)
0.50	0.50	(0.529, 0.470)	(0.705, 0.294)
0.75	0.25	(0.50, 0.50)	(0.666, 0.333)
1	0	(0.545, 0.454)	(0.636, 0.363)

V	V'	W	W'
3.75	3.25	3.75	3.25
4.42	4.07	4.42	4.11
5.08	4.91	5.08	4.97
5.75	5.75	5.31	5.87
6.40	6.59	6.40	6.68

In Maeda (2000), using the possibility measure approach, Maeda introduced a Nash-equilibrium solution for a bi-matrix game $(S^m, S^n, \tilde{A}, \tilde{B})$ with fuzzy payoffs. We shall be denoting the bi-matrix game with fuzzy payoffs by BMGFP.

Definition 8 An element $(x^*, y^*) \in S^m \times S^n$ is said to be a $(v, w)-$ possibility Nash equilibrium solution of BMGFP if the following inequalities hold.

$$\text{Poss}(x^{*\mathrm{T}}\tilde{A}y^* \geq v) \geq \text{Poss}(x^\mathrm{T}\tilde{A}y^* \geq v), \quad \forall\, x \in S^m,$$
$$\text{Poss}(x^{*\mathrm{T}}\tilde{B}y^* \geq w) \geq \text{Poss}(x^\mathrm{T}\tilde{B}y \geq w), \quad \forall\, y \in S^n.$$

Definition 9 By setting $\text{Poss}(x^{*^T} \tilde{A} y^* \geq v) = \gamma$ and $\text{Poss}(x^{*^T} \tilde{B} y^* \geq w) = \delta$, $\gamma, \delta \in [0, 1]$, an element $(x^*, y^*) \in S^m \times S^n$ is said to be a (γ, δ)-possibility Nash-equilibrium solution of the game BMGFP if the following hold.

$$x^{*^T} A_\gamma^R y^* = v \geq x^T A_\gamma^R y^*, \quad \forall\, x \in S^m,$$
$$x^{*^T} B_\delta^R y^* = w \geq x^{*^T} B_\delta^R y, \quad \forall\, y \in S^n,$$

where A_γ^R is the matrix having entries as right end-values of the γ-cut of the respective entries of the matrix $\tilde{A}$, and an analogous interpretation for B_δ^R.

Instead, if we consider all entries in the two payoffs matrices to be trapezoidal intuitionistic fuzzy numbers, $\tilde{A} = [\tilde{a}_{ij}] = \langle [a_{ij}, c_{ij}, l_{ij}, r_{ij}], \quad [a_{ij}, c_{ij}, p_{ij}, q_{ij}] \rangle$ and $\tilde{B} = [\tilde{b}_{ij}] = \langle [b_{ij}, d_{ij}, m_{ij}, n_{ij}], [b_{ij}, d_{ij}, t_{ij}, s_{ij}] \rangle$, then, to determine the (γ, δ)-Nash-equilibrium point, one has to solve the following inequalities

$$(\tilde{a}_{ij})_\gamma^R y \leq v, \quad i = 1, \ldots, m, \, j = 1, \ldots, n,$$
$$(\tilde{b}_{ij})_\delta^R x \leq w, \quad i = 1, \ldots, m, \, j = 1, \ldots, n.$$

The following theorem by Maeda (2000) provides the necessary and sufficient conditions for (γ, δ)-possibility Nash equilibrium solution.

Theorem 2 For $\gamma, \delta \in [0, 1]$, an element $(x^*, y^*) \in S^m \times S^n$ is a (γ, δ)-possibility Nash equilibrium solution of BMGFP if and only if the element $(x^*, y^*, x^{*^T} A_\gamma^R y^*, x^{*^T} B_\delta^R y^*)$ is an optimal solution of the following quadratic programming problem:

(QPP-M) $\max x^T (A_\gamma^R + B_\delta^R) y - v - w$

subject to
$$A_\gamma^R y \leq v,$$
$$B_\delta^R x \leq w,$$
$$x \in S^m, \, y \in S^n.$$

There could be situations when, for a game, Maeda's approach fails to yield the desired equilibrium solution while our proposed approach results in successfully computing the (α, β)-intuitionistic fuzzy measure equilibrium solution. To clarify our point, we first make an observation. Since the model suggested by Maeda (2000) considers only the possibility Nash-equilibrium solution for the fuzzy matrix games, to make a correct and viable comparative analysis, we must assume, $\lambda_1 = \lambda_2 = 1$ in our fuzzy case optimization model (QPP-9).

Table 7 For $\alpha = \beta = 0.5\, \mathbf{I}_2$, the (α, β)-intuitionistic fuzzy measure equilibrium solutions for the intuitionistic fuzzy matrix game between two commerce retailers

λ_1	λ_2	(x_1^*, x_2^*)	(y_1^*, y_2^*)
0	1	(0.631, 0.369)	(0.25, 0.75)
0.25	0.75	(0.631, 0.369)	(0.227, 0.773)
0.50	0.50	(0.631, 0.369)	(0.20, 0.80)
0.75	0.25	(0.631, 0.369)	(0.20, 0.80)
1	0	(0.673, 0.327)	(0.20, 0.80)

V	V'	W	W'
0.225	0.181	0.170	0.173
0.232	0.210	0.164	0.165
0.24	0.24	0.158	0.158
0.252	0.267	0.151	0.148
0.265	0.295	0.147	0.140

Furthermore, if we assume the fuzzy bi-matrix game be a zero-sum game, then the resulting model is as follows:

(QPP-10) $\max f(x, y, v, w)$
$$= \frac{1}{2} \sum_{i=1}^m \sum_{j=1}^n x_i (\alpha_i r_{ij} + \beta_j l_{ij}) y_j - v - w$$

subject to
$$\sum_{j=1}^n \alpha_i (2c_{ij} + r_{ij}) y_j \leq 2v, \quad i = 1, \ldots, m,$$
$$\sum_{i=1}^m \beta_j (2a_{ij} - l_{ij}) x_i \geq -2w, \quad j = 1, \ldots, n,$$
$$\sum_{i=1}^m x_i = \sum_{j=1}^n y_j = 1,$$
$$x_i \geq 0, \, y_j \geq 0, \quad i = 1, \ldots, m, \, j = 1, \ldots, n.$$

We now present an example to demonstrate the case when the equilibrium solution applying Maeda's approach could not be ascertained. Consider the following payoffs matrix for player I.

$$\tilde{A} = \begin{pmatrix} \langle [2, 3, 1, 2] \rangle & \langle [-1.5, -1, 0.5, 2] \rangle \\ \langle [-1.5, -1, 0.5, 4] \rangle & \langle [1, 1.5, 0.8, 1.5] \rangle \end{pmatrix},$$

and the payoffs matrix for player II is $\tilde{B} = -\tilde{A}$. Taking $\gamma = \delta = 0.6$, then model (QPP-M) results in the following problem:

$$\max 2.2\,x_1y_1 + 2.3\,x_2y_1 + 1.5\,x_1y_2$$
$$+ 1.42\,x_2y_2 - v - w$$
subject to
$$3.8y_1 - 0.2y_2 \le v,$$
$$0.6y_1 + 2.1y_2 \le v,$$
$$-1.6x_1 + 1.7x_2 \le w,$$
$$1.7x_1 - 0.68x_2 \le w,$$
$$x_1 + x_2 = 1,$$
$$y_1 + y_2 = 1,$$
$$x_1,\,x_2,\,y_1,\,y_2 \ge 0.$$

We obtain the optimal solution $(x^* = (0.4190, 0.5809),$ $y^* = (0.4181, 0.5818))$. The corresponding optimal payoffs are $V = 1.472$ and $W = 0.3172$.

It is important to note here that $x^{*^{\mathrm{T}}} A_{0.6}^R y^* \ne V$ and $x^{*^{\mathrm{T}}}(-A)_{0.6}^R y^* \ne W$. Hence, by Theorem 2, (x^*, y^*) is not a $(0.6, 0.6)-$ possibility Nash equilibrium solution of the considered BMGFP in the sense of Maeda.

By considering model (QPP-10) with $\boldsymbol{\alpha} = \boldsymbol{\beta} = 0.6\,\mathbf{I}_2$, have to solve the following program:

$$\max 0.15\,x_1y_1 + 0.225\,x_2y_1$$
$$+ 0.125\,x_1y_2 + 0.115\,x_2y_2 - v - w$$
subject to
$$1.5y_1 - 0.35y_2 \le v,$$
$$-0.05y_1 + 0.795y_2 \le v,$$
$$1.25x_1 - 0.325x_2 \ge -w,$$
$$-0.525x_1 + 0.63x_2 \ge -w,$$
$$x_1 + x_2 = 1,$$
$$y_1 + y_2 = 1,$$
$$x_1,\,x_2,\,y_1,\,y_2 \ge 0.$$

The optimal solution is $(x^* = (0.206, 0.794), y^* = (0.425, 0.575))$. The expected optimal payoffs for two players are $V = 0.4359$ and $W = 0$.

Note that the $(\boldsymbol{\alpha}, \boldsymbol{\beta})$-possibility equilibrium solution from our proposed methodology is different than the (γ, δ)-possibility Nash equilibrium solution in Maeda (2000) for the considered zero-sum bi-matrix fuzzy game.

We now compare our solution approach with the method proposed by Seikh et al. (2015c) for solving bi-matrix intuitionistic games. There are a few noticeable differences between our proposed framework and methodology than the one formulated in Seikh et al. (2015c).

(i)   Seikh et al. (2015c) proposed a new ranking function as a combination of the value index and ambiguity index to develop their non-linear model (model number (14) on p. 164) for computing the Nash equilibrium of the bi-matrix game. While we have used a combination of the possibility measure and the necessity measure to define the expectation of the intuitionistic fuzzy measure and applied it to formulate the equivalent quadratic program to computing (α, β)-intuitionistic fuzzy measure equilibrium.

(ii)   The equivalent crisp model (model number (14) on p. 164) by Seikh et al. (2015c) involves non-linear quadratic form in the constraints while the objective function is linear, while the model (QPP) in the present paper has linear constraints and quadratic form in the objective function. The difference is due to the different approaches used in defining equilibrium solutions. Seikh et al. (2015c) considered double fuzzy inequalities and hence required the two players to provide many more tolerances parameters for their respective intuitionistic fuzzy goals. On the other hand, in the present paper, we avoided all such parameters and used the expectation of both the membership and the non-membership of the intuitionistic fuzzy values of the players.

(iii)   Though both our approach as well as the approach by Seikh et al. (2015c) are not able to describe the membership and the non-membership functions explicitly for the optimal values of the two players in the game. But the critical point to observe is that the final values u^* and v^* for the two players in the game by Seikh et al. (2015c) provide absolutely no explicit information on non-membership of these values. In our scheme, we can get the fuzzy measure expected values of the membership as well as the non-membership V, V' and W, W' of the optimal values for the two players in the game.

(iv)   The payoffs in the bi-matrix games by Seikh et al. (2015c) are considered as triangular intuitionistic fuzzy numbers while we worked with the trapezoidal intuitionistic payoffs. For numeric comparison, we solve our Example 1 using the scheme and the model suggested in Seikh et al. (2015c). But before that we have to develop model (14) in Seikh et al. (2015c) to the case of trapezoidal intuitionistic fuzzy numbers.

Let $\tilde{a} = \langle [b, c, l, r], [b, c, p, q] \rangle \in \mathrm{TrIFN}(\mathbb{b}\mathbb{b}(R))$. On the lines of Seikh et al. (2015c) the membership and the non-membership of the value index and the ambiguity index for $\tilde{a}$ respectively are given by

$$G_\mu(\tilde{a}) = \frac{3b + 3c + r - l}{6},$$

$$G_v(\tilde{a}) = \frac{3b + 3c + q - p}{6},$$

$$H_\mu(\tilde{a}) = \frac{3c - 3b + r + l}{3},$$

$$H_v(\tilde{a}) = \frac{3c - 3b + p + q}{3}.$$

Using the ranking function

$$R(\tilde{a}) = V\left(\tilde{a}, \frac{1}{2}\right) - A\left(\tilde{a}, \frac{1}{2}\right)$$

$$= \frac{G_v(\tilde{a}) + G_\mu(\tilde{a})}{2} - \frac{H_\mu(\tilde{a}) + H_v(\tilde{a})}{2},$$

we get,

$$R(\tilde{a}) = \frac{18b - 6c - 3l - r - 3p - q}{12}. \tag{5}$$

Since in our presented approach we have not used the double fuzzy inequalities, therefore, for comparison, we set all the tolerances parameters in model (14) in Seikh et al. (2015c) zero. The resultant non-linear program (NLP) is thus as follows:

(NLP) $\max u + v$

subject to

$$\sum_{j=1}^{n} R(\tilde{a}_{ij})y_j \le u, \quad i = 1, \ldots, m,$$

$$\sum_{i=1}^{m} R(\tilde{b}_{ij})x_i \le v, \quad j = 1, \ldots, n,$$

$$\sum_{j=1}^{n} x_i R(\tilde{a}_{ij})y_j \ge u,$$

$$\sum_{j=1}^{n} x_i R(\tilde{b}_{ij})y_j \ge v,$$

$$\sum_{i=1}^{m} x_i = \sum_{j=1}^{n} y_j = 1,$$

$$x_i \ge 0, \ y_j \ge 0, \quad i = 1, \ldots, m, \ j = 1, \ldots, n.$$

We now recall the payoffs matrices from the media marketing Example 1, and apply ranking function formula in (5), the (NLP) problem to solve becomes as follows:

$\max u + v$

subject to

$5.167y_1 + 7.833y_2 \le u,$

$5.55y_1 + 3.167y_2 \le u,$

$5.167x_1 + 3.167x_2 \le v,$

$3.167x_1 + 5.667x_2 \le v,$

$5.167x_1y_1 + 7.83x_1y_2 + 5.55x_2y_1 + 3.167x_2y_2 \ge u,$

$5.167x_1y_1 + 3.167x_1y_2 + 3.167x_2y_1 + 5.667x_2y_2 \ge v,$

$x_1 + x_2 = 1,$

$y_1 + y_2 = 1,$

$x_1, x_2, y_1, y_2 \ge 0.$

The optimal solution is $(x^* = (0.5555852, 0.4444148), y^* = (0.9333347, 0.0666653))$, and the optimal values for players are $u^* = 5.344450$ and $v^* = 4.277787$. From Tables 1, 2 and 3, we can notice the optimal solutions and the optimal expected membership and non-membership values for the two players corresponding to $\lambda_1 = \lambda_2 = 0.5$.

When compared with Seikh et al. (2015c), our proposed approach may be considered a better contribution on two accounts. First, our approach is more informative in providing optimal values of the expected membership and non-membership of the values for the two players. Second, variations in the proportions of the actual realizations of the expected values from the game for the two players, modeled in the form of matrices $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, provide flexibility to incorporate more subjective evaluations than the conventional techniques of treating the expected values of the game from different fuzzy measures in the same vein.

Furthermore, the approach presented by Nan et al. (2017) and their model number (16) on p. 3729 is almost the same as model number (13), pp. 163–164, by Seikh et al. (2015c) except the objective function formulation and parameters used in the resolution of the double fuzzy inequalities. Both these researches considered the payoffs and goals of the players to be triangular intuitionistic fuzzy numbers and converted the targets into the intuitionistic fuzzy inequality constraints utilizing tolerances parameters for the memberships and the non-memberships degrees. Our approach, on the other hand, is different in treating the intuitionistic fuzzy goals of the players which do not ask for the additional tolerance parameters. We have instead used a combination of the possibility and necessity expectations to define a generalized notion of (α, β)-intuitionistic fuzzy measure equilibrium solutions. A comparative analysis is not straight forward between the approaches yet as we have already included the empirical study of Example 1 using Seikh et al. (2015c) approach, the same remains valid for the model in Nan et al. (2017), and hence not included explicitly to avoid a repeat with no new knowledge.

6 Conclusions

In this paper, we study a class of intuitionistic fuzzy bi-matrix games with trapezoidal intuitionistic fuzzy numbers and define a new intuitionistic fuzzy measure equilibrium solutions for such a class of games. The proposed solution concept embeds in it the subjective perspectives of the two players in the form of parameters λ_1 and λ_2. It also assigns weights or proportions to the actual realizations of the payoffs on selecting the strategies in the form of the diagonal matrices $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ for player I and player II,

respectively. The equivalent crisp optimization models are presented to compute the $(\boldsymbol{\alpha}, \boldsymbol{\beta})$-intuitionistic fuzzy measure equilibrium solutions of the intuitionistic fuzzy bi-matrix games with payoffs matrices of two players involve trapezoidal intuitionistic fuzzy numbers. Some special cases for specific values of the parameters $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, λ_1, λ_2, with corresponding interpretations, are also discussed. We also present illustrative a few real-life examples to demonstrate the proposed solution methodology. A comparative analysis is included with some of the relevant existing research to highlight the distinct features of our proposed method.

The proposed bi-matrix game model involves certain parameters namely $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, λ_1, λ_2 to be supplied by the players. The λ parameter is used to depict the behavior of the player towards the game. If the player thought-process is negative and he is more worried about the lower values of the gain on choosing an action from his strategy set, then he is pessimistic while if the player plays with a positive attitude, and looks ahead towards higher values in gain from the game then he is optimistic. Thus, the parameter λ describes how an individual chooses benefit and losses (or, in general, risk) when the outcomes are unknown. The value of λ can be set to 1/2 in inconclusive situations. Similarly, the values in the matrix $\boldsymbol{\alpha}$ can be considered as a kind of priorities that the Player I attaches to his strategies when he wishes to use only the possibility measure. If the player has a prior information on a strategy to be preferable than the others then this information can be translated in form of diagonal matrix $\boldsymbol{\alpha}$ with each α_i as a kind of pre-decided priority weight attached to the i-th strategy when the player is using the possibility measure to compute the expected payoff from the game. A similar explanation can be given for $1 - \alpha_i$ when the player wishes to only use the necessity measure in calculating the expected value of the game. If there is no clear choice on using the possibility or necessity measures, then one can work with the convex combination and can take matrix $\boldsymbol{\alpha} = 0.5I_m$.

However, our proposed approach exhibits a limitation by describing only the expected intuitionistic fuzzy measure values of the optimal payoffs of two players and not able to explain their membership and non-membership functions. Also, the computational complexity is high compared to its counterpart as the proposed method requires the players to supply not only the trapezoidal intuitionistic numbers in their payoff matrices but also have to strategically decide on the diagonal matrices $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. The illustrative examples indicate that the optimal solutions and the expected values of the two players from the game are sensitive to the choice of the parameters, and hence their correct decision is critical. In the absence of any concrete mechanism to estimate them, the players can stick to choose $0.5I$ for both these matrices in the possibility and necessity components proportions mix for evaluating the expected values. In future, an attempt could be made to design a methodology which can provide a complete characterization of optimal values of two players involved in playing the intuitionistic fuzzy payoffs non-cooperative game or interval intuitionistic fuzzy payoffs. Another interesting approach to study matrix game by Ammar and Brikaa (2018) under rough fuzzy sets could be investigated for further research in a bi-matrix framework with a combination of intuitionistic fuzzy multi-granulation rough sets Huang et al. (2014).

References

Aggarwal A, Dubey D, Chandra S, Mehra A (2012a) Application of Atanassov's I-fuzzy set theory to matrix games with fuzzy goals and fuzzy payoffs. Fuzzy Inf Eng 4:401–414

Aggarwal A, Mehra A, Chandra S (2012b) Application of linear programming with I-fuzzy sets to matrix games with I-fuzzy goals. Fuzzy Optim Decis Mak 11:465–480

Ammar ES, Brikaa MG (2018) On solution of constraint matrix games under rough interval approach. Granul Comput. https://doi.org/10.1007/s41066-018-0123-4

An JJ, Li DF, Nan JX (2017) A mean-area ranking based non-linear programming approach to solve intuitionistic fuzzy bi-matrix games. J Intell Fuzzy Syst 33:563–573

Angelov PP (1997) Optimization in an intuitionistic fuzzy environment. Fuzzy Sets Syst 86:299–306

Atanassov KT (1986) Intuitionistic fuzzy sets. Fuzzy Sets Syst 20:87–96

Atanassov KT (1989) More on intuitionistic fuzzy sets. Fuzzy Sets Syst 33:37–45

Atanassov KT (1994) New operations defined over the intuitionistic fuzzy sets. Fuzzy Sets Syst 61:137–142

Atanassov KT (1999) Intuitionistic fuzzy sets: theory and applications. Physica-Verlag HD, Heidelberg

Bandyopadhyay S, Nayak PK, Pal M (2013) Solution of matrix game with triangular intuitionistic fuzzy pay-off using score function. Open J Optim 2:9–15

Bector CR, Chandra S, Vijay V (2004) Duality in linear programming with fuzzy parameters and matrix games with fuzzy pay-offs. Fuzzy Sets Syst 146:253–269

Campos L (1989) Fuzzy linear programming models to solve fuzzy matrix games. Fuzzy Sets Syst 32:275–289

Chakraborty D, Jana DK, Roy TK (2014) A new approach to solve intuitionistic fuzzy optimization problem using possibility, necessity and credibility measure. Int J Eng Math 2014:1–12

Chen SM, Chang CH (2015) A novel similarity measure between Atanassov's intuitionistic fuzzy sets based on transformation techniques with applications to pattern recognition. Inf Sci 291:96–114

Chen SM, Chang YC (2011) Weighted fuzzy rule interpolation based on GA-based weight-learning techniques. IEEE Trans Fuzzy Syst 19:729–744

Chen SM, Tanuwijaya K (2011) Fuzzy forecasting based on high-order fuzzy logical relationships and automatic clustering techniques. Expert Syst Appl 38:15,425–15,437

Chen SM, Cheng SH, Chiou CH (2016) Fuzzy multiattribute group decision making based on intuitionistic fuzzy sets and evidential reasoning methodology. Inf Fusion 27:215–227

Cheng SH, Chen SM, Jian WS (2016) Fuzzy time series forecasting based on fuzzy logical relationships and similarity measures. Inf Sci 327:272–287

De SK, Biswas R, Roy AR (2001) An application of intuitionistic fuzzy sets in medical diagnosis. Fuzzy Sets Syst 117:209–213

Dubois D, Gottwald S, Hajek P, Kacprzyk J, Prade H (2005) Terminological difficulties in fuzzy set theory—the case of intuitionistic fuzzy sets. Fuzzy Sets Syst 156:485–491

Falsafain A, Taheri SM, Mashinchi M (2008) Fuzzy estimation of parameters in statistical models. Int J Math Comput Sci 2:79–85

Fan M, Zou P, Li SR, Wu CC (2014) A fast approach to bimatrix games with intuitionistic fuzzy payoffs. Sci World J 2014:1–6

Garai T, Chakraborty D, Roy TK (2018) Possibility–necessity–credibility measures on generalized intuitionistic fuzzy number and their applications to multi-product manufacturing system. Granul Comput 3:285–299

Garg H, Kaur G (2018) Novel distance measures for cubic intuitionistic fuzzy sets and their applications to pattern recognitions and medical diagnosis. Granul Comput. https://doi.org/10.1007/s41066-018-0140-3

Grzegorzewski P, Mrówka E (2005) Some notes on (Atanassov's) intuitionistic fuzzy sets. Fuzzy Sets Syst 156:492–495

Huang B, Guo CX, Yl Zhuang, Li HX, Zhou XZ (2014) Intuitionistic fuzzy multigranulation rough sets. Inf Sci 277:299–320

Inuiguchi M, Ramík J, Tanino T, Vlach M (2003) Satisficing solutions and duality in interval and fuzzy linear programming. Fuzzy Sets Syst 135:151–177

Khan I, Aggarwal A, Mehra A (2016) Solving I-fuzzy bi-matrix games with I-fuzzy goals by resolving indeterminacy. J Uncertain Syst 10:204–222

Khan I, Aggarwal A, Mehra A, Chandra S (2017) Solving matrix games with Atanassov's I-fuzzy goals via indeterminacy resolution approach. J Inf Optim Sci 38:259–287

Kumar S, Gangwar SS (2016) Intuitionistic fuzzy time series: an approach for handling nondeterminism in time series forecasting. IEEE Trans Fuzzy Syst 24:1270–1281

Larbani M (2009) Solving bimatrix games with fuzzy payoffs by introducing nature as a third player. Fuzzy Sets Syst 160:657–666

Li DF (1999) A fuzzy multi-objective approach to solve fuzzy matrix games. J Fuzzy Math 7:907–912

Li DF (2005) Multiattribute decision making models and methods using intuitionistic fuzzy sets. J Comput Syst Sci 70:73–85

Li DF (2008) Extension of the LINMAP for multiattribute decision making under Atanassov's intuitionistic fuzzy environment. Fuzzy Optim Decis Mak 7:17–34

Li DF (2010a) Linear programming method for MADM with interval-valued intuitionistic fuzzy sets. Expert Syst Appl 37:5939–5945

Li DF (2010b) Multiattribute decision making method based on generalized OWA operators with intuitionistic fuzzy sets. Expert Syst Appl 37:8673–8678

Li DF (2010c) TOPSIS-based nonlinear-programming methodology for multiattribute decision making with interval-valued intuitionistic fuzzy sets. IEEE Trans Fuzzy Syst 18:299–311

Li DF (2012) A fast approach to compute fuzzy values of matrix games with payoffs of triangular fuzzy numbers. Eur J Oper Res 223:421–429

Li DF, Chuntian C (2002) New similarity measures of intuitionistic fuzzy sets and application to pattern recognitions. Pattern Recognit Lett 23:221–225

Li DF, Chen GH, Huang ZG (2010) Linear programming method for multiattribute group decision making using IF sets. Inf Sci 180:1591–1609

Liu B (2006) A survey of credibility theory. Fuzzy Optim Decis Mak 5:387–408

Liu HW, Wang GJ (2007) Multi-criteria decision-making methods based on intuitionistic fuzzy sets. Eur J Oper Res 179:220–233

Liu P, Chen SM (2017) Group decision making based on Heronian aggregation operators of intuitionistic fuzzy numbers. IEEE Trans Cybern 47:2514–2530

Maeda T (2000) Characterization of the equilibrium strategy of the bimatrix game with fuzzy payoff. J Math Anal Appl 251:885–896

Maeda T (2003) On characterization of equilibrium strategy of two-person zero-sum games with fuzzy payoffs. Fuzzy Sets Syst 139:283–296

Mangasarian OL, Stone H (1964) Two-person nonzero-sum games and quadratic programming. J Math Anal Appl 9:348–355

Nan JX, Li DF, Zhang MJ (2010) A lexicographic method for matrix games with payoffs of triangular intuitionistic fuzzy numbers. Int J Comput Intell Syst 3:280–289

Nan JX, Wei CL, Li DF (2016) Pareto optimal strategies for matrix games with payoffs of intuitionistic fuzzy sets. In: Li D-F, Yang XG, Uetz M, Xu GJ (eds) Game theory and applications, China GTA 2016, China-Dutch GTA 2016. Springer, Berlin, pp 148–161. https://doi.org/10.1007/978-981-10-6753-2_11

Nan JX, Li DF, An JJ (2017) Solving bi-matrix games with intuitionistic fuzzy goals and intuitionistic fuzzy payoffs. J Intell Fuzzy Syst 33:3723–3732

Nayak PK, Pal M (2010) Bi-matrix games with intuitionistic fuzzy goals. Iran J Fuzzy Syst 7:65–79

Nehi HM (2010) A new ranking method for intuitionistic fuzzy numbers. Int J Fuzzy Syst 12:80–86

Nishizaki I, Sakawa M (2013) Fuzzy and multiobjective games for conflict resolution, studies in fuzziness and soft computing, vol 64. Physica-Verlag, Heidelberg

Papageorgiou EI, Iakovidis DK (2013) Intuitionistic fuzzy cognitive maps. IEEE Trans Fuzzy Syst 21:342–354

Ramík J (2005) Duality in fuzzy linear programming: some new concepts and results. Fuzzy Optim Decis Mak 4:25–39

Ramík J (2006) Duality in fuzzy linear programming with possibility and necessity relations. Fuzzy Sets Syst 157:1283–1302

Seikh MR, Nayak PK, Pal M (2013) Matrix games in intuitionistic fuzzy environment. Int J Math Oper Res 5:693–708

Seikh MR, Nayak PK, Pal M (2015a) An alternative approach to solve bi-matrix games with intuitionistic fuzzy goals. Int J Fuzzy Comput Model 1:362–381

Seikh MR, Nayak PK, Pal M (2015b) Matrix games with intuitionistic fuzzy pay-offs. J Inf Optim Sci 36:159–181

Seikh MR, Nayak PK, Pal M (2015c) Solving bi-matrix games with pay-offs of triangular intuitionistic fuzzy numbers. Eur J Pure Appl Math 8:153–171

Seikh MR, Nayak PK, Pal M (2016) Aspiration level approach to solve matrix games with I-fuzzy goals and I-fuzzy pay-offs. Pac Sci Rev A Nat Sci Eng 18:5–13

Verma T, Kumar A, Kacprzyk J (2015) A novel approach to the solution of matrix games with payoffs expressed by trapezoidal intuitionistic fuzzy numbers. J Autom Mob Robot Intell Syst 9:25–46

Vidyottama V, Chandra S, Bector CR (2004) Bimatrix games with fuzzy goals and fuzzy payoffs. Fuzzy Optim Decis Mak 3:327–344

Vijay V, Mehra A, Chandra S, Bector CR (2007) Fuzzy matrix games via a fuzzy relation approach. Fuzzy Optim Decis Mak 6:299–314

Vlachos IK, Sergiadis GD (2007) Intuitionistic fuzzy information—applications to pattern recognition. Pattern Recognit Lett 28:197–206

Vlachos IK, Sergiadis GD (2009) Intuitionistic fuzzy image processing. In: Encyclopedia of artificial intelligence, ch 143, pp 967–974. https://doi.org/10.4018/978-1-59904-849-9

Wang Z, Li KW, Wang W (2009) An approach to multiattribute decision making with interval-valued intuitionistic fuzzy assessments and incomplete weights. Inf Sci 179:3026–3040

Xu C, Meng F, Zhang Q (2017) PN equilibrium strategy for matrix games with fuzzy payoffs. J Intell Fuzzy Syst 32:2195–2206

Xu Z, Chen J, Wu J (2008) Clustering algorithm for intuitionistic fuzzy sets. Inf Sci 178:3775–3790

Yang J, Fei W, Li DF (2016) Non-linear programming approach to solve bi-matrix games with payoffs represented by I-fuzzy numbers. Int J Fuzzy Syst 18:492–503

Zadeh LA (1965) Fuzzy sets. Inf Control 8:338–353