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ORIGINAL PAPER

Hesitant probabilistic fuzzy set based time series forecasting method

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Abstract

Uncertainties due to randomness and fuzziness coexist in the system simultaneously. Recently probabilistic fuzzy set has gained attention of researchers to handle both types of uncertainties simultaneously in a single framework. In this paper, we introduce hesitant probabilistic fuzzy sets in time series forecasting to address the issues of non-stochastic non-determinism along with both types of uncertainties and propose a hesitant probabilistic fuzzy set based time series forecasting method. We also propose an aggregation operator that uses membership grades, weights and immediate probability to aggregate hesitant probabilistic fuzzy elements to fuzzy elements. Advantages of the proposed forecasting method are that it includes both type of uncertainties and non-stochastic hesitation in a single framework and also enhance the accuracy in forecasted outputs. The proposed method has been implemented to forecast the historical enrolment student's data at University of Alabama and share market prizes of State Bank of India (SBI) at Bombay stock exchange (BSE), India. The effectiveness of the proposed method has been examined and tested using error measures.

Keywords Probabilistic and non-probabilistic uncertainties · Hesitant probabilistic fuzzy set · Fuzzy time series forecasting · Immediate probability

1 Introduction

Time series forecasting has been an important area of research since age. Profound applications of time series forecasting are found in many fields that includes finance, engineering, medicines and management science, etc. Regression analysis, autoregressive integration moving average, simple moving average and simple exponential smoothing are few statistical models which are commonly used in conventional time series forecasting. However, these models touch the issue of probabilistic uncertainty of time series data, but fail to include non-probabilistic uncertainty that arises due to inaccuracies in measurement and linguistic representation. Need of fuzzy sets (Zadeh 1965) was felt in time series forecasting to overcome the limitations of conventional time series forecasting methods and to arrive a realistic results with higher accuracy rate in forecasted output with linguistic representation of time series data.

Song and Chissom (1993a, b, 1994) proposed time series forecasting models based on fuzzy sets (Zadeh 1965) to deal uncertainty in time series forecasting that arises due to vague, inaccurate and linguistic representation of time series data. Chen (1996) proposed simple arithmetic operators rather than complex max–min compositions operators used by Song and Chissom (1993a, b, 1994). Afterwards, many researchers (Chen and Hwang 2000; Huarng 2001; Song 2003; Lee and Chou 2004; Liu 2007; Cheng et al. 2008, 2016; Huarng and Yu 2006; Chen et al. 2009; Chen and Tanuwijaya 2011) proposed various fuzzy time series forecasting models with the innovation either in partitioning the universe of discourse or in fuzzy logical relations to enhance the accuracy in forecast. Chen and Chen (2014), Chen and Chen (2015), Chen and Phuong (2017), Wang and Mishra (2018) proposed various forecasting methods using granular computing, adaptive and intelligent fuzzy time series forecasting models. Chen and Chen (2011), Chen (2014), Ye et al. (2016), Yolcu et al. (2016), Kocak (2017) and Efendi et al. (2018) have proposed the high order fuzzy time series forecasting method based on fuzzy logic relations for stock trading. Recently, Bas et al. (2018) proposed ridge regression for forecasting using type 1 fuzzy function.

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Granular computing is intended as a convergence of numerous modeling approaches (Pedrycz and Chen 2011, 2015a, b). Various researchers (Livi and Sadeghian 2016; Wilke and Portmann 2016; Liu and Cocea 2017; D'Aniello et al. 2017) have used granular computing approach in modeling and computing with uncertainty, human-data interaction, machine learning and approximate reasoning. Deng et al. (2016) and Maciel et al. (2016) proposed time series forecasting models using multi-granularity and granular analytics.

Although fuzzy time series methods have achieved great success in forecasting in environment of non-probabilistic uncertainty, but failed to handle non-determinism. Non-determinism in fuzzy time series forecasting occurs due to hesitation. This hesitation is non-probabilistic and is due to single function in fuzzy set for both membership and non-membership and cannot be handled by random probability distribution. To deal with this non-probabilistic non-determinism in fuzzy time series forecasting, many researchers (Joshi and Kumar 2012a, b; Gangwar and Kumar 2014; Kumar and Gangwar 2015, 2016; Wang et al. 2016) developed intuitionistic fuzzy sets (Atanassov 1986) based time series forecasting models. Another non-probabilistic non-determinism in fuzzy time series forecasting occurs when time series data can be fuzzified using multiple valid fuzzification methods. Since difficulty of creating a common membership grade is not due to margin of error or possible distribution values, therefore, this non-determinism cannot be handled using IFS and type-2 fuzzy sets. Torra and Narukawa (2009) and Torra (2010) introduced the hesitant fuzzy set (HFS) as a new generalization of fuzzy sets to address this particular non-determinism. HFS provides an effective tool to eliminate the compromise among the membership grades of time series datum during fuzzification using multiple fuzzy sets. Bisht and Kumar (2016) proposed HFS based fuzzy time series forecasting model and claimed it's out performance in financial time series forecasting. Recently, hesitant fuzzy linguistic sets have also been used by many researchers (Chen and Hong 2014; Lee and Chen 2015a, b; Joshi and Kumar 2018a, b; Joshi et al. 2018) in decision-making problems.

Probabilistic and non-probabilistic uncertainties are two conceptually different kinds of uncertainties which occur simultaneously in the system. One of the main advantages of fuzzy time series forecasting methods is their ability to handle non-stochastic uncertainty. However, these forecasting models do not possess the capabilities to handle stochastic uncertainties. Meghdadi (2001) introduced probabilistic fuzzy set (PFS) to consider both uncertainties in a single framework. Due to its main advantage of combining interpretability of fuzzy set with statistical properties, Liu and Li (2005) proposed a probabilistic fuzzy logic system for the modeling and control problems. Applications of PFS were explored by many researchers (Almeida et al. 2009; Hinojosa et al. 2011; Li and Huang 2012; Huang et al. 2012; Fialho et al. 2016) in various fields where probabilistic uncertainty plays equal and important role as non-probabilistic uncertainty. Xu and Zhou (2017) associated probabilistic to the elements of HFS and defined hesitant probabilistic fuzzy set (HPFS).

The motivation and contribution of this paper are to propose a novel time series forecasting method that can include both stochastic and non-stochastic uncertainties in hesitant fuzzy environment. Since profound applications of HPFS are found in decision making problem (Zhou and Xu 2017; Li and Wang 2017; Ding et al. 2017) to include both types of uncertainties, therefore, we develop a novel time series forecasting method using HPFS for the same reasons. Advantage of proposed forecasting method is its ability to handle uncertainties that are caused by randomness and fuzziness simultaneously and also increases flexibility of using more than one fuzzification method. Another advantage of proposed forecasting method is that it addresses issue of non-statistical non-determinism (hesitation) which arises due to the presence of multiple valid fuzzification methods for time series data. In this paper, non-determinism is included using two different methods of discretization of universe of discourse with equal and unequal length partitions. HPFS is constructed using a probability distribution function that associates probabilities to possible membership grades of time series data in multiple fuzzy sets. We propose an aggregation operator to aggregate the hesitant probabilistic fuzzy elements (HPFEs) using weights and immediate probabilities. Performance of proposed forecasting method is tested using bench mark problem of data of University of Alabama enrolments. As statistical uncertainty is an inherent characteristic of financial time series data, performance of proposed method is also tested on a financial time series data of SBI share price at BSE, India.

Rest of the paper is organized as follows: Basic definitions of fuzzy set, PFS, fuzzy time series, HFS and HPFS are presented in Sect. 2. In Sect. 3, we define the max min composition operator and aggregation operator and also include few examples to understand max–min composition operator and aggregation process for HPFS. This section also includes algorithm of proposed HPFS-based time series forecasting method. Efficiency of proposed forecasting method is tested using dataset of University of Alabama enrolments and SBI share prices in Sect. 4. This section also includes comparison of forecasted enrolments and SBI prices with few other existing forecasting methods. Finally, conclusions are presented in Sect. 5.

2 Preliminaries

In this section, we review the definitions of fuzzy set (Zadeh 1965), PFS (Liu and Li 2005) and fuzzy time series (Song and Chissom 1993a, b; Chen 1996). Definitions of HFS

(Torra and Narukawa 2009; Torra 2010), HPFS (Xu and Zhou 2017) are also reviewed in this section.

Definition 1 (Zadeh 1965) Let $U = \{u_1, u_2, u_3, ..., u_n\}$ be a finite and fixed universe of discourse. A fuzzy set A on $U = \{u_1, u_2, u_3, ..., u_n\}$ is defined as follows:

$$A = \{\langle u, \mu_A(u)\rangle | \forall u \in U\} \tag{1}$$

Here $\mu_A : U \rightarrow [0, 1]$ and $\mu_A(u)$ represents degree of membership of u in A.

Definition 2 (Liu and Li 2005) Probabilistic fuzzy set (PFS) $\tilde{A}$ for a variable $u \in U$ and its fuzzy membership grade $\mu \in [0,1]$ can be expressed by a probability space $(V_u, \wp, P)$. Here V_u and $\wp$ are the set of all possible events $\{\mu \in [0,1]\}$and σ field respectively. The probability P is defined on $\wp$ for all element event E_i in V_u satisfies the following conditions:

$$P(E_i) \geqslant 0, \quad P\left(\sum E_i\right) = \sum P(E_i) \quad P(V_u) = 1 \tag{2}$$

Here E_i is corresponding to an event $\mu = \mu_i \subseteq [0,1]$ and $P(E_i)$ is probability for the event E_i. $\tilde{A}$ can also be expressed as the union of finite sub probability space as follows:

$$\tilde{A} \equiv \bigcup_{u \in U} (V_u, \wp, P). \tag{3}$$

Definition 3 (Song and Chissom 1993a, b; Chen 1996) Let $Y(t)(t = ..., 0, 1, 2, ...)$ be subset of real numbers. A fuzzy time series $F(t)$ on $Y(t)$ is a collection of fuzzy sets $f_i(t)(i = 1, 2, ...)$. If there exists a fuzzy relation$R(t-1, t)$ such that $F(t) = F(t-1) \circ R(t-1, t)$, ($\circ$ is the max–min composition operator) then relation $F(t-1) \rightarrow F(t)$ indicates that $F(t)$ is caused by only $F(t-1)$. This is called first-order model of fuzzy time series forecasting model $F(t)$. If $F(t)$ is caused by$F(t-1), F(t-2), ..., F(t-n)$, then this fuzzy relationship is an nth-order fuzzy time series$F(t-n), ..., F(t-2), F(t-1) \rightarrow F(t)$.

Definition 4 (Torra and Narukawa 2009; Torra 2010) Let $U = \{u_1, u_2, u_3, ..., u_n\}$ be a fixed set and $h_A : U \rightarrow P\ [0, 1]$ be a function from U to the collection of subsets of $[0, 1]$. An HFS H on U is a mathematical object of following form:

$$H = \{\langle u, h_A(u)\rangle | \forall u \in U\} \tag{4}$$

Here $h_A(u)$ is a collection of membership degrees of an element $u \in U$to the set H in $[0, 1]$. Elements of HFS are called hesitant fuzzy element (HFE). Basic operations of union, intersection and complement on HFEs are defined as follows:

- $h_1{}^c = \{1 - \gamma | \gamma \in h_1\}$
- $h_1 \cup h_2 = \{\gamma_1 \vee \gamma_2 | \gamma_1 \in h_1, \gamma_2 \in h_2\}$
- $h_1 \cap h_2 = \{\gamma_1 \wedge \gamma_2 | \gamma_1 \in h_1, \gamma_2 \in h_2\}$

Here $\vee$ and $\wedge$ are max and min operators.

Definition 5 (Xu and Zhou 2017) Let R be a fixed set. HPFS on R is expressed as $H_p = \{\langle \bar{h}(\gamma_i / p_i)\rangle / \gamma_i, p_i\}$ where $\bar{h}(\gamma_i / p_i)$ is a set of elements in γ_i / p_i $\gamma_i \in R$, $0 \leqslant \gamma_i \leqslant 1$, $i = 1, 2, ..., \#\bar{h}$, where $\#\bar{h}$ is the number of possible elements in $\bar{h}(\gamma_i / p_i)$. $p_i \in [0, 1]$ is the hesitant probability of γ_i and $\sum_{i=1}^{\#\bar{h}} p_i = 1$.

3 Proposed hesitant probabilistic fuzzy time series forecasting method

In the proposed forecasting method, time series data are fuzzified using two different fuzzification methods to construct HFS. HFS is converted into HPFS using a suitable probability distribution function. Proposed method utilizes a novel immediate probabilities based aggregation operator to aggregate HPFEs to fuzzy set and max–min composition operator on fuzzy logical relations which are defined as follows:

3.1 Max–min composition operator for fuzzy set

Let R_1 and R_2 be two fuzzy relations on fuzzy sets A_1, A_2, A_3 such that $R_1 \subseteq A_1 \times A_2$ and $R_2 \subseteq A_2 \times A_3$. The max min composition $(R_1 \circ R_2)$ of two relations R_1 and R_2 is expressed by the relation from A_1 to A_3 as follows:

For $(a, b) \in A_1 \times A_2$, $(b, c) \in A_2 \times A_3$

$$\begin{aligned}\mu_{R_1 \circ R_2}(a, c) &= \max_b \left[\min(\mu_{R_1}(a, b), \mu_{R_2}(b, c))\right] \\ &= \bigvee_b \left[\mu_{R_1}(a, b) \wedge \mu_{R_2}(b, c)\right]\end{aligned} \tag{5}$$

Here $\vee$, $\wedge$ are maximum and minimum operations, respectively.

Max–min composition operation is illustrated by following example.

Example 1 If $R_1 = \begin{bmatrix} 0.21 & 0.51 & 0.19 \\ 0.37 & 0.92 & 0.76 \\ 0.71 & 0.97 & 0.39 \\ 0.91 & 0.42 & 0.81 \end{bmatrix}_{4\times3}$ and $R_2 = \begin{bmatrix} 0.29 & 0.97 & 0.86 \\ 0.35 & 0.75 & 0.49 \\ 0.17 & 0.29 & 0.68 \end{bmatrix}_{3\times3}$ are two fuzzy relations, then $R_1 \circ R_2 =$

$$[c_{ij}];\ (i = 1, 2, 3, 4 \text{ and } j = 1, 2, 3) = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \\ c_{41} & c_{42} & c_{43} \end{bmatrix}_{4\times3}$$

$$R_1 \circ R_2 = \begin{bmatrix} 0.21 & 0.51 & 0.19 \\ 0.37 & 0.92 & 0.76 \\ 0.71 & 0.97 & 0.39 \\ 0.91 & 0.42 & 0.81 \end{bmatrix} \circ \begin{bmatrix} 0.29 & 0.97 & 0.86 \\ 0.35 & 0.75 & 0.49 \\ 0.17 & 0.29 & 0.68 \end{bmatrix} = \begin{bmatrix} 0.35 & 0.51 & 0.49 \\ 0.35 & 0.75 & 0.68 \\ 0.35 & 0.75 & 0.71 \\ 0.35 & 0.91 & 0.86 \end{bmatrix}$$

$$c_{11} = \max[\min(0.21,\ 0.29), \min(0.51,\ 0.35), \min(0.19,\ 0.17)] = \max[0.21,\ 0.35,\ 0.17] = 0.35$$

3.2 Aggregation operator

Let H_p be a HPFS on reference set U and let $h_{Hp} : U \to P[0,1]$ be a function that determines HPFEs. A mapping $O : P[0,1] \to [0,1]$ which is defined as follows is an aggregation operator that gives a fuzzy set $H_{A_i} = \{\langle u, O(h_H(u))\rangle | \forall u_i \in U\}$

$$O\{\mu_{A_i}\} = \frac{\prod_{i=1}^{n}(1+(1-\omega_i)\mu_{A_i})^{\hat{p}_{A_i}} - \prod_{i=1}^{n}(1-\mu_{A_i})^{\hat{p}_{A_i}}}{\prod_{i=1}^{n}(1+(1-\omega_i)\mu_{A_i})^{\hat{p}_{A_i}} + (1-\omega_{im})\prod_{i=1}^{n}(1-\mu_{A_i})^{\hat{p}_{A_i}}} \tag{6}$$

Here $\hat{p}_{A_i}$ and ω_i are immediate probability (Yager et al. 1995) and weights of the membership grades μ_{A_i} and are defined as follows:

$$\hat{p}_{A_i} = \frac{\omega_i \cdot p_{A_i}(\mu_{A_i})}{\sum_{i=1}^{n}\omega_i \cdot p_{A_i}(\mu_{A_i})} \tag{7}$$

$$\omega_i = \frac{d_i}{\sum_{i=1}^{n} d_i} \tag{8}$$

Immediate probabilities and weights of the membership grades satisfy the condition of $\sum_{i=1}^{n}\hat{p}_{A_i} = 1$ and $\sum_{i=1}^{n}\omega_i = 1$. ω_{im} is weight of maximum membership grades μ_{A_i}. $p_{A_i}(\mu_{A_i})$ is probability of the membership grades μ_{A_i} and is calculated using following Gaussian probability distribution function (Huang et al. 2012).

$$p_{A_i}(\mu_{A_i}) = \frac{\xi_i}{\sqrt{2\pi}\varsigma_i}\left(e^{-\frac{(u_k-(\mu_{A_i}-1)\xi_i-m_i)}{2\varsigma_i^2}} + e^{-\frac{(u_k-(1-\mu_{A_i})\xi_i-m_i)}{2\varsigma_i^2}}\right), \quad (i = 1, 2, \ldots, n) \tag{9}$$

Here μ_{A_i}, ξ_i, ς_i and m_i are the membership grades, width, standard deviation, and mean of the fuzzy sets A_i, respectively.

Aggregation operator (Eq. 6) satisfies following property:

$$\min\{\mu_{A_i}\} \leqslant O\{\mu_{A_i}\} \leqslant \max\{\mu_{A_i}\}; \forall \mu_{A_i} \in [0,1]$$

Following example illustrates the process of construction of HPFS and aggregation of HPFEs using proposed aggregation operator.

Example 2 Let $H = \{\langle 1219, \{0.429, 0.473\}\rangle, \langle 1123, \{0.739, 0.774\}\rangle\}$ be a HFS on reference set $U = \{1219, 1123\}$ which is constructed using two fuzzy sets $A_1 = [732, 1042, 1352]$ and $A_2 = [732, 1051, 1370]$. Using weights $\omega_1 = 0.493$, $\omega_2 = 0.507$, standard deviation of (732, 1042, 1352) and Eq. (9) probability of $\mu_{A_1}(1219) = 0.429$ is calculated as follows:

$$p_{A_1}(0.429) = \frac{310}{\sqrt{2 \times 3.14 \times 253.11}} \times \left(\mathrm{e}^{-\frac{(1219-(0.429-1)\times 310-1042)}{2\times 64066.67}} + \mathrm{e}^{-\frac{(1219-(1-0.429)\times 310-1042)}{2\times 64066.67}}\right) = 0.673$$

Similarly, other probabilities of membership grades are calculated and following HPFS is obtained.

$$H_p = \{\langle 1219, \{0.429(0.673), 0.473(0.686)\}\rangle, \langle 1123, \{0.739(0.897), 0.774(0.908)\}\rangle\}$$

Using Eq. (7) corresponding immediate probabilities of membership grades are computed as follows:

$$\hat{p}_{A_1}(0.429) = \frac{0.493 \times 0.673}{(0.493 \times 0.673 + 0.507 \times 0.686)} = 0.488 \quad \text{and} \quad \hat{p}_{A_1}(0.473) = \frac{0.507 \times 0.686}{(0.493 \times 0.673 + 0.507 \times 0.686)} = 0.512,$$

$$\hat{p}_{A_2}(0.739) = \frac{0.493 \times 0.897}{(0.493 \times 0.897 + 0.507 \times 0.908)} = 0.49 \quad \text{and} \quad \hat{p}_{A_2}(0.774) = \frac{0.507 \times 0.908}{(0.493 \times 0.897 + 0.507 \times 0.908)} = 0.51$$

Using weights, immediate probabilities of membership grades and Eq. (6) HPFEs are aggregated to H_{A_1} and H_{A_2} as follows:

$$H_{A_1} = \frac{((((1+(1-0.493)\times 0.429)^{0.488} \times (1+(1-0.507)\times 0.473)^{0.512}) - ((1-0.429)^{0.488} \times (1-0.473)^{0.512})))}{((((1+(1-0.493)\times 0.429)^{0.488} \times (1+(1-0.507)\times 0.473)^{0.512}) + (1-0.507)((1-0.429)^{0.488} \times (1-0.473)^{0.512})))} = 0.453$$

$$H_{A_2} = \frac{((((1+(1-0.493)\times 0.739)^{0.49} \times (1+(1-0.507)\times 0.774)^{0.51}) - ((1-0.739)^{0.49} \times (1-0.774)^{0.51})))}{((((1+(1-0.493)\times 0.739)^{0.49} \times (1+(1-0.507)\times 0.774)^{0.51}) + (1-0.507)((1-0.739)^{0.49} \times (1-0.774)^{0.51})))} = 0.758$$

Finally aggregated fuzzy set $H_A = \{\langle 1219, 0.453\rangle, \langle 1123, 0.758\rangle\}$ is obtained.

3.3 Algorithm of proposed HPFS-based time series forecasting method

Proposed HPFS-based time series forecasting method uses following algorithm.

Algorithm for proposed HPFS-based time series forecasting method

1. Define the universe of discourse and include hesitancy by constructing different types of fuzzy sets to fuzzify time series data.
2. Assign weights to membership grades of time series data in different types of fuzzy sets.
3. Assign probability to membership grades using Gaussian probability distribution function.
4. Calculate the immediate probability of membership grades and determine aggregated fuzzy set using aggregation operator.
5. Use max–min operations on FLR to have fuzzified outputs and defuzzify them numerical forecast by centroid average formula.

Each step of above algorithm is further described in detail as follows:

Step 1: Define universe of discourse as $U = [D_{\min} - \sigma, D_{\max} + \sigma]$, where $D_{\min}$ and $D_{\max}$ are the minimum and maximum observed value and σ is the standard deviation of the data. Fuzzify time series data using more than one valid fuzzification methods. In this paper, universe of discourse is partitioned into equal and unequal intervals and length of unequal intervals are determined using CPDA approach. Collection $\langle u, \mu_1(u), \mu_2(u)\rangle$ is an HFE; where $\mu_1(u)$ and $\mu_2(u)$ are membership grades of a time series datum (u) in fuzzy sets with equal intervals A_{e_i} and unequal intervals A_{ue_i}, respectively.

Step 2: Compute the weights to the triangular membership function ω_i using following expression.

$$\omega_i = \frac{d_i}{\sum_{i=1}^{n} d_i}$$

Here d_i is length of corresponding intervals of fuzzy sets A_i.

Step 3: Take membership grade as random variable and use following probability distribution function (Huang et al. 2012) to associate probabilities.

$$p_{A_i}(\mu_{A_i}) = \frac{\xi_i}{\sqrt{2\pi}\varsigma_i}\left(\mathrm{e}^{-\frac{(u_k-(\mu_{A_i}-1)\xi_i-m_i)}{2\varsigma_i^2}} + \mathrm{e}^{-\frac{(u_k-(1-\mu_{A_i})\xi_i-m_i)}{2\varsigma_i^2}}\right), \ (i = 1,2, ..., n)$$

Hereμ_{A_i}, ξ_i, ς_i and m_i are, respectively, membership grades, width, standard deviation and mean of data that lies in fuzzy sets A_i.

Step 4: Compute immediate probability of membership grades for fuzzy sets A_i is $\hat{p}_{A_i}$ using following expression.

$$\hat{p}_{A_i} = \frac{\omega_i \cdot p_{A_i}(\mu_{A_i})}{\sum_{i=1}^{n} \omega_i \cdot p_{A_i}(\mu_{A_i})}$$

Apply aggregation operator (Eq. 6) to have aggregated fuzzy set. Time series data is again fuzzified using following simple algorithm.

```
for i = 1 to m (end of time series data)
  for j = 1to n (end of intervals)
    choose
    μri=max(μ1i,μ2i,...,μji ),1 ≤ r ≤ n
    if HAr is aggregated fuzzy set corresponding to μri then assign aggregated fuzzy set HAr to ui
   end if
  end for
end for
```

Step 5: Fuzzy logical relations (FLRs) are defined on fuzzy sets that are obtained using aggregation of HPFEs and is denoted as $H_{A_i} \rightarrow H_{A_j}$, where H_{A_i} is the fuzzy production of the year n as current state and H_{A_j} is the fuzzy production of the year $n+1$ as next state.

Use max–min operations (Eq. 5) on FLR to have fuzzified outputs and defuzzify them numerical forecast by following centroid average formula:

$$\text{Numerical forecast} = \frac{\sum_{i=1}^{n} f_i c_i}{\sum_{i=1}^{n} f_i} \tag{10}$$

Here f_i is fuzzified output and c_i is average of centroids for equal and unequal intervals.

For error measure RMSE and AFE are the general tools in fuzzy time series forecasting. RMSE, AFE, correlation coefficient and coefficient of determination are applied to estimate the execution of forecasting model. Following error measures, correlation coefficient and coefficient of determination are defined as:

$$\text{RMSE} = \sqrt{\frac{\sum_{i=1}^{n} (O_i - F_i)^2}{n}} \tag{11}$$

$$\text{Forecasting error(in\%)} = \frac{|F_i - O_i|}{O_i} \times 100 \quad (12)$$

$$\text{AFE(in\%)} = \frac{\text{sum of forecasting error}}{n} \quad (13)$$

$$\text{Coefficient of correlation}(R) = \frac{n\sum O_i F_i - \left(\sum O_i\right)\left(\sum F_i\right)}{\sqrt{n\left(\sum O_i^2\right) - \left(\sum O_i\right)^2}\sqrt{n\left(\sum F_i^2\right) - \left(\sum F_i\right)^2}} \quad (14)$$

$$\text{Coefficient of determination} = R^2 \quad (15)$$

Here O_i and F_i denote the observed and forecasted time series data, n is the number of data points and σ is standard deviation of the given data. Positive and negative value of R indicates positive and negative linear correlation, respectively, between forecasted and observed time series data and it is lies between -1 and 1 and R^2 is a non-negative value.

4 Experimental results

In this section, we apply the proposed fuzzy time series forecasting method in the hesitant probabilistic fuzzy environment for forecasting the enrolments of University of Alabama (Song and Chissom 1993b) and the SBI share prizes at BSE, India (Joshi and Kumar 2012a).

4.1 Forecasting historical enrolments data with proposed method

In this subsection the proposed HPFS-based forecasting method is implemented on historical data of the enrolments at University of Alabama (Table 1).

Step 1: $D_{\min}$ and $D_{\max}$ are the observed from Table 1 to define universe of discourse for historical data of student's enrolments of University of Alabama. Using standard deviation $\sigma = 1775$, universe of discourse for University of Alabama enrolment time series data is defined as $U = [11,280,\ 21,112]$. The lower and upper bound of probabilities for historical enrolments data of University of Alabama are computed using CPDA approach. The universe of discourse has been partitioned into 14 unequal intervals as given in Table 2.

Fourteen equal length intervals of universe of discourse are as follows:

Table 1 Historical enrolments data of University of Alabama

Year	Enrolment	Year	Enrolment
1971	13,055	1982	15,433
1972	13,563	1983	15,497
1973	13,867	1984	15,145
1974	14,696	1985	15,163
1975	15,460	1986	15,984
1976	15,311	1987	16,859
1977	15,603	1988	18,150
1978	15,861	1989	18,970
1979	16,807	1990	19,328
1980	16,919	1991	19,337
1981	16,388	1992	18,876

Table 2 Lower and upper bound of probability for historical enrolments data of University of Alabama

Intervals	Cumulative probability		Universe of discourse U			Length of interval
	P_{LB}	P_{UB}	Lower bound	Midpoint	Upper bound	
ue_1	0	0.071428571	11,280	12,436.7	13,593.39	2313.392
ue_2	0.035714286	0.142857143	12,994.31	13,646.78	14,299.24	1304.931
ue_3	0.107142857	0.214285714	13,989.87	14,389.45	14,789.02	799.155
ue_4	0.178571429	0.285714286	14,559.72	14,874.67	15,189.62	629.9016
ue_5	0.25	0.357142857	14996.96	15270.65	15544.34	547.3805
ue_6	0.321428571	0.428571429	15,371.1	15,622.88	15,874.66	503.5593
ue_7	0.392857143	0.5	15,711.59	15,952.89	16,194.18	482.587
ue_8	0.464285714	0.571428571	16,035.07	16,274.39	16513.7	478.6371
ue_9	0.535714286	0.642857143	16,353.3	16,598.66	16,844.02	490.7236
ue_{10}	0.607142857	0.714285714	16,676.77	16,937.75	17,198.74	521.9721
ue_{11}	0.678571429	0.785714286	17,017.26	17,308.3	17,599.34	582.0773
ue_{12}	0.75	0.857142857	17,391.4	17,740.26	18,089.12	697.7184
ue_{13}	0.821428571	0.928571429	17,828.64	18,311.81	18,794.97	966.3292
ue_{14}	0.892857143	1	18,398.5	19,755.25	21,112	2713.505

$$e_1 = [11,280,11,982.29],\quad e_2 = [11,982.29,12,684.57],\quad e_3 = [12,684.57,13,386.86],$$
$$e_4 = [13,386.86,14,089.14],\quad e_5 = [14,089.14,14,791.43],\quad e_6 = [14,791.43,15,493.71],$$
$$e_7 = [15,493.71,16,196],\quad e_8 = [16,196,16,898.29],\quad e_9 = [16,898.29,17,600.57],$$
$$e_{10} = [17,600.57,18,302.86],\quad e_{11} = [18,302.86,19,005.14],\quad e_{12} = [19,005.14,19,707.43],$$
$$e_{13} = [19,707.43,20,409.71],\quad e_{14} = [20,409.71,21,112]$$

The historical enrolments data of University of Alabama follows the normal distribution; universe of discourse is partitioned into 14 unequal lengths of intervals using CPDA approach.

Following 14 triangular fuzzy sets $A_{e_i}(i = 1,2,3,...,14)$ are constructed in accordance with equal length intervals.

$$A_{e_1} = [11,\ 280,\ 11,\ 982.29, 12,\ 684.57],\ A_{e_2} = [11,\ 982.29,\ 12, 684.57,\ 13,\ 386.86],\ A_{e_3} = [12,\ 684.57,\ 13,\ 386.86,\ 14,\ 089.14],$$
$$A_{e_4} = [13,\ 386.86,\ 14,\ 089.14,\ 14,\ 791.43],\ A_{e_5} = [14,\ 089.14,\ 14,\ 791.43,\ 15,\ 493.71],\ A_{e_6} = [14,\ 791.43,\ 15,\ 493.71,\ 16,\ 196],$$
$$A_{e_7} = [15,\ 493.71,\ 16,\ 196,\ 16,\ 898.29], A_{e_8} = [16,\ 196,\ 16,\ 898.29,\ 17,\ 600.57], A_{e_9} = [16,\ 898.29,\ 17,\ 600.57,\ 18,\ 302.86],$$
$$A_{e_{10}} = [17,\ 600.57,\ 18,\ 302.86,\ 19,\ 005.14], A_{e_{11}} = [18,\ 302.86,\ 19,\ 005.14,\ 19,\ 707.43],\ A_{e_{12}} = [19,\ 005.14,\ 19,\ 707.43,\ 20,\ 409.71],$$
$$A_{e_{13}} = [19,\ 707.43,\ 20,\ 409.71,\ 21,\ 112],\quad A_{e_{14}} = [20,\ 409.71,\ 21,\ 112,\ 21,\ 112].$$

Following 14 triangular fuzzy sets $A_{ue_i}(i = 1,2,3,...,14)$ are constructed in accordance with unequal length intervals.

$$A_{ue_1} = [11,\ 280,\ 12,\ 436.72,\ 13,\ 593.39], A_{ue_2} = [12,\ 994.31,\ 13,\ 646.78,\ 14,\ 299.24], A_{ue_3} = [13,\ 989.87,\ 14,\ 389.45,\ 14,\ 789.02],$$
$$A_{ue_4} = [14,\ 559.72,\ 14,\ 874.67,\ 15,\ 189.62], A_{ue_5} = [14,\ 996.96,\ 15,\ 270.65,\ 15,\ 544.34], A_{ue_6} = [15,\ 371.11,\ 15,\ 622.88,\ 15,\ 874.66],$$
$$A_{ue_7} = [15,\ 711.59,\ 15,\ 952.89,\ 16,\ 194.18], A_{ue_8} = [16,\ 035.07,\ 16,\ 274.39,\ 16,\ 513.71], A_{ue_9} = [16,\ 353.30,\ 16,\ 598.66,\ 16,\ 844.02],$$
$$A_{ue_{10}} = [16,\ 676.77,\ 16,\ 937.3,\ 17,\ 198.74], A_{ue_{11}} = [17,\ 017.26,\ 17,\ 308.30,\ 17,\ 599.34], A_{ue_{12}} = [17,\ 391.40,\ 17,\ 740.26,\ 18,\ 089.12],$$
$$A_{ue_{13}} = [17,\ 828.64,\ 18,\ 311.81,\ 18,\ 794.97], A_{ue_{14}} = [18,\ 398.50,\ 19,\ 755.25,\ 21,\ 112].$$

Step 2: Weights of equal and unequal intervals are calculated using Eq. (8) and are shown in Table 3.

Step 3: Membership grades of each enrolment in triangular fuzzy sets with equal and unequal intervals are calculated. Probabilities that are to be associated with the membership grades are also computed to construct the 14 HPFSs $H_{pi}(i = 1,2,3,...,14)$ (Table 4). Immediate probability of membership grades are computed using probability and weights of equal and unequal intervals. HPFEs are aggregated using proposed aggregation operator (Eq. 6) to have 14 fuzzy sets $H_{A_i}(i = 1,2,3,...,14)$ (Table 5).

Step 4: University of Alabama enrolment time series data are fuzzified using the algorithm for fuzzification which is described in the previous Sect. 3. FLRs are determined and fuzzy logical relationship groups (FLRGs) are computed (Table 6).

Step 5: Using max–min operations (Eq. 5) on FLR and forecast the time series data of University of Alabama from Eq. (10). A sample of calculation for the year 1979 enrolment forecast is as follows:

Table 3 Weights of the equal and unequal intervals

Weights			
For equal intervals	For unequal intervals	For equal intervals	For unequal intervals
0.37778	0.622220197	0.74584	0.25416
0.518387	0.48161284	0.741083	0.258917
0.637362	0.362638033	0.729063	0.270937
0.690386	0.30961414	0.707005	0.292995
0.719573	0.280427253	0.668115	0.331885
0.736098	0.263901888	0.592421	0.407579
0.744278	0.255721512	0.2056	0.7944

Table 4 HPFS of membership grades of enrolment for equal and unequal intervals

Actual enrolments	H_{p1}	H_{p2}	H_{p3}	H_{p4}
13,055	{0/0,0.4655/0.6961}	{0.4725/0.7008,0.093/0.5302}	{0.5275/0.7388,0/0}	{0/0, 0/0}
13,563	{0/0,0.0263/0.5172}	{0/0,0.8716/0.9539}	{0.7492/0.8934,0/0}	{0.2508/0.5795,0/0}
13,867	{0/0, 0/0}	{0/0,0.6625/0.836}	{0.3163/0.609, 0/0}	{0.6837/0.8507,0/0}
14,696	{0/0, 0/0}	{0/0, 0/0}	{0/0,0.2328/0.5723}	{0.1359/0.5408,0.4327/0.6748}
15,460	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}
15,311	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}
15,603	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}
15,861	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}
16,807	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}
16,919	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}
16,388	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}
15,433	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}
15,497	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}
15,145	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0.1417/05423}
15,163	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0.0845/0.5283}
15,984	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}
16,859	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}
18,150	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}
18,970	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}
19,328	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}
19,337	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}
18,876	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}

Actual enrolments	H_{p5}	H_{p6}	H_{p7}
13,055	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}
13,563	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}
13,867	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}
14,696	{0.8641/0.9511,0/0}	{0/0, 0/0}	{0/0, 0/0}
15,460	{0.048/0.521,0.3082/0.605}	{0.952/0.9741,0.3531/0.628}	{0/0, 0/0}
15,311	{0.2602/0.5833,0.8526/0.9466}	{0.7398/0.8876, 0/0}	{0/0, 0/0}
15,603	{0/0, 0/0}	{0.8444/0.9432,0.921/0.9684}	{0.1556/0.5463,0/0}
15,861	{0/0, 0/0}	{0.477/0.7039,0.0543/0.5221}	{0.523/0.7357,0.6192/0.805}
16,807	{0/0, 0/0}	{0/0, 0/0}	{0.13/0.5392, 0/0}
16,919	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}
16,388	{0/0, 0/0}	{0/0, 0/0}	{0.7266/0.8793, 0/0}
15,433	{0.0865/0.5287,0.4068/0.6588}	{0.9135/0.9666,0.2458/0.5775}	{0/0, 0/0}
15,497	{0/0, 0.173/0.5515}	{0.9953/0.9774, 0.5/0.7196}	{0.0047/0.5137, 0/0}
15,145	{0.4965/0.7172,0.5409/0.7484}	{0.5035/0.722, 0/0}	{0/0, 0/0}
15,163	{0.471/0.7, 0.6067/0.796}	{0.5291/0.74, 0/0}	{0/0, 0/0}
15,984	{0/0, 0/0}	{0.3019/0.602, 0/0}	{0.6981/0.861,0.8711/0.9537}
16,859	{0/0, 0/0}	{0/0, 0/0}	{0.0559/0.5224, 0/0}
18,150	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}
18,970	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}
19,328	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}
19,337	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}
18,876	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}

Actual enrolments	H_{p8}	H_{p9}	H_{p10}	H_{p11}
13,055	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}
13,563	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}

Table 4 (continued)

Actual enrolments	H_{p8}	H_{p9}	H_{p10}	H_{p11}
13,867	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}
14,696	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}
15,460	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}
15,311	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}
15,603	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}
15,861	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}
16,807	{0.87/0.9533,0/0}	{0/0,0.1509/0.5449}	{0/0,0.499/0.7189}	{0/0, 0/0}
16,919	{0.9705/0.9762,0/0}	{0.0295/0.5177,0/0}	{0/0,0.9281/0.9699}	{0/0, 0/0}
16,388	{0.2734/0.589,0.5253/0.7373}	{0/0,0.1414/0.5423}	{0/0, 0/0}	{0/0, 0/0}
15,433	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}
15,497	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}
15,145	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}
15,163	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}
15,984	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}
16,859	{0.9441/0.9729, 0/0}	{0/0, 0/0}	{0/0,0.6982/0.8606}	{0/0, 0/0}
18,150	{0/0, 0/0}	{0.2177/0.5666,0/0}	{0.7823/0.9127,0/0}	{0/0, 0/0}
18,970	{0/0, 0/0}	{0/0, 0/0}	{0.05/0.5213, 0/0}	{0.95/0.9738,0/0}
19,328	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}	{0.5403/0.748,0/0}
19,337	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}	{0.5275/0.7388,0/0}
18,876	{0/0, 0/0}	{0/0, 0/0}	{0.1839/0.555, 0/0}	{0.8161/0.9303,0/0}

Actual enrolments	H_{p12}	H_{p13}	H_{p14}
13,055	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}
13,563	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}
13,867	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}
14,696	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}
15,460	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}
15,311	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}
15,603	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}
15,861	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}
16,807	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}
16,919	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}
16,388	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}
15,433	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}
15,497	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}
15,145	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}
15,163	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}
15,984	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}
16,859	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}
18,150	{0/0, 0/0}	{0/0,0.6651/0.8378}	{0/0, 0/0}
18,970	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0.4212/0.6676}
19,328	{0.4597/0.6923,0/0}	{0/0, 0/0}	{0/0, 0.6851/0.8517}
19,337	{0.4725/0.7008,0/0}	{0/0, 0/0}	{0/0, 0.6917/0.8562}
18,876	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0.3519/0.6274}

For enrolment of year 1978, hesitant probabilistic fuzzy sets are $H_{p6} = \langle 15861, \{0.477014(0.70386), 0.054253(0.522124)\}\rangle$ and $H_{p7} = \langle 15861, \{0.522986(0.735674), 0.619184(0.805042)\}\rangle$.

Here 0.477014, 0.054253, 0.522986, 0.619184 are the membership grades of the enrolment 15,861 in fuzzy sets $A_{e_6}, A_{ue_6}, A_{e_7}, A_{ue_7}$ respectively. 0.736098 and 0.263902 are the weights in Table 3 of A_{e_6}, A_{ue_6}. Immediate probabilities, $\hat{p}_{A_{e_6}}$ and $\hat{p}_{A_{ue_6}}$ for enrolment of the year 1978 are computed as follows:

Table 5 Aggregated hesitant fuzzified data of University of Alabama

Actual enrolments	H_{A1}	H_{A2}	H_{A3}	H_{A4}	H_{A5}	H_{A6}	H_{A7}	H_{A8}	H_{A9}	H_{A10}	H_{A11}	H_{A12}	H_{A13}	H_{A14}	Aggregated hesitant fuzzified
13,055	0.465457	0.334098	0.527461	0	0	0	0	0	0	0	0	0	0	0	H_{A3}
13,563	0.026275	0.871597	0.749186	0.250814	0	0	0	0	0	0	0	0	0	0	H_{A2}
13,867	0	0.662478	0.316314	0.683686	0	0	0	0	0	0	0	0	0	0	H_{A4}
14,696	0	0	0.232804	0.272812	0.864117	0	0	0	0	0	0	0	0	0	H_{A5}
15,460	0	0	0	0	0.145938	0.916317	0	0	0	0	0	0	0	0	H_{A6}
15,311	0	0	0	0	0.615294	0.739829	0	0	0	0	0	0	0	0	H_{A6}
15,603	0	0	0	0	0	0.893859	0.155614	0	0	0	0	0	0	0	H_{A6}
15,861	0	0	0	0	0	0.398007	0.607941	0	0	0	0	0	0	0	H_{A7}
16,807	0	0	0	0	0	0	0.129984	0.870016	0.15088168	0.498997	0	0	0	0	H_{A8}
16,919	0	0	0	0	0	0	0	0.970504	0.02949552	0.928138	0	0	0	0	H_{A8}
16,388	0	0	0	0	0	0	0.726607	0.403437	0.14143607	0	0	0	0	0	H_{A7}
15,433	0	0	0	0	0.219475	0.865708	0	0	0	0	0	0	0	0	H_{A6}
15,497	0	0	0	0	0.17298	0.986486	0.004679	0	0	0	0	0	0	0	H_{A6}
15,145	0	0	0	0.141681	0.532961	0.503458	0	0	0	0	0	0	0	0	H_{A5}
15,163	0	0	0	0.08453	0.543022	0.529089	0	0	0	0	0	0	0	0	H_{A5}
15,984	0	0	0	0	0	0.301871	0.799093	0	0	0	0	0	0	0	H_{A7}
16,859	0	0	0	0	0	0	0.05594	0.94406	0	0.698241	0	0	0	0	H_{A8}
18,150	0	0	0	0	0	0	0	0	0.21765663	0.782343	0	0	0.66511	0	H_{A10}
18,970	0	0	0	0	0	0	0	0	0	0.050041	0.949959	0	0	0.42123	H_{A11}
19,328	0	0	0	0	0	0	0	0	0	0	0.540277	0.459723	0	0.685095	H_{A14}
19,337	0	0	0	0	0	0	0	0	0	0	0.527461	0.472539	0	0.691729	H_{A14}
18,876	0	0	0	0	0	0	0	0	0	0.183889	0.816111	0	0	0.351947	H_{A11}

$$\hat{p}_{A_{e_6}} = \frac{0.736098 \times 0.70386}{0.736098 \times 0.70386 + 0.263902 \times 0.522124} = 0.789923,$$

$$\hat{p}_{A_{ue_6}} = \frac{0.263902 \times 0.522124}{0.736098 \times 0.70386 + 0.263902 \times 0.522124} = 0.210077$$

Membership grades of the enrolment of the year 1978 (15,861) in H_{p6} are aggregated using Eq. (6) as follows:

$$\frac{((((1 + (1 - 0.736098) \times 0.477014)^{0.789923} \times (1 + (1 - 0.263902) \times 0.054253)^{0.210077}) - ((1 - 0.477014)^{0.789923} \times (1 - 0.054253)^{0.210077})))}{((((1 + (1 - 0.736098) \times 0.477014)^{0.789923} \times (1 + (1 - 0.263902) \times 0.054253)^{0.210077}) + (1 - 0.477014) \times ((1 - 0.738507)^{0.789923} \times (1 - 0.054253)^{0.210077})))} = 0.398$$

Immediate probabilities $\hat{p}_{A_{e_7}}$ and $\hat{p}_{A_{ue_7}}$ for enrolment of the year 1978 are computed using weights 0.744278 and 0.255722 as follows:

$$\hat{p}_{A_{e_7}} = \frac{0.744278 \times 0.735674}{0.744278 \times 0.735674 + 0.255722 \times 0.805042} = 0.726755,$$

$$\hat{p}_{A_{ue_7}} = \frac{0.255722 \times 0.805042}{0.744278 \times 0.735674 + 0.255722 \times 0.805042} = 0.273245$$

Membership grades of the enrolment of the year 1978 (15,861) in H_{p7} are aggregated using Eq. (6) as follows:

$$\frac{((((1 + (1 - 0.744278) \times 0.522986)^{0.726755} \times (1 + (1 - 0.255722) \times 0.619184)^{0.273245}) - ((1 - 0.522986)^{0.726755} \times (1 - 0.619184)^{0.273245})))}{((((1 + (1 - 0.744278) \times 0.522986)^{0.726755} \times (1 + (1 - 0.255722) \times 0.619184)^{0.273245}) + (1 - 0.744278)((1 - 0.522986)^{0.726755} \times (1 - 0.619184)^{0.273245})))} = 0.6079$$

Table 6 FLRs and FLRGs for the data of University of Alabama

FLRs				
$H_{A3} \to H_{A2}$	$H_{A2} \to H_{A4}$	$H_{A4} \to H_{A5}$	$H_{A5} \to H_{A6}$	$H_{A6} \to H_{A6}$
$H_{A6} \to H_{A6}$	$H_{A6} \to H_{A7}$	$H_{A7} \to H_{A8}$	$H_{A8} \to H_{A8}$	$H_{A8} \to H_{A7}$
$H_{A7} \to H_{A6}$	$H_{A6} \to H_{A6}$	$H_{A6} \to H_{A5}$	$H_{A5} \to H_{A5}$	$H_{A5} \to H_{A7}$
$H_{A7} \to H_{A8}$	$H_{A8} \to H_{A10}$	$H_{A10} \to H_{A11}$	$H_{A11} \to H_{A14}$	$H_{A14} \to H_{A14}$
$H_{A14} \to H_{A11}$				
FLRGs				
$H_{A2} \to H_{A4}$				
$H_{A3} \to H_{A2}$				
$H_{A4} \to H_{A5}$				
$H_{A5} \to H_{A5}, H_{A6}, H_{A7}$				
$H_{A6} \to H_{A5}, H_{A6}, H_{A7}$				
$H_{A7} \to H_{A6}, H_{A8}$				
$H_{A8} \to H_{A7}, H_{A8}, H_{A10}$				
$H_{A10} \to H_{A11}$				
$H_{A11} \to H_{A14}$				
$H_{A14} \to H_{A11}, H_{A14}$				

Since max(0.398, 0.6079) = 0.6079, hence H_{A_7} is assigned to 15,861, i.e., the enrolment of the year 1978.

FLR $H_{A_7} \to H_{A_8}$ is used to forecast the enrolment for the year 1978. Following row vector is obtained using max–min operation (Eq. 5).

0	0	0	0	0.615294	0.799093	0.155614	0.799093	0.150882	0.799093	0	0	0	0

Since the centroids of the fuzzy set with equal interval A_{e_1} and unequal intervals A_{ue_1} are 11,982.29 and 12,436.7, respectively, therefore, average centroid is

$$c_1 = \frac{11,982.29 + 12,436.7}{2} = 12,209.49$$

Similarly other average centroids $c_2, c_3, \ldots, c_{14}$ are calculated.

Numerical forecast for the year 1979 is calculated using Eq. (10) and is as follows:

$$\frac{0.615294 \times 15031.04 + 0.799093 \times 15558.3 + 0.155614 \times 16074.44 + 0.799093 \times 16586.34 + 0.150882 \times 17099.62 + 0.799093 \times 17620.31}{0.615294 + 0.799093 + 0.155614 + 0.799093 + 0.150882 + 0.799093} = 16298.77$$

Other enrolments of University of Alabama are also computed in similar way and are shown in Table 7. Proposed hesitant probabilistic fuzzy time series forecasting method is compared with previous proposed forecasting method of Chen (1996), Cheng et al. (2006, 2008), Huarng (2001), Joshi and Kumar (2012a), Kumar and Gangwar (2016), Lee and Chou (2004), Qiu et al. (2011), Song and Chissom (1993a, b, 1994) and Yolcu et al. (2009) using error measures, correlation coefficient, and coefficient of determination (Table 15).

4.2 Proposed method for forecasting market price of SBI share

In this subsection, proposed forecasting method is implemented on market prices of SBI share at BSE, India (Table 8).

Step 1: $D_{\min}$ and $D_{\max}$ are observed from Table 8 to define universe of discourse for data of SBI share prices. Using standard deviation $\sigma = 391.07$, universe of discourse for SBI share price is defined as $U =$ [741.18, 2891.07]. The lower and upper bound of probabilities for market prices

Table 7 Forecasted enrolments

Actual enrolment	Song and Chissom (1993a)	Chen (1996)	Huarng (2001)	Lee and Chou (2004)	Song and Chissom (1994)	Cheng et al. (2006)	Cheng et al. (2008)	Yolcu et al. (2009)	Qiu et al. (2011)	Joshi and Kumar (2012a)	Kumar and Gangwar (2016)	Bisht and Kumar (2016)	Proposed method
13,055	–	–	–	–	–	–	–	–	–	–	–	–	–
13,563	14,000	14,000	–	14,025	–	14,230	14,242	14,031.35	14,195	14,250	–	13,595.67	13,680.75
13,867	14,000	14,000	–	14,568	–	14,230	14,242	14,795.36	14,424	14,246	13,963	13,814.75	13,844.43
14,696	14,000	14,000	14,000	14,568	–	14,230	14,242	14,795.36	14,593	14,246	13,963	14,929.79	14,951.36
15,460	15,500	15,500	15,500	15,654	14,700	15,541	15,474.3	14,795.36	15,589	15,491	14,867	15541.27	15532.34
15,311	16,000	16,000	15,500	15,654	14,800	15,541	15474.3	16,406.57	15,645	15,491	15,287	15,540.62	15,533.19
15,603	16,000	16,000	16,000	15,654	15,400	15,541	15,474.3	16,406.57	15,634	15,491	15,376	15,540.62	15,533.19
15,861	16,000	16,000	16,000	15,654	15,500	16,196	15474.3	16,406.57	16,100	16,345	15,376	15,540.62	15,533.19
16,807	16,000	16,000	16,000	16,197	15,500	16,196	16,146.5	16,406.57	16,188	16,345	15,376	16,254.5	16,298.77
16,919	16,813	16,833	17,500	17,283	16,800	16,196	16,988.3	17,315.29	17,077	15,850	16,523	17,040.41	17,113.79
16,388	16,813	16,833	16,000	17,283	16,200	17,507	16,988.3	17315.29	17,105	15,850	16,606	17,040.41	17,113.79
15,433	16,789	16,833	16,000	16,197	16,400	16,196	16,146.5	17,315.29	16,369	15,850	17,519	16,254.5	16,298.77
15,497	16,000	16,000	16,000	15,654	16,800	15,541	15,474.3	16,406.57	15,643	15,450	16,606	15540.62	15533.19
15,145	16,000	16,000	15,500	15,654	16,400	15,541	15474.3	16406.57	15,648	15,450	15,376	15540.62	15533.19
15,163	16,000	16,000	16,000	15,654	15,500	15,541	15,474.3	16,406.57	15,622	15,491	15,376	15,541.27	15,532.34
15,984	16,000	16,000	16,000	15,654	15,500	15,541	15,474.3	16,406.57	15,623	15,491	15,287	15,541.27	15,532.34
16,859	16,000	16,000	16,000	16,197	15,500	16,196	16,146.5	16,406.57	16,231	16,345	15,287	16,254.5	16,298.77
18,150	16,813	16,833	17,500	17,283	16,800	17,507	16,988.3	17,315.29	17,090	17,950	16,523	17,040.41	17,113.79
18,970	19,000	19,000	19,000	18,369	19,300	18,872	19,144	19132.79	18,325	18,961	17,519	18,902.3	18,741.35
19,328	19,000	19,000	19,000	19,454	17,800	18,872	19,144	19,132.79	19,000	18,961	19,500	19,357.3	19,190.44
19,337	19,000	19,000	19,500	19,454	19,300	18,872	19,144	19,132.79	19,000	18,961	19,000	19,168.56	18,972.15
18,876	–	19,000	19,000	–	19,600	18,872	19,144	19,132.79	19,000	18,961	19,500	19,168.56	18,972.15

Table 8 Actual market prices of SBI share at BSE, India

Months	SBI prices	Months	SBI prices
Apr-08	1819.95	Apr-09	1355
May-08	1840	May-09	1891
Jun-08	1496.7	Jun-09	1935
Jul-08	1567.5	Jul-09	1840
Aug-08	1638.9	Aug-09	1886
Sep-08	1618	Sep-09	2235
Oct-08	1569.9	Oct-09	2500
Nov-08	1375	Nov-09	2394
Dec-08	1325	Dec-09	2374
Jan-09	1376.4	Jan-10	2315
Feb-09	1205.9	Feb-10	2059.95
Mar-09	1132.25	Mar-10	2120.05

of SBI share at BSE, India are computed using CPDA approach. The universe of discourse has been partitioned into 14 unequal intervals as given in Table 9.

Fourteen equal length intervals of universe of discourse are as follows:

$$e_1 = [741.18,\ 894.75],\ e_2 = [894.75,\ 1048.31], e_3 = [1048.31,\ 1201.87], e_4 = [1201.87,\ 1355.44],$$
$$e_5 = [1355.44,\ 1509],\ e_6 = [1509,\ 1662.56], e_7 = [1662.56,\ 1816.13],\ e_8 = [1816.13,\ 1969.69],$$
$$e_9 = [1969.69,\ 2123.25], e_{10} = [2123.25,\ 2276.81], e_{11} = [2276.81,\ 2430.38],$$
$$e_{12} = [2430.38,\ 2583.94], e_{13} = [2583.94,\ 2737.5], e_{14} = [2737.5,\ 2891.07]$$

Universe of discourse of market prices of SBI share at BSE, India is partitioned into 14 unequal lengths of intervals using CPDA approach.

Step 2: Weights of equal and unequal intervals are calculated using Eq. (8) and are shown in Table 10.

Table 10 Weights of the equal and unequal intervals

Weights			
For equal intervals	For unequal intervals	For equal intervals	For unequal intervals
0.3941504	0.60585	0.744406	0.255594
0.5165022	0.483498	0.739632	0.260368
0.6356153	0.364385	0.727569	0.272431
0.6887699	0.31123	0.705439	0.294561
0.7180469	0.281953	0.666439	0.333561
0.734629	0.265371	0.590597	0.409403
0.742839	0.257161	0.198734	0.801266

Step 3: Membership grades of SBI share price in triangular fuzzy sets with equal and unequal intervals are calculated. Probabilities that are to be associated with the membership grades are also computed to construct the 14 HPFSs $H_{pi}(i = 1, 2, 3, ..., 14)$(Table 11). Immediate probability of membership grades are computed using probability and weights of equal and unequal intervals. HPFEs are aggregated using proposed aggregation operator (Eq. 6) to have 14 fuzzy sets $H_{A_i}(i = 1, 2, 3, ..., 14)$(Table 12).

Step 4: SBI share price data are fuzzified using the algorithm for fuzzification which is described in the previous Sect. 3. FLRs are determined and fuzzy logical relationship groups (FLRGs) are computed (Table 13).

Table 9 Lower bound and upper bound of probability for market prices of SBI share at BSE, India

Intervals	Cumulative probability		Universe of discourse U			Length of interval
	P_{LB}	P_{UB}	Lower bound	Midpoint	Upper bound	
ue_1	0	0.071428571	741.1846	977.2266	1213.269	472.084
ue_2	0.035714286	0.142857143	1081.28	1225.031	1368.781	287.5006
ue_3	0.107142857	0.214285714	1300.62	1388.654	1476.688	176.0687
ue_4	0.178571429	0.285714286	1426.169	1495.558	1564.948	138.779
ue_5	0.25	0.357142857	1522.501	1582.8	1643.099	120.5981
ue_6	0.321428571	0.428571429	1604.931	1660.403	1715.874	110.9434
ue_7	0.392857143	0.5	1679.948	1733.109	1786.271	106.3229
ue_8	0.464285714	0.571428571	1751.215	1803.941	1856.667	105.4526
ue_9	0.535714286	0.642857143	1821.327	1875.385	1929.442	108.1155
ue_{10}	0.607142857	0.714285714	1892.594	1950.094	2007.594	115.0001
ue_{11}	0.678571429	0.785714286	1967.611	2031.732	2095.853	128.2424
ue_{12}	0.75	0.857142857	2050.04	2126.901	2203.761	153.7203
ue_{13}	0.821428571	0.928571429	2146.373	2252.823	2359.273	212.9002
ue_{14}	0.892857143	1	2271.922	2581.494	2891.065	619.1434

Table 11 HPFS of Membership grades of the data of share market prices of SBI for equal and unequal intervals

Actual SBI prices	H_{p1}	H_{p2}	H_{p3}	H_{p4}
1819.95	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}
1840	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}
1496.7	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}	{0.0801/0.5273,0.9835/0.9771}
1567.5	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}
1638.9	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}
1618	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}
1569.9	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}
1375	{0/0, 0/0}	{0/0, 0/0}	{0/0,0.8449/0.9434}	{0.8726/0.9542, 0/0}
1325	{0/0, 0/0}	{0/0, 0.3046/0.6033}	{0.1982/0.5598, 0.2769/0.5906}	{0.8018/0.9231, 0/0}
1376.4	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0.8608/0.9499}	{0.8635/0.9509, 0/0}
1205.9	{0/0, 0.0312/0.518}	{0/0, 0.8669/0.9522}	{0.9738/0.9764, 0/0}	{0.0262/0.5171, 0/0}
1132.25	{0/0,0.3432/0.6227}	{0.4534/0.6882, 0.3546/0.6288}	{0.5466/0.7525, 0/0}	{0/0, 0/0}
1355	{0/0, 0/0}	{0/0, 0.0959/0.5308}	{0.0028/0.5135, 0.6177/0.804}	{0.9972/0.9974, 0/0}
1891	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}
1935	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}
1840	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}
1886	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}
2235	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}
2500	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}
2394	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}
2374	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}
2315	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}
2059.95	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}
2120.05	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}

Actual SBI prices	H_{p5}	H_{p6}
1819.95	{0/0, 0/0}	{0/0, 0/0}
1840	{0/0, 0/0}	{0/0, 0/0}
1496.7	{0.9199/0.9681, 0/0}	{0/0, 0/0}
1567.5	{0.619/0.8049, 0.7463/0.8916}	{0.381/0.6435, 0/0}
1638.9	{0.1541/0.5458, 0.0696/0.5251}	{0.8459/0.9439, 0.6124/0.8001}
1618	{0.2902/0.5965, 0.4162/0.6646}	{0.7098/0.8683, 0.2356/0.5734}
1569.9	{0.6034/0.7936, 0.7861/0.9147}	{0.3966/0.6527, 0/0}
1375	{0.1274/0.5385, 0/0}	{0/0, 0/0}
1325	{0/0, 0/0}	{0/0, 0/0}
1376.4	{0.1365/0.5409, 0/0}	{0/0, 0/0}
1205.9	{0/0, 0/0}	{0/0, 0/0}
1132.25	{0/0, 0/0}	{0/0, 0/0}
1355	{0/0, 0/0}	{0/0, 0/0}
1891	{0/0, 0/0}	{0/0, 0/0}
1935	{0/0, 0/0}	{0/0, 0/0}
1840	{0/0, 0/0}	{0/0, 0/0}
1886	{0/0, 0/0}	{0/0, 0/0}
2235	{0/0, 0/0}	{0/0, 0/0}
2500	{0/0, 0/0}	{0/0, 0/0}
2394	{0/0, 0/0}	{0/0, 0/0}
2374	{0/0, 0/0}	{0/0, 0/0}
2315	{0/0, 0/0}	{0/0, 0/0}
2059.95	{0/0, 0/0}	{0/0, 0/0}
2120.05	{0/0, 0/0}	{0/0, 0/0}

Table 11 (continued)

Actual SBI prices	H_{p7}	H_{p8}	H_{p9}	H_{p10}
1819.95	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}
1840	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}
1496.7	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}
1567.5	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}
1638.9	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}
1618	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}
1569.9	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}
1375	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}
1325	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}
1376.4	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}
1205.9	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}
1132.25	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}
1355	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}
1891	{0.5124/0.7282, 0/0}	{0.4876/0.711, 0/0}	{0/0,0.7111/0.8692}	{0/0, 0/0}
1935	{0.2259/0.5697, 0/0}	{0.7741/0.9081, 0/0}	{0/0, 0/0}	{0/0, 0.7375/0.8862}
1840	{0.8445/0.9433, 0/0}	{0.1555/0.5462, 0.3161/0.6089}	{0/0, 0.3454/0.6239}	{0/0, 0/0}
1886	{0.545/0.7513, 0/0}	{0.455/0.6892, 0/0}	{0/0, 0.8036/0.9241}	{0/0, 0/0}
2235	{0/0, 0/0}	{0/0, 0/0}	{0.2723/0.5885, 0/0}	{0.7277/0.88, 0/0}
2500	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}
2394	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}	{0.2369/0.5739,0/0}
2374	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}	{0.3671/0.6357,0/0}
2315	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}	{0.7513/0.8947,0/0}
2059.95	{0/0, 0/0}	{0.4122/0.6621, 0/0}	{0.5878/0.7823, 0/0}	{0/0, 0/0}
2120.05	{0/0, 0/0}	{0.0208/0.5163, 0/0}	{0.9792/0.9768, 0/0}	{0/0, 0/0}

Actual SBI prices	H_{p11}	H_{p12}	H_{p13}	H_{p14}
1819.95	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}
1840	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}
1496.7	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}
1567.5	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}
1638.9	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}
1618	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}
1569.9	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}
1375	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}
1325	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}
1376.4	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}
1205.9	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}
1132.25	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}
1355	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}
1891	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}
1935	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}
1840	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}
1886	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0/0}
2235	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0.8326/0.938}	{0/0, 0/0}
2500	{0.5466/0.7525, 0/0}	{0.4534/0.6882, 0/0}	{0/0, 0/0}	{0/0, 0.7368/0.8857}
2394	{0.7631/0.9017, 0/0}	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0.3943/0.6513}
2374	{0.6329/0.8149, 0/0}	{0/0, 0/0}	{0/0, 0/0}	{0/0, 0.3297/0.6157}
2315	{0.2487/0.5786, 0/0}	{0/0, 0/0}	{0/0, 0.4159/0.6643}	{0/0, 0.1392/0.5416}
2059.95	{0/0, 0.5599/0.7621}	{0/0, 0.1289/0.5389}	{0/0, 0/0}	{0/0, 0/0}
2120.05	{0/0, 0/0}	{0/0, 0.9109/0.9659}	{0/0, 0/0}	{0/0, 0/0}

Table 12 Aggregate hesitant fuzzified data of share prices of SBI

Months	Actual SBI prices	H_{A1}	H_{A2}	H_{A3}	H_{A4}	H_{A5}	H_{A6}	H_{A7}	H_{A8}	H_{A9}	H_{A10}	H_{A11}	H_{A12}	H_{A13}	H_{A14}	Aggregated hesitant fuzzified
Apr-08	1819.95	0	0	0	0	0	0	0.975092	0.322739	0	0	0	0	0	0	H_{A7}
May-08	1840	0	0	0	0	0	0	0.844526	0.174584	0.34543	0	0	0	0	0	H_{A7}
Jun-08	1496.7	0	0	0	0.820618	0.919908	0	0	0	0	0	0	0	0	0	H_{A5}
Jul-08	1567.5	0	0	0	0	0.618269	0.380957	0	0	0	0	0	0	0	0	H_{A5}
Aug-08	1638.9	0	0	0	0	0.137178	0.81728	0	0	0	0	0	0	0	0	H_{A6}
Sep-08	1618	0	0	0	0	0.356465	0.653813	0	0	0	0	0	0	0	0	H_{A6}
Oct-08	1569.9	0	0	0	0	0.629162	0.396586	0	0	0	0	0	0	0	0	H_{A5}
Nov-08	1375	0	0	0.844901	0.872601	0.127399	0	0	0	0	0	0	0	0	0	H_{A4}
Dec-08	1325	0	0.304563	0.21186	0.801799	0	0	0	0	0	0	0	0	0	0	H_{A4}
Jan-09	1376.4	0	0	0.860804	0.863484	0.136516	0	0	0	0	0	0	0	0	0	H_{A4}
Feb-09	1205.9	0.031217	0.866917	0.973778	0.026222	0	0	0	0	0	0	0	0	0	0	H_{A3}
Mar-09	1132.25	0.343238	0.406243	0.546614	0	0	0	0	0	0	0	0	0	0	0	H_{A3}
Apr-09	1355	0	0.095867	0.341173	0.997159	0	0	0	0	0	0	0	0	0	0	H_{A4}
May-09	1891	0	0	0	0	0	0	0.512415	0.487585	0.711135	0	0	0	0	0	H_{A9}
Jun-09	1935	0	0	0	0	0	0	0.225887	0.774113	0	0.7375	0	0	0	0	H_{A8}
Jul-09	1840	0	0	0	0	0	0	0.844526	0.174584	0.34543	0	0	0	0	0	H_{A7}
Aug-09	1886	0	0	0	0	0	0	0.544975	0.455025	0.803629	0	0	0	0	0	H_{A9}
Sep-09	2235	0	0	0	0	0	0	0	0	0.272291	0.727709	0	0	0.83257	0	H_{A13}
Oct-09	2500	0	0	0	0	0	0	0	0	0	0	0.546614	0.453386	0	0.736753	H_{A14}
Nov-09	2394	0	0	0	0	0	0	0	0	0	0.236884	0.763116	0	0	0.394345	H_{A11}
Dec-09	2374	0	0	0	0	0	0	0	0	0	0.367124	0.632876	0	0	0.32974	H_{A11}
Jan-10	2315	0	0	0	0	0	0	0	0	0	0.751332	0.248668	0	0.415904	0.139154	H_{A10}
Feb-10	2059.95	0	0	0	0	0	0	0	0.412214	0.587786	0	0.559929	0.12893	0	0	H_{A9}
Mar-10	2120.05	0	0	0	0	0	0	0	0.020844	0.979156	0	0	0.91087	0	0	H_{A9}

Table 13 FLRs and FLRGs for data of share price of SBI

FLRs				
$H_{A7} \rightarrow H_{A7}$	$H_{A7} \rightarrow H_{A5}$	$H_{A5} \rightarrow H_{A5}$	$H_{A5} \rightarrow H_{A6}$	$H_{A6} \rightarrow H_{A6}$
$H_{A6} \rightarrow H_{A5}$	$H_{A5} \rightarrow H_{A4}$	$H_{A4} \rightarrow H_{A4}$	$H_{A4} \rightarrow H_{A4}$	$H_{A4} \rightarrow H_{A3}$
$H_{A3} \rightarrow H_{A3}$	$H_{A3} \rightarrow H_{A4}$	$H_{A4} \rightarrow H_{A9}$	$H_{A9} \rightarrow H_{A8}$	$H_{A8} \rightarrow H_{A7}$
$H_{A7} \rightarrow H_{A9}$	$H_{A9} \rightarrow H_{A13}$	$H_{A13} \rightarrow H_{A14}$	$H_{A14} \rightarrow H_{A11}$	$H_{A11} \rightarrow H_{A11}$
$H_{A11} \rightarrow H_{A10}$	$H_{A10} \rightarrow H_{A9}$	$H_{A9} \rightarrow H_{A9}$		

FLRGs
$H_{A3} \rightarrow H_{A3}, H_{A4}$
$H_{A4} \rightarrow H_{A3}, H_{A4}, H_{A9}$
$H_{A5} \rightarrow H_{A4}, H_{A5}, H_{A6}$
$H_{A6} \rightarrow H_{A5}, H_{A6}$
$H_{A7} \rightarrow H_{A5}, H_{A7}, H_{A9}$
$H_{A8} \rightarrow H_{A7}$
$H_{A9} \rightarrow H_{A8}, H_{A9}, H_{A13}$
$H_{A10} \rightarrow H_{A9}$
$H_{A11} \rightarrow H_{A10}, H_{A11}$
$H_{A13} \rightarrow H_{A14}$
$H_{A14} \rightarrow H_{A11}$

Table 14 Forecasted SBI share prices data

Months	Actual SBI prices	Chen (1996)	Huarng (2001)	Pathak and Singh (2011)	Joshi and Kumar (2012a)	Kumar and Gangwar (2016)	Bisht and Kumar (2016)	Proposed method
Apr-08	1819.95	–	–	–	–	–	–	–
May-08	1840	1900	1855	1770	1777.8	1725.98	1877.657	1860.08
Jun-08	1496.7	1900	1855	1832.5	1865.7	1725.98	1877.657	1860.08
Jul-08	1567.5	1500	1575	1470	1531.5	1512.39	1466.36	1452.59
Aug-08	1638.9	1500	1505	1570	1531.5	1512.39	1466.36	1452.59
Sep-08	1618	1600	1610	1670	1777.8	1574.35	1533.504	1544.29
Oct-08	1569.9	1600	1610	1603.33	1531.5	1574.35	1533.504	1544.29
Nov-08	1375	1500	1505	1670	1531.5	1512.39	1466.36	1452.59
Dec-08	1325	1433	1482	1382.5	1504.23	1305.52	1520.652	1682.31
Jan-09	1376.4	1433	1365	1332.5	1504.23	1665.9	1520.652	1682.31
Feb-09	1205.9	1433	1482	1332.5	1504.23	1305.52	1520.652	1682.31
Mar-09	1132.25	1433	1155	1195	1258.23	1294.27	1144.718	1264.98
Apr-09	1355	1300	1365	1145	1258.23	1294.27	1322.446	1264.98
May-09	1891	1433	1482	1357.5	1504.23	1665.9	1520.652	1682.31
Jun-09	1935	190	1890	1882.5	1865.71	2006.51	1877.657	2138.21
Jul-09	1840	1900	1890	1970	1883.93	2006.51	1895.491	1853.54
Aug-09	1886	1900	1855	1470	1865.71	1725.98	1877.657	1860.08
Sep-09	2235	1900	1855	1970	1865.71	2006.51	1877.657	2138.21
Oct-09	2500	2300	2485	2245	2142.04	2520	2311.382	2466.99
Nov-09	2394	2300	2415	2470	2245.65	2420	2374.204	2328.48
Dec-09	2374	2300	2345	2395	2191.75	2365.99	2352.723	2321.66
Jan-10	2315	2300	2205	2395	2191.75	2365.99	2352.723	2321.66
Feb-10	2059.95	2300	2205	2295	2142.04	2020	2311.382	2070.4
Mar-10	2120.05	2100	2135	2070	1883.93	2120	2166.247	2138.21

Table 15 Error measure, correlation coefficient and coefficient of determination verification of propose method in forecasted enrolments of University of Alabama

Model	RMSE	AFE	R	R^2
Song and Chissom (1993a)	650.4	3.22	0.9173	0.8414
Song and Chissom (1994)	880.73	3.75	0.8317	0.6917
Chen (1996)	638.36	3.11	0.9262	0.8579
Huarng (2001)	476.97	2.36	0.9467	0.8962
Lee and Chou (2004)	501.28	2.67	0.9542	0.9105
Cheng et al. (2006)	511.04	2.66	0.9548	0.9117
Cheng et al. (2008)	478.45	2.39	0.9587	0.9192
Yolcu et al. (2009)	805.17	4.29	0.9121	0.83
Qiu et al. (2011)	511.33	2.65	0.9599	0.9219
Joshi and Kumar (2012a)	433.76	2.24	0.9688	0.9387
Kumar and Gangwar (2016)	493.56	2.33	0.9594	0.9254
Bisht and Kumar (2016)	428.63	1.94	0.9667	0.9346
Chen (2014)	421	–	–	–
Proposed method	431.64	2.04	0.9674	0.9359

Table 16 Error measure, correlation coefficient and coefficient of determination verification of propose method in forecasted data of share prices data of SBI at BSE, India

Model	RMSE	AFE	R	R^2
Chen (1996)	187.26	8.26	0.8839	0.7813
Huarng (2001)	164.04	6.29	0.911	0.8314
Pathak and Singh (2011)	205.96	8.95	0.8685	0.7544
Joshi and Kumar (2012a)	200.17	9.52	0.882	0.778
Kumar and Gangwar (2016)	134.28	6.3	0.9446	0.8924
Bisht and Kumar (2016)	179.03	7.86	0.9001	0.8101
Proposed method	182.98	8.65	0.896	0.8028

Step 5: Using max–min operations (Eq. 5) on FLR and forecast the time series data of SBI share price from Eq. (10) (Table 14).

We compare forecasted enrolments and SBI share prices using error measures, correlation coefficient and coefficient of determination with few existing fuzzy time series forecasting methods. RMSE and AFE in forecasting both enrolments and SBI share prices are found less compared to previous forecasting methods. Additionally, actual and forecasted data (Tables 15, 16) are found highly correlated compared to previous forecasting methods proposed by various researchers.

5 Conclusions

Probabilistic and non-probabilistic uncertainties occur in the system simultaneously. This paper contributes a methodology for a novel HPFS-based time series forecasting model to include both probabilistic and non-probabilistic uncertainties along with hesitant information. In proposed forecasting method, HFS is converted into HPFS by assigning probability to membership grades using Gaussian probability distribution function. We also propose an aggregation operator that uses immediate probability, weights and membership grades to aggregate HPFEs of HPFS. To verify the performance of proposed forecasting method, it is implemented to time series data of University of Alabama and SBI share price at BSE India. RMSE and AFE in forecasting both University of Alabama enrolments and SBI share price using proposed HPFS-based time series forecasting method are found less than that of previous forecasted methods. Even though RMSE in forecasting University of Alabama using proposed forecasting method is slightly higher than that of the methods proposed by Chen (2014) and Bisht and Kumar (2016), but inherent characteristic of HPFS of handling probabilistic uncertainty is an added advantage of proposed forecasting method. It is also observed that coefficient of correlation between actual and forecasted data is almost in accordance with the methods proposed by Bisht and Kumar (2016) and Joshi and Kumar (2012a) (Table 15). Reduced RMSE and AFE in forecasting SBI share prices (Table 16) confirm the out performance of proposed forecasting method over other forecasting methods that were proposed by Chen (1996), Pathak and Singh (2011), Joshi and Kumar (2012a). Although proposed HPFS based forecasting method slightly under perform in forecasting SBI share compared to the methods proposed by Huarng (2001), Kumar and Gangwar (2016), but it is competent enough to handle both types of uncertainties of financial time series forecasting.

As HPFS includes the prominent characteristic of both hesitant and probabilistic fuzzy set, it makes proposed HPFS-based forecasting method more competent in handling both probabilistic and non-probabilistic uncertainties along with non-deterministic hesitation that may arise due to multiple fuzzification of time series data.

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