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ORIGINAL PAPER

Novel distance measures for cubic intuitionistic fuzzy sets and their applications to pattern recognitions and medical diagnosis

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Abstract

Cubic intuitionistic fuzzy set (IFS), handles the uncertainties by characterizing them into membership and non-membership interval in form of interval-valued IFS and further the degree of agreement, as well as disagreement corresponding to these intervals, is given in the form of an IFS. Under this environment, some series of distance measures based on Hamming, Euclidean, and Hausdorff metrics are proposed. Various relations among them are derived. The practical relevance of our work is justified by giving two real-life examples, one on medical diagnosis and other on pattern recognition. Further, comparison analysis has been done with the existing decision-making approaches and the advantages of the proposed approach are highlighted.

Keywords Decision-making · Intuitionistic fuzzy set · Cubic intuitionistic fuzzy set · Distance measures · Interval-valued IFS

1 Introduction

Fuzzy set (FS) theory, introduced by Zadeh (1965), is one of the most successful theories to represent the uncertainty in the data. Since then, many scholars have utilized the FS to handle the uncertainties in the real fields (Chen and Tanuwijaya 2011; Chen et al. 2001, 2012a; Garg 2018b; Garg and Ansha 2018; Wang and Chen 2008). However, FS considers only the degree of agreement (also named as membership) of each object which is lying between [0,1] and considers that degree of disagreement (also named as non-membership) is the complement of the agreement. But mere considering of the degree of agreement was not fetching appropriate precise results. Later, Atanassov (1986) extended the theory of FSs to an IFS which is characterized by a degree of membership and non-membership functions. Subsequently, Atanassov and Gargov (1989) presented the concept of interval-valued IFS (IVIFS), which is a further extension of IFSs, where the membership degrees are represented by interval-valued intuitionistic fuzzy numbers (IVIFNs).

With economic and social developments, decision-making (DM) has been widely applied in various fields such as aggregation operators (AOs), pattern recognition, medical diagnosis, and clustering analysis, etc. However, in the past few decades, researchers have applied IFSs and IVIFSs to such fields. For instance, Garg (2016a, 2017b) presented some interactive AOs for different intuitionistic fuzzy numbers (IFNs). Garg (2018c) presented some geometric AOs for IVIFNs. Liu and Wang (2018) presented some ordered weighted AOs for solving DM problems. Mahmood et al. (2018) presented some hybrid AOs for solving DM problems under the triangular IFNs. Apart from these, several other AOs are developed by the scholars in the literature to solve the DM problems by using IFNs and/or IVIFNs information (Chen and Chang 2015; Chen et al. 2012b, 2016; Liu et al. 2017).

However, in the field of an information measure, the concept of distance, similarity, correlation, etc., are paying more attention by the scholars to solve the DM problems. Under it, Chen (1997) presented the similarity measure between the vague sets. Based on Hausdorff distance, Hung and Yang (2004) presented some similarity measures for IFSs. Garg (2016b) presented a generalized score function to rank the

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different IVIFNs. Szmidt and Kacprzyk (2000) presented the distance and similarity measures between the IFSs. Singh and Garg (2017) developed the distance measures between the type-2 IFSs. Garg and Kumar (2018a) presented new similarity measures for IFSs using set pair analysis. Xu (2007) presented similarity measures between the two IVIFSs and applied them to solve the pattern recognition problems. Dugenci (2016) presented some distance measures for IVIFSs and applied them to solve the group DM problems. Park et al. (2009), Wei et al. (2011) presented the correlation coefficients for IVIFSs. Garg (2017a) presented the distance and similarity measures for an intuitionistic multiplicative preference relation. Zhang et al. (2014) presented some entropy measures based on the distance measures for IVIFSs to solve the DM problems. Rani and Garg (2017) presented the family of the distance measures for the complex IFS, which is an extension of IFSs. Kumar and Garg (2018a, b) presented a technique for order preference with respect to the similarity to the ideal solution (TOPSIS) approach for solving the DM problems by using connection number of the set pair analysis theory. However, apart from them, some other theories for solving the DM problems by using distance or similarity measures are presented in the literature. For them, we refer to Arora and Garg (2018b), Garg (2018b), Huang and Li (2018), Jamkhaneh and Garg (2018) and Mitchell (2003).

Since all these facilitate the uncertainties to a great extent, but still they cannot withstand the situations where the decision-maker has to consider the falsity corresponding to the truth value ranging over an interval. Jun et al. (2012) combined the theory of the interval-valued FS with the FS and presented the concept of a cubic fuzzy set (CFS). After it, Mahmood et al. (2016) extended the CFS into the cubic hesitant FS by combining interval-valued hesitant FS and hesitant FS to solve the DM problems. Since CFSs considered only acceptance region into the analysis, but it is quite clear that the degree of rejection/non-membership plays an equivalent role during the performance analysis of any DM problems. So, by keeping these facts in mind, Kaur and Garg (2018a, c) introduced the idea of the cubic intuitionistic fuzzy set (CIFS), which is a hybrid set and formed by combining the features of IVIFSs and IFSs. Clearly, CIFS contains more information to represent the data in terms of IVIFNs and IFNs simultaneously. Furthermore, it allows us to consider the non-membership corresponding to the membership interval expressed in the form of IVIFS. Also, they have a fundamental characteristic of clubbing the information varying over different time spans. For example, suppose at the start of a financial year, a stock market analyst estimates the return on his investment to be 60–70% tending towards profit state and 10–15% tending towards the loss state. However, at the end of the financial year, the analyst found his returns to be 55% agreeing to the profit estimate and 35% disagreeing towards the loss estimate. Thus, a P-order CIFS is formed as $(\langle[0.60, 0.70], [0.10, 0.15]\rangle, \langle 0.55, 0.35\rangle)$. On the other hand, if at end of the financial year, the analyst found the returns to be 30% disagreeing to the profit estimate and 20% agreeing to the loss state then an R-order CIFS is formed as $\big(\langle[0.60, 0.70], [0.10, 0.15]\rangle, \langle 0.30, 0.20\rangle\big)$. Thus, CIFSs have a powerful ability to express the uncertainty and fuzzy decision process more precisely and objectively during the DM process. Keeping the advantages of this set, recently, Kaur and Garg (2018b) presented some generalized AOs for cubic IFNs to solve the DM problems. Garg and Kaur (2018) presented a TOPSIS method based on the distance measures under CIFS environment to solve the group DM problems. However, to the best of authors knowledge, no work has been conducted so far on the information measures for the different CIFSs. Thus, motivated by the aforementioned studies and the advantages of CIFSs, in this manuscript, we present a novel series of non-weighted as well as weighted distance measures to process the available information. In the existing DM theories, the evaluation is done by processing the information confined to one-time frame only whereas using the distances on CIFSs the analysis can be done varying over two time spans simultaneously. Based on all these features, the objectives of this work are:

1. To extend the theory of the cubic set to CIFS, by considering the degree of the rejection into the analysis and hence develops some series of distance measures for them.
2. To discuss some desirable properties on the proposed distance measures.
3. To propose an efficient decision-making approach based on the developed measures and illustrate them with real-life cases associated with medical diagnosis and pattern recognition.
4. To compare the outcomes of the proposed approach with the existing approaches and to justify the proposed approach's adaptability to the real-life cases.

To achieve the first objective, in this article, we represent the information towards the different objects under the CIFSs where the preferences related to each element are represented in the form of cubic intuitionistic fuzzy numbers (CIFNs). Then, we present a family of the distance measures based on Hamming, Euclidean and Hausdorff metrics. The desirable characteristics and relations between the various proposed measures are investigated to achieve the second objective. The third objective is achieved by establishing a DM approach based on the proposed measures to solve the problems associated with the medical diagnosis and pattern recognition under CIFS environment. The obtained results are compared with the existing approaches under IVIFSs for fulfilling the objective 4.

The rest of the manuscript is summarized as follows. Section 2 presents some basic concepts of IVIFSs, CFSs, and CIFSs. In Sect. 3, we present a family of distance measures between the two CIFSs and their desirable relations. Section 4 describes an approach based on the proposed measures to solve the DM problems followed by an illustrative example. Finally, a concrete conclusion is drawn in Sect. 5.

2 Preliminaries

Some basic concepts on IVIFS, CFS and CIFSs are summarized here over the universal set U.

Definition 1 (Atanassov 1986) An IFS A in a set U is given as:

$$A = \{(x, \zeta_A(x), \vartheta_A(x)) \mid x \in U\}, \tag{1}$$

where ζ_A and ϑ_A are the mappings from U to [0, 1] such that $0 \le \zeta_A(x) \le 1$ and $0 \le \vartheta_A(x) \le 1$ and $0 \le \zeta_A(x) + \vartheta_A(x) \le 1$. We denote this pair as $A = \langle \zeta_A, \vartheta_A \rangle$ and called as IFN.

Definition 2 (Atanassov and Gargov 1989) An IVIFS A in U is given as

$$A = \{\langle x, [\zeta_A^-(x), \zeta_A^+(x)], [\vartheta_A^-(x), \vartheta_A^+(x)] \rangle \mid x \in U\}, \tag{2}$$

where $0 \le \zeta_A^-(x) \le \zeta_A^+(x) \le 1$, $0 \le \vartheta_A^-(x) \le \vartheta_A^+(x) \le 1$ and $\zeta_A^+(x) + \vartheta_A^+(x) \le 1$. This pair is often called as IVIFN.

Definition 3 (Jun et al. 2012) A CFS $\mathcal{A}$ in U is given as

$$\mathcal{A} = \{(x, A_F(x), \lambda_F(x)) \mid x \in U\}, \tag{3}$$

where $A_F(x) = [A^-(x), A^+(x)]$ and $\lambda_F(x)$, respectively, represents the interval-valued FS and FS in $x \in U$. We denote these pairs as $\mathcal{A} = \langle A_F, \lambda_F \rangle$.

Definition 4 (Jun et al. 2012) The complement of the CFS $\mathcal{A}$ is defined to be the cubic fuzzy set $\mathcal{A}^c = \{x, \langle A^c(x), 1 - \lambda(x) \rangle \mid x \in U\}$.

Definition 5 (Kaur and Garg 2018a, c) A CIFS $\mathcal{A}$ is an ordered pair given by

$$\mathcal{A} = \{\langle x, A(x), \lambda(x) \rangle \mid x \in U\}, \tag{4}$$

where $A = \{x, \langle [\zeta^-(x), \zeta^+(x)], [\vartheta^-(x), \vartheta^+(x)] \rangle \mid x \in U\}$ represents the IVIFS over U while $\lambda(x) = \{x, \langle \zeta(x), \vartheta(x) \rangle \mid x \in U\}$ represents an IFS. We denote this pair as $f = (A, \lambda)$, where $A = \langle [\zeta^-, \zeta^+], [\vartheta^-, \vartheta^+] \rangle$ and $\lambda = \langle \zeta, \vartheta \rangle$ and called as CIFN.

Remark 1 The special cases are to be considered from Eq. (4), which are summarized as follows:

(i) A CIFS $\mathcal{A}$ in which $A(x) = \langle [0,0],[1,1] \rangle$ and $\lambda(x) = \langle 1, 0 \rangle$ for all $x \in U$ is denoted by $\ddot{0}$.
(ii) A CIFS $\mathcal{A}$ in which $A(x) = \langle [1,1],[0,0] \rangle$ and $\lambda(x) = \langle 0, 1 \rangle$ for all $x \in U$ is denoted by $\ddot{1}$.
(iii) A CIFS $\mathcal{A}$ in which $A(x) = \langle [0,0],[1,1] \rangle$ and $\lambda(x) = \langle 0, 1 \rangle$ for all $x \in U$ is denoted by $\hat{0}$.
(iv) A CIFS $\mathcal{A}$ in which $A(x) = \langle [1,1],[0,0] \rangle$ and $\lambda(x) = \langle 1, 0 \rangle$ for all $x \in U$ is denoted by $\hat{1}$.

Definition 6 (Kaur and Garg 2018a, c) For any CIFNs $\alpha = \left(\left\langle [\zeta^-, \zeta^+], [\vartheta^-, \vartheta^+] \right\rangle, \left\langle \zeta, \vartheta \right\rangle\right)$ and $\alpha_i = \left(\left\langle [\zeta_i^-, \zeta_i^+], [\vartheta_i^-, \vartheta_i^+] \right\rangle, \left\langle \zeta_i, \vartheta_i \right\rangle\right)$, $i \in \Lambda$, the following operations have been defined as follows:

(i) $\alpha^c = \left(\langle [\vartheta^-, \vartheta^+], [\zeta^-, \zeta^+] \rangle, \langle \vartheta, \zeta \rangle\right)$;
(ii) (Equality) $\alpha_1 = \alpha_2 \Leftrightarrow [\zeta_1^-, \zeta_1^+] = [\zeta_2^-, \zeta_2^+]$, $[\vartheta_1^-, \vartheta_1^+] = [\vartheta_2^-, \vartheta_2^+]$, $\zeta_1 = \zeta_2$ and $\vartheta_1 = \vartheta_2$;
(iii) (R-order) $\alpha_1 \subseteq \alpha_2$ if $[\zeta_1^-, \zeta_1^+] \subseteq [\zeta_2^-, \zeta_2^+]$,$[\vartheta_1^-, \vartheta_1^+] \supseteq [\vartheta_2^-, \vartheta_2^+]$, $\zeta_1 \ge \zeta_2$ and $\vartheta_1 \le \vartheta_2$;
(iv) (R-union): $\bigcup \alpha_i = \left(\left\langle \left[\sup\limits_{i\in\Lambda} \zeta_i^-, \sup\limits_{i\in\Lambda} \zeta_i^+\right], \left[\inf\limits_{i\in\Lambda} \vartheta_i^-, \inf\limits_{i\in\Lambda} \vartheta_i^+\right] \right\rangle, \left\langle \inf\limits_{i\in\Lambda} \zeta_i, \sup\limits_{i\in\Lambda} \vartheta_i \right\rangle\right)$;
(v) (R-intersection): $\bigcap \alpha_i = \left(\left\langle \left[\inf\limits_{i\in\Lambda} \zeta_i^-, \inf\limits_{i\in\Lambda} \zeta_i^+\right], \left[\sup\limits_{i\in\Lambda} \vartheta_i^-, \sup\limits_{i\in\Lambda} \vartheta_i^+\right] \right\rangle, \left\langle \sup\limits_{i\in\Lambda} \zeta_i, \inf\limits_{i\in\Lambda} \vartheta_i \right\rangle\right)$.

3 Proposed distance measures between CIFSs

Let $\Phi(U)$ denote the set of all CIFSs defined on U.

Definition 7 For CIFSs $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$, distance measure is a real-valued function $d : \Phi(U) \times \Phi(U) \to [0, 1]$ satisfying the following properties:

(P1) $0 \le d(\mathcal{A}, \mathcal{B}) \le 1$;
(P2) $d(\mathcal{A}, \mathcal{B}) = 0$ if and only if $\mathcal{A} = \mathcal{B}$;
(P3) $d(\mathcal{A}, \mathcal{B}) = d(\mathcal{B} \mathcal{A})$;
(P4) If $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{C}$, then $d(\mathcal{A}, \mathcal{B}) \le d(\mathcal{A}, \mathcal{C})$ and $d(\mathcal{B}, \mathcal{C}) \le d(\mathcal{A}, \mathcal{C})$.

For CIFSs $\mathcal{A} = (\langle [\zeta_A^-(x), \zeta_A^+(x)], [\vartheta_A^-(x), \vartheta_A^+(x)] \rangle, \langle [\zeta_A(x), \vartheta_A(x)] \rangle)$ and $\mathcal{B} = (\langle [\zeta_B^-(x), \zeta_B^+(x)], [\vartheta_B^-(x), \vartheta_B^+(x)] \rangle, \langle [\zeta_B(x), \vartheta_B(x)] \rangle)$ over the universal set $U = \{x_1, x_2, \ldots, x_n\}$, we define some series of the distance measures as follows:

(i) Hamming distance measure:

$$d_1(\mathcal{A},\mathcal{B}) = \frac{1}{6}\sum_{i=1}^{n}\left(\begin{array}{c} \left|\zeta_A^-(x_i)-\zeta_B^-(x_i)\right| + \left|\zeta_A^+(x_i)-\zeta_B^+(x_i)\right| + \left|\vartheta_A^-(x_i)-\vartheta_B^-(x_i)\right| \\ +\left|\vartheta_A^+(x_i)-\vartheta_B^+(x_i)\right| + \left|\zeta_A(x_i)-\zeta_B(x_i)\right| + \left|\vartheta_A(x_i)-\vartheta_B(x_i)\right| \end{array}\right); \tag{5}$$

(ii) Normalized Hamming distance measure:

$$d_2(\mathcal{A},\mathcal{B}) = \frac{1}{6n}\sum_{i=1}^{n}\left(\begin{array}{c} \left|\zeta_A^-(x_i)-\zeta_B^-(x_i)\right| + \left|\zeta_A^+(x_i)-\zeta_B^+(x_i)\right| + \left|\vartheta_A^-(x_i)-\vartheta_B^-(x_i)\right| \\ +\left|\vartheta_A^+(x_i)-\vartheta_B^+(x_i)\right| + \left|\zeta_A(x_i)-\zeta_B(x_i)\right| + \left|\vartheta_A(x_i)-\vartheta_B(x_i)\right| \end{array}\right); \tag{6}$$

(iii) Euclidean distance measure:

$$d_3(\mathcal{A},\mathcal{B}) = \left(\frac{1}{6}\sum_{i=1}^{n}\left(\begin{array}{c} \left|\zeta_A^-(x_i)-\zeta_B^-(x_i)\right|^2 + \left|\zeta_A^+(x_i)-\zeta_B^+(x_i)\right|^2 + \left|\vartheta_A^-(x_i)-\vartheta_B^-(x_i)\right|^2 \\ +\left|\vartheta_A^+(x_i)-\vartheta_B^+(x_i)\right|^2 + \left|\zeta_A(x_i)-\zeta_B(x_i)\right|^2 + \left|\vartheta_A(x_i)-\vartheta_B(x_i)\right|^2 \end{array}\right);\right)^{1/2} \tag{7}$$

(iv) Normalized Euclidean distance measure:

$$d_4(\mathcal{A},\mathcal{B}) = \left(\frac{1}{6n}\sum_{i=1}^{n}\left(\begin{array}{c} \left|\zeta_A^-(x_i)-\zeta_B^-(x_i)\right|^2 + \left|\zeta_A^+(x_i)-\zeta_B^+(x_i)\right|^2 + \left|\vartheta_A^-(x_i)-\vartheta_B^-(x_i)\right|^2 \\ +\left|\vartheta_A^+(x_i)-\vartheta_B^+(x_i)\right|^2 + \left|\zeta_A(x_i)-\zeta_B(x_i)\right|^2 + \left|\vartheta_A(x_i)-\vartheta_B(x_i)\right|^2 \end{array}\right).\right)^{1/2} \tag{8}$$

From them, the following results are obtained.

Theorem 1 *The distance d_2, between two CIFSs $\mathcal{A}$ and $\mathcal{B}$ satisfies the following properties* (P1)–(P4):

(P1) $0 \le d_2(\mathcal{A},\mathcal{B}) \le 1$;
(P2) $d_2(\mathcal{A},\mathcal{B}) = 0$ *if and only if* $\mathcal{A} = \mathcal{B}$;
(P3) $d_2(\mathcal{A},\mathcal{B}) = d_2(\mathcal{B},\mathcal{A})$;
(P4) *If* $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{C}$, *then* $d_2(\mathcal{A},\mathcal{B}) \le d_2(\mathcal{A},\mathcal{C})$ *and* $d_2(\mathcal{B},\mathcal{C}) \le d_2(\mathcal{A},\mathcal{C})$, *where* $\mathcal{C} \in \Phi(U)$.

Proof For a collection of CIFSs $\mathcal{A}$ and $\mathcal{B}$, we have

(P1) By definition of d_2, we have $d_2(\mathcal{A},\mathcal{B}) \ge 0$, so for arbitrary CIFSs $\mathcal{A}$ and $\mathcal{B}$, it is enough to show that $d_2(\mathcal{A},\mathcal{B}) \le 1$. For two CIFSs $\mathcal{A}$ and $\mathcal{B}$, we get $0 \le \zeta_A^-(x_i), \zeta_A^+(x_i), \vartheta_A^-(x_i), \vartheta_A^+(x_i) \le 1$, $0 \le \zeta_A(x_i), \vartheta_A(x_i) \le 1$ and $0 \le \zeta_B^-(x_i), \zeta_B^+(x_i) \le 1$, $0 \le \vartheta_B^-(x_i), \vartheta_B^+(x_i) \le 1$ and $0 \le \zeta_B(x_i), \vartheta_B(x_i) \le 1$. This implies that $0 \le \left|\zeta_A^-(x_i)-\zeta_B^-(x_i)\right| \le 1$, $0 \le \left|\zeta_A^+(x_i)-\zeta_B^+(x_i)\right| \le 1$, $0 \le \left|\vartheta_A^-(x_i)-\vartheta_B^-(x_i)\right| \le 1$, $0 \le \left|\vartheta_A^+(x_i)-\vartheta_B^+(x_i)\right| \le 1$, and $0 \le \left|\zeta_A(x_i)-\zeta_B(x_i)\right| \le 1$, $0 \le \left|\vartheta_A(x_i)-\vartheta_B(x_i)\right| \le 1$. Therefore,

$$0 \le \left(\begin{array}{c} \left|\zeta_A^-(x_i)-\zeta_B^-(x_i)\right| + \left|\zeta_A^+(x_i)-\zeta_B^+(x_i)\right| + \left|\vartheta_A^-(x_i)-\vartheta_B^-(x_i)\right| \\ +\left|\vartheta_A^+(x_i)-\vartheta_B^+(x_i)\right| + \left|\zeta_A(x_i)-\zeta_B(x_i)\right| + \left|\vartheta_A(x_i)-\vartheta_B(x_i)\right| \end{array}\right) \le 6$$

which implies that $d_2(\mathcal{A},\mathcal{B}) \le 1$.

(P2) For two CIFSs $\mathcal{A}$ and $\mathcal{B}$, assume that $d_2(\mathcal{A},\mathcal{B}) = 0$ which implies that

$$\frac{1}{6n}\sum_{i=1}^{n}\left(\begin{array}{c} \left|\zeta_A^-(x_i)-\zeta_B^-(x_i)\right| + \left|\zeta_A^+(x_i)-\zeta_B^+(x_i)\right| + \left|\vartheta_A^-(x_i)-\vartheta_B^-(x_i)\right| \\ +\left|\vartheta_A^+(x_i)-\vartheta_B^+(x_i)\right| + \left|\zeta_A(x_i)-\zeta_B(x_i)\right| + \left|\vartheta_A(x_i)-\vartheta_B(x_i)\right| \end{array}\right) = 0$$

if and only if, for all i,

$$|\zeta_A^-(x_i)-\zeta_B^-(x_i)|=0, |\zeta_A^+(x_i)-\zeta_B^+(x_i)|=0, |\vartheta_A^-(x_i)-\vartheta_B^-(x_i)|=0,$$
$$|\vartheta_A^+(x_i)-\vartheta_B^+(x_i)|=0, |\zeta_A(x_i)-\zeta_B(x_i)|=0, |\vartheta_A(x_i)-\vartheta_B(x_i)|=0$$

which is equivalent to

$$\zeta_A^-(x_i)=\zeta_B^-(x_i), \zeta_A^+(x_i)=\zeta_B^+(x_i), \vartheta_A^-(x_i)=\vartheta_B^-(x_i),$$
$$\vartheta_A^+(x_i)=\vartheta_B^+(x_i), \zeta_A(x_i)=\zeta_B(x_i), \vartheta_A(x_i)=\vartheta_B(x_i)$$

Thus, $d_2(\mathcal{A},\mathcal{B})=0$ implies that $\mathcal{A}=\mathcal{B}$.

(P3) For two CIFSs $\mathcal{A}$ and $\mathcal{B}$,

$$\begin{aligned} d_2(\mathcal{A},\mathcal{B}) &= \frac{1}{6n}\sum_{i=1}^{n}\begin{pmatrix} |\zeta_A^-(x_i)-\zeta_B^-(x_i)| + |\zeta_A^+(x_i)-\zeta_B^+(x_i)| + |\vartheta_A^-(x_i)-\vartheta_B^-(x_i)| \\ +|\vartheta_A^+(x_i)-\vartheta_B^+(x_i)| + |\zeta_A(x_i)-\zeta_B(x_i)| + |\vartheta_A(x_i)-\vartheta_B(x_i)| \end{pmatrix} \\ &= \frac{1}{6n}\sum_{i=1}^{n}\begin{pmatrix} |\zeta_B^-(x_i)-\zeta_A^-(x_i)| + |\zeta_B^+(x_i)-\zeta_A^+(x_i)| + |\vartheta_B^-(x_i)-\vartheta_A^-(x_i)| \\ +|\vartheta_B^+(x_i)-\vartheta_A^+(x_i)| + |\zeta_B(x_i)-\zeta_A(x_i)| + |\vartheta_B(x_i)-\vartheta_A(x_i)| \end{pmatrix} \\ &= d_2(\mathcal{B},\mathcal{A}). \end{aligned}$$

Hence, $d_2(\mathcal{A},\mathcal{B})=d_2(\mathcal{B},\mathcal{A})$.

(P4) If $\mathcal{A}\subseteq\mathcal{B}\subseteq\mathcal{C}$, then $[\zeta_A^-(x_i),\zeta_A^+(x_i)]\subseteq[\zeta_B^-(x_i),\zeta_B^+(x_i)]\subseteq[\zeta_C^-(x_i),\zeta_C^+(x_i)]$, $[\vartheta_A^-(x_i),\vartheta_A^+(x_i)]\supseteq[\vartheta_B^-(x_i),\vartheta_B^+(x_i)]\supseteq[\vartheta_C^-(x_i),\vartheta_C^+(x_i)]$. Also, $\zeta_A(x_i)\geq\zeta_B(x_i)\geq\zeta_C(x_i)$ and $\vartheta_A(x_i)\leq\vartheta_B(x_i)\leq\vartheta_C(x_i)$. Therefore, $|\zeta_A^-(x_i)-\zeta_B^-(x_i)|\leq|\zeta_A^-(x_i)-\zeta_C^-(x_i)|$, $|\zeta_A^+(x_i)-\zeta_B^+(x_i)|\leq|\zeta_A^+(x_i)-\zeta_C^+(x_i)|$, $|\vartheta_A^-(x_i)-\vartheta_B^-(x_i)|\leq|\vartheta_A^-(x_i)-\vartheta_C^-(x_i)|$, $|\vartheta_A^+(x_i)-\vartheta_B^+(x_i)|\leq|\vartheta_A^+(x_i)-\vartheta_C^+(x_i)|$, $|\zeta_A(x_i)-\zeta_B(x_i)|\leq|\zeta_A(x_i)-\zeta_C(x_i)|$, and $|\vartheta_A(x_i)-\vartheta_B(x_i)|\leq|\vartheta_A(x_i)-\vartheta_C(x_i)|$. Thus,

$$\begin{aligned} d_2(\mathcal{A},\mathcal{C}) &= \frac{1}{6n}\sum_{i=1}^{n}\begin{pmatrix} |\zeta_A^-(x_i)-\zeta_C^-(x_i)| + |\zeta_A^+(x_i)-\zeta_C^+(x_i)| + |\vartheta_A^-(x_i)-\vartheta_C^-(x_i)| \\ +|\vartheta_A^+(x_i)-\vartheta_C^+(x_i)| + |\zeta_A(x_i)-\zeta_C(x_i)| + |\vartheta_A(x_i)-\vartheta_C(x_i)| \end{pmatrix} \\ &\geq \frac{1}{6n}\sum_{i=1}^{n}\begin{pmatrix} |\zeta_A^-(x_i)-\zeta_B^-(x_i)| + |\zeta_A^+(x_i)-\zeta_B^+(x_i)| + |\vartheta_A^-(x_i)-\vartheta_B^-(x_i)| \\ +|\vartheta_A^+(x_i)-\vartheta_B^+(x_i)| + |\zeta_A(x_i)-\zeta_B(x_i)| + |\vartheta_A(x_i)-\vartheta_B(x_i)| \end{pmatrix} \\ &\geq d_2(\mathcal{A},\mathcal{B}). \end{aligned}$$

Similarly, $d_2(\mathcal{A},\mathcal{C})\geq d_2(\mathcal{B},\mathcal{C})$.
Hence, d_2 is a valid distance measure.

Theorem 2 *For two CIFSs $\mathcal{A}$ and $\mathcal{B}$, the distance measure $d_4(\mathcal{A},\mathcal{B})$ satisfies the properties* (P1)–(P4) *as described in Definition 7.*

Proof For CIFSs $\mathcal{A}$ and $\mathcal{B}$, we have

(P1) Since $0\leq\zeta_A^-(x_i),\zeta_A^+(x_i),\vartheta_A^-(x_i),\vartheta_A^+(x_i)\leq 1$, $0\leq\zeta_A(x_i),\vartheta_A(x_i)\leq 1$, $0\leq\zeta_B^-(x_i)$, $\zeta_B^+(x_i)$, $\vartheta_B^-(x_i)$, $\vartheta_B^+(x_i)\leq 1$ and $0\leq\zeta_B(x_i),\vartheta_B(x_i)\leq 1$. This implies that $0\leq|\zeta_A^-(x_i)-\zeta_B^-(x_i)|^2\leq 1$, $0\leq|\zeta_A^+(x_i)-\zeta_B^+(x_i)|^2\leq 1$, $0\leq|\vartheta_A^-(x_i)-\vartheta_B^-(x_i)|^2\leq 1$, $0\leq|\vartheta_A^+(x_i)-\vartheta_B^+(x_i)|^2\leq 1$, $0\leq|\zeta_A(x_i)-\zeta_B(x_i)|^2\leq 1$, $0\leq|\vartheta_A(x_i)-\vartheta_B(x_i)|^2\leq 1$. Thus, it follows that $0\leq d_4(\mathcal{A},\mathcal{B})\leq 1$.

(P2) For two CIFSs $\mathcal{A}$ and $\mathcal{B}$, assume that $d_4(\mathcal{A},\mathcal{B})=0$ which implies that

$$\begin{aligned} &\left(\frac{1}{6n}\sum_{i=1}^{n}\begin{pmatrix} |\zeta_A^-(x_i)-\zeta_B^-(x_i)|^2 + |\zeta_A^+(x_i)-\zeta_B^+(x_i)|^2 + |\vartheta_A^-(x_i)-\vartheta_B^-(x_i)|^2 \\ +|\vartheta_A^+(x_i)-\vartheta_B^+(x_i)|^2 + |\zeta_A(x_i)-\zeta_B(x_i)|^2 + |\vartheta_A(x_i)-\vartheta_B(x_i)|^2 \end{pmatrix}\right)^{1/2} = 0 \\ &\Leftrightarrow |\zeta_A^-(x_i)-\zeta_B^-(x_i)|=0, |\zeta_A^+(x_i)-\zeta_B^+(x_i)|=0, |\vartheta_A^-(x_i)-\vartheta_B^-(x_i)|=0, \\ &\quad |\vartheta_A^+(x_i)-\vartheta_B^+(x_i)|=0, |\zeta_A(x_i)-\zeta_B(x_i)|=0, |\vartheta_A(x_i)-\vartheta_B(x_i)|=0 \\ &\Leftrightarrow \zeta_A^-(x_i)=\zeta_B^-(x_i), \zeta_A^+(x_i)=\zeta_B^+(x_i), \vartheta_A^-(x_i)=\vartheta_B^-(x_i), \\ &\quad \vartheta_A^+(x_i)=\vartheta_B^+(x_i), \zeta_A(x_i)=\zeta_B(x_i), \vartheta_A(x_i)=\vartheta_B(x_i) \quad \text{for all } i \\ &\Leftrightarrow A=B \end{aligned}$$

(P3) For any two numbers a and b, we have $|a-b| = |b-a|$. Thus, we have $d_4(\mathcal{A},\mathcal{B}) = d_4(\mathcal{B},\mathcal{A})$.

(P4) If $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{C}$, then $[\zeta_A^-(x_i), \zeta_A^+(x_i)] \subseteq [\zeta_B^-(x_i), \zeta_B^+(x_i)] \subseteq [\zeta_C^-(x_i), \zeta_C^+(x_i)]$, $[\vartheta_A^-(x_i), \vartheta_A^+(x_i)] \supseteq [\vartheta_B^-(x_i), \vartheta_B^+(x_i)] \supseteq [\vartheta_C^-(x_i), \vartheta_C^+(x_i)]$. Also, $\zeta_A(x_i) \geq \zeta_B(x_i) \geq \zeta_C(x_i)$ and $\vartheta_A(x_i) \leq \vartheta_B(x_i) \leq \vartheta_C(x_i)$. Therefore,

$$
\begin{aligned}
&\left|\zeta_A^-(x_i) - \zeta_B^-(x_i)\right|^2 \leq \left|\zeta_A^-(x_i) - \zeta_C^-(x_i)\right|^2, \left|\zeta_A^+(x_i) - \zeta_B^+(x_i)\right|^2 \leq \left|\zeta_A^+(x_i) - \zeta_C^+(x_i)\right|^2,\\
&\left|\vartheta_A^-(x_i) - \vartheta_B^-(x_i)\right|^2 \leq \left|\vartheta_A^-(x_i) - \vartheta_C^-(x_i)\right|^2, \left|\vartheta_A^+(x_i) - \vartheta_B^+(x_i)\right|^2 \leq \left|\vartheta_A^+(x_i) - \vartheta_C^+(x_i)\right|^2,\\
&\left|\zeta_A(x_i) - \zeta_B(x_i)\right|^2 \leq \left|\zeta_A(x_i) - \zeta_C(x_i)\right|^2, \left|\vartheta_A(x_i) - \vartheta_B(x_i)\right|^2 \leq \left|\vartheta_A(x_i) - \vartheta_C(x_i)\right|^2.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
d_4(\mathcal{A},\mathcal{C}) &= \left(\frac{1}{6n}\sum_{i=1}^{n}\left(\begin{array}{c}\left|\zeta_A^-(x_i) - \zeta_C^-(x_i)\right|^2 + \left|\zeta_A^+(x_i) - \zeta_C^+(x_i)\right|^2 + \left|\vartheta_A^-(x_i) - \vartheta_C^-(x_i)\right|^2\\ + \left|\vartheta_A^+(x_i) - \vartheta_C^+(x_i)\right|^2 + \left|\zeta_A(x_i) - \zeta_C(x_i)\right|^2 + \left|\vartheta_A(x_i) - \vartheta_C(x_i)\right|^2\end{array}\right)\right)^{1/2}\\
&\geq \left(\frac{1}{6n}\sum_{i=1}^{n}\left(\begin{array}{c}\left|\zeta_A^-(x_i) - \zeta_B^-(x_i)\right|^2 + \left|\zeta_A^+(x_i) - \zeta_B^+(x_i)\right|^2 + \left|\vartheta_A^-(x_i) - \vartheta_B^-(x_i)\right|^2\\ + \left|\vartheta_A^+(x_i) - \vartheta_B^+(x_i)\right|^2 + \left|\zeta_A(x_i) - \zeta_B(x_i)\right|^2 + \left|\vartheta_A(x_i) - \vartheta_B(x_i)\right|^2\end{array}\right)\right)^{1/2}\\
&= d_4(\mathcal{A},\mathcal{B}).
\end{aligned}
$$

Similarly, $d_4(\mathcal{A},\mathcal{C}) \geq d_4(\mathcal{B},\mathcal{C})$. Hence, d_4 is a valid distance measure.

Example 1 Let $\mathcal{A}$ and $\mathcal{B}$ be two known pattern expressed as $\mathcal{A} = \{(x_1, (\langle[0.2, 0.4], [0.3, 0.5]\rangle, \langle 0.3, 0.4\rangle)), (x_2, (\langle[0.3, 0.4], [0.2, 0.6]\rangle, \langle 0.1, 0.3\rangle))\}$ and $\mathcal{B} = \{(x_1, (\langle[0.3, 0.4], [0.1, 0.2]\rangle, \langle 0.25, 0.40\rangle)), (x_2, (\langle[0.15, 0.25], [0.30, 0.45]\rangle, \langle 0.4, 0.2\rangle))\}$, then

$$
\begin{aligned}
d_2(\mathcal{A},\mathcal{B}) &= \frac{1}{6\times 2}\left(\begin{array}{c}|0.20-0.30| + |0.40-0.40| + |0.30-0.10| + |0.50-0.20|\\ +|0.30-0.25| + |0.40-0.40| + |0.30-0.15| + |0.40-0.25|\\ +|0.20-0.30| + |0.60-0.45| + |0.10-0.40| + |0.30-0.20|\end{array}\right)\\
&= 0.1333
\end{aligned}
$$

and

$$
\begin{aligned}
d_4(\mathcal{A},\mathcal{B}) &= \left(\frac{1}{6\times 2}\left(\begin{array}{c}|0.20-0.30|^2 + |0.40-0.40|^2 + |0.30-0.10|^2 + |0.50-0.20|^2\\ +|0.30-0.25|^2 + |0.40-0.40|^2 + |0.30-0.15|^2 + |0.40-0.25|^2\\ +|0.20-0.30|^2 + |0.60-0.45|^2 + |0.10-0.40|^2 + |0.30-0.20|^2\end{array}\right)\right)^{1/2}\\
&= 0.1633
\end{aligned}
$$

Theorem 3 *The measures d_1 and d_3 satisfies the following inequalities:*

(i) $0 \leq d_1 \leq n$,
(ii) $0 \leq d_3 \leq \sqrt{n}$.

Proof Let $\mathcal{A}$ and $\mathcal{B}$ be any two CIFSs. Clearly, $d_1(\mathcal{A},\mathcal{B}) = nd_2(\mathcal{A},\mathcal{B})$. By Theorem 1, we have $0 \leq d_2(\mathcal{A},\mathcal{B}) \leq 1$. It implies that $0 \leq \frac{d_1(\mathcal{A},\mathcal{B})}{n} \leq 1$ and hence $0 \leq d_1(\mathcal{A},\mathcal{B}) \leq n$. Similarly, we can prove that $0 \leq d_3(\mathcal{A},\mathcal{B}) \leq \sqrt{n}$. As $\mathcal{A}$ and $\mathcal{B}$ are arbitrary, so we get $0 \leq d_1 \leq n$ and $0 \leq d_3 \leq \sqrt{n}$.

Theorem 4 *The measures d_1, d_3 and d_2, d_4 satisfy the following inequalities:*

(i) $d_3 \leq \sqrt{d_1}$
(ii) $d_4 \leq \sqrt{d_2}$

Proof Let $\mathcal{A}$ and $\mathcal{B}$ be two CIFSs.

(i) Since $0 \leq \zeta_A^-(x_i), \zeta_A^+(x_i), \vartheta_A^-(x_i), \vartheta_A^+(x_i) \leq 1$, $0 \leq \zeta_A(x_i), \vartheta_A(x_i) \leq 1$, $0 \leq \zeta_B^-(x_i), \zeta_B^+(x_i), \vartheta_B^-(x_i), \vartheta_B^+(x_i) \leq 1$ and $0 \leq \zeta_B(x_i), \vartheta_B(x_i) \leq 1$. Then, by using the property, $\eta^2 < \eta$, for any $\eta \in [0, 1]$,

we have $\left|\zeta_A^-(x_i) - \zeta_B^-(x_i)\right|^2 \le \left|\zeta_A^-(x_i) - \zeta_B^-(x_i)\right|$,
$\left|\zeta_A^+(x_i) - \zeta_B^+(x_i)\right|^2 \le \left|\zeta_A^+(x_i) - \zeta_B^+(x_i)\right|$,
$\left|\vartheta_A^-(x_i) - \vartheta_B^-(x_i)\right|^2 \le \left|\vartheta_A^-(x_i) - \vartheta_B^-(x_i)\right|$,
$\left|\vartheta_A^+(x_i) - \vartheta_B^+(x_i)\right|^2 \le \left|\vartheta_A^+(x_i) - \vartheta_B^+(x_i)\right|$,$\left|\zeta_A(x_i) - \zeta_B(x_i)\right|^2 \le \left|\zeta_A(x_i) - \zeta_B(x_i)\right|$ and $\left|\vartheta_A(x_i) - \vartheta_B(x_i)\right|^2 \le \left|\vartheta_A(x_i) - \vartheta_B(x_i)\right|$, for all i. This implies that,

$$\begin{pmatrix} \left|\zeta_A^-(x_i) - \zeta_B^-(x_i)\right|^2 + \left|\zeta_A^+(x_i) - \zeta_B^+(x_i)\right|^2 + \left|\vartheta_A^-(x_i) - \vartheta_B^-(x_i)\right|^2 \\ +\left|\vartheta_A^+(x_i) - \vartheta_B^+(x_i)\right|^2 + \left|\zeta_A(x_i) - \zeta_B(x_i)\right|^2 + \left|\vartheta_A(x_i) - \vartheta_B(x_i)\right|^2 \end{pmatrix} \le \begin{pmatrix} \left|\zeta_A^-(x_i) - \zeta_B^-(x_i)\right| + \left|\zeta_A^+(x_i) - \zeta_B^+(x_i)\right| + \left|\vartheta_A^-(x_i) - \vartheta_B^-(x_i)\right| \\ +\left|\vartheta_A^+(x_i) - \vartheta_B^+(x_i)\right| + \left|\zeta_A(x_i) - \zeta_B(x_i)\right| + \left|\vartheta_A(x_i) - \vartheta_B(x_i)\right| \end{pmatrix}.$$

Therefore, for all i, we have

$$\frac{1}{6}\sum_{i=1}^{n}\begin{pmatrix} \left|\zeta_A^-(x_i) - \zeta_B^-(x_i)\right|^2 + \left|\zeta_A^+(x_i) - \zeta_B^+(x_i)\right|^2 + \left|\vartheta_A^-(x_i) - \vartheta_B^-(x_i)\right|^2 \\ +\left|\vartheta_A^+(x_i) - \vartheta_B^+(x_i)\right|^2 + \left|\zeta_A(x_i) - \zeta_B(x_i)\right|^2 + \left|\vartheta_A(x_i) - \vartheta_B(x_i)\right|^2 \end{pmatrix} \le \frac{1}{6}\sum_{i=1}^{n}\begin{pmatrix} \left|\zeta_A^-(x_i) - \zeta_B^-(x_i)\right| + \left|\zeta_A^+(x_i) - \zeta_B^+(x_i)\right| + \left|\vartheta_A^-(x_i) - \vartheta_B^-(x_i)\right| \\ +\left|\vartheta_A^+(x_i) - \vartheta_B^+(x_i)\right| + \left|\zeta_A(x_i) - \zeta_B(x_i)\right| + \left|\vartheta_A(x_i) - \vartheta_B(x_i)\right| \end{pmatrix},$$

which implies that $d_3(\mathcal{A}, \mathcal{B}) \le \sqrt{d_1(\mathcal{A}, \mathcal{B})}$. Since, $\mathcal{A}$ and $\mathcal{B}$ are arbitrary CIFSs thus, $d_3 \le \sqrt{d_1}$ is true for all CIFSs.

(ii) This can be proceeded in the similar manner as above.

As in practical situations, many times we have to deal with such situations in which various CIFSs may have weights assigned to them. So, taking into account weights $\kappa_i (i = 1, 2, \ldots, n)$, where each $\kappa_i > 0$ and $\sum_{i=1}^n \kappa_i = 1$, we define weighted Hamming as well as weighted Euclidean distances between two CIFSs $\mathcal{A}$ and $\mathcal{B}$ as follows:

(i) Weighted Hamming distance measure:

$$d_5(\mathcal{A}, \mathcal{B}) = \frac{1}{6}\sum_{i=1}^{n}\kappa_i\begin{pmatrix} \left|\zeta_A^-(x_i) - \zeta_B^-(x_i)\right| + \left|\zeta_A^+(x_i) - \zeta_B^+(x_i)\right| + \left|\vartheta_A^-(x_i) - \vartheta_B^-(x_i)\right| \\ +\left|\vartheta_A^+(x_i) - \vartheta_B^+(x_i)\right| + \left|\zeta_A(x_i) - \zeta_B(x_i)\right| + \left|\vartheta_A(x_i) - \vartheta_B(x_i)\right| \end{pmatrix} \tag{9}$$

(ii) Weighted Euclidean distance measure:

$$d_6(\mathcal{A}, \mathcal{B}) = \left(\frac{1}{6}\sum_{i=1}^{n}\kappa_i\begin{pmatrix} \left|\zeta_A^-(x_i) - \zeta_B^-(x_i)\right|^2 + \left|\zeta_A^+(x_i) - \zeta_B^+(x_i)\right|^2 + \left|\vartheta_A^-(x_i) - \vartheta_B^-(x_i)\right|^2 \\ +\left|\vartheta_A^+(x_i) - \vartheta_B^+(x_i)\right|^2 + \left|\zeta_A(x_i) - \zeta_B(x_i)\right|^2 + \left|\vartheta_A(x_i) - \vartheta_B(x_i)\right|^2 \end{pmatrix}\right)^{1/2} \tag{10}$$

Especially, when $\kappa_i = 1/n$, for $i = 1, 2, \ldots, n$, then distances d_5 and d_6 reduces to d_2 and d_4, respectively.

Theorem 5 *The weighted measures $d_k(\mathcal{A}, \mathcal{B}), (k = 5, 6)$ satisfy the following properties for $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \Phi(U)$:*

(P1) $0 \le d_k(\mathcal{A}, \mathcal{B}) \le 1$;
(P2) $d_k(\mathcal{A}, \mathcal{B}) = 0$ *if and only if* $\mathcal{A} = \mathcal{B}$;
(P3) $d_k(\mathcal{A}, \mathcal{B}) = d_k(\mathcal{B}, \mathcal{A})$.
(P4) *If* $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{C}$ *then* $d_k(\mathcal{A}, \mathcal{B}) \le d_k(\mathcal{A}, \mathcal{C})$ *and* $d_k(\mathcal{B}, \mathcal{C}) \le d_k(\mathcal{A}, \mathcal{C})$.

Proof Since $\kappa_i > 0$ and $\sum_{i=1}^n \kappa_i = 1$, then we can easily get the proof of above theorem. Hence, we omit it here.

Theorem 6 *The measures d_5 and d_1 satisfy the inequality $d_5 \le d_1$.*

Proof Since $\kappa_i > 0$ and $\sum_{i=1}^n \kappa_i = 1$, then for any two CIFSs $\mathcal{A}$ and $\mathcal{B}$, we have

$$\begin{aligned} d_5(\mathcal{A}, \mathcal{B}) &= \frac{1}{6}\sum_{i=1}^{n}\kappa_i\begin{pmatrix} \left|\zeta_A^-(x_i) - \zeta_B^-(x_i)\right| + \left|\zeta_A^+(x_i) - \zeta_B^+(x_i)\right| + \left|\vartheta_A^-(x_i) - \vartheta_B^-(x_i)\right| \\ +\left|\vartheta_A^+(x_i) - \vartheta_B^+(x_i)\right| + \left|\zeta_A(x_i) - \zeta_B(x_i)\right| + \left|\vartheta_A(x_i) - \vartheta_B(x_i)\right| \end{pmatrix} \\ &\le \frac{1}{6}\sum_{i=1}^{n}\begin{pmatrix} \left|\zeta_A^-(x_i) - \zeta_B^-(x_i)\right| + \left|\zeta_A^+(x_i) - \zeta_B^+(x_i)\right| + \left|\vartheta_A^-(x_i) - \vartheta_B^-(x_i)\right| \\ +\left|\vartheta_A^+(x_i) - \vartheta_B^+(x_i)\right| + \left|\zeta_A(x_i) - \zeta_B(x_i)\right| + \left|\vartheta_A(x_i) - \vartheta_B(x_i)\right| \end{pmatrix} \\ &= d_1(\mathcal{A}, \mathcal{B}). \end{aligned}$$

Since, $\mathcal{A}$ and $\mathcal{B}$ are arbitrary, therefore we obtain $d_5 \le d_1$.

Theorem 7 *The measures d_6 and d_3 satisfy the inequality $d_6 \le d_3$.*

Proof Similar to Theorem 6, so we omit here.

Theorem 8 *The measures d_6 and d_1 satisfy the inequality $d_6 \le \sqrt{d_1}$.*

Proof Since, $0 \le \zeta_A^-(x_i), \zeta_A^+(x_i), \vartheta_A^-(x_i), \vartheta_A^+(x_i) \le 1$ and $0 \le \zeta_B^-(x_i),\ \zeta_B^+(x_i),\ \vartheta_B^-(x_i),\ \vartheta_B^+(x_i) \le 1$. and hence $0 \le \left|\zeta_A^-(x_i) - \zeta_B^-(x_i)\right| \le 1$. Now for any real number $\eta \in [0,1]$, we know that $|\eta|^2 \le |\eta|$. Thus, it follows that $\left|\zeta_A^-(x_i) - \zeta_B^-(x_i)\right|^2 \le \left|\zeta_A^-(x_i) - \zeta_B^-(x_i)\right|$. Similarly, $\left|\zeta_A^+(x_i) - \zeta_B^+(x_i)\right|^2 \le \left|\zeta_A^+(x_i) - \zeta_B^+(x_i)\right|$, $\left|\vartheta_A^+(x_i) - \vartheta_B^+(x_i)\right|^2 \le \left|\vartheta_A^+(x_i) - \vartheta_B^+(x_i)\right|$, $\left|\zeta_A(x_i) - \zeta_B(x_i)\right|^2 \le \left|\zeta_A(x_i) - \zeta_B(x_i)\right|$ and $\left|\vartheta_A(x_i) - \vartheta_B(x_i)\right|^2 \le \left|\vartheta_A(x_i) - \vartheta_B(x_i)\right|$. Now, for $\kappa_i > 0$ such that $\sum_{i=1}^n \kappa_i = 1$, we have

$$\begin{aligned} d_6(\mathcal{A},\mathcal{B}) &= \left(\frac{1}{6}\sum_{i=1}^n \kappa_i\left(\begin{array}{l} \left|\zeta_A^-(x_i) - \zeta_B^-(x_i)\right|^2 + \left|\zeta_A^+(x_i) - \zeta_B^+(x_i)\right|^2 + \left|\vartheta_A^-(x_i) - \vartheta_B^-(x_i)\right|^2 \\ + \left|\vartheta_A^+(x_i) - \vartheta_B^+(x_i)\right|^2 + \left|\zeta_A(x_i) - \zeta_B(x_i)\right|^2 + \left|\vartheta_A(x_i) - \vartheta_B(x_i)\right|^2 \end{array}\right)\right)^{1/2} \\ &\le \left(\frac{1}{6}\sum_{i=1}^n \left(\begin{array}{l} \left|\zeta_A^-(x_i) - \zeta_B^-(x_i)\right| + \left|\zeta_A^+(x_i) - \zeta_B^+(x_i)\right| + \left|\vartheta_A^-(x_i) - \vartheta_B^-(x_i)\right| \\ + \left|\vartheta_A^+(x_i) - \vartheta_B^+(x_i)\right| + \left|\zeta_A(x_i) - \zeta_B(x_i)\right| + \left|\vartheta_A(x_i) - \vartheta_B(x_i)\right| \end{array}\right)\right)^{1/2} \\ &\le \sqrt{d_1(\mathcal{A},\mathcal{B})} \end{aligned}$$

As $\mathcal{A}$ and $\mathcal{B}$ are arbitrary, so we get $d_6 \le \sqrt{d_1}$.

Theorem 9 *The measures d_6 and d_5 have the inequality $d_6 \le \sqrt{d_5}$.*

Proof Similar to Theorem 8.

Next, we propose Hausdorff measures for two CIFSs $\mathcal{A}$ and $\mathcal{B}$ as follows:

(i) Hausdorff hamming distance measure:

$$d_1^H(\mathcal{A},\mathcal{B}) = \frac{1}{6}\sum_{i=1}^n\left(\max\left(\begin{array}{l} \left|\zeta_A^-(x_i) - \zeta_B^-(x_i)\right|, \left|\zeta_A^+(x_i) - \zeta_B^+(x_i)\right|, \left|\vartheta_A^-(x_i) - \vartheta_B^-(x_i)\right|, \\ \left|\vartheta_A^+(x_i) - \vartheta_B^+(x_i)\right|, \left|\zeta_A(x_i) - \zeta_B(x_i)\right|, \left|\vartheta_A(x_i) - \vartheta_B(x_i)\right| \end{array}\right)\right); \tag{11}$$

(ii) Hausdorff normalized hamming distance measure:

$$d_2^H(\mathcal{A},\mathcal{B}) = \frac{1}{6n}\sum_{i=1}^n\left(\max\left(\begin{array}{l} \left|\zeta_A^-(x_i) - \zeta_B^-(x_i)\right|, \left|\zeta_A^+(x_i) - \zeta_B^+(x_i)\right|, \left|\vartheta_A^-(x_i) - \vartheta_B^-(x_i)\right|, \\ \left|\vartheta_A^+(x_i) - \vartheta_B^+(x_i)\right|, \left|\zeta_A(x_i) - \zeta_B(x_i)\right|, \left|\vartheta_A(x_i) - \vartheta_B(x_i)\right| \end{array}\right)\right); \tag{12}$$

(iii) Hausdorff Euclidean distance measure:

$$d_3^H(\mathcal{A},\mathcal{B}) = \left(\frac{1}{6}\sum_{i=1}^n\left(\max\left(\begin{array}{l} \left|\zeta_A^-(x_i) - \zeta_B^-(x_i)\right|^2, \left|\zeta_A^+(x_i) - \zeta_B^+(x_i)\right|^2, \left|\vartheta_A^-(x_i) - \vartheta_B^-(x_i)\right|^2, \\ \left|\vartheta_A^+(x_i) - \vartheta_B^+(x_i)\right|^2, \left|\zeta_A(x_i) - \zeta_B(x_i)\right|^2, \left|\vartheta_A(x_i) - \vartheta_B(x_i)\right|^2 \end{array}\right)\right)\right)^{1/2}; \tag{13}$$

(iv) Hausdorff normalized Euclidean distance measure:

$$d_4^H(\mathcal{A},\mathcal{B}) = \left(\frac{1}{6n}\sum_{i=1}^n\left(\max\left(\begin{array}{l} \left|\zeta_A^-(x_i) - \zeta_B^-(x_i)\right|^2, \left|\zeta_A^+(x_i) - \zeta_B^+(x_i)\right|^2, \left|\vartheta_A^-(x_i) - \vartheta_B^-(x_i)\right|^2, \\ \left|\vartheta_A^+(x_i) - \vartheta_B^+(x_i)\right|^2, \left|\zeta_A(x_i) - \zeta_B(x_i)\right|^2, \left|\vartheta_A(x_i) - \vartheta_B(x_i)\right|^2 \end{array}\right)\right)\right)^{1/2}; \tag{14}$$

(v) Hausdorff weighted hamming distance measure:

$$d_5^H(\mathcal{A},\mathcal{B})=\left(\frac{1}{6}\sum_{i=1}^{n}\kappa_i\max\begin{pmatrix}\left|\zeta_A^-(x_i)-\zeta_B^-(x_i)\right|,\left|\zeta_A^+(x_i)-\zeta_B^+(x_i)\right|,\left|\vartheta_A^-(x_i)-\vartheta_B^-(x_i)\right|,\\ \left|\vartheta_A^+(x_i)-\vartheta_B^+(x_i)\right|,\left|\zeta_A(x_i)-\zeta_B(x_i)\right|,\left|\vartheta_A(x_i)-\vartheta_B(x_i)\right|\end{pmatrix}\right); \tag{15}$$

(vi) Hausdorff weighted Euclidean distance measure:

$$d_6^H(\mathcal{A},\mathcal{B})=\left(\frac{1}{6}\sum_{i=1}^{n}\kappa_i\max\begin{pmatrix}\left|\zeta_A^-(x_i)-\zeta_B^-(x_i)\right|^2,\left|\zeta_A^+(x_i)-\zeta_B^+(x_i)\right|^2,\left|\vartheta_A^-(x_i)-\vartheta_B^-(x_i)\right|^2,\\ \left|\vartheta_A^+(x_i)-\vartheta_B^+(x_i)\right|^2,\left|\zeta_A(x_i)-\zeta_B(x_i)\right|^2,\left|\vartheta_A(x_i)-\vartheta_B(x_i)\right|^2\end{pmatrix}\right)^{1/2}. \tag{16}$$

Theorem 10 *For $\mathcal{A},\mathcal{B},\mathcal{C}\in\Phi(U)$, the measures $d_k^H(\mathcal{A},\mathcal{B}),(k=2,4)$ satisfies the following properties.*

(P1) $0\le d_k^H(\mathcal{A},\mathcal{B})\le 1$;
(P2) $d_k^H(\mathcal{A},\mathcal{B})=0$ *if and only if* $A=B$;
(P3) $d_k^H(\mathcal{A},\mathcal{B})=d_k^H(\mathcal{B},\mathcal{A})$;
(P4) *If* $\mathcal{A}\subseteq\mathcal{B}\subseteq\mathcal{C}$ *then* $d_k^H(\mathcal{A},\mathcal{B})\le d_k^H(\mathcal{A},\mathcal{C})$ *and* $d_k^H(\mathcal{B},\mathcal{C})\le d_k^H(\mathcal{A},\mathcal{C})$.

Proof Let $\mathcal{A},\mathcal{B},\mathcal{C}\in\Phi(U)$ be any three CIFSs. Then, for $q=1,2$, we have

(P1) Since $\mathcal{A}$ and $\mathcal{B}$ are two CIFSs, so by the definitions of CIFSs, we have $0\le|\zeta_A^-(x_i)-\zeta_B^-(x_i)|^q\le 1$, $0\le|\zeta_A^+(x_i)-\zeta_B^+(x_i)|^q\le 1, 0\le|\vartheta_A^-(x_i)-\vartheta_B^-(x_i)|^q\le 1$, $0\le|\vartheta_A^+(x_i)-\vartheta_B^+(x_i)|^q\le 1$, $0\le|\zeta_A(x_i)-\zeta_B(x_i)|^q\le 1$ and $0\le|\vartheta_A(x_i)-\vartheta_B(x_i)|^q\le 1$ which implies that $0\le\max\left(\left|\zeta_A^-(x_i)-\zeta_B^-(x_i)\right|^q,\left|\zeta_A^+(x_i)-\zeta_B^+(x_i)\right|^q,\left|\vartheta_A^-(x_i)-\vartheta_B^-(x_i)\right|^q,\left|\vartheta_A^+(x_i)-\vartheta_B^+(x_i)\right|^q,\left|\zeta_A(x_i)-\zeta_B(x_i)\right|^q,\left|\vartheta_A(x_i)-\vartheta_B(x_i)\right|^q\right)\le 1$. Thus, $0\le d_k^H(\mathcal{A},\mathcal{B})\le 1$.

(P2) Assume that $d_k^H(\mathcal{A},\mathcal{B})=0$ if and only if

$$\begin{aligned}
&\max\begin{pmatrix}\left|\zeta_A^-(x_i)-\zeta_B^-(x_i)\right|^q,\left|\zeta_A^+(x_i)-\zeta_B^+(x_i)\right|^q,\left|\vartheta_A^-(x_i)-\vartheta_B^-(x_i)\right|^q,\\ \left|\vartheta_A^+(x_i)-\vartheta_B^+(x_i)\right|^q,\left|\zeta_A(x_i)-\zeta_B(x_i)\right|^q,\left|\vartheta_A(x_i)-\vartheta_B(x_i)\right|^q\end{pmatrix}=0\\
&\Leftrightarrow \left|\zeta_A^-(x_i)-\zeta_B^-(x_i)\right|^q=0,\left|\zeta_A^+(x_i)-\zeta_B^+(x_i)\right|^q=0,\left|\vartheta_A^-(x_i)-\vartheta_B^-(x_i)\right|^q=0,\\
&\quad\left|\vartheta_A^+(x_i)-\vartheta_B^+(x_i)\right|^q=0,\left|\zeta_A(x_i)-\zeta_B(x_i)\right|^q=0,\left|\vartheta_A(x_i)-\vartheta_B(x_i)\right|^q=0 \qquad \text{for all } i\\
&\Leftrightarrow \zeta_A^-(x_i)=\zeta_B^-(x_i),\zeta_A^+(x_i)=\zeta_B^+(x_i),\vartheta_A^-(x_i)=\vartheta_B^-(x_i),\\
&\quad\vartheta_A^+(x_i)=\vartheta_B^+(x_i),\zeta_A(x_i)=\zeta_B(x_i),\vartheta_A(x_i)=\vartheta_B(x_i) \qquad \text{for all } i,\\
&\Leftrightarrow \mathcal{A}=\mathcal{B}
\end{aligned}$$

(P3) For two CIFSs $\mathcal{A}$ and $\mathcal{B}$, we have

$$\begin{aligned}
d_k^H(\mathcal{A},\mathcal{B})&=\left(\frac{1}{6n}\sum_{i=1}^{n}\max\begin{pmatrix}\left|\zeta_A^-(x_i)-\zeta_B^-(x_i)\right|^q,\left|\zeta_A^+(x_i)-\zeta_B^+(x_i)\right|^q,\left|\vartheta_A^-(x_i)-\vartheta_B^-(x_i)\right|^q,\\ \left|\vartheta_A^+(x_i)-\vartheta_B^+(x_i)\right|^q,\left|\zeta_A(x_i)-\zeta_B(x_i)\right|^q,\left|\vartheta_A(x_i)-\vartheta_B(x_i)\right|^q\end{pmatrix}\right)^{1/q}\\
&=\left(\frac{1}{6n}\sum_{i=1}^{n}\max\begin{pmatrix}\left|\zeta_B^-(x_i)-\zeta_A^-(x_i)\right|^q,\left|\zeta_B^+(x_i)-\zeta_A^+(x_i)\right|^q,\left|\vartheta_B^-(x_i)-\vartheta_A^-(x_i)\right|^q,\\ \left|\vartheta_B^+(x_i)-\vartheta_A^+(x_i)\right|^q,\left|\zeta_B(x_i)-\zeta_A(x_i)\right|^q,\left|\vartheta_B(x_i)-\vartheta_A(x_i)\right|^q\end{pmatrix}\right)^{1/q}\\
&=d_k^H(\mathcal{B},\mathcal{A})
\end{aligned}$$

(P4) If $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{C}$, then $[\zeta_A^-(x_i), \zeta_A^+(x_i)] \subseteq [\zeta_B^-(x_i), \zeta_B^+(x_i)] \subseteq [\zeta_C^-(x_i), \zeta_C^+(x_i)]$, $[\vartheta_A^-(x_i), \vartheta_A^+(x_i)] \supseteq [\vartheta_B^-(x_i), \vartheta_B^+(x_i)] \supseteq [\vartheta_C^-(x_i), \vartheta_C^+(x_i)]$. Also, $\zeta_A(x_i) \geq \zeta_B(x_i) \geq \zeta_C(x_i)$ and $\vartheta_A(x_i) \leq \vartheta_B(x_i) \leq \vartheta_C(x_i)$. Therefore, $\left|\zeta_A^-(x_i) - \zeta_B^-(x_i)\right| \leq \left|\zeta_A^-(x_i) - \zeta_C^-(x_i)\right|$, $\left|\zeta_A^+(x_i) - \zeta_B^+(x_i)\right| \leq \left|\zeta_A^+(x_i) - \zeta_C^+(x_i)\right|$, $\left|\vartheta_A^-(x_i) - \vartheta_B^-(x_i)\right| \leq \left|\vartheta_A^-(x_i) - \vartheta_C^-(x_i)\right|$, $\left|\vartheta_A^+(x_i) - \vartheta_B^+(x_i)\right| \leq \left|\vartheta_A^+(x_i) - \vartheta_C^+(x_i)\right|$, $\left|\zeta_A(x_i) - \zeta_B(x_i)\right| \leq \left|\zeta_A(x_i) - \zeta_C(x_i)\right|$, $\left|\vartheta_A(x_i) - \vartheta_B(x_i)\right| \leq \left|\vartheta_A(x_i) - \vartheta_C(x_i)\right|$. Thus,

$$
\begin{aligned}
d_2^H(\mathcal{A},\mathcal{C}) &= \frac{1}{6n}\sum_{i=1}^{n} \max\begin{pmatrix} \left|\zeta_A^-(x_i) - \zeta_C^-(x_i)\right|, \left|\zeta_A^+(x_i) - \zeta_C^+(x_i)\right|, \left|\vartheta_A^-(x_i) - \vartheta_C^-(x_i)\right|, \\ \left|\vartheta_A^+(x_i) - \vartheta_C^+(x_i)\right|, \left|\zeta_A(x_i) - \zeta_C(x_i)\right|, \left|\vartheta_A(x_i) - \vartheta_C(x_i)\right| \end{pmatrix} \\
&\geq \frac{1}{6n}\sum_{i=1}^{n} \max\begin{pmatrix} \left|\zeta_A^-(x_i) - \zeta_B^-(x_i)\right|, \left|\zeta_A^+(x_i) - \zeta_B^+(x_i)\right|, \left|\vartheta_A^-(x_i) - \vartheta_B^-(x_i)\right|, \\ \left|\vartheta_A^+(x_i) - \vartheta_B^+(x_i)\right|, \left|\zeta_A(x_i) - \zeta_B(x_i)\right|, \left|\vartheta_A(x_i) - \vartheta_B(x_i)\right| \end{pmatrix} \\
&= d_2^H(\mathcal{A},\mathcal{B}).
\end{aligned}
$$

Hence, $d_2^H(\mathcal{A},\mathcal{C}) \geq d_2^H(\mathcal{A},\mathcal{B})$. Similarly, $d_2^H(\mathcal{A},\mathcal{C}) \geq d_2^H(\mathcal{B},\mathcal{C})$.

Theorem 11 *The measures d_5^H and d_6^H are also the valid distance measures.*

Proof Similar to Theorem 10.

Theorem 12 *The Hausdorff measures d_1^H and d_5^H satisfy the inequality $d_5^H \leq d_1^H$.*

Proof For two CIFSs $\mathcal{A}$ and $\mathcal{B}$ and $\kappa_i > 0$ be such that $\sum_{i=1}^{n} \kappa_i = 1$. Then, we have

$$
\begin{aligned}
d_5^H(\mathcal{A},\mathcal{B}) &= \frac{1}{6}\sum_{i=1}^{n} \kappa_i \max\begin{pmatrix} \left|\zeta_A^-(x_i) - \zeta_B^-(x_i)\right|, \left|\zeta_A^+(x_i) - \zeta_B^+(x_i)\right|, \left|\vartheta_A^-(x_i) - \vartheta_B^-(x_i)\right|, \\ \left|\vartheta_A^+(x_i) - \vartheta_B^+(x_i)\right|, \left|\zeta_A(x_i) - \zeta_B(x_i)\right|, \left|\vartheta_A(x_i) - \vartheta_B(x_i)\right| \end{pmatrix} \\
&\leq \frac{1}{6}\sum_{i=1}^{n} \max\begin{pmatrix} \left|\zeta_A^-(x_i) - \zeta_B^-(x_i)\right|, \left|\zeta_A^+(x_i) - \zeta_B^+(x_i)\right|, \left|\vartheta_A^-(x_i) - \vartheta_B^-(x_i)\right|, \\ \left|\vartheta_A^+(x_i) - \vartheta_B^+(x_i)\right|, \left|\zeta_A(x_i) - \zeta_B(x_i)\right|, \left|\vartheta_A(x_i) - \vartheta_B(x_i)\right| \end{pmatrix} \\
&= d_1^H(\mathcal{A},\mathcal{B}).
\end{aligned}
$$

As $\mathcal{A}$ and $\mathcal{B}$ are arbitrary CIFSs, so we get $d_5^H \leq d_1^H$.

Theorem 13 *The Hausdorff measures d_3^H and d_6^H satisfy the inequality $d_6^H \leq d_3^H$.*

Proof Similar to Theorem 12.

Theorem 14 *The measure d_1^H, d_2^H, d_3^H and d_4^H satisfy the following relations:*

(i) $d_3^H \leq \sqrt{d_1^H}$;

(ii) $d_4^H \leq \sqrt{d_2^H}$.

Table 1 Computed distance measures values for medical diagnosis problem

Measures	Measurement values of $\mathcal{P}$ from				Ranking
	$\mathcal{Q}_1$	$\mathcal{Q}_2$	$\mathcal{Q}_3$	$\mathcal{Q}_4$	
d_1	0.6833	0.5833	0.4417	0.5000	$\mathcal{Q}_3 \succ \mathcal{Q}_4 \succ \mathcal{Q}_2 \succ \mathcal{Q}_1$
d_2	0.1708	0.1458	0.1104	0.1250	$\mathcal{Q}_3 \succ \mathcal{Q}_4 \succ \mathcal{Q}_2 \succ \mathcal{Q}_1$
d_3	0.4601	0.3591	0.3011	0.3124	$\mathcal{Q}_3 \succ \mathcal{Q}_4 \succ \mathcal{Q}_2 \succ \mathcal{Q}_1$
d_4	0.2300	0.1795	0.1506	0.1562	$\mathcal{Q}_3 \succ \mathcal{Q}_4 \succ \mathcal{Q}_2 \succ \mathcal{Q}_1$
d_1^H	0.2667	0.1967	0.1600	0.1683	$\mathcal{Q}_3 \succ \mathcal{Q}_4 \succ \mathcal{Q}_2 \succ \mathcal{Q}_1$
d_2^H	0.0667	0.0492	0.0400	0.0421	$\mathcal{Q}_3 \succ \mathcal{Q}_4 \succ \mathcal{Q}_2 \succ \mathcal{Q}_1$
d_3^H	0.3317	0.2420	0.2164	0.2082	$\mathcal{Q}_4 \succ \mathcal{Q}_3 \succ \mathcal{Q}_2 \succ \mathcal{Q}_1$
d_4^H	0.1658	0.1210	0.1082	0.1041	$\mathcal{Q}_4 \succ \mathcal{Q}_3 \succ \mathcal{Q}_2 \succ \mathcal{Q}_1$

Table 2 Distance measurement values using weighted distance measures

Distance measures	Measurement values of $\mathcal{P}$ from				Ranking
	$\mathcal{Q}_1$	$\mathcal{Q}_2$	$\mathcal{Q}_3$	$\mathcal{Q}_4$	
d_5	0.1825	0.1445	0.1318	0.1057	$\mathcal{Q}_4 \succ \mathcal{Q}_3 \succ \mathcal{Q}_2 \succ \mathcal{Q}_1$
d_6	0.2529	0.1816	0.1776	0.1403	$\mathcal{Q}_4 \succ \mathcal{Q}_3 \succ \mathcal{Q}_2 \succ \mathcal{Q}_1$
d_5^H	0.0725	0.0498	0.0500	0.0397	$\mathcal{Q}_4 \succ \mathcal{Q}_2 \succ \mathcal{Q}_3 \succ \mathcal{Q}_1$
d_6^H	0.1796	0.1226	0.1312	0.0983	$\mathcal{Q}_4 \succ \mathcal{Q}_2 \succ \mathcal{Q}_3 \succ \mathcal{Q}_1$

Proof We will prove part (i) only, while other can be proven similarly.

Let $\mathcal{A}$ and $\mathcal{B}$ be two CIFSs and for any number $\eta \in [0,1]$, we know that $|\eta|^2 \leq |\eta|$. Thus, by the definition of d_3^H, we have $\left|\zeta_A^-(x_i) - \zeta_B^-(x_i)\right|^2 \leq \left|\zeta_A^-(x_i) - \zeta_B^-(x_i)\right|$, $\left|\zeta_A^+(x_i) - \zeta_B^+(x_i)\right|^2 \leq \left|\zeta_A^+(x_i) - \zeta_B^+(x_i)\right|$, $\left|\vartheta_A^-(x_i) - \vartheta_B^-(x_i)\right|^2 \leq \left|\vartheta_A^-(x_i) - \vartheta_B^-(x_i)\right|$, $\left|\vartheta_A^+(x_i) - \vartheta_B^+(x_i)\right|^2 \leq \left|\vartheta_A^+(x_i) - \vartheta_B^+(x_i)\right|$, $\left|\zeta_A(x_i) - \zeta_B(x_i)\right|^2 \leq \left|\zeta_A(x_i) - \zeta_B(x_i)\right|$ and $\left|\vartheta_A(x_i) - \vartheta_B(x_i)\right|^2 \leq \left|\vartheta_A(x_i) - \vartheta_B(x_i)\right|$. Thus,

$$\begin{aligned} d_3^H(\mathcal{A},\mathcal{B}) &= \left(\frac{1}{6}\sum_{i=1}^n \max\left(\begin{array}{l} \left|\zeta_A^-(x_i)-\zeta_B^-(x_i)\right|^2, \left|\zeta_A^+(x_i)-\zeta_B^+(x_i)\right|^2, \left|\vartheta_A^-(x_i)-\vartheta_B^-(x_i)\right|^2, \\ \left|\vartheta_A^+(x_i)-\vartheta_B^+(x_i)\right|^2, \left|\zeta_A(x_i)-\zeta_B(x_i)\right|^2, \left|\vartheta_A(x_i)-\vartheta_B(x_i)\right|^2 \end{array}\right)\right)^{1/2} \\ &\leq \left(\frac{1}{6}\sum_{i=1}^n \max\left(\begin{array}{l} \left|\zeta_A^-(x_i)-\zeta_B^-(x_i)\right|, \left|\zeta_A^+(x_i)-\zeta_B^+(x_i)\right|, \left|\vartheta_A^-(x_i)-\vartheta_B^-(x_i)\right|, \\ \left|\vartheta_A^+(x_i)-\vartheta_B^+(x_i)\right|, \left|\zeta_A(x_i)-\zeta_B(x_i)\right|, \left|\vartheta_A(x_i)-\vartheta_B(x_i)\right| \end{array}\right)\right)^{1/2} \\ &= \sqrt{d_1^H(\mathcal{A},\mathcal{B})}. \end{aligned}$$

Since, $\mathcal{A}$ and $\mathcal{B}$ are arbitrary, so we get $d_3^H \leq \sqrt{d_1^H}$.

Theorem 15 *The measures d_2^H and d_2 satisfy the inequality $d_2^H \leq d_2$.*

Proof Since for any positive numbers $a_i (i = 1, 2, \ldots, n)$, we know that $\max_i\{a_i\} \leq \sum_{i=1}^n a_i$ and hence by the definition of d_2^H, we get

$$\begin{aligned} d_2^H(\mathcal{A},\mathcal{B}) &= \frac{1}{6n}\sum_{i=1}^n \max\left(\begin{array}{l} \left|\zeta_A^-(x_i)-\zeta_B^-(x_i)\right|, \left|\zeta_A^+(x_i)-\zeta_B^+(x_i)\right|, \left|\vartheta_A^-(x_i)-\vartheta_B^-(x_i)\right|, \\ \left|\vartheta_A^+(x_i)-\vartheta_B^+(x_i)\right|, \left|\zeta_A(x_i)-\zeta_B(x_i)\right|, \left|\vartheta_A(x_i)-\vartheta_B(x_i)\right| \end{array}\right) \\ &\leq \frac{1}{6n}\sum_{i=1}^n \left(\begin{array}{l} \left|\zeta_A^-(x_i)-\zeta_B^-(x_i)\right| + \left|\zeta_A^+(x_i)-\zeta_B^+(x_i)\right| + \left|\vartheta_A^-(x_i)-\vartheta_B^-(x_i)\right| \\ + \left|\vartheta_A^+(x_i)-\vartheta_B^+(x_i)\right| + \left|\zeta_A(x_i)-\zeta_B(x_i)\right| + \left|\vartheta_A(x_i)-\vartheta_B(x_i)\right| \end{array}\right) \\ &= d_2(\mathcal{A},\mathcal{B}). \end{aligned}$$

Since, $\mathcal{A}$ and $\mathcal{B}$ are arbitrary, therefore, $d_2^H \leq d_2$.

Theorem 16 *The measures d_k^H and $d_k, (k = 1, 2, \ldots, 6)$ satisfy the inequality $d_k^H \leq d_k$.*

Proof Similar to Theorem 15.

Theorem 17 *The measures d_5^H and d_1 satisfy the inequality $d_5^H \leq d_1$.*

Proof Since from the above theorems, we get $d_5^H \leq d_5$ and $d_5 \leq d_1$. Therefore, $d_5^H \leq d_1$.

Theorem 18 *The measures d_6^H and d_3 satisfy the inequality $d_6^H \leq d_3$.*

Proof From Theorem 16, we get $d_6^H \leq d_6$ while from Theorem 7, we get $d_6 \leq d_3$. Hence, $d_6^H \leq d_3$.

Theorem 19 *Distance measures d_5, d_1, d_1^H and d_3, d_6, d_3^H satisfy the following relations:*

(i) $d_1 \geq \frac{d_5+d_1^H}{2}$ *and* $d_1 \geq \sqrt{d_5 \cdot d_1^H}$;

(ii) $d_3 \geq \frac{d_6+d_3^H}{2}$ *and* $d_3 \geq \sqrt{d_6 \cdot d_3^H}$.

Proof Since $d_1^H \leq d_1$ and $d_5 \leq d_1$. So, by adding these inequalities, we get $\frac{d_5+d_1^H}{2} \leq d_1$ while by multiplying we get $\sqrt{d_5 \cdot d_1^H} \leq d_1$.

Also, $d_3^H \leq d_3$ and $d_6 \leq d_3$. So, by adding these inequalities, we get $\frac{d_6+d_3^H}{2} \leq d_3$ and by multiplying, we get $\sqrt{d_6 \cdot d_3^H} \leq d_3$.

Table 3 Comparative study of Example 1

Existing studies	Measurement values of $\mathcal{P}$ from				Ranking
	Q_1	Q_2	Q_3	Q_4	
Dugenci (2016)	0.2677	0.2888	0.3439	0.2751	$Q_1 \succ Q_4 \succ Q_2 \succ Q_3$
Park et al. (2009)	0.9196	0.9005	0.9222	0.8563	$Q_3 \succ Q_1 \succ Q_2 \succ Q_4$
Wei et al. (2011)	0.9270	0.9181	0.9263	0.8969	$Q_1 \succ Q_3 \succ Q_2 \succ Q_4$
Zhang et al. (2014)	0.1369	0.1444	0.1381	0.1620	$Q_1 \succ Q_3 \succ Q_2 \succ Q_4$
Xu (2007)	0.8701	0.8643	0.8473	0.8621	$Q_1 \succ Q_2 \succ Q_4 \succ Q_3$

Table 4 Proposed measures results for pattern recognition

Distance measures	Measurement values of $\mathcal{P}$ from				Ranking
	C_1	C_2	C_3	C_4	
d_1	0.5967	0.2117	0.5900	0.7683	$C_2 \succ C_3 \succ C_1 \succ C_4$
d_2	0.1492	0.0529	0.1475	0.1921	$C_2 \succ C_3 \succ C_1 \succ C_4$
d_3	0.4472	0.1439	0.3969	0.4741	$C_2 \succ C_3 \succ C_1 \succ C_4$
d_4	0.2236	0.0720	0.1985	0.2371	$C_2 \succ C_3 \succ C_1 \succ C_4$
d_1^H	0.2633	0.0683	0.2500	0.2783	$C_2 \succ C_3 \succ C_1 \succ C_4$
d_2^H	0.0658	0.0171	0.0625	0.0696	$C_2 \succ C_3 \succ C_1 \succ C_4$
d_3^H	0.3452	0.0967	0.3175	0.3437	$C_2 \succ C_3 \succ C_4 \succ C_1$
d_4^H	0.1726	0.0483	0.1588	0.1719	$C_2 \succ C_3 \succ C_4 \succ C_1$

Table 5 Measurement values corresponding to the weighted distance measures for pattern recognition

Distance measures	Measurement values of $\mathcal{P}$ from				Ranking
	C_1	C_2	C_3	C_4	
d_5	0.1491	0.0552	0.1504	0.1867	$C_2 \succ C_1 \succ C_3 \succ C_4$
d_6	0.2271	0.0765	0.2050	0.2320	$C_2 \succ C_3 \succ C_1 \succ C_4$
d_5^H	0.0679	0.0186	0.0651	0.0689	$C_2 \succ C_3 \succ C_1 \succ C_4$
d_6^H	0.1773	0.0526	0.1651	0.1700	$C_2 \succ C_3 \succ C_4 \succ C_1$

Table 6 Comparative analysis of pattern recognition example

Existing approaches	Measurement values of $\mathcal{P}$ from				Ranking
	C_1	C_2	C_3	C_4	
Dugenci (2016)	0.4352	0.1882	0.3299	0.5301	$C_2 \succ C_3 \succ C_1 \succ C_4$
Park et al. (2009)	0.8614	0.9748	0.9141	0.7814	$C_2 \succ C_3 \succ C_1 \succ C_4$
Wei et al. (2011)	0.8801	0.9904	0.9267	0.7897	$C_2 \succ C_3 \succ C_1 \succ C_4$
Zhang et al. (2014)	0.1760	0.0630	0.1397	0.2336	$C_2 \succ C_3 \succ C_1 \succ C_4$
Xu (2007)	0.7412	0.9219	0.8343	0.6855	$C_2 \succ C_3 \succ C_1 \succ C_4$

4 Decision-making based on the proposed distance measure of CIFSs

In this section, a decision-making method by using the above defined distance measures for CIFSs has been presented followed by an illustrative example for demonstrating the approach.

4.1 Decision-making approach

For this, assume that there are m alternatives, denoted by $A_1, A_2, \dots, A_m$, which are evaluated by an expert under the set of n criteria, denoted by $C_1, C_2, \dots, C_n$, and gave his preferences in the form of the CIFNs $\alpha_{pq} = \big(\langle[\zeta^-_{pq}, \zeta^+_{pq}], [\vartheta^-_{pq}, \vartheta^+_{pq}]\rangle, \langle\zeta_{pq}, \vartheta_{pq}\rangle\big)$; $p = 1, 2, \dots, m$; $q = 1, 2, \dots, n$. Thus, the rating values corresponding to each alternative are summarized in terms of CIFNs as follows:

$$A_p = \left\{ \big(x_q, \langle[\zeta^-_{pq}(x_q), \zeta^+_{pq}(x_q)], [\vartheta^-_{pq}(x_q), \vartheta^+_{pq}(x_q)]\rangle, \langle\zeta_{pq}(x_q), \vartheta_{pq}(x_q)\rangle\big) \mid x_q \in U \right\}.$$

Let $\kappa_q (q = 1, 2, \dots, n)$ be the weight of the criteria C_q such that $\kappa_q > 0$ and $\sum_{q=1}^{n} \kappa_q = 1$. Then, the following steps are proposed to solve the DM problems using proposed measures:

Step 1 Collect all the information corresponding to each alternative in terms of CIFNs and hence an overall decision matrix D is expressed as

$$D = \begin{array}{c} \\ A_1 \\ A_2 \\ \vdots \\ A_m \end{array} \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & \dots & C_n \end{array} \\ \begin{pmatrix} \alpha_{11} & \alpha_{12} & \dots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \dots & \alpha_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{m1} & \alpha_{m2} & \dots & \alpha_{mn} \end{pmatrix} \end{array} \tag{17}$$

Step 2 Utilize the proposed distance measures 'd' to compute the measurement value of each alternative from its reference set.

Step 3 Find the index value of $r_p = \arg\min_{1\le p\le m}\{d\}$ and hence select the best one(s) accordingly.

4.2 Numerical examples

Two examples, related to pattern recognition and medical diagnosis are taken to demonstrate the approach.

Example 2 Consider a set of diseases $\mathcal{Q} = \{\mathcal{Q}_1$(Viral fever), $\mathcal{Q}_2$(Malaria), $\mathcal{Q}_3$(Typhoid), $\mathcal{Q}_4$ (Stomach problem)$\}$ and a set of symptoms $S = \{s_1$(Temperature), s_2(Headache), s_3 (Stomach-ache), s_4(Cough)$\}$ which are represented in the form of CIFSs (where the IVIFS set represents the probable disease during initial stage, i.e., before diagnosis and IFS represents the corresponding measure after diagnosis) as below:

$$\begin{aligned}
\mathcal{Q}_1 = \big\{ &\big(s_1, \big(\langle[0.10, 0.20], [0.30, 0.60]\rangle, \langle 0.40, 0.20\rangle\big)\big), \big(s_2, \big(\langle[0.25, 0.30], [0.45, 0.50]\rangle, \langle 0.60, 0.30\rangle\big)\big), \\
&\big(s_3, \big(\langle[0.30, 0.45], [0.20, 0.25]\rangle, \langle 0.10, 0.80\rangle\big)\big), \big(s_4, \big(\langle[0.40, 0.50], [0.10, 0.30]\rangle, \langle 0.30, 0.70\rangle\big)\big)\big\} \\
\mathcal{Q}_2 = \big\{ &\big(s_1, \big(\langle[0.15, 0.45], [0.25, 0.30]\rangle, \langle 0.40, 0.60\rangle\big)\big), \big(s_2, \big(\langle[0.20, 0.25], [0.30, 0.35]\rangle, \langle 0.15, 0.20\rangle\big)\big), \\
&\big(s_3, \big(\langle[0.45, 0.60], [0.20, 0.25]\rangle, \langle 0.29, 0.63\rangle\big)\big), \big(s_4, \big(\langle[0.16, 0.20], [0.25, 0.30]\rangle, \langle 0.15, 0.30\rangle\big)\big)\big\} \\
\mathcal{Q}_3 = \big\{ &\big(s_1, \big(\langle[0.20, 0.30], [0.25, 0.35]\rangle, \langle 0.15, 0.25\rangle\big)\big), \big(s_2, \big(\langle[0.30, 0.40], [0.35, 0.45]\rangle, \langle 0.35, 0.40\rangle\big)\big), \\
&\big(s_3, \big(\langle[0.15, 0.25], [0.30, 0.35]\rangle, \langle 0.18, 0.35\rangle\big)\big), \big(s_4, \big(\langle[0.19, 0.36], [0.20, 0.40]\rangle, \langle 0.30, 0.25\rangle\big)\big)\big\} \\
\mathcal{Q}_4 = \big\{ &\big(s_1, \big(\langle[0.14, 0.28], [0.15, 0.30]\rangle, \langle 0.20, 0.30\rangle\big)\big), \big(s_2, \big(\langle[0.16, 0.32], [0.17, 0.34]\rangle, \langle 0.18, 0.20\rangle\big)\big), \\
&\big(s_3, \big(\langle[0.50, 0.55], [0.20, 0.30]\rangle, \langle 0.40, 0.30\rangle\big)\big), \big(s_4, \big(\langle[0.30, 0.40], [0.50, 0.55]\rangle, \langle 0.35, 0.55\rangle\big)\big)\big\}.
\end{aligned}$$

Suppose a patient $\mathcal{P}$, has been evaluated by an expert in order to find that his symptoms has maximum compliance with that of diseases $\mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Q}_3$ or $\mathcal{Q}_4$. For it, they have recorded the preferences of patient $\mathcal{P}$ with respect to all the symptoms in terms of CIFN represented by the following set:

$$\mathcal{P}=\Big\{\Big(s_1,\big(\langle[0.20,0.30],[0.40,0.50]\rangle,\langle 0.10,0.40\rangle\big)\Big),\Big(s_2,\big(\langle[0.30,0.40],[0.10,0.60]\rangle,\langle 0.20,0.40\rangle\big)\Big),$$
$$\Big(s_3,\big(\langle[0.40,0.50],[0.20,0.30]\rangle,\langle 0.60,0.30\rangle\big)\Big),\Big(s_4,\big(\langle[0.10,0.50],[0.20,0.30]\rangle,\langle 0.40,0.30\rangle\big)\Big)\Big\}.$$

Now, the aim of the problem is to identify the disease of patient $\mathcal{P}$ among Q_1, Q_2, Q_3, Q_4. For it, the proposed non-weighted measures are utilized to compute the measurement values of $Q_k (k = 1, 2, 3, 4)$ from $\mathcal{P}$. The results corresponding to these measures values are summarized in Table 1 and conclude that the Q_3 is the disease by which patient $\mathcal{P}$ suffers.

If we assign weights 0.10, 0.30, 0.45 and 0.15 to set $s_k (k = 1, 2, 3, 4)$ and utilize weighted measures d_5, d_6, d_5^H and d_6^H, we get their measurement values and summarize in Table 2. Based on step 3 of proposed approach, we get that the patient $\mathcal{P}$ suffers from the disease Q_4.

To compare the proposed results with the existing approaches (Dugenci 2016; Park et al. 2009; Wei et al. 2011; Xu 2007; Zhang et al. 2014) under IVIFS environment, we first convert the preferences of the decision-maker from CIFNs to IVIFNs. For it, we have taken the fuzzy judgment as zero in CIFS so that it gets reduced to IVIFS and then the various existing approaches have been utilized to compute the most desirable diseases. The result corresponding to their approaches is summarized in Table 3. From it, the relativity of CIFS to the real-life scenario can be noticed. Since it facilitates us to study the symptoms for a prolonged time period and relating it to the existing IVIFS environment, it has been found that under it, the patient is found to be suffering from Q_1 disease, i.e., Viral fever, but on analyzing the symptoms and its effect after diagnosis (under CIFS environment), the patient is found to be suffering from disease Q_3, i.e., Typhoid whereas under the weighted criteria, the patient is diagnosed with disease Q_4, i.e., stomach problem.

The comparison analysis clearly shows that from the information available during the initial stage (recorded in form of IVIFS) shows the patient to be suffering from disease Q_1, i.e., Viral Fever. However, if we consider the information of post-diagnosis period clubbed together with the initial symptoms, i.e., in form of CIFS, then the patient is diagnosed with the disease Q_3, i.e., Typhoid and from Q_4, i.e., stomach problem under the suitable weighted criteria. This shows the effect of the time frame during which the information is recorded from the patient and CIFS allows us to handle the complete opinion of a patient of both pre-diagnosis and post-diagnosis stages. Thus, the results obtained from our approach show a significant difference from the existing ones because, in the prevailing approaches, the decision is made only on the basis of initial symptoms assessed before the diagnosis while our approach takes into account both the pre-diagnosis as well as the post-diagnosis stages.

Example 3 Consider four known patterns C_1, C_2, C_3 and C_4 whose rating values are given as

$$C_1=\big\{\big(x_1,\big(\langle[0.15,0.30],[0.35,0.40]\rangle,\langle 0.20,0.65\rangle\big)\big),\big(x_2,\big(\langle[0.13,0.25],[0.40,0.45]\rangle,\langle 0.30,0.60\rangle\big)\big),$$
$$\big(x_3,\big(\langle[0.30,0.45],[0.25,0.30]\rangle,\langle 0.55,0.33\rangle\big)\big),\big(x_4,\big(\langle[0.10,0.30],[0.25,0.35]\rangle,\langle 0.11,0.20\rangle\big)\big)\big\}$$
$$C_2=\big\{\big(x_1,\big(\langle[0.10,0.15],[0.35,0.40]\rangle,\langle 0.40,0.20\rangle\big)\big),\big(x_2,\big(\langle[0.15,0.22],[0.27,0.30]\rangle,\langle 0.15,0.60\rangle\big)\big),$$
$$\big(x_3,\big(\langle[0.40,0.45],[0.21,0.33]\rangle,\langle 0.16,0.40\rangle\big)\big),\big(x_4,\big(\langle[0.50,0.60],[0.15,0.20]\rangle,\langle 0.35,0.28\rangle\big)\big)\big\},$$
$$C_3=\big\{\big(x_1,\big(\langle[0.14,0.25],[0.35,0.65]\rangle,\langle 0.10,0.40\rangle\big)\big),\big(x_2,\big(\langle[0.35,0.45],[0.15,0.20]\rangle,\langle 0.30,0.50\rangle\big)\big),$$
$$\big(x_3,\big(\langle[0.45,0.55],[0.15,0.25]\rangle,\langle 0.20,0.80\rangle\big)\big),\big(x_4,\big(\langle[0.30,0.50],[0.10,0.30]\rangle,\langle 0.20,0.35\rangle\big)\big)\big\},$$
$$C_4=\big\{\big(x_1,\big(\langle[0.30,0.35],[0.25,0.45]\rangle,\langle 0.20,0.30\rangle\big)\big),\big(x_2,\big(\langle[0.20,0.55],[0.40,0.45]\rangle,\langle 0.20,0.45\rangle\big)\big),$$
$$\big(x_3,\big(\langle[0.15,0.25],[0.20,0.35]\rangle,\langle 0.60,0.20\rangle\big)\big),\big(x_4,\big(\langle[0.10,0.29],[0.40,0.50]\rangle,\langle 0.30,0.40\rangle\big)\big)\big\}$$

Consider an unknown pattern $\mathcal{P}$, which will be recognized as:

$$\mathcal{P}=\big\{\big(x_1,\big(\langle[0.10,0.30],[0.35,0.45]\rangle,\langle 0.60,0.10\rangle\big)\big),\big(x_2,\big(\langle[0.15,0.20],[0.25,0.29]\rangle,\langle 0.18,0.66\rangle\big)\big),$$
$$\big(x_3,\big(\langle[0.44,0.50],[0.20,0.30]\rangle,\langle 0.18,0.35\rangle\big)\big),\big(x_4,\big(\langle[0.60,0.70],[0.20,0.30]\rangle,\langle 0.40,0.25\rangle\big)\big)\big\}.$$

In these preference values, the IVIFN in each CIFN shows the probable interval of belongingness as well as non-belongingness of the pattern P to the classes C_1, C_2, C_3 and C_4 using Pattern-classifier 1. However, the IFS portion, shows the extent of agree-ness and disagree-ness shown by the Pattern-classifier 2 to that of Classifier 1. Now, the aim of the problem is to classify the pattern $\mathcal{P}$ with $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ and $\mathcal{C}_4$.

Initially, by using non-weighted measures d_1, d_2, d_3, d_4, d_1^H, d_2^H and d_3^H, the measurement values of $\mathcal{P}$ from $\mathcal{C}_k$, $(k = 1, 2, 3, 4)$ are computed and their results are summarized in Table 4. Thus, based on the maximum recognition principle, it has been concluded that unknown pattern $\mathcal{P}$ belongs to the pattern $\mathcal{C}_2$. Later on, if we assign weight 0.33, 0.23, 0.24 and 0.20 to x_1, x_2, x_3 and x_4, then by using weighted measures d_5, d_6, d_5^H and d_6^H, the measurement values of each pattern $\mathcal{C}_k (k = 1, 2, 3, 4)$ from the unknown pattern $\mathcal{P}$ are computed and summarized in Table 5. From these tabulated values, we see that the pattern $\mathcal{P}$ can be classified to the class of pattern $\mathcal{C}_2$.

To compare the proposed results with the existing approaches (Dugenci 2016; Park et al. 2009; Wei et al. 2011; Xu 2007; Zhang et al. 2014) under IVIFS environment, we have taken the fuzzy judgments as zero in CIFS so that it gets reduced to IVIFS and hence their corresponding results are summarized in Table 6. It is clearly seen from their measurement values that the unknown pattern $\mathcal{P}$ recognized with the known pattern $\mathcal{C}_2$ and their results coincide with the proposed approach's result.

Noticeably, although the proposed approach's results coincide with the results of existing approaches, still the results obtained from the proposed method are superior because they bring the surety of the pattern $\mathcal{P}$, to be classified under Class $\mathcal{C}_2$, verified by two pattern-classifiers, whereas the existing approaches only give an outlook of the pattern recognized by the classifier 1 only. Moreover, by employing the proposed approach it is also found that both the Pattern-classifiers 1 and 2 recognizes the pattern $\mathcal{P}$ to be similar to class $\mathcal{C}_2$.

5 Conclusion

In the present manuscript, we have extended the theory of the IFS to the cubic IFS which is an extension of the cubic set. In this set, preference corresponding to an element is expressed by means of the IVIFS and IFS which shows the importance of IVIFS to get the more appropriate results through IFS. Then, based on it, some family of distance measures based on Hamming, Euclidean, and Hausdorff measures, have been proposed along with their desired relations. These measures are advantageous over distance measures associated with CFS in the manner that these enhance our scope of analysis beyond the membership degrees. Instead of remaining confined to perform all the analysis sticking to only the membership portion, distance measures on CIFSs provide us with a wide perspective of handling the non-membership degrees too. Unlike all the existing and developed distance measures, measures in CIFSs allow us to consider the degree of disagreement (in form of IFS values) corresponding to the agreed interval region (in form of IVIFS) which clearly incline our results more towards the practically feasible values. To demonstrate the efficiency of the proposed operators, two illustrative examples have been taken into account. From the studies, it has been concluded that these distance measures employed together with the decision-making approach can model the uncertainties in a much better way than that of the existing approaches and provide us with a deep analysis of the real-life situations. As in the medical recognition example, the proposed approach not only provides us with the disease to which the patient is prone to, but also analyzes the patient's state both during the pre-diagnosis and post-diagnosis states and in the pattern recognition example, it classifies the pattern to its most similar class and meanwhile provides us with the verification of uniform working of both the classifiers 1 and 2. Thus, it is clear that using the proposed method, we reach the desired results along with some additional information about the real-life scenarios. In the future, the result of this paper can be extended to a linguistic environment under the Pythagorean, neutrosophic and other uncertain and fuzzy environment (Arora and Garg 2018a; Chen et al. 2012a, 2016; Garg 2018a; Garg and Kumar 2018b; Garg and Nancy 2018; Liu et al. 2018, 2017; Rani and Garg 2018; Singh and Garg 2018; Wang and Chen 2008).

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