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ORIGINAL PAPER

Multi-level granularity in formal concept analysis

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Abstract
Formal concept analysis is a tool for data analysis and knowledge processing, and granular computing is a methodology for knowledge discovery in database. Applying granular computing into some data analysis theories, such as formal concept analysis, is a new trend in recent years. The multi-level granular analysis is an essential work in formal concept analysis. In order to combine formal concept analysis and granular computing properly, some basic and specific granules in the framework of formal concept analysis are required. By collecting and organizing the existing results in the theory of lattices and formal concept analysis, we firstly extract five types of granules on the basis of concept lattices from different perspectives and levels. They include the granules induced by objects and attributes, respectively, and the granules induced by both objects and attributes simultaneously. Then, we discuss the granules' relationships and explain their semantics. The main contribution of this study is providing some specific granules, which are practical and can be used conveniently in formal concept analysis.

Keywords Concept lattice · Granular computing · Object · Attribute · Granule

1 Introduction

The concept of granulation plays an essential role in human cognition, especially in the realm of everyday reasoning (Pedrycz 2013).

Originally, the notion of granulation can be rooted in the concept of a linguistic variable, which was introduced by Zadeh (1973). Zadeh (1979) contributed the first paper on information granularity. After that, the term of granular computing (GrC) was introduced by Lin (1997, 1998) and Zadeh (1997).

Granular computing is a computational model for information processing. A granule is a block constituted by some objects through an indiscernibility relation, a similarity relation, or a functional relation. The process to find granules is called information granulating. The finer the granule is, the more knowledge we have. Therefore, to classify and describe granules in an appropriate way are interesting and central problems in granular computing. At present, granular computing is well-known in formation, transformation, synthesis, and decomposition of granules. Granular computing allows us to consider problems using granularity in various levels. This leads to a new research area and attracts more and more scholars, because it is a good method for instructing our thoughts and actions.

Pedrycz (2001, 2002) introduced some research topics about granular computing, and then, he proposed many different granular computing techniques and discussed several principles of GrC (Pedrycz and Chen 2011, 2015a, b; Pedrycz 2013). Pedrycz (2014) suggested that granular computing has emerged as a unified and coherent platform of constructing, describing, and processing information granules, which can be treated as a definition of GrC from the perspective of description. Besides these contribution, Bargiela and Pedrycz have a deep research about GrC (Bargiela and Pedrycz 2003, 2005a, b; Pedrycz 2005; Bargiela and Pedrycz 2008). Yao (2000, 2002) discussed some basic problems, and described granules from the perspective of logic. Yao (2016b) also showed his triarchic theory of granular computing in. Miao et al. (2012) researched from the perspective of set theory. Liang et al. (2015) discussed the feasibility of granular computing in big data, and showed that GrC is a new approach and methodology in processing

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big data. Ciucci (2016) reviewed the definition of orthopairs and a hierarchy on orthopairs in the light of granular computing. Fujita et al. (2016) claimed the advances of

Granular computing. Additionally, many significant results were obtained (Huang and Li 2018; Wang et al. 2017; Min and Xu 2016; Xu and Wang 2016; Peters and Weber 2016; Dubois and Prade 2016).

Actually, we often divide a complicated problem or information into several simple parts on the basis of the characteristics or properties of these information in our daily lives. The reason why we do like this is because of the limitation of our cognitive competence. After solving with the simple parts one by one, the results are combined to be the final one. In this process, each part can be treated as a granule. This idea treating a part as a granule is easy to understand and is widely used in our research and daily lives. That means, granular computing is common used in our daily lives naturally. However, the idea is only a methodology and has few concrete definitions or models.

Furthermore, we often have some complicated problems that cannot be resolved by only one granule. In order to solve these problems, Qian et al. (2010) presented the multi-granulation in Rough set theory. On the basis of multi-granulation, many research achievements were obtained in the framework of Rough sets (Qian et al. 2014; Lin et al. 2013; Yao and She 2016; She et al. 2017; Dai and Tian 2013; Zhang et al. 2015; Yao et al. 2014).

For formal concept analysis (FCA), considering granular computing and multi-granulation analysis in FCA is also necessary.

Formal concept analysis, a tool for data analysis, was proposed by German mathematician Wille (1982) and improved by Ganter and Wille (1999). The formal context is the basis of FCA, which can be represented in a cross table. Wille defined a formal concept (concept, shortly) on the basis of a formal context, and construct their structure using proper partial orders and operators. The structure is a lattice, which is called a concept lattice by Wille, and it is the basis of FCA theory. A concept lattice can visualize the potential information hidden in the formal context. That is one of the important reasons why FCA is used in data analysis and data mining.

The main research topics of FCA include attribute reduction (Zhang et al. 2005; Dias and Vieira 2015; Singh et al. 2017; Singh and Kumar 2017), combination with other uncertainty analysis approaches, such as Rough set theory and Three-way decisions (Kumar 2012; Wang and Liu 2008; Yao 2016a; Qi et al. 2014, 2016; Chen et al. 2014; Ma and Mi 2016), and others (Belohlavek and Vychodil 2010; Shao et al. 2014; Ren et al. 2017). Additionally, FCA plays an important role and is adapted to a variety of applications (Belohlavek et al. 2011; Poelmans et al. 2013; Kaytoue et al. 2011; Codocedo and Napoli 2015; Tonella 2003; Li et al. 2017).

There are many useful results in Granule analysis in FCA. For example, by using stability index, Zhi and Li (2016) classified the power set of objects into three categories, and named them atomic granules, basic granules, and composite granules, respectively. Then, they proposed methods for describing the three categories. Particularly, each kind of granule in (Zhi and Li 2016) is a subset of the object set. Li and Wu (2017b) investigated the main research topics of granular computing approach for FCA from different perspectives, such as a granular computing model based on Galois connection, object/attribute granule, granular rule, granular reduct, granular concept and learning, and concept granular computing systems. Li et al. (2017a) proposed a family of tripartition of the object set, which is related to three-way decisions and can be considered as multi-granularity in FCA. Loia et al. (2018) studied the data granulating in FCA method.

Other researches related to GrC and FCA include attribute granulating, granule transformation, GrC based on incomplete data (Wu et al. 2009; Shao and Leung 2014; Xu and Li 2016; Li et al. 2015, 2016; Belohlavek et al. 2014; Huang et al. 2017; Gong et al. 2017).

Different from the aforementioned existing works, this paper provides and discusses some specific granules in different semantics with structural complexity.

We firstly use the equivalence class of an object in a formal context as a granule since a formal context can be considered as an information system. Moreover, a formal concept reflecting objects and attributes can be treated as a kind of granule, which is the fundamental component to construct the concept lattice. It is easy to see that the two kinds of granules show different semantics. One is from the perspective of objects and reflect the equivalence class, the other is from the perspective of both object and attribute and reflect the lattice structure. Therefore, to confirm some specific granules in FCA is feasible and meaningful. Motivated by the above analysis, this paper extracts some granules from different levels and angles. Since all the granules can be calculated, proper granules can be chosen and applied in accordance with the real situations and people's objectives. That is, a multi-level granularity in FCA is proposed in this paper.

The structure of this paper is as follows. Section 2 briefly reviews some basic notions of FCA. Section 3 proposes five types of granules in FCA with respect to objects and discusses their relationships. In parallel, Sect. 4 proposes five types of granules from the perspective of attributes. Section 5 gives another two granules with more information and semantics. Finally, the paper is concluded with a summary in Sect. 6.

2 Preliminaries

In this section, we give the associated definitions in FCA and the original idea of this paper.

2.1 Notions in FCA

Definition 2.1 (Ganter and Wille 1999) A formal context (G, M, I) consists of two sets G and M and a relation I between G and M. The elements of G are called objects and the elements of M are called attributes of the context. In order to express that an object g is in a relation I with an attribute m, we write gIm or $(g, m) \in I$ and read it as "the object g has the attribute m".

A formal context can be represented by a cross table, in which a cross in row g and column m means that the object g has the attribute m.

If there exists $g \in G$ such that g has all the attributes in M or g does not have any attribute in M, we think the object g is meaningless. Similarly, we are not interested in such attribute that is in a relation with all objects or not in a relation with any object. A formal context that does not have such objects and attributes is called canonical. All formal contexts in this paper are canonical.

Example 2.1 Table 1 is a revised version of a commonly used formal context named "Living Beings and Water" in Ganter and Wille (1999). In the original formal context, the attribute a (means "needs water to live") is possessed by all the objects, therefore it is deleted to ensure that the formal context is canonical. Now, we denote the revised data as (G, M, I) and show it in Table 1. In which, the object set G includes eight objects, namely, 1: leech, 2: bream, 3: frog, 4: dog, 5: spike-weed, 6: reed, 7: bean, 8: maize. The attribute set M includes eight attributes, they are: b: lives in water, c: lives no land, d: needs chlorophyll to produce foods, e: two seed leaves, f: one seed leaf, g: can move around, h: has limbs, i: suckles its offspring.

Based on the formal context (G, M, I), Wille and Ganter (Wille 1982; Ganter and Wille 1999) defined a pair of dual operators for $A \subseteq G$ and $B \subseteq M$ by:

$$A^{\uparrow} = \{m \in M | gIm \quad \text{for all} \quad g \in A\},$$
$$B^{\downarrow} = \{g \in G | gIm \quad \text{for all} \quad m \in B\}.$$

The detailed properties about the dual operators can be found in Ganter and Wille (1999).

By using the operators, the canonical formal context can be explained by the following statements: $\forall g \in G, g^{\uparrow} \neq \emptyset, g^{\uparrow} \neq M$, and $\forall m \in M, m^{\downarrow} \neq \emptyset, m^{\downarrow} \neq G$.

A formal context (G, M, I) is called clarified if $g^{\uparrow} = h^{\uparrow}$ implies $g = h$ and $m^{\downarrow} = n^{\downarrow}$ implies $m = n$ for any $g, h \in G$ and $m, n \in M$. In brief, a formal context is a clarified context if and only if it has neither same rows nor same columns. For simplicity, the formal contexts discussed in this paper are also clarified.

Suppose $A \subseteq G$, $B \subseteq M$, if $A^{\uparrow} = B$ and $B^{\downarrow} = A$, then (A, B) is called a formal concept, where A is called the extent of the formal concept, and B is called the intent of the formal concept.

The family of all formal concepts of (G, M, I) forms a complete lattice, which is called the concept lattice and is denoted by $L(G, M, I)$. For any $(A_1, B_1), (A_2, B_2) \in L(G, M, I)$, the partial order is defined by:

$$(A_1, B_1) \leqslant (A_2, B_2) \Leftrightarrow A_1 \subseteq A_2 (\Leftrightarrow B_1 \supseteq B_2),$$

and the infimum and supremum of (A_1, B_1) and (A_2, B_2) are defined by:

$$(A_1, B_1) \wedge (A_2, B_2) = (A_1 \cap A_2, (B_1 \cup B_2)^{\downarrow\uparrow}),$$
$$(A_1, B_1) \vee (A_2, B_2) = ((A_1 \cup A_2)^{\uparrow\downarrow}, B_1 \cap B_2).$$

For any $g \in G$, a pair $(g^{\uparrow\downarrow}, g^{\uparrow})$ is a formal concept and is called an object concept. Similarly, for any $m \in M$, a pair $(m^{\downarrow}, m^{\downarrow\uparrow})$ is also a formal concept and is called an attribute concept. We denote the set of object concepts of $L(G, M, I)$ as $\mathcal{OB}(L)$ and the set of attribute concepts of $L(G, M, I)$ as $\mathcal{AT}(L)$.

Table 1 A formal context (G, M, I)

G	b	c	d	e	f	g	h	i
1	×					×		
2	×					×	×	
3	×	×				×	×	
4		×				×	×	×
5	×		×		×			
6	×	×	×		×			
7		×	×	×				
8		×	×		×			

Example 2.2 The formal context in Example 2.1 has 19 formal concepts, which form a lattice shown in Fig. 1. To describe the concept lattice clearly, each formal concept in Fig. 1 is indexed by a number. They are: 1. $(\emptyset, M)$, 2. $(4, cghi)$, 3. $(3, bcgh)$, 4. $(7, cde)$, 5. $(6, bcdf)$, 6. $(34, cgh)$, 7. $(23, bgh)$, 8. $(123, bg)$, 9. $(36, bc)$, 10. $(678, cd)$, 11. $(68, cdf)$, 12. $(56, bdf)$, 13. $(234, gh)$, 14. $(568, df)$, 15. $(1234, g)$, 16. $(34678, c)$, 17. $(12356, b)$, 18. $(5678, d)$, 19. $(G, \emptyset)$. For simplicity, these formal concepts are denoted by the marked numbers when we mention them in the rest of this paper.

In this concept lattice, the set of object concepts is $\mathcal{OB}(L) = \{2, 3, 4, 5, 7, 8, 11, 12\}$, the set of attribute concepts is $\mathcal{AT}(L) = \{2, 4, 13, 14, 15, 16, 17, 18, 19\}$.

2.2 Notions in theory of lattices

Since a concept lattice is a lattice, we introduce an important notion in theory of lattices, it will be one of the granules in our research.

Definition 2.2 (Davey and Priestley 2002) Let L be a lattice. An element $x \in L$ is join-irreducible if

(i) $x \neq 0$ (in case L has a zero),
(ii) $x = a \vee b$ implies $x = a$ or $x = b$ for all $a, b \in L$.

A meet-irreducible element is defined dually.

The order relation in a lattice is often denoted by $\leq$, and we write $a < b$ for $a \leq b$ and $a \neq b$.

Definition 2.3 (Ganter and Wille 1999) a is called a lower neighbour of b if $a < b$ and there is no element of c fulfilling $a < c < b$. In this case, b is an upper neighbour of a.

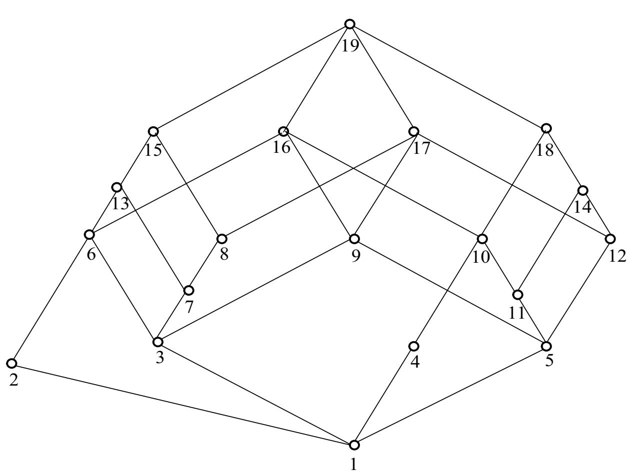

Fig. 1 Concept lattice of Example 2.1

In this paper, we call the set of all upper neighbours of a the upper-neighbourhood of a.

The upper neighbours of the zero element in a lattice are called the atoms; they are always join-irreducible (if exist). The coatoms, i.e., the lower neighbours of the unit element, are always meet-irreducible.

We call the join-irreducible (meet-irreducible) elements and atoms (coatoms) in a concept lattice $L(G, M, I)$ join-irreducible (meet-irreducible) concepts and atom (coatom) concepts, respectively, and denote the set of join-irreducible (meet-irreducible) concepts as $\mathcal{J}(L)$ ($\mathcal{M}(L)$) and the set of atom (coatom) concepts as $\mathcal{A}(L)$ ($\mathcal{CA}(L)$).

Example 2.3 In Example 2.2, the set of join-irreducible concepts is $\mathcal{J}(L) = \{2, 3, 4, 5, 7, 8, 11, 12\}$, the set of meet-irreducible concepts is $\mathcal{M}(L) = \{2, 4, 13, 14, 15, 16, 17, 18\}$, the set of atom concepts is $\mathcal{A}(L) = \{2, 3, 4, 5\}$, and the set of coatom concepts is $\mathcal{CA}(L) = \{15, 16, 17, 18\}$.

2.3 Notions in information systems

A formal context can be considered as an information system when the table is expressed by a two-value table. That is, the cross in the table is replaced by the number 1 and the space is replaced by the number 0. Under this situation, we can analyse the formal context from the perspective of information systems.

2.3.1 Object/attribute equivalence class

A basic notion in information systems is the equivalence class. There are two kinds of equivalence class in a formal context that is treated as an information system. One is object equivalence class, and the other is attribute equivalence class. An object equivalence class is constituted by the objects that have the same attributes, that is, the objects that have the same values for each attribute. Therefore, the objects with the same row can compose an object equivalence class. Dually, the attribute equivalence class is composed by the attributes with the same column.

For a formal context (G, M, I), the equivalence class of an object g is denoted by $\mathcal{OBE}(g)$, and the equivalence class of an attribute m is denoted by $\mathcal{ATE}(m)$. Thus, we use $\mathcal{OBE}(L)$ and $\mathcal{ATE}(L)$ to denote the set of all object equivalence classes and the set of attribute equivalence classes of the concept lattice $L(G, M, I)$, respectively.

In Example 2.1, since the formal context is a clarified context, each singleton set that contains one object is an object equivalence class, and each singleton set containing one attribute is an attribute equivalence class. Thus, we have $\mathcal{OBE}(L) = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}\}$, and $\mathcal{ATE}(L) = \{\{b\}, \{c\}, \{d\}, \{e\}, \{f\}, \{g\}, \{h\}, \{i\}\}$.

2.3.2 Object/attribute-induced pair

Although equivalence class is the most important and basic notion, it is composed by objects or attributes singularly. What we want to find and discuss is a set that can show the information about objects and attributes simultaneously. Therefore, we introduce the notion of object/attribute-induced pair.

For an object $g \in G$, we generalize it into a pair $(g, g^{\uparrow})$ using the operator $\uparrow$ in FCA. We call such pair an object-induced pair, and denote it as $\mathcal{OBI}(g)$. Similarly, for an attribute $m \in M$, an attribute-induced pair is the pair $(m^{\downarrow}, m)$, and is denoted by $\mathcal{ATI}(m)$. It should be noticed that most of these pairs are just pairs with the form of a pair (object set, attribute set) rather than formal concepts except atom concepts and coatom concepts. For a concept lattice $L(G, M, I)$, the set of object-induced pairs is denoted by $\mathcal{OBI}(L)$ and the set of attribute-induced pairs is denoted by $\mathcal{ATI}(L)$.

The terms of object/attribute-induced pair were proposed firstly by Qi et al. (2005) to research the transformation between equivalence classes and extents of formal concepts. Later, Wei and Wan (2016) used such pairs to produce object/property pictorial diagram for a formal context, and studied a kind of granule transformation. The definitions of object/property pictorial diagram of a formal context are as follows.

Definition 2.4 (Wei and Wan 2016) Let (G, M, I) be a formal context, $L(G, M, I)$ be its concept lattice. Denote $\mathcal{OBI}(L) = \{(g, g^{\uparrow}) | g \in G\}$, $\mathcal{ATI}(L) = \{(m^{\downarrow}, m) | \ m \in M\}$. For any $g_i, g_j \in G$, if $g_i^{\uparrow} \subseteq g_j^{\uparrow}$, then we denote $(g_i, g_i^{\uparrow}) \leq (g_j, g_j^{\uparrow})$, and call $(\mathcal{OBI}(L), \leq)$ the object pictorial diagram of (G, M, I). Dually, for any $m_s, m_t \in M$, if $m_s^{\downarrow} \subseteq m_t^{\downarrow}$, then we denote $(m_s^{\downarrow}, m_s) \leq (m_t^{\downarrow}, m_t)$, and call $(\mathcal{ATI}(L), \leq)$ the property pictorial diagram of (G, M, I).

Example 2.4 For the formal context shown in Example 2.1, we have $\mathcal{OBI}(L) = \{(1, bg), (2, bgh), (3, bcgh), (4, \ cghi), (5, bdf), (6, bcdf), (7, cde), (8, cdf)\}$, $\mathcal{ATI}(L) = \{(12356, b), (34678, c), (5678, d), (7, e), (568, f), (1234, \ g), (234, h), (4, i)\}$. The Hasse graphs of $\mathcal{OBI}(L)$ and $\mathcal{ATI}(L)$ are given in Figs. 2 and 3, respectively.

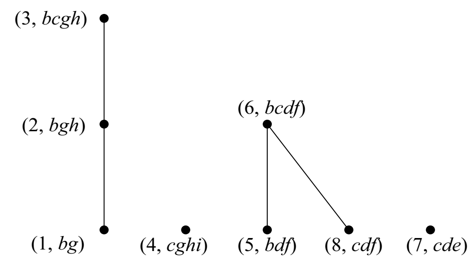

Fig. 2 Object pictorial diagram of Example 2.1

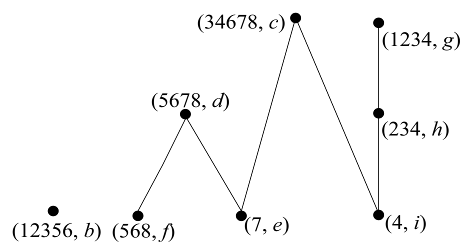

Fig. 3 Property pictorial diagram of Example 2.1

3 Multi-granule analysis from the viewpoint of objects

For a formal context (G, M, I) and its concept lattice $L(G, M, I)$, the previous section gives ten kinds of sets, or, five pairs of sets. We review them in order: (1) the set of object concepts $\mathcal{OB}(L)$ and the set of attribute concepts $\mathcal{AT}(L)$, (2) the set of join-irreducible concepts $\mathcal{J}(L)$ and the set of meet-irreducible concepts $\mathcal{M}(L)$, (3) the set of atom concepts $\mathcal{A}(L)$ and the set of coatom concepts $\mathcal{CA}(L)$, (4) the object equivalence class $\mathcal{OBE}(L)$ and the attribute equivalence class $\mathcal{ATE}(L)$, (5) the set of object-induced pair $\mathcal{OBI}(L)$ and the set of attribute-induced pair $\mathcal{ATI}(L)$. These sets have different meanings and functions. Generally, they are different sets, but some of them are included by others, some of them have intersection. We discuss their relationships in the following subsections.

Essentially, every element in these sets can be treated as a granule with different levels from different perspectives in formal concept analysis.

We give new names shown in Table 2 for the five pairs of sets mentioned above. These names use granular description that help us describe them clearly and conveniently.

In the ten granules, there are five of them oriented to objects, since they are related to objects. They are: object concepts, join-irreducible concepts, atom concepts, object equivalence class, and object-induced pair. The other five granules are related to attributes.

We can discuss these granules from the viewpoint of objects (i.e., extents) or attributes (i.e., intents). This section first investigates it from the objects' viewpoint, and then investigates from attributes' perspective.

3.1 Relationship among $\mathcal{A}(L)$, $\mathcal{J}(L)$, and $\mathcal{OB}(L)$

From the definition of atom, the following theorem can be induced directly.

Table 2 Granules in a formal context (G, M, I)

	Set of such things	Symbol	Name of granule
1	Object concept	$\mathcal{OB}(L)$	Elementary granule
1′	Attribute concept	$\mathcal{AT}(L)$	Elementary granule
2	Join-irreducible concept	$\mathcal{J}(L)$	Essential granule
2′	Meet-irreducible concept	$\mathcal{M}(L)$	Essential granule
3	Atom concept	$\mathcal{A}(L)$	Atomic granule
3′	Coatom concept	$\mathcal{CA}(L)$	Atomic granule
4	Object equivalence class	$\mathcal{OBE}(L)$	Classified granule
4′	Attribute equivalence class	$\mathcal{ATE}(L)$	Classified granule
5	Object-induced pair	$\mathcal{OBI}(L)$	Pictorial granule
5′	Attribute-induced pair	$\mathcal{ATI}(L)$	Pictorial granule

Theorem 3.1 *For any lattice L, we have* $\mathcal{A}(L) \subseteq \mathcal{J}(L)$.

For a clarified and canonical formal context (G, M, I), Theorem 3.1 indicates that each atom concept in the concept lattice $L(G, M, I)$ is a join-irreducible concept.

Theorem 3.2 *For any concept lattice* $L(G, M, I)$, *we have* $\mathcal{J}(L) \subseteq \mathcal{OB}(L)$.

Proof From the theory of lattices, we have the following result: $Q \subseteq L$ is join-dense in a lattice L if and only if $\mathcal{J}(L) \subseteq Q$. From the Basic Theorem on Concept Lattices (Ganter and Wille 1999), we also know that the set of object concepts $\mathcal{OB}(L)$ is join-dense in the concept lattice $L(G, M, I)$. Therefore, we have $\mathcal{J}(L) \subseteq \mathcal{OB}(L)$. □

Theorem 3.2 shows that every join-irreducible concept is an object concept.

Therefore, the relationship among the three kinds of concepts can be described in Fig. 4.

Example 3.1 For the formal context shown in Example 2.1, we can obtain the following results from Examples 2.2 and 2.3: $\mathcal{OB}(L) = \{2, 3, 4, 5, 7, 8, 11, 12\}$, $\mathcal{J}(L) = \{2, 3, 4, 5, 7, 8, 11, 12\}$, $\mathcal{A}(L) = \{2, 3, 4, 5\}$. It is easy to get $\mathcal{A}(L) \subseteq \mathcal{J}(L) = \mathcal{OB}(L)$, which illustrates Fig. 4.

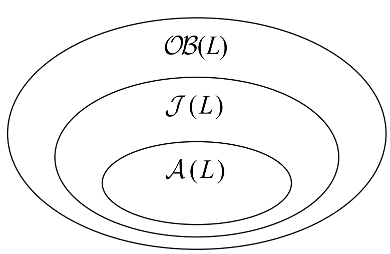

Fig. 4 Results of Theorems 3.1 and 3.2

Based on the relationship among $\mathcal{A}(L)$, $\mathcal{J}(L)$, and $\mathcal{OB}(L)$, we can obtain the following results from the quantity viewpoint.

Corollary 3.1 *For any concept lattice* $L(G, M, I)$, *we have* $|\mathcal{A}(L)| \leq |\mathcal{J}(L)| \leq |\mathcal{OB}(L)|$.

3.2 Relationship among $\mathcal{OBE}(L)$, $\mathcal{OBI}(L)$ and $\mathcal{A}(L)$

Generally, an object-induced pair in a formal context (G, M, I) is a pair (a certain object g, a set of attributes $g^{\uparrow}$), and an object equivalence class is a set $[g]_R$, where R is an equivalence relation on G. However, when we consider a clarified context, the object equivalence class of an object g is a singleton set $\{g\}$. Now, we generalize the form of object equivalence class into a pair (objects, attributes) and denote it as $\mathcal{OBE}_P(g)$, thus, it must be $(g, g^{\uparrow})$. So, in this case, every object's equivalence class is the same with the second part of the object-induced pair. Denoting all the generalized form of object equivalence class as $\mathcal{OBE}_P(L)$, we obtain the following theorems.

Theorem 3.3 *Suppose* (G, M, I) *is a clarified formal context, and* $L(G, M, I)$ *is its concept lattice. For each* $g \in G$, *we have* $\mathcal{OBE}_P(g) = \mathcal{OBI}(g)$; *thus,* $\mathcal{OBE}_P(L) = \mathcal{OBI}(L)$.

Theorem 3.4 *Suppose* (G, M, I) *is a clarified formal context, and* $L(G, M, I)$ *is its concept lattice. For each atom concept* (X, B), $\{B|(X, B) \in \mathcal{A}(L)\} \subseteq \{g^{\uparrow}|(g, g^{\uparrow}) \in \mathcal{OBI}(L)\} = \{g^{\uparrow}|(g, g^{\uparrow}) \in \mathcal{OBE}_P(L)\}$ *holds.*

Proof Since any atom concept must be an object concept, for an atom concept (X, B), there must exist an object g_0 such that $g_0^{\uparrow\downarrow} = X$ and $g_0^{\uparrow} = B$. Furthermore, $g_0^{\uparrow}$ is the second element of the object-induced pair produced by g_0. Thus, the result is obtained. □

Theorems 3.3 and 3.4 are shown in Fig. 5, and Examples 2.3 and 2.4 confirm the results.

It is easy to obtain the following result based on the above theorems.

Corollary 3.2 *For any concept lattice* $L(G, M, I)$, $|\mathcal{A}(L)| \leq |\mathcal{OBE}(L)| = |\mathcal{OBI}(L)|$ *holds.*

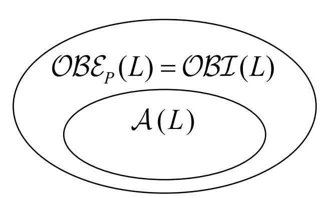

Fig. 5 Results of Theorems 3.3 and 3.4

We give a result presented in Wei and Wan (2016) to complete our discussion.

Theorem 3.5 (Wei and Wan 2016) *For any* $(g, g^{\uparrow}) \in \mathcal{OBI}(L)$, *if* $|g^{\uparrow}| = 1$, *then* $(g^{\uparrow\downarrow}, g^{\uparrow}) \in \mathcal{AT}(L)$.

This result reveals the relationship between an object-induced pair and an attribute concept.

3.3 Granule chain and level from the viewpoint of objects

This section explains the meanings of the granule names shown in Table 2.

When we consider the granules in a formal context from the viewpoint of objects, the first granule we think about naturally is the object equivalence class, since it is the most elementary unit in a data set.

Since the formal context in this paper is clarified context, every equivalence class is a singleton set containing one object. However, such singleton set has two disadvantages. One is that the related attribute information of this object is not reflected, the other is that its form is not the same as a formal concept. Thus, we generalize the singleton set into a pair: (an object, a set of attributes), in which the second part contains the attributes possessed by this object. Such pair can be considered as a kind of classified granule.

For a clarified formal context, the previous generalized pair is just the object-induced pair. Since the object-induced pair can form a pictorial graph, it can be treated as a pictorial granule.

In these object-induced pairs, atom concepts are special cases. In a concept lattice, atom concepts are at the bottom of the lattice; they are the upper neighbours of the bottom element $(\emptyset, M)$. In other words, atom concepts are basic in the lattice structure. Therefore, we call atom concepts atomic granules.

Since atom concepts are join-irreducible concepts and each formal concept can be expressed by the join of join-irreducible concepts, join-irreducible elements are very important in a lattice structure. Thus, the join-irreducible concept is another granule. We call the join-irreducible concept essential granule.

Moreover, we can consider a more complicated concept: the object concept. In some situations, an object concept is the simplest formal concept in a concept lattice since it is produced by an object g using the form $(g^{\uparrow\downarrow}, g^{\uparrow})$ directly. If we want to find a formal concept by using an object, the corresponding object concept will be the first concept we consider. Furthermore, each formal concept can be expressed by the join of object concepts, which leads the object concept to an important role, and we consider it as an elementary granule.

Finally, we have the following granule chain shown in Fig. 6 with respect to objects.

From the granule level's perspective, the relationship among these granules is shown in Fig. 7.

4 Multi-granule analysis from the viewpoint of attributes

In parallel with Sect. 3, this section gives granules and shows their relationships from the viewpoint of attributes. Since the status of objects and attributes are similar and dual, the results in this section are given directly and all the proofs are omitted.

4.1 Relationship among $\mathcal{CA}(L)$, $\mathcal{M}(L)$ and $\mathcal{AT}(L)$

Similar to Theorems 3.1 and 3.2, the following theorem is induced directly by using the definition of coatom and the Basic Theorem on Concept Lattices.

Theorem 4.1 *For any concept lattice* L, *we have* $\mathcal{CA}(L) \subseteq \mathcal{M}(L) \subseteq \mathcal{AT}(L)$.

The theorem suggests that for any formal context (G, M, I) and its concept lattice $L(G, M, I)$, each coatom concept is a meet-irreducible concept, and each meet-irreducible concept is an attribute concept. The relationship among $\mathcal{CA}(L)$, $\mathcal{M}(L)$, and $\mathcal{AT}(L)$ is described in Fig. 8.

Corollary 4.1 *For any concept lattice* $L(G, M, I)$, *we have* $|\mathcal{CA}(L)| \leq |\mathcal{M}(L)| \leq |\mathcal{AT}(L)|$.

4.2 Relationship among $\mathcal{ATE}(L)$, $\mathcal{ATI}(L)$ and $\mathcal{CA}(L)$

In a clarified formal context, an attribute equivalence class is a singleton set $\{m\}$. We also generalize the form of attribute equivalence class into a pair (objects, attribute) and denote it as $\mathcal{ATE}_P(m)$, which must be $(m^{\downarrow}, m)$. We denote all the

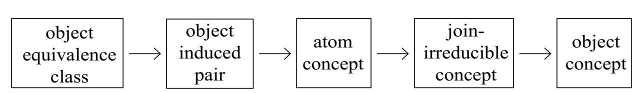

Fig. 6 Granule chain with respect to objects

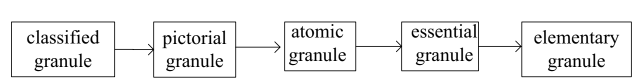

Fig. 7 Granule level with respect to objects

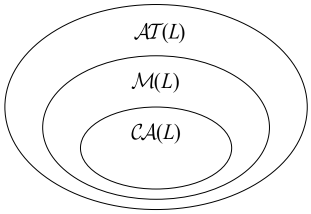

Fig. 8 Result of Theorem 4.1

generalized form of attribute equivalence class as $\mathcal{ATE}_P(L)$, and obtain the following theorems.

Theorem 4.2 *Suppose* (G, M, I) *is a clarified formal context, and* $L(G, M, I)$ *is its concept lattice. For each* $m \in M$, *we have* $\mathcal{ATE}_P(m) = \mathcal{ATI}(m)$; *thus,* $\mathcal{ATE}_P(L) = \mathcal{ATI}(L)$.

Theorem 4.3 *Suppose* (G, M, I) *is a clarified formal context, and* $L(G, M, I)$ *is its concept lattice. For each coatom concept* (X, B), $\{X|(X, B) \in \mathcal{CA}(L)\} \subseteq \{m^{\downarrow}|(m^{\downarrow}, m) \in \mathcal{ATI}(L)\} = \{m^{\downarrow}|(m^{\downarrow}, m) \in \mathcal{ATE}_P(g)\,(L)\}$ *holds.*

The results of Theorems 4.2 and 4.3 are illustrated in Fig. 9.

Furthermore, the relationship among their quantity is as follows.

Corollary 4.2 *For any concept lattice* $L(G, M, I)$, $|\mathcal{CA}(L)| \leq |\mathcal{ATE}(L)| = |\mathcal{ATI}(L)|$ *holds.*

The following theorem reveals the relationship between an attribute-induced pair and an object concept.

Theorem 4.4 (Wei and Wan 2016) *For any* $(m^{\downarrow}, m) \in \mathcal{ATI}(L)$, *if* $|m^{\downarrow}| = 1$, *then* $(m^{\downarrow}, m^{\downarrow\uparrow}) \in \mathcal{OB}(L)$.

4.3 Granule chain and level from the viewpoint of attributes

Granule chain and level from the viewpoint of attributes is similar to Sect. 3.3, we just give the final results and omit the analysis process. The granule chain is shown in Fig. 10, and granule level is the same as Fig. 7 since each granule's type is the same as the granules from the viewpoint of objects.

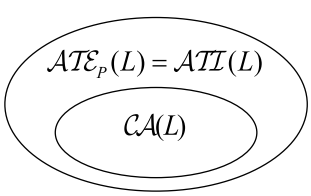

Fig. 9 Results of Theorems 4.2 and 4.3

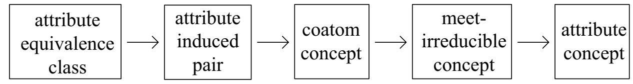

Fig. 10 Granule chain with respect to attributes

5 Integrated granules

Sections 3 and 4 investigate the granules from the viewpoints of objects and attributes, respectively. Actually, we can consider granules from both objects and attributes simultaneously. In this section, we discuss formal concepts and sublattices, by treating them as common granules and lattice granules, respectively. We call them integrated granules since they show the information about objects and attributes simultaneously. To some extent, these two kinds of granules are more complicated in structure and more meaningful in semantics than the granules proposed in previous sections.

5.1 A formal concept as a granule

This section discusses the relationship between the common granule and the previous granules in the object/attribute granule chain.

5.1.1 Relationship between the common granule and the object granule chain

In formal concept analysis, we have the following important lemma, which is contained in the Basic Theorem on Concept Lattices (Ganter and Wille 1999) and shows that $\{(g^{\uparrow\downarrow}, g^{\uparrow})|g \in G\}$ is join-dense in $L(G, M, I)$.

Lemma 5.1 *Suppose* (G, M, I) *is a formal context, and* $L(G, M, I)$ *is its concept lattice. Then,* $(X, B) = \vee_{g\in X}(g^{\uparrow\downarrow}, g^{\uparrow})$ *holds for any concept* (X, B).

Since Lemma 5.1 shows the relationship between a formal concept and its extent, and the object concept is the final granule in object granule chain we discussed above, Lemma 5.1 also reveals the relationship between a common granule and each granule in the object granule chain.

The following theorem shows the relationship between object-induced pairs and join-irreducible concepts.

Theorem 5.1 (Wei and Wan 2016) *Let* (G, M, I) *be a formal context,* $(\mathcal{OBI}(L), \leq)$ *be its object pictorial diagram, and* $Max(\mathcal{OBI}(L))$ *be the set of maximal elements in* $(\mathcal{OBI}(L), \leq)$. *Thus,*

(1) $Max(\mathcal{OBI}(L)) \subseteq \mathcal{J}(L)$;

(2) *if* $(g, g^{\uparrow}) \notin Max(\mathcal{OBI}(L))$ *and* $\cap_{g_j \in UN_g} g_j^{\uparrow} - g^{\uparrow} \neq \emptyset$, *then* $(g^{\uparrow\downarrow}, g^{\uparrow}) \in \mathcal{J}(L)$, *where* UN_g *is the upper-neighbourhood of* g.

Corollary 5.1 (Wei and Wan 2016) *Let* (G, M, I) *be a formal context,* $(\mathcal{OBI}(L), \leq)$ *be its object pictorial diagram. For any* $(g, g^{\uparrow}) \in \mathcal{OBI}(L)$, UN_g *is the upper-neighbourhood of* g. *Thus,*

(1) *if* $|UN_g| \leq 1$, *then* $(g^{\uparrow\downarrow}, g^{\uparrow}) \in \mathcal{J}(L)$;
(2) *if* $|UN_g| \geq 2$ *and* $\cap_{g_j \in UN_g} g_j^{\uparrow} - g^{\uparrow} \neq \emptyset$, *then* $(g^{\uparrow\downarrow}, g^{\uparrow}) \in \mathcal{J}(L)$.

5.1.2 Relationship between the common granule and the attribute granule chain

The following lemma from the Basic Theorem on Concept Lattices (Ganter and Wille 1999) shows that $\{(m^{\downarrow}, m^{\downarrow\uparrow}) | m \in M\}$ is meet-dense in $L(G, M, I)$.

Lemma 5.2 *Suppose* (G, M, I) *is a formal context,* $L(G, M, I)$ *is its concept lattice. Then,* $(X, B) = \wedge_{m \in B}(m^{\downarrow}, m^{\downarrow\uparrow})$ *holds for any concept* (X, B).

Since the result reveals the relationship between a formal concept and its intent, attribute concepts is the final granule in attribute granule chain, Lemma 5.2 also shows the relationship between a common granule and each granule in the attribute granule chain.

Then, the relationship between attribute-induced pairs and meet-irreducible concepts is given in Theorem 5.2 and Corollary 5.2.

Theorem 5.2 (Wei and Wan 2016) *Let* (G, M, I) *be a formal context,* $(\mathcal{ATI}(L), \leq)$ *be its property pictorial diagram, and* $Max(\mathcal{ATI}(L))$ *be the set of maximal elements in* $(\mathcal{ATI}(L), \leq)$. *Thus,*

(1) $Max(\mathcal{ATI}(L)) \subseteq \mathcal{M}(L)$;
(2) *if* $(m^{\downarrow}, m) \notin Max(\mathcal{ATI}(L))$ *and* $\cap_{m_t \in UN_m} m_t^{\downarrow} - m^{\downarrow} \neq \emptyset$, *then* $(m^{\downarrow}, m^{\downarrow\uparrow}) \in \mathcal{M}(L)$, *where* UN_m *is the upper-neighbourhood of* m.

Corollary 5.2 (Wei and Wan 2016) *Let* (G, M, I) *be a formal context,* $(\mathcal{ATI}(L), \leq)$ *be its property pictorial diagram. For any* $(m^{\downarrow}, m) \in \mathcal{ATI}(L)$, UN_m *is the upper-neighbourhood of* m. *Thus,*

(1) *if* $|UN_m| \leq 1$, *then* $(m^{\downarrow}, m^{\downarrow\uparrow}) \in \mathcal{M}(L)$;
(2) *if* $|UN_m| \geq 2$ *and* $\cap_{m_t \in UN_m} m_t^{\downarrow} - m^{\downarrow} \neq \emptyset$, *then* $(m^{\downarrow}, m^{\downarrow\uparrow}) \in \mathcal{M}(L)$.

5.2 A sublattice as a granule

The concept of sublattice given in Definition 5.1 is a common notion in the theory of lattices.

Definition 5.1 (Davey and Priestley 2002) Let L be a lattice and $\emptyset \neq S \subseteq L$. Then S is a sublattice of L if $a, b \in S$ imply $a \vee b \in S$ and $a \wedge b \in S$.

A sublattice is a part of concept lattice, and it reveals some information about the lattice or the original data. Meanwhile, it is also a lattice, which means that it has a good structure and some good properties.

Since there may have many sublattices in a lattice, to find a specific one with meaningful semantics is important. Luckily, the sublattice generated by a nonempty set $S \subseteq L$ is a special one.

Theorem 5.3 (Davey and Priestley 2002) *Let* L *be a lattice. For each* $S \subseteq L(S \neq \emptyset)$, *let* $\langle S \rangle = \bigcap\{K \in \mathtt{Sub}_0 L | S \subseteq K\}$, *then,* $\langle S \rangle$ *is the smallest sublattice of* L *which contains* S. *Where* $\mathtt{Sub}L$ *is the collection of all sublattices of* L *and* $\mathtt{Sub}_0 L = \mathtt{Sub}L \cup \{\emptyset\}$, *both are ordered by inclusion.*

The sublattice $\langle S \rangle$ is called the sublattice generated by S. Compared to other sublattices, this is the smallest sublattice containing S.

All the granules in Sects. 3 and 4 are easy to obtain. However, the definition of the sublattice generated by S in terms of set-intersection does not give a viable method for calculating $\langle S \rangle$ in a finite lattice. A special calculation method was given by Davey and Priestley (2002) by adding elements to S. Therefore, computing a sublattice is not as easy as other granules.

5.3 Semantics differences among the granules

The granules in FCA are proposed from the perspectives of single object and single attribute, respectively, which suggests that there are two types of granules. One regards to objects, the other regards to attributes. And then, the granules' form is generalized and more meaningful granules are introduced by combining the information of objects and attributes together. Finally, more complicated granules, such as formal concepts and sublattices, are given to show more information of the formal context. The detailed semantics of each granules are discussed as follows.

For single object/attribute, due to the clarified formal context, the single object/attribute represents the corresponding object/attribute equivalence class, which has information for classification.

Using operators from FCA, the object/attribute equivalence class can be generalized to a pair (an object set, an attribute set). We call them an object/attribute-induced pair, which shows the information of a formal context intuitively. Most of these pairs are not formal concepts, but there are some special granules: atom/coatom concepts that reflect the atomic characteristics of formal concepts.

Then, the most basic and important notions, i.e., join/meet-irreducible concepts, that can determine the structure of a concept lattice are considered. Due to their importance in lattice structure, we call them essential granules.

Taking into account of join/meet-irreducible concepts and the Basic Theorem in FCA, we consider object/attribute concepts as elementary granules. Object/Attribute concepts not only have concise meaning but also are easy to compute.

Object/Attribute concepts are special formal concepts, so we consider general formal concepts as common granules in FCA. Further, we promote these granules to a higher level, sublattice, which has a lattice structure. The sublattice is the only one granule that has structure, while other granules do not have.

Therefore, the granules we proposed have different semantics and complexities, and they reflect different levels and information contained in a formal context.

5.4 An example

We consider Example 2.2 again to explain the idea shown in Sect. 5.3 from the perspective of objects. We choose the object 3 to start discussion.

The singleton set {3} is an equivalence class on the object set *G* in this formal context, and its object-induced pair is (3, *bcgh*). This pair is the object concept produced by the object 3, and it is not only an atom concept but also a join-irreducible concept (this concept is marked as No.3 in Example 2.2).

Therefore, this concept has different status and plays different roles. As an object-induced pair, the concept shows the original binary relation between the object 3 and the attributes. As an atom concept, the concept is an upper neighbour of the bottom and shows the basic information about the lattice. As a join-irreducible concept, the concept is indispensable to construct the lattice. As an object concept, the concept is a kind of important basic concept to form a common concept, and the method is shown in Lemma 5.1.

There are three common concepts related to the concept No.3. They are No.6, No.7, and No.9. These three concepts reflect general information about the lattice since they can be created by other concepts.

Furthermore, the minimal sublattice containing the four concepts is the set that not only contains these four concepts but also contains No.13, No.16, No.17, and No.19. The sublattice that is marked as the solid points is shown in Fig. 11. A sublattice is a kind of composite information, and it has the lattice structure. It seems a "reduced" version of the original concept lattice.

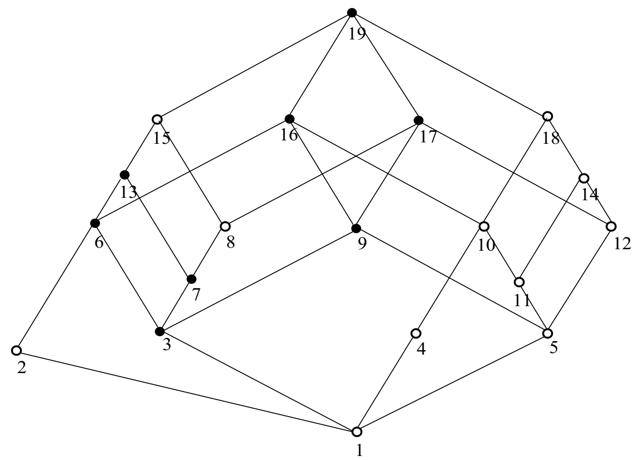

Fig. 11 A marked sublattice

6 Conclusion

In this paper, we review relevant literatures and give five types of singular specific granules and two types of integrated granules in the framework of FCA. We explain them from the perspective of granular computing. All the granules are existing notions in FCA or theory of lattices, they are concrete and easy to understand.

We investigate the granules and their relationships from two perspectives. One is the source producing granules, the other is complexity of granule structure. The granules with respect to source are shown in Fig. 12. The chain shows the relationships between the lower granules and the higher ones. The granules with respect to structure complexity are shown in Fig. 13, which reveals their different levels.

These different granules not only are meaningful and correlated but also can be calculated and transformed. This is consistent with the essential idea of granular computing.

The contributions of this paper include proposing different granules with different meaning and giving a multi-level granularity in FCA, which is an essential work when we study FCA using the idea of GrC.

Actually, these granules we proposed in this paper can be used solely or combined together according to different situations and purposes. In addition, the measure and quantification of these granules are profound because they can directly tell us which granule is "bigger" or "smaller". Different from the granular transformation, we can further discuss the composition and decomposition of these granules. All these topics are important in the research of FCA and GrC, and need to be studied in the future.

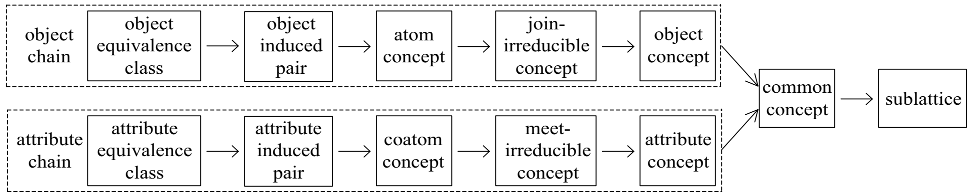

Fig. 12 Granule chain

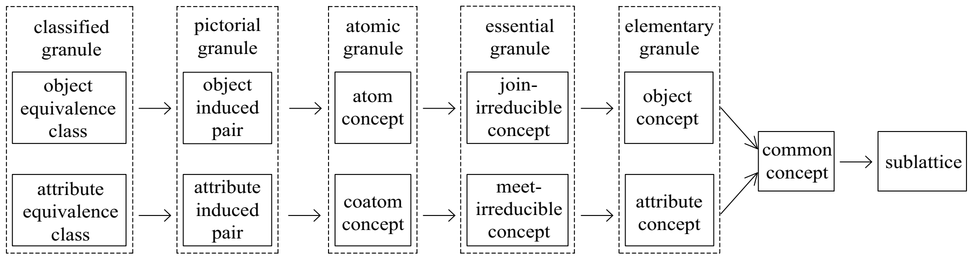

Fig. 13 Granule level

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