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ORIGINAL PAPER

Hybrid aggregation operators based on Pythagorean hesitant fuzzy sets and their application to group decision making

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Abstract
Pythagorean hesitant fuzzy set (PHFS) is a powerful tool to deal with uncertainty and vagueness. Therefore, based on Pythagorean hesitant fuzzy information in this paper we develop hybrid aggregation operators for Pythagorean hesitant fuzzy information namely, Pythagorean hesitant fuzzy hybrid weighted averaging operator, Pythagorean hesitant fuzzy hybrid weighted geometric operator. These developed operators can weigh both the argument and their ordered positions. Additionally, some numerical examples are given to illustrate the developed operators. Moreover we develop a multi-attribute group decision making approach based on the proposed operators. Finally, we give a numerical example to show the effectiveness and flexibility of the proposed method.

Keywords Pythagorean hesitant fuzzy sets · PHFHWA operator · PHFHWG operator · Multi-attribute group decision making (MAGDM)

1 Introduction

Fuzzy sets (Zadeh 1965) are considered as an important tool to solve multi-attribute decision making (MADM) problems (Bellman and Zadeh 1970; Yager 1997) and can also be applied to fuzzy logic and approximate reasoning (Zadeh 1975a, b), pattern recognition (Pedrycz 1990) and decision making based on granular computing is studied in Pedrycz and Chen (2011, 2015a, b). Wang and Chen (2008) introduced a new approach for evaluating students' answer scripts using fuzzy numbers associated with degrees of confidence of the evaluator. Based on automatic clustering techniques and fuzzy logical relationships Chen et al. (2009) developed an approach for forecast enrollments. Chen and Chen (2011) introduced a new forecasting method based on high-order fuzzy logical relationships to forecast the Taiwan Stock Exchange Capitalization Weighted Stock Index (TAIEX). In Chen et al. (2014) the authors proposed a new method for group decision making with incomplete fuzzy preference relations based on the constructed modified consistency matrices of experts which satisfy the additive consistency and the order consistency with consistency degrees. Chen and Chen (2011) proposed fuzzy risk analysis based on the proposed fuzzy ranking method, where the evaluating values are represented by generalized fuzzy numbers. Zulueta and Garcia (2018) proposed dynamic multi-attribute decision making (DMADM) problems with correlated periods, in which the attribute assessment values take the form of 2-tuple linguistic values. Mandal and Ranadive (2018) proposed multi-granulation interval-valued fuzzy preference relation probabilistic rough sets (MG-IVFPR-PRSs).

However, the fuzzy set theory is still confronted with some limitations when decision makers intend to deal with some uncertain information induced from several sources of vagueness. To address this situation, a series of generalizations of fuzzy set theory were proposed over the past years (Atanassov 1986; Torra 2010; Zhu et al. 2012; Yager 2013).

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Among the various extension forms of fuzzy set theory, intuitionistic fuzzy set (IFS), due to Atanassov is generally assumed as an intuitively straightforward extension of fuzzy set theory (Atanassov 1986). Since the establishment of IFS, it has been successfully applied in many areas of decision making problems. Chen and Chang (2015) proposed similarity measure between IFSs based on transformation techniques and apply the proposed similarity measure between AIFSs to deal with pattern recognition problems. Xu (2007) developed intuitionistic fuzzy weighted averaging (IFWA) operator, intuitionistic fuzzy ordered weighted averaging (IFOWA) operator and intuitionistic fuzzy hybrid weighted averaging (IFHWA) operators for MADM problems. Xu and Yager (2006) developed some geometric operator's namely intuitionistic fuzzy weighted geometric (IFWG) operator, the intuitionistic fuzzy ordered weighted geometric (IFOWG) operator and the intuitionistic fuzzy hybrid weighted geometric (IFHWG) operator. Tang and Meng (2018) proposed linguistic intuitionistic fuzzy Hamacher aggregation (LIFGA) operators to deal with MADM problems. Jamkhaneh and Garg (2017) developed some new operation for generalized IFS to deal with the MADM problems.

Since the IFS fulfill the condition that the sum of its memberships degrees is less than or equal to 1. However, the decision makers deals with the situation of particular attributes that the sum of its memberships degrees is greater than 1. To overcome this situation Yager (2013) introduced the notion of Pythagorean fuzzy set (PFS), which has been proved useful to deal with uncertain information in decision making procedures. PFS satisfy the condition that the square sum of membership degree and a nonmembership degree is less than or equal to 1. Since the introduction of the PFS, many scholars have explored the PFS from different facts and obtained plenty of meaningful results. For instance Yager (2014) developed various aggregation operators, namely, Pythagorean fuzzy weighted average (PFWA) operator, Pythagorean fuzzy weighted geometric (PFWG) operator, Pythagorean fuzzy power weighted average (PFPWA) operator, and Pythagorean fuzzy power weighted geometric (PFPWG) operator to aggregate the Pythagorean fuzzy numbers. The aggregation operators discussed above or of the same priority level. However, in real group decision making problems the attribute and decision makers may have different priority level. Due to this short coming Khan et al. (2018a) developed Pythagorean fuzzy prioritized operator for MADM problem. Rahman et al. (2017a) developed a MADM approach based on Pythagorean fuzzy weighted geometric aggregation operator (PFWG) for plant location selection. Rahman et al. (2017b) developed interval-valued Pythagorean fuzzy geometric (IVPFG) operator for MADM problems. Rahman et al. (2018) developed interval-valued Pythagorean fuzzy ordered weighted averaging (IVPFOWA) operator for MADM problems. Based on Choquet integral in Khan at al. 2018a the authors developed interval-valued Pythagorean fuzzy (IVPF) TOPSIS method to deal with MADM problem. Extended the concept of traditional gray relational analysis (GRA) method Khan and Abdullah developed interval-valued Pythagorean fuzzy GRA method for MADM problems (Khan and Abdullah 2018).

Hesitant fuzzy sets (HFSs) which are another extension of fuzzy sets are extremely useful in handling situations where decision makers are hesitate in providing their preferences with regard to objects in a decision making process and have provided a theory for solving MCDM problems in certain situations. HFS was first introduced by Torra and Narukawa (2009) and Torra (2010), and permits the membership degree of an element to be a set of several possible values between 0 and 1. Xia and Xu (2011) developed a series of aggregation operators under hesitant fuzzy environment namely hesitant fuzzy weighted averaging (HFWA) operator, hesitant fuzzy weighted geometric (HFWG) operator, hesitant fuzzy ordered weighted averaging (HFOWA) operator, hesitant fuzzy ordered weighted averaging (HFOWG) operator, generalized hesitant fuzzy weighted averaging (GHFWA) operator, generalized hesitant fuzzy weighted geometric (GHFWG) operator, generalized hesitant fuzzy ordered weighted averaging (GHFOWA) operator, generalized hesitant fuzzy ordered weighted geometric (GHFOWG) operator. Indeed according to the decision makers (DMs) preferences it is possible to extend the shorter HFN by adding any of its values until it is equal in length to the longer 1. Therefore, due to the varied preferences of DMs this may lead to a different optimal alternative. In Liao and Xu (2015) introduced the concept of hesitant fuzzy hybrid weighted averaging (HFHWA) operator, hesitant fuzzy hybrid weighted geometric aggregation operator (HFHWG). Lee and Chen (2015a) developed a fuzzy group decision making method based on the likelihood-based comparison relations of hesitant fuzzy linguistic term sets. Chen and Hong (2014) presented a new method for multi-criteria linguistic decision making based on hesitant fuzzy linguistic term sets using the pessimistic attitude and the optimistic attitude of the decision-maker. Lee and Chen (2015b) proposed a new fuzzy decision making method and propose a new fuzzy group decision making method based on the proposed likelihood-based comparison relations of hesitant fuzzy linguistic term sets and also developed hesitant fuzzy linguistic weighted average (HFLWA) operator, the proposed hesitant fuzzy linguistic weighted geometric

(HFLWG) operator, the proposed hesitant fuzzy linguistic ordered weighted average (HFLOWA) operator, and the proposed hesitant fuzzy linguistic ordered weighted geometric (HFLOWG) operator of hesitant fuzzy linguistic term sets.

Zhu et al. (2012) developed the concept of dual hesitant fuzzy set (HFS) and also discussed their basic operations and properties. In Peng et al. (2014) the authors applied the concept of Intuitionistic hesitant fuzzy set (IHFS) to group decision making problems using fuzzy cross-entropy. However, there may be a situation where the decision maker may provide the degree of membership and nonmembership of a particular attribute in such a way that their sum is greater than 1. To overcome this shortcoming Khan et al. (2017), introduced the concept of Pythagorean hesitant fuzzy set, generalized the concept of intuitionistic hesitant fuzzy set under the restriction that the square sum of its membership degrees is less than or equal to 1. The authors discussed some basic operational laws and developed Pythagorean hesitant fuzzy weighted averaging (PHFWA) operator and Pythagorean hesitant fuzzy weighted geometric (PHFWG) operator under Pythagorean hesitant fuzzy environments. Khan et al. (2018a) developed Pythagorean hesitant fuzzy ordered weighted averaging (PHFOWA) operator and Pythagorean hesitant fuzzy ordered weighted geometric (PHFOWG) operator for MADM problems.

Since the aggregating operators by Khan et al. (2017, 2018a) developed under Pythagorean hesitant fuzzy environment cannot weight both the argument and their ordered positions. Therefore, motivating by the idea presented in Xu and Da (2003) in this paper we develop Pythagorean hesitant fuzzy hybrid weighted averaging (PHFHWA) operator and Pythagorean hesitant fuzzy hybrid weighted geometric (PHFHWG) operator to deal with MADM problems. The main focus of this paper is the aggregation techniques given to aggregate the values for each alternative under the attributes. To do this, we organize the remainder of the paper as:

In Sect. 2, we discuss some basic definitions and properties. In Sect. 3 we develop some aggregation operators for Pythagorean hesitant fuzzy information namely, Pythagorean hesitant fuzzy hybrid weighted averaging (PHFHWA) operator, Pythagorean hesitant fuzzy hybrid weighted geometric (PHFHWG) operator. Based on the developed operators in Sect. 4 we give an application to MADM with Pythagorean hesitant fuzzy information. Concluding remarks are made in Sect. 5.

2 Basic concepts

Zhu et al. (2012) initiated the concept of dual hesitant fuzzy set (DHFS) in which each element in the DHFS is expressed by $\hat{h} = \langle \Lambda_{\hat{h}}, \Gamma_{\hat{h}} \rangle$. DHFS has its greatest use in practical multiple attribute decision making (MADM) problems and the academic research has achieved great development. However, in the some practical problems the sum of membership degree and nonmembership degree to which an alternative satisfying an attribute provided by decision maker (DM) may be bigger than 1, but their square sum is less than or equal 1. Therefore, Khan et al. (2017) developed Pythagorean hesitant fuzzy set (PHFS) characterized by a membership degree and nonmembership degree, satisfies the condition that the square sum of its membership degree and nonmembership degree is less than or equal to 1. It is defined as follows:

Definition 1 (Khan et al. 2017) Let X be a fixed set. A Pythagorean hesitant fuzzy set abbreviated as PHFS P_H in X is an object with the following notion:

$$P_H = \left\{ \left\langle x, \Lambda_{P_H}(x), \Gamma_{P_H}(x) | x \in X \right\rangle \right\}, \tag{1}$$

where $\Lambda_{P_H}(x)$ and $\Gamma_{P_H}(x)$ are mappings from X to $[0, 1]$, denoting a possible degree of membership and nonmembership degree of element $x \in X$ in P_H, respectively, and for each element $x \in X$, $\forall\ h_{P_H}(x) \in \Lambda_{P_H}(x)$, $\exists h'_{P_H}(x) \in \Gamma_{P_H}(x)$ such that $0 \preccurlyeq h^2_{P_H}(x) + h'^2_{P_H}(x) \preccurlyeq 1$, and $\forall\ h'_{P_H}(x) \in \Gamma_{P_H}(x)$, $\exists h_{P_H}(x) \in \Lambda_{P_H}(x)$ such that $0 \preccurlyeq h^2_{P_H}(x) + h'^2_{P_H}(x) \preccurlyeq 1$. For any PHFS $P_H = \left\{ \left\langle x, \Lambda_{P_H}(x), \Gamma_{P_H}(x) | x \in X \right\rangle \right\}$ and for all $x \in X$, $\Pi_{P_H}(x) = \cup_{h_{P_H} \in \Lambda_{P_H}(x), h'_{P_H}(x) \in \Gamma_{P_H}(x)} \sqrt{1 - h^2_{P_H} - h'^2_{P_H}}$ is said to be the degree of indeterminacy of x to P_H, where $1 - h^2_{P_H} - h'^2_{P_H} \succcurlyeq 0$.

Moreover, PHFS(X) denotes the set of all elements of PHFSs. If X has only one element $\langle x, \Lambda_{P_H}(x), \Gamma_{P_H}(x) \rangle$ is said to be Pythagorean hesitant fuzzy number and is denoted by $\hat{h} = \langle \Lambda_{\hat{h}}, \Gamma_{\hat{h}} \rangle$ for convenience. We denote the set of all PHFNs by PHFNS. For all $x \in X$ if $\Lambda_{P_H}(x)$ and $\Gamma_{P_H}(x)$ have only element. Then the PHFSs become PFSs. If the nonmembership degree is $\{0\}$, then PHFSs become a HFSs.

Definition 2 (Khan et al. 2017) For any three PFN $\hat{h} = \langle \Lambda_{\hat{h}}, \Gamma_{\hat{h}} \rangle$, $\hat{h}_1 = \left\langle \Lambda_{\hat{h}_1}, \Gamma_{\hat{h}_1} \right\rangle$, $\hat{h}_2 = \left\langle \Lambda_{\hat{h}_2}, \Gamma_{\hat{h}_2} \right\rangle$, and $\lambda \succ 0$. The following operational laws are valid.

(1) $\hat{h}_1 \cup \hat{h}_2 = \left\langle \max\left\{ \Lambda_{\hat{h}_1}, \Lambda_{\hat{h}_2} \right\}, \min\left\{ \Gamma_{\hat{h}_1}, \Gamma_{\hat{h}_2} \right\} \right\rangle$,

(2) $\hat{h}_1 \cap \hat{h}_2 = \left\langle \min\left\{ \Lambda_{\hat{h}_1}, \Lambda_{\hat{h}_2} \right\}, \max\left\{ \Gamma_{\hat{h}_1}, \Gamma_{\hat{h}_2} \right\} \right\rangle$,

(3) $\hat{h}^c = \langle \Gamma_{\hat{h}}, \Lambda_{\hat{h}} \rangle$,

(4) $\hat{h}_1 \oplus \hat{h}_2 = \left\langle \cup_{h_{\hat{h}_1} \in \Lambda_{\hat{h}_1}, h_{\hat{h}_2} \in \Lambda_{\hat{h}_2}} \left\{ \sqrt{h^2_{\hat{h}_1} + h^2_{\hat{h}_2} - h^2_{\hat{h}_1} h^2_{\hat{h}_2}} \right\}, \cup_{h'_{\hat{h}_1} \in \Gamma_{\hat{h}_1}, h'_{\hat{h}_2} \in \Gamma_{\hat{h}_2}} \left\{ h'_{\hat{h}_1} h'_{\hat{h}_2} \right\} \right\rangle,$

(5) $\hat{h}_1 \otimes \hat{h}_2 = \left\langle \cup_{h_{\hat{h}_1} \in \Lambda_{\hat{h}_1}, h_{\hat{h}_2} \in \Lambda_{\hat{h}_2}} \left\{ h_{\hat{h}_1} h_{\hat{h}_2} \right\}, \cup_{h'_{\hat{h}_1} \in \Gamma_{\hat{h}_1}, h'_{\hat{h}_2} \in \Gamma_{\hat{h}_2}} \left\{ \sqrt{h'^2_{\hat{h}_1} + h'^2_{\hat{h}_2} - h'^2_{\hat{h}_1} h'^2_{\hat{h}_2}} \right\} \right\rangle,$

(6) $\lambda\hat{h} = \left\langle \cup_{h_{\hat{h}} \in \Lambda_{\hat{h}}} \left\{ \sqrt{1 - (1 - (h_{\hat{h}})^2)^{\lambda}} \right\}, \cup_{h'_{\hat{h}} \in \Gamma_{\hat{h}}} \left\{ (h'_{\hat{h}})^{\lambda} \right\} \right\rangle,$

$\lambda \succ 0,$

(7) $\hat{h}^{\lambda} = \left\langle \cup_{h_{\hat{h}} \in \Lambda_{\hat{h}}} \{ h^{\lambda}_{\hat{h}} \}, \cup_{h'_{\hat{h}} \in \Gamma_{\hat{h}}} \left\{ \sqrt{1 - \left(1 - (h'_{\hat{h}})^2\right)^{\lambda}} \right\} \right\rangle,$

$\lambda \succ 0.$

To compare PHFNs Khan et al. (2017) introduced the concept of score function and accuracy function as follows:

Definition 3 (Khan et al. 2017) Let $\hat{h} = (\Lambda_{\hat{h}}, \Gamma_{\hat{h}})$ be a PHFN. Then we define the score function $S(\hat{h})$ and the accuracy function $A(\hat{h})$ is defined as follows:

$$S(\hat{h}) = \left(\frac{1}{l_{h_{\hat{h}} \in \Lambda_{\hat{h}}}} \sum_{h_{\hat{h}} \in \Lambda_{\hat{h}}} h_{\hat{h}} \right)^2 - \left(\frac{1}{l_{h'_{\hat{h}} \in \Gamma_{\hat{h}}}} \sum_{h'_{\hat{h}} \in \Gamma_{\hat{h}}} h'_{\hat{h}} \right)^2 \tag{2}$$

$$A(\hat{h}) = \left(\frac{1}{l_{h_{\hat{h}} \in \Lambda_{\hat{h}}}} \sum_{h_{\hat{h}} \in \Lambda_{\hat{h}}} h_{\hat{h}} - s(\hat{h}) \right)^2 + \left(\frac{1}{l_{h'_{\hat{h}} \in \Gamma_{\hat{h}}}} \sum_{h'_{\hat{h}} \in \Gamma_{\hat{h}}} h'_{\hat{h}} - s(\hat{h}) \right)^2, \tag{3}$$

where $S(\hat{h}) \in [-1, 1]$, $l_{h_{\hat{h}}}$ denotes the number of elements in $\Lambda_{\hat{h}}$ and $l_{h'_{\hat{h}}}$ denotes the number of elements in $\Gamma_{\hat{h}}$. Here we can see that $S(\hat{h})$ is just the mean value in statistics, and $A(\hat{h})$ is just the standard variance, which reflects the accuracy function between all values in the PHFN $\hat{h}$ and their mean value. Inspired by this idea, based on the score $S(\hat{h})$ and the accuracy function $A(\hat{h})$, we can compare and rank, two PHFNs as follows:

Definition 4 (Khan et al. 2017) Let $\hat{h}_1$ and $\hat{h}_2$ be two PHFNs, $S(\hat{h}_1)$ be the score of $\hat{h}_1$, $S(\hat{h}_2)$ be the score of $\hat{h}_2$, and $A(\hat{h}_1)$ be the deviation degree of $\hat{h}_1$, $A(\hat{h}_2)$ be the accuracy function of $\hat{h}_2$. Then

1. If $S(\hat{h}_1) \prec S(\hat{h}_2)$, then $\hat{h}_1 \prec \hat{h}_2$.
2. If $S(\hat{h}_1) \succ S(\hat{h}_2)$, then $\hat{h}_1 \succ \hat{h}_2$.
3. If $S(\hat{h}_1) = S(\hat{h}_2)$, then $\hat{h}_1 \sim \hat{h}_2$.
   i. If $A(\hat{h}_1) \prec A(\hat{h}_2)$, then $\hat{h}_1 \prec \hat{h}_2$.
   ii. If $A(\hat{h}_1) \succ A(\hat{h}_2)$, then $\hat{h}_1 \succ \hat{h}_2$.
   iii. If $A(\hat{h}_1) = A(\hat{h}_2)$, then $\hat{h}_1 \sim \hat{h}_2$.

Based on the operational laws we defined the following aggregation operators under Pythagorean hesitant fuzzy environments.

Definition 5 (Khan et al. 2017) Let $\hat{h}_i = (\Lambda_{\hat{h}_i}, \Gamma_{\hat{h}_i})(i = 1, 2, 3, \ldots, n)$ be a collection of all $PHFN's$, and $w = (w_1, w_2, \ldots, w_n)^T$ be the weight vector of $\hat{h}_i\,(i = 1, 2, 3, \ldots, n)$ with $w_i \geq 0\,(i = 1, 2, 3, \ldots, n)$, where $w_i \in [0, 1]$ and $\sum_{i=1}^{n} w_i = 1$. Then Pythagorean hesitant fuzzy weighted averaging ($PHFWA$) operator is a mapping

$PHFWA : PHFN^n \to PHFN$ can be defined as:

$$PHFWA(\hat{h}_1, \hat{h}_2, \ldots, \hat{h}_n) = \bigoplus_{i=1}^{n} (w_i \hat{h}_i) = \left\langle \begin{array}{l} \cup_{h_{\hat{h}_1} \in \Lambda_{\hat{h}_1}, h_{\hat{h}_2} \in \Lambda_{\hat{h}_2}, \ldots, h_{\hat{h}_n} \in \Lambda_{\hat{h}_n}} \left\{ \sqrt{1 - \prod_{i=1}^{n} (1 - h^2_{\hat{h}_i})^{w_i}} \right\}, \\ \cup_{h'_{\hat{h}_1} \in \Gamma_{\hat{h}_1}, h'_{\hat{h}_1} \in \Gamma_{\hat{h}_2}, \ldots h'_{\hat{h}_n} \in \Gamma_{\hat{h}_n}} \left\{ \prod_{i=1}^{n} \left(h'_{\hat{h}_i} \right)^{w_i} \right\} \end{array} \right\rangle \tag{4}$$

and the PHFWA operator is said to be a Pythagorean hesitant fuzzy weighted averaging operator.

Definition 6 (Khan et al. 2017) Let $\hat{h}_i = \left(\Lambda_{\hat{h}_i}, \Gamma_{\hat{h}_i} \right)$ $(i = 1, 2, 3, \ldots, n)$ be a collection of all PHFNs, and $w = (w_1, w_2, \ldots, w_n)$ be the weight vector of $\hat{h}_i (i = 1, 2, 3, \ldots, n)$ with $w_i \geqslant 0 (i = 1, 2, \ldots, n)$ such that $w_i \in [0, 1]$ and $\sum_{i=1}^{n} w_i = 1$. Then, Pythagorean hesitant fuzzy ordered weighted geometric ($PHFWG$) operator is a mapping

$PHFWG : PHFN^n \to PHFN$ can be defined as:

$$PHFWG(\hat{h}_1, \hat{h}_2, \ldots, \hat{h}_n) = \bigotimes_{i=1}^{n} (\hat{h}_i^{w_i}) = \left\langle \begin{array}{l} \cup_{h_{\hat{h}_1} \in \Lambda_{\hat{h}_1}, h_{\hat{h}_2} \in \Lambda_{\hat{h}_2}, \ldots, h_{\hat{h}_n} \in \Lambda_{\hat{h}_n}} \left\{ \prod_{i=1}^{n} \left(h_{\hat{h}_i} \right)^{w_i} \right\}, \\ \cup_{h'_{\hat{h}_1} \in \Gamma_{\hat{h}_1}, h'_{\hat{h}_2} \in \Gamma_{\hat{h}_2}, \ldots, h'_{\hat{h}_n} \in \Gamma_{\hat{h}_n}} \left\{ \sqrt{1 - \prod_{i=1}^{n} \left(1 - h'^2_{\hat{h}_i} \right)^{w_i}} \right\} \end{array} \right\rangle \tag{5}$$

and the PHFWG operator is said to be a Pythagorean hesitant fuzzy weighted geometric operator.

Definition 7 (Khan et al. 2018b) Let $\hat{h}_i = \left(\Lambda_{\hat{h}_i}, \Gamma_{\hat{h}_i} \right)$ $(i = 1, 2, 3, \ldots, n)$ be a collection of all $PHFN's$, and

$w = (w_1, w_2, \ldots, w_n)^T$ be the weight vector of $\hat{h}_i (i = 1, 2, 3, \ldots, n)$ with $w_i \geqslant 0\,(i = 1, 2, \ldots n)$, where $w_i \in [0,\ 1]$ and $\sum_{i=1}^{n} w_i = 1$. Then Pythagorean hesitant fuzzy ordered weighted averaging (*PHFOWA*) operator is a mapping

PHFOWA : $PHFN^n \rightarrow PHFN$ can be defined as:

$$PHFOWA(\hat{h}_1, \hat{h}_2, \ldots, \hat{h}_n) = \overset{n}{\underset{i=1}{\oplus}} \left(w_i \hat{h}_{\sigma(i)}\right) = \left\langle \begin{array}{l} \cup_{h_{\hat{h}_{\sigma(1)}} \in \Lambda_{\hat{h}_{\sigma(1)}}, h_{\hat{h}_{\sigma(2)}} \in \Lambda_{\hat{h}_{\sigma(2)}}, \ldots, h_{\hat{h}_{\sigma(n)}} \in \Lambda_{\hat{h}_{\sigma(n)}}} \left\{ \sqrt{1 - \prod_{i=1}^{n} \left(1 - h^2_{\hat{h}_{\sigma(i)}}\right)^{w_i}} \right\}, \\ \cup_{h'_{\hat{h}_{\sigma(1)}} \in \Gamma_{\hat{h}_{\sigma(1)}}, h'_{\hat{h}_{\sigma(2)}} \in \Gamma_{\hat{h}_{\sigma(2)}}, \ldots, h'_{\hat{h}_{\sigma(n)}} \in \Gamma_{\hat{h}_{\sigma(n)}}} \left\{ \prod_{i=1}^{n} \left(h'_{\hat{h}_{\sigma(i)}}\right)^{w_i} \right\} \end{array} \right\rangle \tag{6}$$

Definition 8 (Khan et al. 2018b) Let $\hat{h}_i = \left(\Lambda_{\hat{h}_i}, \Gamma_{\hat{h}_i}\right)$ $(i = 1, 2, 3, \ldots, n)$ be a collection of all PHFNs $\hat{h}_{\sigma(i)}$ be the largest in them, $w = (w_1, w_2, \ldots, w_n)$ be the weight vector of $\hat{h}_i\, (i = 1, 2, 3, \ldots, n)$ with $w_i \geqslant 0\, (i = 1, 2, 3, \ldots, n)$ such that $w_i \in [0, 1]$ and $\sum_{i=1}^{n} w_i = 1$. Then Pythagorean hesitant fuzzy ordered weighted geometric (*PHFOWG*) operator is a mapping

PHFOWG : $PHFN^n \rightarrow PHFN$ can be defined as:

$$PHFOWG(\hat{h}_1, \hat{h}_2, \ldots, \hat{h}_n) = \overset{n}{\underset{i=1}{\otimes}} \left(\hat{h}^{w_i}_{\sigma(i)}\right) = \left\langle \begin{array}{l} \cup_{h_{\hat{h}_{\sigma(1)}} \in \Lambda_{\hat{h}_{\sigma(1)}}, h_{\hat{h}_{\sigma(2)}} \in \Lambda_{\hat{h}_{\sigma(2)}}, \ldots, h_{\hat{h}_{\sigma(n)}} \in \Lambda_{\hat{h}_{\sigma(n)}}} \left\{ \prod_{i=1}^{n} \left(h_{\hat{h}_{\sigma(i)}}\right)^{w_i} \right\}, \\ \cup_{h'_{\hat{h}_{\sigma(1)}} \in \Gamma_{\hat{h}_{\sigma(1)}}, h'_{\hat{h}_{\sigma(2)}} \in \Gamma_{\hat{h}_{\sigma(2)}}, \ldots, h'_{\hat{h}_{\sigma(n)}} \in \Gamma_{\hat{h}_{\sigma(n)}}} \left\{ \sqrt{1 - \prod_{i=1}^{n} \left(1 - h'^2_{\hat{h}_{\sigma(i)}}\right)^{w_i}} \right\} \end{array} \right\rangle \tag{7}$$

3 Pythagorean hesitant fuzzy hybrid aggregation operators

From Definitions 5 to 8, we know that the *PHFWA* operator and *PHFWG* operator weighs only the Pythagorean hesitant fuzzy numbers, respectively, while the PHFOWA operator and *PHFOWG* operator weighs only the ordered positions of the Pythagorean hesitant fuzzy numbers, respectively, instead of weighing the Pythagorean hesitant fuzzy numbers themselves. In the following, we develop a *PHFHWA* operator, *PHFHWG* operator which weighs both the given Pythagorean hesitant fuzzy number and its ordered position.

Definition 9 Let $\hat{h}_i = \left(\Lambda_{\hat{h}_i}, \Gamma_{\hat{h}_i}\right) (i = 1, 2, 3, \ldots, n)$ be a collection of all $PHFN's$, and $w = (w_1, w_2, \ldots, w_n)^T$ be the weight vector of $\hat{h}_i (i = 1, 2, 3, \ldots, n)$ with $w_i \geqslant 0 (i = 1, 2, \ldots, n)$ where $w_i \in [0, 1]$ and $\sum_{i=1}^{n} w_i = 1$, n is the balancing coefficient which plays a role of balance. Then Pythagorean hesitant fuzzy hybrid weighted averaging *PHFHWA* operator is a mapping *PHFHWA*: $PHFN^n \rightarrow PHFN$ with an aggregation-associated vector $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ such that $\omega_i \in [0, 1]$, $\sum_{i=1}^{n} \omega_i = 1$ and can be defined as follows:

$$\begin{aligned} PHFHWA(\hat{h}_1, \hat{h}_2, \ldots, \hat{h}_n) &= \overset{n}{\underset{i=1}{\oplus}} \left(\omega_i \dot{\hat{h}}_{\sigma(i)}\right) \\ &= \left(\omega_1 \dot{\hat{h}}_{\sigma(1)} \oplus \omega_1 \dot{\hat{h}}_{\sigma(2)} \oplus \cdots \oplus \omega_n \dot{\hat{h}}_{\sigma(n)}\right) \end{aligned} \tag{8}$$

and the mapping *PHFHWA* is said to be a Pythagorean hesitant fuzzy hybrid weighted averaging operator where $\dot{\hat{h}}_{\sigma(i)}$ is the *i*th largest of $\dot{\hat{h}}_{\sigma(i)} = nw_k \hat{h}_k (k = 1, 2, \ldots, n)$.

Using the different manifestation of weighting vector, the *PHFHWA* operator can be reduced into some special cases. For instance, if the associated-weighting vector $\omega = \left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)^T$, then the *PHFHWA* operator reduces to the PHFWA operator; if $w = \left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)^T$, then the *PHFHWA* operator reduces to the *PHFOWA* operator. It must be pointed out that the weighing operation of the ordered position can be synchronized with the weighing operation of the given importance by the *PHFHWA* operator. This characteristic is different from the *PHFHWA* operator.

Theorem 1 *Let $\hat{h}_i = \left(\Lambda_{\hat{h}_i}, \Gamma_{\hat{h}_i}\right)(i = 1, 2, 3, \ldots, n)$ be a collection of all PHFNs, and $w = (w_1, w_2, \ldots, w_n)^T$ be the weight vector of $\hat{h}_i(i = 1, 2, 3, \ldots, n)$ with $w_i \geqslant 0(i = 1, 2, \ldots, n)$ where $w_i \in [0, 1]$ and $\sum_{i=1}^{n} w_i = 1$. Then the aggregation result using PHFHWA operator with an aggregation-associated vector $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ such that $\omega_i \in [0, 1]$, $\sum_{i=1}^{n} \omega_i = 1$ is also a PHFN and*

$$PHFHWA(\hat{h}_1, \hat{h}_2, \ldots, \hat{h}_n) = \left(\overset{n}{\underset{i=1}{\oplus}} \left(\omega_i \dot{\hat{h}}_{\sigma(i)}\right)\right) = \left\langle \begin{array}{l} \cup_{\dot{h}_{\dot{\hat{h}}_{\sigma(i)}} \in \Lambda_{\dot{\hat{h}}_{\sigma(i)}}, \dot{h}_{\dot{\hat{h}}_{\sigma(i)}} \in \Lambda_{\dot{\hat{h}}_{\sigma(i)}}, \ldots, \dot{h}_{\dot{\hat{h}}_{\sigma(i)}} \in \Lambda_{\dot{\hat{h}}_{\sigma(i)}}} \sqrt{1 - \prod_{i=1}^{n} \left(1 - \dot{h}^2_{\hat{h}_i}\right)^{\omega_i}}, \\ \cup_{\dot{h}'_{\dot{\hat{h}}_{\sigma(i)}} \in \Gamma_{\dot{\hat{h}}_{\sigma(i)}}, \dot{h}'_{\dot{\hat{h}}_{\sigma(i)}} \in \Gamma_{\dot{\hat{h}}_{\sigma(i)}}, \ldots, \dot{h}'_{\dot{\hat{h}}_{\sigma(i)}} \in \Gamma_{\dot{\hat{h}}_{\sigma(i)}}} \left\{ \prod_{i=1}^{n} \left(\dot{h}'_{\dot{\hat{h}}_{\sigma(i)}}\right)^{\omega_i} \right\} \end{array} \right\rangle \tag{9}$$

where $\dot{\hat{h}}_{\sigma(i)}$ is the ith largest of $\dot{\hat{h}}_{\sigma(i)} = nw_k\hat{h}_k(k = 1, 2, \ldots, n)$.

Proof By mathematical induction we prove that Eq. (9) hold for all n. For this first we show that Eq. (9) holds for $n = 2$. Since,

$$\omega_1 \dot{\hat{h}}_{\sigma(1)} = \left\langle \cup_{\dot{h}_{\dot{\hat{h}}_{\sigma(1)}} \in \Lambda_{\dot{\hat{h}}_{\sigma(1)}}} \left\{ \sqrt{1 - \left(1 - \left(\dot{h}_{\dot{\hat{h}}_{\sigma(1)}}\right)^2\right)^{\omega_1}} \right\}, \cup_{\dot{h}'_{\dot{\hat{h}}_{\sigma(1)}} \in \Gamma_{\dot{\hat{h}}_{\sigma(1)}}} \left\{ \left(\dot{h}'_{\dot{\hat{h}}_{\sigma(1)}}\right)^{\omega_1} \right\} \right\rangle$$

and

$$\omega_2 \dot{\hat{h}}_{\sigma(2)} = \left\langle \cup_{\dot{h}_{\dot{\hat{h}}_{\sigma(2)}} \in \Lambda_{\dot{\hat{h}}_{\sigma(2)}}} \left\{ \sqrt{1 - \left(1 - \left(\dot{h}_{\dot{\hat{h}}_{\sigma(2)}}\right)^2\right)^{\omega_2}} \right\}, \cup_{\dot{h}'_{\dot{\hat{h}}_{\sigma(2)}} \in \Gamma_{\dot{\hat{h}}_{\sigma(2)}}} \left\{ \left(\dot{h}'_{\dot{\hat{h}}_{\sigma(2)}}\right)^{\omega_2} \right\} \right\rangle$$

So,

$$\begin{aligned} PHFHWA(\hat{h}_1, \hat{h}_2) &= \omega_1 \dot{\hat{h}}_{\sigma(1)} \oplus \omega_2 \dot{\hat{h}}_{\sigma(2)} \\ &= \left\langle \cup_{\dot{h}_{\dot{\hat{h}}_{\sigma(1)}} \in \Lambda_{\dot{\hat{h}}_{\sigma(1)}}} \left\{ \sqrt{1 - \left(1 - \left(\dot{h}_{\dot{\hat{h}}_{\sigma(1)}}\right)^2\right)^{\omega_1}} \right\}, \cup_{\dot{h}'_{\dot{\hat{h}}_{\sigma(1)}} \in \Gamma_{\dot{\hat{h}}_{\sigma(1)}}} \left\{ \left(\dot{h}'_{\dot{\hat{h}}_{\sigma(1)}}\right)^{\omega_1} \right\} \right\rangle \\ &\oplus \left\langle \cup_{\dot{h}_{\dot{\hat{h}}_{\sigma(2)}} \in \Lambda_{\dot{\hat{h}}_{\sigma(2)}}} \left\{ \sqrt{1 - \left(1 - \left(\dot{h}_{\dot{\hat{h}}_{\sigma(2)}}\right)^2\right)^{\omega_2}} \right\}, \cup_{\dot{h}'_{\dot{\hat{h}}_{\sigma(2)}} \in \Gamma_{\dot{\hat{h}}_{\sigma(2)}}} \left\{ \left(\dot{h}'_{\dot{\hat{h}}_{\sigma(2)}}\right)^{\omega_2} \right\} \right\rangle \\ &= \left\langle \begin{array}{l} \cup_{\dot{h}_{\dot{\hat{h}}_{\sigma(1)}} \in \Lambda_{\dot{\hat{h}}_{\sigma(1)}}, \dot{h}_{\dot{\hat{h}}_{\sigma(2)}} \in \Lambda_{\dot{\hat{h}}_{\sigma(2)}}} \sqrt{1 - \left(1 - \dot{h}^2_{\dot{\hat{h}}_{\sigma(1)}}\right)^{\omega_1} + 1 - \left(1 - \dot{h}^2_{\dot{\hat{h}}_{\sigma(2)}}\right)^{\omega_2} - \left(\left(1 - \dot{h}^2_{\dot{\hat{h}}_{\sigma(1)}}\right)^{w_1} \left(1 - \dot{h}^2_{\dot{\hat{h}}_{\sigma(2)}}\right)^{\omega_2}\right)}, \\ \cup_{\dot{h}'_{\dot{\hat{h}}_{\sigma(1)}} \in \Gamma_{\dot{\hat{h}}_{\sigma(1)}}, \dot{h}'_{\dot{\hat{h}}_{\sigma(2)}} \in \Gamma_{\dot{\hat{h}}_{\sigma(2)}}} \left\{ \left(\dot{h}'_{\dot{\hat{h}}_{\sigma(1)}}\right)^{w_1} \left(\dot{h}'_{\dot{\hat{h}}_{\sigma(2)}}\right)^{\omega_2} \right\} \end{array} \right\rangle \\ &= \left\langle \cup_{\dot{h}_{\dot{\hat{h}}_{\sigma(1)}} \in \Lambda_{\dot{\hat{h}}_{\sigma(1)}}, \dot{h}_{\dot{\hat{h}}_{\sigma(2)}} \in \Lambda_{\dot{\hat{h}}_{\sigma(2)}}} \left\{ \sqrt{1 - \prod_{i=1}^{2} \left(1 - \dot{h}^2_{\dot{\hat{h}}_{\sigma(1)}}\right)^{\omega_i}} \right\}, \cup_{\dot{h}'_{\dot{\hat{h}}_{\sigma(1)}} \in \Gamma_{\dot{\hat{h}}_{\sigma(1)}}, \dot{h}'_{\dot{\hat{h}}_{\sigma(2)}} \in \Gamma_{\dot{\hat{h}}_{\sigma(2)}}} \left\{ \prod_{i=1}^{2} \left(\dot{h}'_{\dot{\hat{h}}_{\sigma(i)}}\right)^{w_i} \right\} \right\rangle. \end{aligned}$$

Thus, the equation is hold for $n = 2$. Suppose the equation is hold for $n = k$, i.e.,

$$PHFHWA(\hat{h}_1, \hat{h}_2, \ldots, \hat{h}_k) = \left\langle \begin{array}{l} \cup_{\dot{h}_{\dot{\hat{h}}_{\sigma(1)}} \in \Lambda_{\dot{\hat{h}}_{\sigma(1)}}, \dot{h}_{\dot{\hat{h}}_{\sigma(2)}} \in \Lambda_{\dot{\hat{h}}_{\sigma(2)}}, \ldots, \dot{h}_{\dot{\hat{h}}_{\sigma(k)}} \in \Lambda_{\dot{\hat{h}}_{\sigma(k)}}} \left\{ \sqrt{1 - \prod_{i=1}^{k} \left(1 - \dot{h}^2_{\dot{\hat{h}}_{\sigma(i)}}\right)^{\omega_i}} \right\}, \\ \cup_{\dot{h}'_{\dot{\hat{h}}_{\sigma(1)}} \in \Gamma_{\dot{\hat{h}}_{\sigma(1)}}, \dot{h}'_{\dot{\hat{h}}_{\sigma(2)}} \in \Gamma_{\dot{\hat{h}}_{\sigma(2)}}, \ldots, \dot{h}'_{\dot{\hat{h}}_{\sigma(k)}} \in \Gamma_{\dot{\hat{h}}_{\sigma(k)}}} \left\{ \prod_{i=1}^{k} \left(\dot{h}'_{\dot{\hat{h}}_{\sigma(i)}}\right)^{\omega_i} \right\} \end{array} \right\rangle$$

We show that the equation is hold for $n = k + 1$. i.e,

$$PHFHWA(\hat{h}_1, \hat{h}_2, \ldots, \hat{h}_{k+1}) = \left\langle \begin{array}{l} \cup_{\dot{h}_{\dot{\hat{h}}_{\sigma(1)}} \in \Lambda_{\dot{\hat{h}}_{\sigma(1)}}, \dot{h}_{\dot{\hat{h}}_{\sigma(2)}} \in \Lambda_{\dot{\hat{h}}_{\sigma(2)}}, \ldots, \dot{h}_{\dot{\hat{h}}_{\sigma(k)}} \in \Lambda_{\dot{\hat{h}}_{\sigma(k)}}} \left\{ \sqrt{1 - \prod_{i=1}^{k} \left(1 - \dot{h}^2_{\dot{\hat{h}}_{\sigma(i)}}\right)^{\omega_i}} \right\}, \\ \cup_{\dot{h}'_{\dot{\hat{h}}_{\sigma(1)}} \in \Gamma_{\dot{\hat{h}}_{\sigma(1)}}, \dot{h}'_{\dot{\hat{h}}_{\sigma(2)}} \in \Gamma_{\dot{\hat{h}}_{\sigma(2)}}, \ldots, \dot{h}'_{\dot{\hat{h}}_{\sigma(k)}} \in \Gamma_{\dot{\hat{h}}_{\sigma(k)}}} \left\{ \prod_{i=1}^{k} \left(\dot{h}'_{\dot{\hat{h}}_{\sigma(i)}}\right)^{\omega_i} \right\} \end{array} \right\rangle$$

$$\oplus \left\langle \cup_{\dot{h}_{\dot{\hat{h}}_{\sigma(k+1)}} \in \Lambda_{\dot{\hat{h}}_{\sigma(k+1)}}} \left\{ \sqrt{1 - \left(1 - \dot{h}^2_{\dot{\hat{h}}_{\sigma(k+1)}}\right)^{\omega_{k+1}}} \right\}, \cup_{\dot{h}'_{\dot{\hat{h}}_{\sigma(k+1)}} \in \Gamma_{\dot{\hat{h}}_{\sigma(k+1)}}} \left\{ \left(\dot{h}'_{\dot{\hat{h}}_{\sigma(k+1)}}\right)^{\omega_{k+1}} \right\} \right\rangle$$

$$= \left\langle \begin{array}{l} \cup_{\dot{h}_{\dot{\hat{h}}_{\sigma(1)}} \in \Lambda_{\dot{\hat{h}}_{\sigma(1)}}, \dot{h}_{\dot{\hat{h}}_{\sigma(2)}} \in \Lambda_{\dot{\hat{h}}_{\sigma(2)}}, \ldots, \dot{h}_{\dot{\hat{h}}_{\sigma(k)}} \in \Lambda_{\dot{\hat{h}}_{\sigma(k)}}, \dot{h}_{\dot{\hat{h}}_{\sigma(k+1)}} \in \Lambda_{\dot{\hat{h}}_{\sigma(k+1)}}} \\ \sqrt{1 - \prod_{i=1}^{k} \left(1 - \dot{h}^2_{\dot{\hat{h}}_{\sigma(i)}}\right)^{\omega_i} + 1 - \left(1 - \dot{h}^2_{\dot{\hat{h}}_{\sigma(k+1)}}\right)^{\omega_{k+1}} - \left(\sqrt{1 - \prod_{i=1}^{k} \left(1 - \dot{h}^2_{\dot{\hat{h}}_{\sigma(i)}}\right)^{\omega_i}} \left(1 - \left(1 - \dot{h}^2_{\dot{\hat{h}}_{\sigma(k+1)}}\right)^{\omega_{k+1}}\right)\right)}, \\ \cup_{\dot{h}'_{\dot{\hat{h}}_{\sigma(1)}} \in \Gamma_{\dot{\hat{h}}_{\sigma(1)}}, \dot{h}'_{\dot{\hat{h}}_{\sigma(2)}} \in \Gamma_{\dot{\hat{h}}_{\sigma(2)}}, \ldots, \dot{h}'_{\dot{\hat{h}}_{\sigma(k)}} \in \Gamma_{\dot{\hat{h}}_{\sigma(k)}}, \dot{h}'_{\dot{\hat{h}}_{\sigma(k+1)}} \in \Gamma_{\dot{\hat{h}}_{\sigma(k+1)}}} \left\{ \prod_{i=1}^{k} \left(\dot{h}'_{\dot{\hat{h}}_{\sigma(i)}}\right)^{\omega_i} \right\} \left\{ \left(\dot{h}'_{\dot{\hat{h}}_{\sigma(k+1)}}\right)^{\omega_{k+1}} \right\} \end{array} \right\rangle$$

$$= \left\langle \begin{array}{l} \cup_{\dot{h}_{\dot{\hat{h}}_{\sigma(1)}} \in \Lambda_{\dot{\hat{h}}_{\sigma(1)}}, \dot{h}_{\dot{\hat{h}}_{\sigma(2)}} \in \Lambda_{\dot{\hat{h}}_{\sigma(2)}}, \ldots, \dot{h}_{\dot{\hat{h}}_{\sigma(k)}} \in \Lambda_{\dot{\hat{h}}_{\sigma(k)}}, \dot{h}_{\dot{\hat{h}}_{\sigma(k+1)}} \in \Lambda_{\dot{\hat{h}}_{\sigma(k+1)}}} \left\{ \sqrt{1 - \prod_{i=1}^{k+1} \left(1 - \dot{h}^2_{\dot{\hat{h}}_{\sigma(i)}}\right)^{\omega_i}} \right\}, \\ \cup_{\dot{h}'_{\dot{\hat{h}}_{\sigma(1)}} \in \Gamma_{\dot{\hat{h}}_{\sigma(1)}}, \dot{h}'_{\dot{\hat{h}}_{\sigma(2)}} \in \Gamma_{\dot{\hat{h}}_{\sigma(2)}}, \ldots, \dot{h}'_{\dot{\hat{h}}_{\sigma(k)}} \in \Gamma_{\dot{\hat{h}}_{\sigma(k)}}, \dot{h}'_{\dot{\hat{h}}_{\sigma(k+1)}} \in \Gamma_{\dot{\hat{h}}_{\sigma(k+1)}}} \left\{ \prod_{i=1}^{k+1} \left(h'_{\dot{\hat{h}}_{\sigma(i)}}\right)^{\omega_i} \right\} \end{array} \right\rangle$$

which is the required result.

Example 1 Let $\hat{h}_1 = \langle\{0.4, 0.7, 0.9\}, \{0.4, 0.8\}\rangle, \hat{h}_2 = \langle\{0.5, 0.6\}, \{0.6, 0.7, 0.8\}\rangle$ and $\hat{h}_3 = \langle\{0.6, 0.7, 0.9\}, \{0.4, 0.7\}\rangle$ be two Pythagorean HFS with $w = (0.3, 0.25, 0.45)^T$ is the weighted vector of $\hat{h}_i (i = 1, 2)$ and the aggregation-associated vector is $\omega = (0.25, 0.4, 0.35)^T$. Then using Pythagorean hesitant fuzzy hybrid weighted averaging operator we can obtain,

$$PHFHWA(\hat{h}_1, \hat{h}_2, \hat{h}_2) = \overset{3}{\underset{i=1}{\oplus}} \left(\omega_i \dot{\hat{h}}_{\sigma(i)}\right) = \left\langle \begin{array}{l} \cup_{\dot{h}_{\dot{\hat{h}}_{\sigma(2)}} \in \Lambda_{\dot{\hat{h}}_{\sigma(2)}}, \dot{h}_{\dot{\hat{h}}_{\sigma(2)}} \in \Lambda_{\dot{\hat{h}}_{\sigma(2)}}} \left\{ \sqrt{1 - \prod_{i=1}^{3} \left(1 - \dot{h}^2_{\dot{\hat{h}}_{\sigma(i)}}\right)^{\omega_i}} \right\}, \\ \cup_{\dot{h}'_{\dot{\hat{h}}_{\sigma(2)}} \in \Gamma_{\dot{\hat{h}}_{\sigma(2)}}, \dot{h}'_{\dot{\hat{h}}_{\sigma(2)}} \in \Gamma_{\dot{\hat{h}}_{\sigma(2)}}} \left\{ \prod_{i=1}^{3} \left(\dot{h}'_{\dot{\hat{h}}_{\sigma(i)}}\right)^{\omega_i} \right\} \end{array} \right\rangle$$

$$= \left\langle \left\{ \begin{array}{l} 0.5009, 0.5305, 0.6116, 0.6322, 0.7492, 0.7611, \\ 0.5533, 0.5784, 0.6483, 0.6663, 0.7705, 0.3812, \\ 0.7091, 0.7233, 0.7646, 0.7756, 0.8419, 0.8490 \end{array} \right\}, \left\{ \begin{array}{l} 0.4616, 0.4806, 0.4978, 0.5924, 0.6168, 0.6389, \\ 0.5575, 0.5805, 0.6012, 0.7155, 0.7451, 0.7716 \end{array} \right\} \right\rangle.$$

Definition 10 Let $\hat{h}_i = \left(\Lambda_{\hat{h}_i}, \Gamma_{\hat{h}_i}\right) (i = 1, 2, 3, \ldots, n)$ be a collection of all $PHFN's$, and $w = (w_1, w_2, \ldots, w_n)^T$ be the weight vector of $\hat{h}_i (i = 1, 2, 3, \ldots, n)$ with $w_i \geqslant 0 (i = 1, 2, 3, \ldots, n)$ where $w_i \in [0, 1]$ and $\sum_{i=1}^{n} w_i = 1$, n is the balancing coefficient which plays a role of balance. Then Pythagorean hesitant fuzzy hybrid weighted geometric ($PHFHWG$) operator is a mapping $PHFHWG : PHFN^n \to PHFN$ with an aggregation-associated vector $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ such that $\omega_i \in [0, 1]$, $\sum_{i=1}^{n} \omega_i = 1$ and can be defined as follows:

$$PHFHWG(\hat{h}_1, \hat{h}_2, \ldots, \hat{h}_n) = \overset{n}{\underset{i=1}{\otimes}} \left(\ddot{\hat{h}}_i^{\omega_i}\right) = \left(\ddot{\hat{h}}_1^{\omega_1} \otimes \ddot{\hat{h}}_2^{\omega_2} \otimes, \ldots, \otimes \ddot{\hat{h}}_n^{\omega_n}\right) \tag{10}$$

and the mapping $PHFHWG$ is said to be a Pythagorean hesitant fuzzy hybrid weighted geometric operator, where $\ddot{\hat{h}}_{\sigma(i)}$ is the ith largest of $\ddot{\hat{h}}_{\sigma(i)} = \hat{h}_k^{nw_k} (k = 1, 2, \ldots, n)$.

Using the different manifestation of weighting vector, the $PHFHWG$ operator can be reduced into some special cases. For instance, if the associated-weighting vector

$\omega = \left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)^T$, then the *PHFHWG* operator reduces to the *PHFWG* operator; if $w = \left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)^T$, then the *PHFHWG* operator reduces to the *PHFOWG* operator. It must be pointed out that the weighing operation of the ordered position can be synchronized with the weighing operation of the given importance by the *PHFHWG* operator. This characteristic is different from the *PHFHWG* operator.

Theorem 2 *Let* $\hat{h}_i = \left(\Lambda_{\hat{h}_i}, \Gamma_{\hat{h}_i}\right)(i = 1, 2, 3, \ldots, n)$ *be a collection of all PHFNs, and* $w = (w_1, w_2, \ldots, w_n)^T$ *be the weight vector of* $\hat{h}_i(i = 1, 2, 3, \ldots, n)$ *with* $w_i \geqslant 0(i = 1, 2, \ldots, n)$ *where* $w_i \in [0, 1]$ *and* $\sum_{i=1}^{n} w_i = 1$. *Then the aggregation result using PHFHWG operator with an aggregation-associated vector* $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ *such that* $\omega_i \in [0, 1]$, $\sum_{i=1}^{n} \omega_i = 1$ *is also a PHFN, and*

$$PHFHWG(\hat{h}_1, \hat{h}_2, \ldots, \hat{h}_n) = \bigotimes_{i=1}^{n} \left(\ddot{\hat{h}}_i^{\omega_i}\right) = \left\langle \begin{array}{c} \cup_{\ddot{h}_{\ddot{\hat{h}}_{\sigma(1)}} \in \Lambda_{\ddot{\hat{h}}_{\sigma(1)}}, \ddot{h}_{\ddot{\hat{h}}_{\sigma(2)}} \in \Lambda_{\ddot{\hat{h}}_{\sigma(2)}}, \ldots, \ddot{h}_{\ddot{\hat{h}}_{\sigma(n)}} \in \Lambda_{\ddot{\hat{h}}_{\sigma(n)}}} \\ \left\{ \prod_{i=1}^{n} \left(\ddot{h}_{\ddot{\hat{h}}_{\sigma(i)}}\right)^{\omega_i} \right\}, \\ \cup_{\ddot{h}'_{\ddot{\hat{h}}_{\sigma(1)}} \in \Gamma_{\ddot{\hat{h}}_{\sigma(1)}}, \ddot{h}'_{\ddot{\hat{h}}_{\sigma(2)}} \in \Gamma_{\ddot{\hat{h}}_{\sigma(2)}}, \ldots, \ddot{h}'_{\ddot{\hat{h}}_{\sigma(n)}} \in \Gamma_{\ddot{\hat{h}}_{\sigma(n)}}} \\ \left\{ \sqrt{1 - \prod_{i=1}^{n} \left(1 - \ddot{h}'^{2}_{\ddot{\hat{h}}_{\sigma(i)}}\right)^{\omega_i}} \right\} \end{array} \right\rangle. \tag{11}$$

where $\ddot{\hat{h}}_{\sigma(i)}$ *is the ith largest of* $\ddot{\hat{h}}_{\sigma(i)} = \hat{h}_k^{nw_k}(k = 1, 2, \ldots, n)$.

Proof By mathematical induction we prove that Eq. (11) holds for all n. For this first we show that Eq. (11) holds for $n = 2$. Since,

$$\ddot{\hat{h}}_1^{\omega_1} = \left\langle \cup_{\ddot{h}_{\ddot{\hat{h}}_{\sigma(1)}} \in \Lambda_{\ddot{\hat{h}}_{\sigma(1)}}} \left\{ \left(\ddot{h}_{\ddot{\hat{h}}_{\sigma(1)}}\right)^{\omega_1} \right\}, \cup_{\ddot{h}'_{\ddot{\hat{h}}_{\sigma(1)}} \in \Gamma_{\ddot{\hat{h}}_{\sigma(1)}}} \left\{ \sqrt{1 - \left(1 - \ddot{h}'^{2}_{\ddot{\hat{h}}_{\sigma(1)}}\right)^{\omega_1}} \right\} \right\rangle$$

and

$$\ddot{\hat{h}}_2^{\omega_2} = \left\langle \cup_{\ddot{h}_{\ddot{\hat{h}}_{\sigma(2)}} \in \Lambda_{\ddot{\hat{h}}_{\sigma(2)}}} \left\{ \left(\ddot{h}_{\ddot{\hat{h}}_{\sigma(2)}}\right)^{\omega_2} \right\}, \cup_{\ddot{h}'_{\ddot{\hat{h}}_{\sigma(1)}} \in \Gamma_{\ddot{\hat{h}}_{\sigma(1)}}} \left\{ \sqrt{1 - \left(1 - \ddot{h}'^{2}_{\ddot{\hat{h}}_{\sigma(2)}}\right)^{\omega_2}} \right\} \right\rangle$$

So,

$$\begin{aligned} PHFHWG(\hat{h}_1, \hat{h}_2) &= \ddot{\hat{h}}_1^{\omega_1} \otimes \ddot{\hat{h}}_2^{\omega_2} \\ &= \left\langle \begin{array}{c} \cup_{\ddot{h}_{\ddot{\hat{h}}_{\sigma(1)}} \in \Lambda_{\ddot{\hat{h}}_{\sigma(1)}}} \left\{ \left(\ddot{h}_{\ddot{\hat{h}}_{\sigma(1)}}\right)^{\omega_1} \right\}, \\ \cup_{\ddot{h}'_{\ddot{\hat{h}}_{\sigma(1)}} \in \Gamma_{\ddot{\hat{h}}_{\sigma(1)}}} \left\{ \sqrt{1 - \left(1 - \ddot{h}'^{2}_{\ddot{\hat{h}}_{\sigma(1)}}\right)^{\omega_1}} \right\} \end{array} \right\rangle \otimes \left\langle \begin{array}{c} \cup_{\ddot{h}_{\ddot{\hat{h}}_{\sigma(2)}} \in \Lambda_{\ddot{\hat{h}}_{\sigma(2)}}} \left\{ \left(\ddot{h}_{\ddot{\hat{h}}_{\sigma(2)}}\right)^{\omega_2} \right\}, \\ \cup_{\ddot{h}'_{\ddot{\hat{h}}_{\sigma(2)}} \in \Gamma_{\ddot{\hat{h}}_{\sigma(2)}}} \left\{ \sqrt{1 - \left(1 - \ddot{h}'^{2}_{\ddot{\hat{h}}_{\sigma(2)}}\right)^{\omega_2}} \right\} \end{array} \right\rangle \\ &= \left\langle \begin{array}{l} \cup_{\ddot{h}_{\ddot{\hat{h}}_{\sigma(1)}} \in \Lambda_{\ddot{\hat{h}}_{\sigma(1)}}, \ddot{h}_{\ddot{\hat{h}}_{\sigma(2)}} \in \Lambda_{\ddot{\hat{h}}_{\sigma(2)}}} \left\{ \left(\ddot{h}_{\ddot{\hat{h}}_{\sigma(1)}}\right)^{\omega_1} \left(\ddot{h}_{\ddot{\hat{h}}_{\sigma(2)}}\right)^{\omega_2} \right\}, \\ \cup_{\ddot{h}'_{\ddot{\hat{h}}_{\sigma(1)}} \in \Gamma_{\ddot{\hat{h}}_{\sigma(1)}}, \ddot{h}'_{\ddot{\hat{h}}_{\sigma(2)}} \in \Gamma_{\ddot{\hat{h}}_{\sigma(2)}}} \sqrt{1 - \left(1 - \ddot{h}'^{2}_{\ddot{\hat{h}}_{\sigma(1)}}\right)^{\omega_1} + 1 - \left(1 - \ddot{h}'^{2}_{\ddot{\hat{h}}_{\sigma(2)}}\right)^{\omega_2} - \left(\left(1 - \ddot{h}'^{2}_{\ddot{\hat{h}}_{\sigma(1)}}\right)^{\omega_1} \left(1 - \ddot{h}'^{2}_{\ddot{\hat{h}}_{\sigma(2)}}\right)^{\omega_2}\right)} \end{array} \right\rangle \\ &= \left\langle \cup_{\ddot{h}_{\ddot{\hat{h}}_{\sigma(1)}} \in \Lambda_{\ddot{\hat{h}}_{\sigma(1)}}, \ddot{h}_{\ddot{\hat{h}}_{\sigma(2)}} \in \Lambda_{\ddot{\hat{h}}_{\sigma(2)}}} \left\{ \prod_{i=1}^{2} \left(\ddot{h}_{\ddot{\hat{h}}_{\sigma(i)}}\right)^{\omega_i} \right\}, \cup_{\ddot{h}'_{\ddot{\hat{h}}_{\sigma(1)}} \in \Gamma_{\ddot{\hat{h}}_{\sigma(1)}}, \ddot{h}'_{\ddot{\hat{h}}_{\sigma(2)}} \in \Gamma_{\ddot{\hat{h}}_{\sigma(2)}}} \left\{ \sqrt{1 - \prod_{i=1}^{2} \left(1 - \ddot{h}'^{2}_{\ddot{\hat{h}}_{\sigma(i)}}\right)^{\omega_i}} \right\} \right\rangle \end{aligned}$$

Thus, the equation is holds for $n = 2$. Suppose the equation is holds for $n = k$,

$$PHFHWG(\hat{h}_1, \hat{h}_2, \dots, \hat{h}_k) = \left\langle \begin{matrix} \cup_{\ddot{h}_{\ddot{\hat{h}}_{\sigma(1)}} \in \Lambda_{\ddot{\hat{h}}_{\sigma(1)}}, \ddot{h}_{\ddot{\hat{h}}_{\sigma(2)}} \in \Lambda_{\ddot{\hat{h}}_{\sigma(2)}}, \dots, \ddot{h}_{\ddot{\hat{h}}_{\sigma(k)}} \in \Lambda_{\ddot{\hat{h}}_{\sigma(k)}}} \left\{ \prod_{i=1}^{k} \left(\ddot{h}_{\ddot{\hat{h}}_{\sigma(i)}} \right)^{\omega_i} \right\}, \\ \cup_{\ddot{h}'_{\ddot{\hat{h}}_{\sigma(1)}} \in \Gamma_{\ddot{\hat{h}}_{\sigma(1)}}, \ddot{h}'_{\ddot{\hat{h}}_{\sigma(2)}} \in \Gamma_{\ddot{\hat{h}}_{\sigma(2)}}, \dots, \ddot{h}'_{\ddot{\hat{h}}_{\sigma(k)}} \in \Gamma_{\ddot{\hat{h}}_{\sigma(k)}}} \left\{ \sqrt{1 - \prod_{i=1}^{k} \left(1 - \ddot{h}'^2_{\ddot{\hat{h}}_{\sigma(i)}} \right)^{\omega_i}} \right\} \end{matrix} \right\rangle$$

Suppose it is hold for $n = k$, we show that the equation is hold for $n = k + 1$. i.e.,

$$PHFHWG(\hat{h}_1, \hat{h}_2, \dots, \hat{h}_{k+1}) = \left\langle \begin{matrix} \cup_{\ddot{h}_{\ddot{\hat{h}}_{\sigma(1)}} \in \Lambda_{\ddot{\hat{h}}_{\sigma(1)}}, \ddot{h}_{\ddot{\hat{h}}_{\sigma(2)}} \in \Lambda_{\ddot{\hat{h}}_{\sigma(2)}}, \dots, \ddot{h}_{\ddot{\hat{h}}_{\sigma(k)}} \in \Lambda_{\ddot{\hat{h}}_{\sigma(k)}}} \left\{ \prod_{i=1}^{k} \left(\ddot{h}_{\ddot{\hat{h}}_{\sigma(i)}} \right)^{\omega_i} \right\}, \\ \cup_{\ddot{h}'_{\ddot{\hat{h}}_{\sigma(1)}} \in \Gamma_{\ddot{\hat{h}}_{\sigma(1)}}, \ddot{h}'_{\ddot{\hat{h}}_{\sigma(2)}} \in \Gamma_{\ddot{\hat{h}}_{\sigma(2)}}, \dots, \ddot{h}'_{\ddot{\hat{h}}_{\sigma(k)}} \in \Gamma_{\ddot{\hat{h}}_{\sigma(k)}}} \left\{ \sqrt{1 - \prod_{i=1}^{k} \left(1 - \ddot{h}'^2_{\ddot{\hat{h}}_{\sigma(i)}} \right)^{\omega_i}} \right\} \end{matrix} \right\rangle$$

$$\otimes \left\langle \cup_{\ddot{h}_{\ddot{\hat{h}}_{\sigma(k+1)}} \in \Lambda_{\ddot{\hat{h}}_{\sigma(k+1)}}} \left\{ \left(\ddot{h}_{\ddot{\hat{h}}_{\sigma(k+1)}} \right)^{\omega_{k+1}} \right\}, \cup_{\ddot{h}'_{\ddot{\hat{h}}_{\sigma(k+1)}} \in \Gamma_{\ddot{\hat{h}}_{\sigma(k+1)}}} \left\{ \sqrt{1 - \left(1 - \ddot{h}'^2_{\ddot{\hat{h}}_{\sigma(k+1)}} \right)^{\omega_{k+1}}} \right\} \right\rangle$$

$$= \left\langle \begin{matrix} \cup_{\ddot{h}_{\ddot{\hat{h}}_{\sigma(1)}} \in \Lambda_{\ddot{\hat{h}}_{\sigma(1)}}, \ddot{h}_{\ddot{\hat{h}}_{\sigma(2)}} \in \Lambda_{\ddot{\hat{h}}_{\sigma(2)}}, \dots, \ddot{h}_{\ddot{\hat{h}}_{\sigma(k)}} \in \Lambda_{\ddot{\hat{h}}_{\sigma(k)}}, \ddot{h}_{\ddot{\hat{h}}_{\sigma(k+1)}} \in \Lambda_{\ddot{\hat{h}}_{\sigma(k+1)}}} \left\{ \prod_{i=1}^{k} \left(\ddot{h}_{\ddot{\hat{h}}_{\sigma(i)}} \right)^{\omega_i} \left(\ddot{h}_{\ddot{\hat{h}}_{\sigma(k+1)}} \right)^{\omega_{k+1}} \right\}, \\ \cup_{\ddot{h}'_{\ddot{\hat{h}}_{\sigma(1)}} \in \Gamma_{\ddot{\hat{h}}_{\sigma(1)}}, \ddot{h}'_{\ddot{\hat{h}}_{\sigma(2)}} \in \Gamma_{\ddot{\hat{h}}_{\sigma(2)}}, \dots, \ddot{h}'_{\ddot{\hat{h}}_{\sigma(k)}} \in \Gamma_{\ddot{\hat{h}}_{\sigma(k)}}, \ddot{h}'_{\ddot{\hat{h}}_{\sigma(k+1)}} \in \Gamma_{\ddot{\hat{h}}_{\sigma(k+1)}}} \left\{ \sqrt{1 - \prod_{i=1}^{k} \left(1 - \ddot{h}'^2_{\ddot{\hat{h}}_{\sigma(i)}} \right)^{\omega_i} + 1 - \left(1 - \ddot{h}'^2_{\ddot{\hat{h}}_{\sigma(k+1)}} \right)^{\omega_{k+1}} - \left(\sqrt{1 - \prod_{i=1}^{k} \left(1 - \ddot{h}'^2_{\ddot{\hat{h}}_{\sigma(i)}} \right)^{\omega_i}} \left(1 - \left(1 - \ddot{h}'^2_{\ddot{\hat{h}}_{\sigma(k+1)}} \right)^{\omega_{k+1}} \right) \right)} \right\} \end{matrix} \right\rangle$$

$$= \left\langle \begin{matrix} \cup_{\ddot{h}_{\ddot{\hat{h}}_{\sigma(1)}} \in \Lambda_{\ddot{\hat{h}}_{\sigma(1)}}, \ddot{h}_{\ddot{\hat{h}}_{\sigma(2)}} \in \Lambda_{\ddot{\hat{h}}_{\sigma(2)}}, \dots, \ddot{h}_{\ddot{\hat{h}}_{\sigma(k+1)}} \in \Lambda_{\ddot{\hat{h}}_{\sigma(k+1)}}} \left\{ \prod_{i=1}^{k+1} \left(\ddot{h}_{\ddot{\hat{h}}_{\sigma(i)}} \right)^{\omega_i} \right\}, \\ \cup_{\ddot{h}'_{\ddot{\hat{h}}_{\sigma(1)}} \in \Gamma_{\ddot{\hat{h}}_{\sigma(1)}}, \ddot{h}'_{\ddot{\hat{h}}_{\sigma(2)}} \in \Gamma_{\ddot{\hat{h}}_{\sigma(2)}}, \dots, \ddot{h}'_{\ddot{\hat{h}}_{\sigma(k+1)}} \in \Gamma_{\ddot{\hat{h}}_{\sigma(k+1)}}} \left\{ \sqrt{1 - \prod_{i=1}^{k+1} \left(1 - \ddot{h}'^2_{\ddot{\hat{h}}_{\sigma(i)}} \right)^{\omega_i}} \right\} \end{matrix} \right\rangle.$$

Hence the equation is true for $n = k + 1$. Therefore, the equation is true for all n.

Example 2 Consider the Pythagorean hesitant fuzzy numbers with weighted vector and aggregation-associated vector given in Example (1). Using Pythagorean hesitant fuzzy weighted geometric operator we obtained.

$$PHFHWG(\hat{h}_1, \hat{h}_2, \hat{h}_3) = \bigotimes_{i=1}^{3} \left(\ddot{\hat{h}}^{\omega_i}_{\sigma(i)} \right) = \left\langle \begin{matrix} \cup_{\ddot{h}_{\ddot{\hat{h}}_{\sigma(1)}} \in \Lambda_{\ddot{\hat{h}}_{\sigma(1)}}, \ddot{h}_{\ddot{\hat{h}}_{\sigma(2)}} \in \Lambda_{\ddot{\hat{h}}_{\sigma(2)}}, \ddot{h}_{\ddot{\hat{h}}_{\sigma(3)}} \in \Lambda_{\ddot{\hat{h}}_{\sigma(3)}}} \left\{ \prod_{i=1}^{3} \left(\ddot{h}_{\ddot{\hat{h}}_{\sigma(i)}} \right)^{\omega_i} \right\}, \\ \cup_{\ddot{h}'_{\ddot{\hat{h}}_{\sigma(1)}} \in \Gamma_{\ddot{\hat{h}}_{\sigma(1)}}, \ddot{h}'_{\ddot{\hat{h}}_{\sigma(2)}} \in \Gamma_{\ddot{\hat{h}}_{\sigma(2)}}, \ddot{h}'_{\ddot{\hat{h}}_{\sigma(3)}} \in \Gamma_{\ddot{\hat{h}}_{\sigma(3)}}} \left\{ \sqrt{1 - \prod_{i=1}^{3} \left(1 - \ddot{h}'^2_{\ddot{\hat{h}}_{\sigma(i)}} \right)^{\omega_i}} \right\} \end{matrix} \right\rangle.$$

Then we have,

$$= \left\langle \left\{ \begin{matrix} 0.5242, 0.5499, 0.6412, 0.6726, 0.7019, 0.7363, \\ 0.5458, 0.5726, 0.6676, 0.7004, 0.7309, 0.7667, \\ 0.5831, 0.6116, 0.7132, 0.7481, 0.7807, 0.8190 \end{matrix} \right\}, \left\{ \begin{matrix} 0.4495, 0.4982, 0.5603, 0.6417, 0.6677, 0.7030, \\ 0.5478, 0.5836, 0.6310, 0.6957, 0.7169, 0.7459 \end{matrix} \right\} \right\rangle.$$

Lemma 1 *Let* $\hat{h}_i \succ 0,\ w_i \succ 0\ \ (i = 1, 2, 3, \ldots n)$ *and* $\sum_{i=1}^{n} w_i = 1$ *Then,* $\prod_{i=1}^{n} \left(\hat{h}_i\right)^{w_i} \preccurlyeq \sum_{i=1}^{n} w_i \hat{h}_i$, *where the equality holds if and only if* $\hat{h}_1 = \hat{h}_2 = \hat{h}_3 = \cdots = \hat{h}_n$.

Theorem 3 *Let* $\hat{h}_i = \left(\Lambda_{\hat{h}_i}, \Gamma_{\hat{h}_i}\right)(i = 1, 2, 3, \ldots, n)$ *be a collection of all PHFNs* $\hat{h}_{\sigma(i)}$ *be the largest in them,* $w = (w_1, w_2, \ldots, w_n)$ *be the weight vector of* $\hat{h}_i (i = 1, 2, 3, \ldots, n)$ *with* $w_i \geqslant 0\, (i = 1, 2, \ldots, n)$ *such that* $w_i \in [0, 1]$ *and* $\sum_{i=1}^{n} w_i = 1$. *Then*

$$PHFHWG(\hat{h}_1, \hat{h}_2, \hat{h}_3, \ldots, \hat{h}_n) \preccurlyeq PHFHWA(\hat{h}_1, \hat{h}_2, \hat{h}_3, \ldots, \hat{h}_n)$$

Proof Proof of the Theorem follows from Lemma [1].

4 Decision making based on Pythagorean hesitant fuzzy hybrid aggregation operators

In this section, we forward a framework for determining attribute weights and the ranking orders for all the alternatives with under Pythagorean hesitant fuzzy environment.

A multi-attribute decision making problem can be stated as a decision matrix whose elements show the evaluation information of all alternatives with respect to an attribute. We construct a Pythagorean hesitant fuzzy decision matrix, whose elements are PHFNs, which are given not only the information that the alternative X_i satisfies the attributes A_j, but also the information that the alternative X_i does not satisfies the attributes A_j may initiate from a doubt between a few different values.

Consider a MADM with anonymity where there is a discrete set of m alternatives $\{X_1, X_2, \ldots, X_m\}$ be a set of alternatives and let $\{A_1, A_2, \ldots, A_n\}$ be a set of attributes whose weighting vector is $w = (w_1, w_2, \ldots, w_n)^T$, where $w_i \geqslant 0, i = 1, 2, \ldots, m$, and $\sum_{i=1}^{m} w_i = 1$. To evaluate the performance of the ith alternative X_i under the jth attribute A_j, let $\{D_1, D_2, \ldots, D_t\}$ be a set of decision makers and $\omega = (\omega_1, \omega_2, \ldots, \omega_t)$ be the weighting vector of the decision makers with $\omega_k \geqslant 0 (k = 1, 2, \ldots, t)$, $\sum_{k=1}^{t} \omega_k = 1$ and is required to provide not only the information that the alternative X_i satisfies the attribute A_j, but also the information that the alternative X_i does not satisfy the attribute A_j. These two part information can be expressed by Λ_{ij} and Γ_{ij} which denote the degrees of membership that the alternative X_i satisfy the attribute A_j and nonmembership that the alternative X_i does not satisfy the attribute A_j, then the performance of the alternative X_i under the attribute A_j can be expressed by an PHFN $\hat{h}_{ij} = \langle \Lambda_{ij}, \Gamma_{ij} \rangle$ with the condition that for all $h_{ij} \in \Lambda_{ij} \exists h'_{ij} \in \Gamma_{ij}$ such that $0 \preccurlyeq (h_{ij})^2 + (h'_{ij})^2 \preccurlyeq 1$, and for all $h_{ij} \in \Gamma_{ij} \exists h'_{ij} \in \Lambda_{ij}$, such that $0 \preccurlyeq (h_{ij})^2 + (h'_{ij})^2 \preccurlyeq 1$ $(i = 1, 2, \ldots, m; j = 1, 2, \ldots, n)$. The Pythagorean hesitant fuzzy decision matrix H, can be written as:

$$H = \begin{bmatrix} \hat{h}_{11} & \hat{h}_{12} & \cdots & \hat{h}_{1n} \\ \vdots & & \ddots & \vdots \\ \hat{h}_{m1} & \hat{h}_{m2} & \cdots & \hat{h}_{mn} \end{bmatrix}.$$

Main steps of the proposed multi-attribute group decision making (MAGDM) problem are as follows:

Step 1 In this step we construct the Pythagorean hesitant fuzzy decision matrices $C = [\hat{h}_{ij}]_{m \times n}$ for decision where $\hat{h}_{ij} = \langle \Lambda_{ij}, \Gamma_{ij} \rangle$ $(i = 1, 2, \ldots, m; j = 1, 2, \ldots, n)$.

If the attribute have two types, such as cost and benefit attributes. Then the Pythagorean hesitant decision matrix can be converted into the normalized Pythagorean hesitant fuzzy decision matrix. $D_N = [\gamma_{ij}]_{m \times n}$, Where $\gamma_{ij} = \begin{cases} \hat{h}_{ij} & \text{if the attribute is of benefit type} \\ \hat{h}_{ij}^c & \text{if the attribute is of cost type} \end{cases}$.

Where $\hat{h}_{ij}^c = \langle \Gamma_{ij}, \Lambda_{ij} \rangle$ $(i = 1, 2, \ldots, m; j = 1, 2, \ldots, n)$. If all the attributes have the same type than there is no need to normalized the decision matrix.

Step 2 Utilize the developed aggregation operators to obtain the PHFN $\hat{h}_i (i = 1, 2, \ldots, m)$ for the alternatives X_i. That is the developed operators to derive the collective overall preference values $\hat{h}_i (i = 1, 2, \ldots, m)$ of the alternative X_i, where $w = (w_1, w_2, \ldots w_n)^T$ is the weighting vector of the attributes and aggregation-associated vector $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$.

Step 3 Using Eqs. (2) we calculate the scores $S(\hat{h}_i)(i = 1, 2, \ldots, m)$ and the deviation degrees $A\left(\hat{h}_i\right)(i = 1, 2, \ldots, m)$ of all the overall values $\hat{h}_i (i = 1, 2, \ldots, m)$.

Step 4 Rank the alternatives $X_i (i = 1, 2, \ldots, m)$ and then select the best one.

5 Illustrative example

To demonstrate the application of the proposed MAGDM method, we give an example shown for talent recruitment problem.

Example 3 Suppose Hazara University Mansehra Pakistan was intended to recruit a dean of Science. The recruitment process was as follows. First, the university released an opening recruitment announcement on the website. Any people who satisfied the basic recruitment conditions could apply for the position using the online application system before the deadline. After receiving applications from candidates at home and abroad, the staffs of the personnel department made a strict selection by checking the application documents. Finally, four candidates X_1, X_2, X_3, X_4 and X_5 entered the interview for further selection based on the following three attributes:

Table 1 Pythagorean hesitant fuzzy decision matrix

	A_1	A_2	A_3
X_1	$\langle\{0.6, 0.7, 0.9\}, \{0.4, 0.5, 0.8\}\rangle$	$\langle\{0.8, 0.9\}, \{0.4, 0.6, 0.7\}\rangle$	$\langle\{0.5, 0.6, 0.8\}, \{0.7, 0.8\}\rangle$
X_2	$\langle\{0.3, 0.7, 0.9\}, \{0.8, 0.9\}\rangle$	$\langle\{0.4, 0.7, 0.8\}, \{0.5, 0.7, 0.9\}\rangle$	$\langle\{0.6, 0.9\}, \{0.4, 0.6, 0.8\}\rangle$
X_3	$\langle\{0.40.9\}, \{0.7, 0.8\}\rangle$	$\langle\{0.5, 0.6, 0.7\}, \{0.7, 0.8\}\rangle$	$\langle\{0.6, 0.7, 0.8\}, \{0.5, 0.8\}\rangle$
X_4	$\langle\{0.5, 0.6, 0.9\}, \{0.6, 0.7, 0.8\}\rangle$	$\langle\{0.4, 0.8\}, \{0.6, 0.7, 0.9\}\rangle$	$\langle\{0.8, 0.9\}, \{0.4, 0.6\}\rangle$
X_5	$\langle\{0.7, 0.9\}, \{0.4, 0.5, 0.8\}\rangle$	$\langle\{0.6, 0.8\}, \{0.5, 0.7\}\rangle$	$\langle\{0.4, 0.5, 0.7\}, \{0.7, 0.9\}\rangle$

A1 IAcademic background and influence.
A2 Leadership.
A3 Research and teaching experiences.

Assume that the attribute weighting vector is $w = (0.25, 0.35, 0.4)^T$ and the associated-weighting vector is $\omega = (0.35, 0.40, 0.25)^T$. Assume that the decision makers D_1, D_2 and D_3 give the decision matrix shown in Table 1. To make a rational decision, the interrelationships between attributes should be considered. In the following, we use the proposed method presented above to solve this MAGDM problem.

Step 1 To avoid influence each other, the decision makers are required to evaluate the five candidates $X_i (i = 1, 2, 3, 4, 5)$ under the above three attributes in anonymity and the decision matrix $C = (\dot{\hat{h}}_{ij})_{m\times n}$ is presented in Table 1, where $\dot{\hat{h}}_{ij} (i = 1, 2, 3, 4, 5)$ and $j = 1, 2, 3)$ are in the form of PHFNs.

For PHFHWA operator

Step 2 Utilize the *PHFHWA* operator to obtain the Pythagorean hesitant fuzzy matrix (see Table 2).

Step 3 Compute the score values $S(\hat{h}_i)(i = 1, 2, 3, 4, 5)$ of $\hat{h}_i (i = 1, 2, 3, 4, 5)$ by Definition 3. The score values for the alternatives are:

$$S(\hat{h}_1) = 0.2176, S(\hat{h}_2) = 0.1038, S(\hat{h}_3) = -0.0214,$$
$$S(\hat{h}_4) = 0.14738, S(\hat{h}_5) = 0.0906.$$

Step 4 Since, $S(\hat{h}_4) \succ S(\hat{h}_1) \succ S(\hat{h}_2) \succ S(\hat{h}_5) \succ S(\hat{h}_3)$. Therefore, $X_1 \succ X_4 \succ X_2 \succ X_5 \succ X_3$ and the most desirable alternative is X_4.

For PHFHWG operator

Step 2′ Utilize the *PHFHWG* operator to obtain the Pythagorean hesitant fuzzy matrix (See Table 3).

Table 2 Pythagorean hesitant fuzzy hybrid averaging matrix

	A_1	A_2	A_3
X_1	$\langle\{0.8111, 0.9084\}, \{0.3821, 0.5849, 0.6876\}\rangle$	$\langle\{0.5334, 0.6297, 0.8439\}, \{0.5030, 0.5946, 0.8459\}\rangle$	$\langle\{0.5403, 0.6439, 0.8406\}, \{0.6518, 0.7651\}\rangle$
X_2	$\langle\{0.6439, 0.9294\}, \{0.3330, 0.5417, 0.7651\}\rangle$	$\langle\{0.4090, 0.7120, 0.8111\}, \{0.4830, 0.6876, 0.8953\}\rangle$	$\langle\{0.2613, 0.6297, 0.8439\}, \{0.8459, 0.9240\}\rangle$
X_3	$\langle\{0.6439, 0.7445, 0.8406\}, \{0.4353, 0.7651\}\rangle$	$\langle\{0.5106, 0.6117, 0.7120\}, \{0.6876, 0.7911\}\rangle$	$\langle\{0.2613, 0.8439\}, \{0.7653, 0.8459\}\rangle$
X_4	$\langle\{0.6406, 0.9294\}, \{0.3330, 0.5417\}\rangle$	$\langle\{0.4090, 0.8111\}, \{0.5849, 0.6876, 0.8953\}\rangle$	$\langle\{0.4405, 0.5334, 0.7316\}, \{0.6817, 0.7653, 0.8459\}\rangle$
X_5	$\langle\{0.6117, 0.8111\}, \{0.4830, 0.6876\}\rangle$	$\langle\{0.6297, 0.8439\}, \{0.5030, 0.5946, 0.8459\}\rangle$	$\langle\{0.4345, 0.5403, 0.7445\}, \{0.6518, 0.8812\}\rangle$

Table 3 Pythagorean hesitant fuzzy hybrid weighted geometric matrix

	A_1	A_2	A_3
X_1	$\langle\{0.7911, 0.8953\}, \{0.4090, 0.6117, 0.7120\}\rangle$	$\langle\{0.6817, 0.7653, 0.9240\}, \{0.3501, 0.4405, 0.7316\}\rangle$	$\langle\{0.4353, 0.5417, 0.7651\}, \{0.7445, 0.8406\}\rangle$
X_2	$\langle\{0.5417, 0.8812\}, \{0.4345, 0.6439, 0.8406\}\rangle$	$\langle\{0.3821, 0.6876, 0.7911\}, \{0.5106, 0.7120, 0.9084\}\rangle$	$\langle\{0.4054, 0.7653, 0.9240\}, \{0.7316, 0.8439\}\rangle$
X_3	$\langle\{0.5030, 0.9240\}, \{0.6297, 0.7316\}\rangle$	$\langle\{0.5417, 0.6518, 0.7651\}, \{0.5403, 0.8406\}\rangle$	$\langle\{0.4830, 0.5849, 0.6876\}, \{0.7120, 0.8111\}\rangle$
X_4	$\langle\{0.7651, 0.8812\}, \{0.4345, 0.6439\}\rangle$	$\langle\{0.5946, 0.6817, 0.8459\}, \{0.5334, 0.6297, 0.7316\}\rangle$	$\langle\{0.3821, 0.7911\}, \{0.6117, 0.7120, 0.9084\}\rangle$
X_5	$\langle\{0.7653, 0.9240\}, \{0.3501, 0.4405, 0.7316\}\rangle$	$\langle\{0.5849, 0.7911\}, \{0.5106, 0.7120\}\rangle$	$\langle\{0.3330, 0.4353, 0.6518\}, \{0.7445, 0.9294\}\rangle$

Step 3′ Compute the score values $S(\hat{h}_i)(i = 1, 2, 3, 4, 5)$ of $\hat{h}_i(i = 1, 2, 3, 4, 5)$ by Definition 3. The score values for the alternatives are:

$$S(\hat{h}_1) = 0.1324, S(\hat{h}_2) = -0.1220, S(\hat{h}_3) = -0.1023,$$
$$S(\hat{h}_4) = 0.0580, S(\hat{h}_5) = -0.0360.$$

Step 4′ Since, $S(\hat{h}_1) \succ S(\hat{h}_4) \succ S(\hat{h}_2) \succ S(\hat{h}_5) \succ S(\hat{h}_3)$. Therefore, $X_1 \succ X_4 \succ X_2 \succ X_5 \succ X_3$ and the most desirable alternative is X_1.

5.1 Comparison analysis

In this subsection we compare our approach to the existing methods of PFNs, introduced by Yager (2013) and HFNs introduced by Torra (2010), which are the special cases of PHFNs to verify the validity and effectiveness of the proposed approach.

5.1.1 A comparison analysis with the existing MCDM method with PFNs

PFNs can be considered as a special case of PHFNs when there is only one element in membership and nonmembership degree. For comparison, the PHNs can be transformed to PFNs by calculating the average value of the membership and nonmembership degrees. After transformation, the Pythagorean information presented in Tables 2 and 3 can be shown in Tables 4 and 5, respectively.

Now we calculate the comprehensive evaluation values using the Pythagorean fuzzy hybrid weighted average (PFHWA) operator and the Pythagorean fuzzy hybrid weighted geometric (PFHWG) operator (Rahman et al. 2016). The score values and the ranking of the alternatives using PFHWA operator and PFHWG operator are given in Table 6, respectively, which are the same as the proposed approach. But PHFSs are more flexible than PFSs because they consider the situations where decision makers would like to use several possible values to express the membership and nonmembership degrees.

Table 4 Pythagorean fuzzy hybrid averaging matrix

	A_1	A_2	A_3
X_1	$\langle 0.86, 0.55\rangle$	$\langle 0.67, 0.65\rangle$	$\langle 0.68, 0.71\rangle$
X_2	$\langle 0.79, 0.55\rangle$	$\langle 0.64, 0.69\rangle$	$\langle 0.58, 0.89\rangle$
X_3	$\langle 0.74, 0.60\rangle$	$\langle 0.61, 0.74\rangle$	$\langle 0.55, 0.81\rangle$
X_4	$\langle 0.79, 0.44\rangle$	$\langle 0.61, 0.72\rangle$	$\langle 0.57, 0.76\rangle$
X_5	$\langle 0.71, 0.59\rangle$	$\langle 0.74, 0.65\rangle$	$\langle 0.57, 0.77\rangle$

Table 5 Pythagorean fuzzy hybrid weighted geometric matrix

	A_1	A_2	A_3
X_1	$\langle 0.84, 0.58\rangle$	$\langle 0.79, 0.50\rangle$	$\langle 0.58, 0.79\rangle$
X_2	$\langle 0.71, 0.64\rangle$	$\langle 0.62, 0.71\rangle$	$\langle 0.69, 0.79\rangle$
X_3	$\langle 0.71, 0.68\rangle$	$\langle 0.65, 0.69\rangle$	$\langle 0.59, 0.76\rangle$
X_4	$\langle 0.79, 0.44\rangle$	$\langle 0.61, 0.72\rangle$	$\langle 0.59, 0.74\rangle$
X_5	$\langle 0.85, 0.51\rangle$	$\langle 0.69, 0.61\rangle$	$\langle 0.47, 0.84\rangle$

5.1.2 A comparison analysis with the existing MCDM method with PFNs,

HFNs can be considered as a special case of PHFNs when there is only membership degree. For comparison, the PHNs can be transformed to HFNs by removing the nonmembership degrees. After transformation, the hesitant fuzzy information can be shown in Tables 6 and 7, respectively.

Now we calculate the comprehensive evaluation values using the hesitant fuzzy hybrid weighted average (HFHWA) operator (Liao and Xu 2015) and the hesitant fuzzy hybrid weighted geometric (HFHWG) operator (Liao and Xu 2015). The score values and the ranking of the alternatives using HFHWA operator and HFHWG operator are given in Table 8, respectively, which are the same as the proposed approach. But PHFSs are more flexible than HFSs because they consider the situations where decision makers would like to use several possible values to express the membership and nonmembership degrees.

6 Conclusion

During the process of solving the real problems, the decision maker always encounters the evaluation information of alternatives which is incomplete, indeterminate and inconsistent. Fortunately, the Pythagorean hesitant fuzzy set PHFS is a better tool to depict this kind of information. Therefore, in

Table 6 hesitant fuzzy hybrid averaging matrix

	A_1	A_2	A_3
X_1	{0.8111, 0.9084}	{0.5334, 0.6297, 0.8439}	{0.5403, 0.6439, 0.8406}
X_2	{0.6439, 0.9294}	{0.4090, 0.7120, 0.8111}	{0.2613, 0.6297, 0.8439}
X_3	{0.6439, 0.7445, 0.8406}	{0.5106, 0.6117, 0.7120}	{0.2613, 0.8439}
X_4	{0.6406, 0.9294}	{0.4090, 0.8111}	{0.4405, 0.5334, 0.7316}
X_5	{0.6117, 0.8111}	{0.6297, 0.8439}	{0.4345, 0.5403, 0.7445}

Table 7 Hesitant fuzzy hybrid weighted geometric matrix

	A_1	A_2	A_3
X_1	{0.7911, 0.8953}	{0.6817, 0.7653, 0.9240}	{0.4353, 0.5417, 0.7651}
X_2	{0.5417, 0.8812}	{0.3821, 0.6876, 0.7911}	{0.4054, 0.7653, 0.9240}
X_3	{0.5030, 0.9240}	{0.5417, 0.6518, 0.7651}	{0.4830, 0.5849, 0.6876}
X_4	{0.7651, 0.8812}	{0.5946, 0.6817, 0.8459}	{0.3821, 0.7911}
X_5	{0.7653, 0.9240}	{0.5849, 0.7911}	{0.3330, 0.4353, 0.6518}

Table 8 Comparison analysis with existing methods

	X_1	X_2	X_3	X_4	X_5	Ranking
PHFHWA	0.2176	0.1038	− 0.0214	0.14737	0.0906	$X_1 \succ X_4 \succ X_2 \succ X_5 \succ X_3$
PHFHWG	0.1324	− 0.1220	− 0.1023	0.0580	− 0.0360	$X_1 \succ X_4 \succ X_2 \succ X_5 \succ X_3$
PFHWA (Rahman et al. 2016)	0.1858	0.0196	− 0.0694	0.0881	0.0540	$X_1 \succ X_4 \succ X_2 \succ X_5 \succ X_3$
PFHWG (Rahman et al. 2016)	0.1629	− 0.062	− 0.0708	0.0063	0.0050	$X_1 \succ X_4 \succ X_2 \succ X_5 \succ X_3$
HFHWA (Liao and Xu 2015)	0.7688	0.7263	0.6744	0.7089	0.7068	$X_1 \succ X_2 \succ X_4 \succ X_5 \succ X_3$
HFHWG (Liao and Xu 2015)	0.7429	0.5751	0.6458	0.7011	0.6659	$X_1 \succ X_4 \succ X_3 \succ X_5 \succ X_2$

this paper we considered Pythagorean hesitant fuzzy information and on the basis of hybrid aggregation operators. Since, we know that the PHFWA operator and PHFWG operator weights only the Pythagorean hesitant fuzzy numbers, respectively, while the PHFOWA operator and PHFOWG operator weights only the ordered positions of the Pythagorean hesitant fuzzy numbers respectively instead of weighting the Pythagorean hesitant fuzzy numbers themselves. We developed PHFHWA operator and PHFHWG operator which weight both the given Pythagorean hesitant fuzzy number and its ordered position. Furthermore, we have given a decision making method based on the developed operators. Moreover, we have given a numerical example to show the validity and effectiveness of the proposed approach. Finally we compared our approach to existing methods.

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