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ORIGINAL PAPER

Fuzzy rough soft set and its application to lattice

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Abstract

In this study, we establish a connection between rough soft set (Shabir et al., Knowl Base Syst 40:72–80, 2013) and fuzzy set. Based on the novel granulation structure called modified soft rough approximation space, fuzzy rough soft set is introduced. The important basic properties of fuzzy rough soft set are studied and supported by illustrative examples. Moreover lattice theory is studied on fuzzy rough soft set. The definitions and propositions presented in this paper enrich the soft set theory, rough set theory and fuzzy set theory, and also extend their application scopes. The paper ends with conclusions having future investigations of the study.

Keywords MSR-approximation space · Rough soft set · Fuzzy rough soft set · Lattice

1 Introduction

At present, uncertainty is an important and interesting topic to the researchers as it has been considered in many situations like engineering, economics, social science, computer science, environmental science, medical science, etc. Fuzzy set theory, probability theory, rough set theory and soft set theory are successfully applied to solve the problems with uncertainties in these areas. The concept of fuzzy set was introduced by Zadeh (1965) and it has been applied to solve the various problems (Akram and Ali 2018; Chen and Chen 2012; Chen and Tanuwijaya 2011; Chen and Chang 2011; Chen et al. 2001; Joshi and Kumar 2018; Li et al. 2018; Liu and Li 2018; Wang and Chen 2008). Fuzzy set allows objects belong to a set or a relation to a given degree ranging between 0 and 1, i.e., a membership function is needed to define it. Pawlak (1982) gave an alternative approach called rough set theory to tackle uncertainties. In this theory, Pawlak described that every rough set is associated with two crisp sets, called lower and upper approximations and are viewed as the set of elements which certainly and possibly belong to the set. Rough set is now a rigorous area of research with applications in various fields, such as knowledge discovery, decision analysis, signal processing, mereology and many other fields. Although, these methods are successfully used to describe uncertainty, each of these theories has inherent difficulties chosen by Molodtsov (1999, 2004). Molodtsov (1999) developed a new concept called soft set which is free from difficulties affecting existing methods to deal with uncertainty. Most of the operations on soft set are defined by Maji et al. (2003) and redefined by Cağman and Cagman and Enginoglu, (2010). Of late, a rapid development of interest in soft set theory and its applications have been found. Maji et al. (2001) also introduced the notion of fuzzy soft set. They (Maji et al. 2002) also discussed the application of soft set theory in a decision-making problem.

From the mathematical point of view, lattice (Davey and Priestley 2001) is a partially ordered set in which any two elements have a unique supremum and an infimum. Lattices can also be characterized as algebraic structures satisfying certain axiomatic identities. Since the two definitions are equivalent, lattice theory can be developed on both order relation and universal algebra.

The study of the algebraic structure of mathematical theory proves itself effective in making the applications more efficient. The study of lattices in rough set theory was initiated by Iwinski (1987). Pomykala and Pomykala (1998) showed that set of rough sets is a stone lattice. Thomas and Nair (2011) introduced the concept of intuitionistic fuzzy sublattices and intuitionistic fuzzy ideals

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of a lattice. Bera and Roy (2013) defined modular lattice in the rough set environment using indiscernibility relation. Bera and Roy (2016) introduced notions of upper and lower soft rough ideals in a lattice. For more information about lattice structure under rough set environment, we refer to various works of the authors (Jarvinen 2007; Li 2010; Liao et al. 2010; Pomykala and Pomykala 1998; Rana and Roy 2011, 2013, 2014, 2015). Similarly, algebraic structure (lattice) of soft set and its hybridization with rough set is an interesting topic to the researchers. For example, Bera et al. (2017) described the soft binary relation as well as soft congruence relation over lattice and defined soft congruence relation on lattice. In addition, the applications of soft set theory in lattice have been found in the papers (Li and Liu 2009; Maji et al. 2002; Roy and Bera 2015a; Nagarajan and Meenambigar 2011; Zhang and Wang 2014).

A connection between the soft set and the rough set has been discussed by Feng et al. (2011). They introduced the notion of the soft rough set, where instead of equivalence classes parameterized subsets of a set is employed to find lower and upper approximations of a subset. But in their discussion, some cases may be occurred, where upper approximation of a non-empty set may be empty. Again upper approximation of a subset X may not be contained in the set X. Although these cases do not occur in (classical) rough set theory but there are some other generalised rough set models (Fan et al. 2017; Li and Xu 2015; Li et al. 2018; Yao and Lin 1997) concerning these situations. After that Shabir et al. (2013) redefined the soft rough set model and called Modified Soft Rough (MSR) set whose lower and upper approximations are different from the (classical) rough set theory and soft set theory. They showed that MSR-sets satisfy all the basic properties of rough sets in one hand, on the other hand in this new model, information granules are finer than soft rough sets. Then, Roy and Bera (2015a) approximated the soft set in the MSR approximation space and defined the notion of rough soft set. In the present paper, an attempt is taken to connect the rough soft set with the fuzzy set in an MSR approximation space.

In this paper, we calculate the measure of roughness of rough soft set in a modified soft rough approximation space and introduce the notion of fuzzy rough soft set. We also define here absolute fuzzy rough soft set and null fuzzy rough soft set. We study the properties like subset, union, intersection on fuzzy rough soft set and provide some examples to analyze the definitions. We also present some propositions on fuzzy rough soft set. An order relation on fuzzy rough soft set is also included on fuzzy rough soft set and its application to lattice is discussed through an example with the help of Hasse diagram. The motivations of this paper are as follows:

(i) To introduce the fuzziness into rough soft set approaches.
(ii) To include the order relation on fuzzy rough soft set and study algebraic structure (lattice theory) on fuzzy rough soft set.

The rest of the paper is structured as follows: Some basic definitions about on fuzzy set, rough set, soft set, modified rough soft set and rough soft set are introduced in Sect. 2. We introduce the notion of fuzzy rough soft set including the definitions and some propositions in Sect. 3. We end Sect. 3 by presenting the Hasse diagram which is an application of fuzzy rough soft set to lattice. Sect. 4 concludes the paper with the future investigations of the proposed study.

2 Preliminaries

Let U be a non-empty set of universe and R be an equivalence relation on U. The pair (U, R) is called Pawlak's approximation space. The equivalence relation R is often called an indiscernibility relation and related to an information system. An indiscernibility relation $R = I(B), B \subseteq A$ is defined as:$(x, y) \in I(B) \Leftrightarrow a(x) = a(y),\ \forall a \in B$, where $x, y \in U$, and $a(x)$ denotes the value of attribute a for object x. When two objects have the same value over a certain group of attributes, we say that they are indiscernible with respect to this group of attributes, or have the same description with respect to the indiscernibility relation. Indiscernibility relation is an equivalence relation. By this equivalence relation, we form equivalence class and all the equivalence classes form a partition of the universe, which are the basic building blocks of universal set called granules. Any subset of objects of the universe is approximated by two sets, called the lower and the upper approximations and can be viewed as the sets of elements which certainly and possibly belong to the set. Pair of two approximations is called Rough set.

Definition 2.1 (Pawlak 1982) Let U be the set of universe and ρ be an equivalence relation on U. The pair (U, ρ) is called Pawlak's approximation space. The lower and upper approximations of $X \subseteq U$ are treated as: $\underline{X} = \{x \in U : [x]_\rho \subseteq X\}$ and $\overline{X} = \{x \in U : [x]_\rho \cap X \neq \phi\}$, where $[x]_\rho$ denotes the equivalence class of $x\,(\in U)$. If $\underline{X} \neq \overline{X}$, then X is said to be rough set over (U, ρ).

Definition 2.2 (Zadeh 1965) A fuzzy set $\tilde{A}$ in X is characterized by a membership function $\mu_{\tilde{A}}(x)$ which associates with each point in X to a real number in the interval [0, 1] with the value of $\mu_{\tilde{A}}(x)$ at x representing the grade of membership $\mu_{\tilde{A}}(x)$ of x in $\tilde{A}$.

A fuzzy set $\tilde{A}$ can be written an $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) : x \in X\}$.

According to Zadeh (1965), intersection, union, and complement of fuzzy set are defined component-wise as follows:

(i) $(\mu_{\tilde{A}} \cap \mu_{\tilde{B}})(x) = \min\{\mu_{\tilde{A}}(x),\ \mu_{\tilde{B}}(x)\}$,
(ii) $(\mu_{\tilde{A}} \cup \mu_{\tilde{B}})(x) = \max\{\mu_{\tilde{A}}(x),\ \mu_{\tilde{B}}(x)\}$,
(iii) $(\mu^c_{\tilde{A}})(x) = 1 - \mu_{\tilde{A}}(x)$,

where $\mu_{\tilde{A}}(x)$ and $\mu_{\tilde{B}}(x)$ are membership functions of two fuzzy sets $\tilde{A}$ and $\tilde{B}$, respectively, in X and $x \in X$; and $\mu^c_{\tilde{A}}(x)$ denotes the complement of the membership function $\mu_{\tilde{A}}(x)$.

Definition 2.3 (Maji et al. 2003) A pair $S = (F, A)$ is called a soft set over U, where $F : A \to P(U)$ denotes a set valued mapping and $P(U)$ is the power set of U.

It is noted that a soft set is a parameterised family of subset of U. Maji et al. (2003) introduced some binary operations such as AND, OR on soft set which are defined as follows:

Definition 2.4 (Maji et al. 2003) Let (G_1, B_1) and (G_2, B_2) be two soft sets over U. Then (G_1, B_1) AND $(G_2, B_2,)$ is denoted by $(G_1, B_1) \wedge (G_2, B_2,)$ and is defined by $(G_1, B_1) \wedge (G_2, B_2) = (H_1, A \times B)$, where $H_1(x, y) = G_1(x) \cap G_2(y), \forall (x, y) \in A \times B$.

Definition 2.5 (Maji et al. 2003) Let (G_1, B_1) and (G_2, B_2) be two fuzzy rough soft sets over U. Then (G_1, B_1) OR $(G_2, B_2,)$ is noted by $(G_1, B_1) \vee (G_2, B_2,)$ and is defined by $(G_1, B_1) \vee (G_2, B_2) = (H_2, A \times B)$, where $H_2(x, y) = G_1(x) \cup G_2(y), \forall (x, y) \in A \times B$.

Definition 2.6 (Shabir et al. 2013) Let (F, A) be a soft set over U and $\psi : U \to P(A)$ be another mapping defined by $\psi(x) = \{a : x \in F(a)\}$. Then the pair (U, ψ) is called the MSR approximation space and for any $X \subseteq U$, lower MSR-approximation, $\underline{X}_\psi$ and upper MSR-approximation, $\overline{X}_\psi$ respectively are defined as follows:

$\underline{X}_\psi = \{x \in U : \psi(x) \neq \psi(y)\ \forall\ y \in X^c\}$, where X^c is the complement of X, i.e., $X^c = U - X$,

$\overline{X}_\psi = \{x \in U : \psi(x) = \psi(y)\ \text{ for some }\ y \in X\}$.

If $\underline{X}_\psi \neq \overline{X}_\psi$, then X is said to be a modified soft rough set.

In the above definition, parameter set A of the soft set (F, A) plays the role in defining the approximations of a subset X of U.

Example 2.1 Let $U = \{p_1, p_2, p_3, p_4, p_5\}$ be the universal set and a set of parameters $A = \{e_1, e_2, e_3, e_4\}$. Let the soft set (F, A) over U is stated in below:

$F(e_1) = \{p_1, p_2, p_4\}$, $F(e_2) = \{p_1, p_2, p_4, p_5\}$, $F(e_3) = \{p_5\}$, $F(e_4) = \{p_2, p_3, p_5\}$. Then from the definition of MSR set $\psi : U \to P(A)$ is constructed as follows:

$\psi(p_1) = \{e_1, e_2\}$, $\psi(p_2) = \{e_1, e_2, e_4\}$, $\psi(p_3) = \{e_4\}$, $\psi(p_4) = \{e_1, e_2\}$, $\psi(p_5) = \{e_2, e_3, p_4\}$. Let $X = \{p_1, p_2, p_3\}$. Then for the MSR approximation space (U, ψ), we can write $\underline{X}_\psi = \{p_2, p_3\}$ and $\overline{X}_\psi = \{p_1, p_2, p_3, p_4\}$. Clearly, $\underline{X}_\psi \neq \overline{X}_\psi$ and hence X is the modified soft rough set.

Definition 2.7 (Roy and Bera 2015a) Let (F, A) be a soft set over U and (U, ψ) be an MSR-approximation space with respect to (F, A). Let (G, B) be another soft set over U. Then (G, B) is said to be rough soft set with respect to the parameter $e \in B$ if $\underline{G(e)}_\psi \neq \overline{G(e)}_\psi$. (G, B) is said to be a full rough soft set or a simply rough one if $\underline{G(e)}_\psi \neq \overline{G(e)}_\psi, \forall\ e \in B$ and it is denoted by $RsG(e_B)$. Therefore, rough soft set with respect to the parameter e is given by $RsG(e) = (\underline{G(e)}_\psi, \overline{G(e)}_\psi)$.

In this definition, we see that one soft set is approximated with respect to another one.

Example 2.2 Considering the universal set U, soft set (F, A) and set valued function $\psi : U \to P(A)$ as given in Example 2.1. Let (G, B) be another soft set over U, where $B = \{e_1, e_2\}$, $G(e_1) = \{p_2, p_5\}$ and $G(e_2) = \{p_1, p_3, p_5\}$. Then the lower MSR approximation and the upper MSR approximation set of (G, B) in (U, ψ) are $\underline{G(e_1)}_\psi = \{p_2, p_5\}$, $\overline{G(e_1)}_\psi = \{p_2, p_5\}$, $\underline{G(e_2)}_\psi = \{p_3, p_5\}$, $\overline{G(e_2)}_\psi = \{p_1, p_3, p_4, p_5\}$. Clearly, (G, B) is rough soft set with respect to the parameter e_2.

Definition 2.8 (Roy and Bera 2015a) Let (F, A) be a soft set over U and (U, ψ) be an MSR-approximation space. Let (G, B) be another soft set over U. Measure of roughness of (G, B) with respect to the parameter $e \in B$ is denoted by $R_{G(e)}$ and is defined as: $R_{G(e)} = \frac{|\underline{G(e)}_\psi|}{|\overline{G(e)}_\psi|}$, where $|\underline{G(e)}_\psi|$ and $|\overline{G(e)}_\psi|$ denote the cardinalities of the sets $\underline{G(e)}_\psi$ and $\overline{G(e)}_\psi$, respectively. Clearly, $0 \leq R_{G(e)} \leq 1$.

3 Fuzzy rough soft set

Here, we discuss for each soft set (G, B) over U there is an associated fuzzy set. It is known that each soft set (G, B) over U, roughness of (G, B) with respect to the parameter $e \in B$, in MSR-approximation space is a number from the interval [0, 1]. Hence, we define a fuzzy set for every soft set. We denote the notation (U, ψ) as a MSR approximation space with respect to A for the soft set (F, A) over U in the paper.

Definition 3.1 Let (*G*, *B*) be a soft set over *U*. Then fuzzy rough soft set of (*G*, *B*) over (U, ψ) is defined as: $\{(G(e), R_{G(e)}) : G(e) \in (G, B)\}$, where $R_{G(e)}$ is the roughness of (*G*, *B*) with respect to the parameter $e \in B$.

To understand the concept presented above, we consider an example as follows:

Example 3.1 Consider in Example 2.2, roughness of (*G*, *B*) is given by $R_{G(e_1)} = 1$ and $R_{G(e_2)} = \frac{1}{2}$. Therefore, the fuzzy rough soft set of (*G*, *B*) is given by $\{(G(e_1), 1), (G(e_2), \frac{1}{2})\}$.

Definition 3.2 A fuzzy rough soft set (*G*, *B*) over (U, ψ) is said to be null fuzzy rough soft set if $R_G(e) = 0, \forall\, e \in B$ and we use the symbol (G_ϕ, B).

Definition 3.3 A fuzzy rough soft set (*G*, *B*) over (U, ψ) is said to be absolute fuzzy rough soft set if $R_G(e) = 1, \forall\, e \in B$ and we denote it by (G_U, B).

Example 3.2 In Example 2.1, let (*G*, *C*) be another soft set over *U* which is defined as: $G(e_1) = \{p_1\}, G(e_2) = \{p_4\}$.

Then in the MSR approximation space (U, ψ), lower and upper MSR approximations of (*G*, *C*) are given by $\underline{G(e_1)}_\psi = \phi$, $\overline{G(e_1)}_\psi = \{p_1, p_4\}$, $\underline{G(e_2)}_\psi = \phi$, $\overline{G(e_2)}_\psi = \{p_1, p_4\}$. Also roughness of (*G*, *C*) is given by $R_{G(e_1)} = 0$ and $R_{G(e_2)} = 0$. So (*G*, *C*) is a null fuzzy rough soft set.

If we consider the soft set (*G*, *D*) over *U* defined as $G(e_1) = \{p_2, p_5\}$ and $G(e_2) = \{p_2, p_3, p_5\}$. Then in the MSR approximation space (U, ψ), $\underline{G(e_1)}_\psi = \{p_2, p_5\}$, $\overline{G(e_1)}_\psi = \{p_2, p_5\}$, $\underline{G(e_2)}_\psi = \{p_2, p_3, p_5\}$, $\overline{G(e_2)}_\psi = \{p_2, p_3, p_5\}$. Now $R_{G(e_1)} = 1$ and $R_{G(e_2)} = 1$. Hence (*G*, *D*) is an absolute fuzzy rough soft set.

Definition 3.4 Let (G_1, B_1) and (G_2, B_2) be two fuzzy rough soft sets over (U, ψ) with membership functions $R_{G_1(e_{B_1})}$ and $R_{G_2(e_{B_2})}$, respectively. (G_1, B_1) is said to be fuzzy rough soft subset of (G_2, B_2) if

(i) $B_1 \subseteq B_2$ and **(ii)** $R_{G_1(e)} = R_{G_2(e)}\, \forall\, e \in B_1$.

We write $(G_1, B_1) \sqsubseteq_F (G_2, B_2)$, where the symbol '$\sqsubseteq_F$' denotes fuzzy rough soft subset.

Definition 3.5 Two fuzzy rough soft sets, (G_1, B_1) and (G_2, B_2) over (U, ψ) is said to be equal if $(G_1, B_1) \sqsubseteq_F (G_2, B_2)$ and $(G_2, B_2) \sqsubseteq_F (G_1, B_1)$.

Proposition 3.1 If (G_1, B_1) is a soft subset of (G_2, B_2) then (G_1, B_1) is a fuzzy rough soft subset of (G_2, B_2).

Proof Let (G_1, B_1) be soft subset of (G_2, B_2), then by

Definition 3.6 (i) $B_1 \subseteq B_2$, and(ii) $G_1(e) = G_2(e)\, \forall\, e \in B_1$.

Therefore, $\underline{G_1(e)}_\psi = \underline{G_2(e)}_\psi$ and $\overline{G_1(e)}_\psi = \overline{G_2(e)}_\psi$, $\forall$ $e \in B_1$. This gives $R_{G_1(e)} = R_{G_2(e)}\, \forall\, e \in B_1$. This completes the proof of the proposition.

From Proposition 3.1, it is clear that every soft subset of a soft set is a fuzzy rough soft subset.

Proposition 3.2 Let (G_1, B_1) and (G_2, B_2) be two fuzzy rough soft sets over (U, ψ) with membership functions $R_{G(e_{B_1})}$ and $R_{G(e_{B_2})}$, respectively. Then $H(e) = (G_1, B_1) \sqcup (G_2, B_2)$, $B_1 \cap B_2 = \phi$ is a fuzzy rough soft set. The membership function of fuzzy rough soft set is denoted by $R_{H(e)}$ and is given as follows:

$$R_{H(e)} = \begin{cases} R_{G(e_{B_1})}, & \text{if } e \in B_1 - B_2, \\ R_{G(e_{B_2})}, & \text{if } e \in B_2 - B_1. \end{cases}$$

Now we introduce some operations such as union, intersection, complement, AND, OR on fuzzy rough soft set.

Definition 3.7 Let (G_1, B_1) and (G_2, B_2) be two fuzzy rough soft sets over (U, ψ) with membership functions $R_{G_1(e_{B_1})}$ and $R_{G_2(e_{B_2})}$, respectively. Then the union of (G_1, B_1) and (G_2, B_2) is defined as $(G_1, B_1) \sqcup_F (G_2, B_2) = (H, C)$, where $C = B_1 \cup B_2$; the symbol '$\sqcup_F$' denotes fuzzy rough soft union, and the membership function is described as follows:

$$R_{H(e)} = \begin{cases} R_{G_1(e_{B_1})}, & \text{if } e \in B_1 - B_2, \\ R_{G_2(e_{B_2})}, & \text{if } e \in B_2 - B_1, \\ \max\{R_{G_1(e)}, R_{G_2(e)}\}, & \text{if } e \in B_1 \cap B_2. \end{cases}$$

Definition 3.8 Let (G_1, B_1) and (G_2, B_2) be two fuzzy rough soft sets over (U, ψ) with membership functions $R_{G_1(e_{B_1})}$ and $R_{G_2(e_{B_2})}$, respectively. The intersection of (G_1, B_1) and (G_2, B_2) is defined as $(G_1, B_1) \sqcap_F (G_2, B_2) = (H, C)$, where $C = B_1 \cap B_2$; the symbol '$\sqcap_F$' means fuzzy rough soft intersection, and the membership function is given by $R_{H(e)} = \min\{R_{G_1(e)}, R_{G_2(e}\}, e \in C$.

Definition 3.9 Complement of a fuzzy rough soft set (G, B) with membership function $R_{G(e_B)}$ is denoted by (G^c, B) and the rough membership function is given by $R_{G^c(e_B)} = 1 - R_{G(e_B)}$.

Definition 3.10 Let (G_1, B_1) and (G_2, B_2) be two fuzzy rough soft sets over (U, ψ) with membership functions $R_{G_1(e_{B_1})}$ and $R_{G_2(e_{B_2})}$, respectively. Then (G_1, B_1) AND (G_2, B_2), denoted by $(G_1, B_1) \wedge_F (G_2, B_2)$, defined by $(G_1, B_1) \wedge_F (G_2, B_2) = (H_1, A \times B)$, where $H_1(x, y) = G_1(x) \cap G_2(y)$ and the membership function is given by

$R_{H_1(x,y)} = \min\{R_{G_1(x)}, R_{G_2(y)}\}, \forall\, (x, y) \in A \times B$.

Definition 3.11 Let (G_1, B_1) and (G_2, B_2) be two fuzzy rough soft sets over (U, ψ) with membership functions $R_{G_1(e_{B_1})}$ and $R_{G_2(e_{B_2})}$, respectively. Then (G_1, B_1) OR (G_2, B_2) is denoted by $(G_1, B_1) \vee_F (G_2, B_2)$ and is defined as $(G_1, B_1) \vee_F (G_2, B_2) = (H_2, A \times B)$, where $H_2(x, y) = G_1(x) \cup G_2(y)$ and the membership function is given by

$R_{H_2(x,y)} = \max\{R_{G_1(x)}, R_{G_2(y)}\}, \forall\, (x, y) \in A \times B$.

Proposition 3.3 Let (G_1, B_1) and (G_2, B_2) be two fuzzy rough soft sets over (U, ψ) with the membership functions $R_{G_1(e_{B_1})}$ and $R_{G_2(e_{B_2})}$, respectively. Then

(i) $(G_1, B_1) \vee_F (G_2, B_2) = (G_2, B_2) \vee_F (G_1, B_1)$
(ii) $(G_1, B_1) \wedge_F (G_2, B_2) = (G_2, B_2) \wedge_F (G_1, B_1)$.

That is fuzzy rough soft set is commutative in respect of the operations $\vee_F$ and $\wedge_F$.

Proposition 3.4 Let (G_1, B_1) and (G_2, B_2) be two fuzzy rough soft sets over (U, ψ) with the membership functions $R_{G_1(e_{B_1})}$ and $R_{G_2(e_{B_2})}$, respectively. Then the following results hold:

(i) $((G_1, B_1) \sqcup_F (G_2, B_2))^c = (G_1, B_1)^c \sqcap_F (G_2, B_2)^c$
(ii) $((G_1, B_1) \sqcap_F (G_2, B_2))^c = (G_1, B_1)^c \sqcup_F (G_2, B_2)^c$.

Proof Case 1: Let $e \in B_1 - B_2$ and $R_{G_1(e)} = p$. Then $R_{G_2(e)} = 0$. Therefore, $R_{(G_1 \sqcup_F G_2)^c(e)} = 1 - p$. Also $R_{(G_1{}^c \sqcap_F G_2{}^c)(e)} = \min\{R_{G_1{}^c(e)}, R_{G_2{}^c(e)}\} = \min\{1 - p, 1\} = 1 - p$.

Case 2: If $e \in B_2 - B_1$, then the proof can be established in similar way as in Case 1.

Case 3: Suppose $e \in B_1 \cap B_2$, then

$$R_{(G_1{}^c \sqcap_F G_2{}^c)(e)} = \min\{R_{G_1{}^c(e)}, R_{G_2{}^c(e)}\}$$
$$= 1 - \max\{R_{G_1(e)}, R_{G_2(e)}\} = R_{(G_1 \sqcup_F G_2)^c(e)}.$$

(ii) Proof is similar to that of proof (i). □

Proposition 3.4 showed that fuzzy rough soft set is satisfied by the D‘Morgan law.

Proposition 3.5 For any two fuzzy rough soft sets (G_1, B_1) and (G_2, B_2) over (U, ψ), the following conditions establish.

(i) $(G_1, B_1) \sqcup_F (G_2, B_2) = (G_2, B_2) \sqcup_F (G_1, B_1)$ and $(G_1, B_1) \sqcap_F (G_2, B_2) = (G_2, B_2) \sqcap_F (G_1, B_1)$.
(ii) $(G_1, B_1) \sqcup_F (\phi, B) = (G_1, B_1)$ and $(G_1, B_1) \sqcap_F (\phi, B) = (\phi, B_2)$, where (ϕ, B) denotes the null fuzzy rough soft set.
(iii) $(\phi, B) \sqcup_F (\phi, B) = (\phi, B), (\phi, B) \sqcap_F (\phi, B) = (\phi, B)$.

Now, we define a binary relation ‘$\asymp$’ on fuzzy rough soft set (G, B) over (U, ψ) as $G(e_1) \asymp G(e_2)$ if and only if $R_{G(e_1)} = R_{G(e_2)}$ for $e_1, e_2 \in B$.

Clearly, ‘$\asymp$’ is an equivalence relation on (G, B). We denote equivalence class of $(G(e_1), R_{G(e_1)})$ by the relation ‘$\asymp$’ as $[G(e_1)]_{\asymp}$.

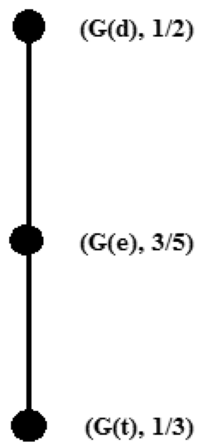
Proposition 3.6 Every fuzzy rough soft set forms a chain by the order relation ‘$\asymp$’.

Proof Measure of roughness of each members of a class $[G(e_1)]_{\asymp}$ is same. Therefore, each element of $[G(e_1)]_{\asymp}$ can be characterized by a unique real number from the interval [0, 1]. Therefore, there is a strict order relation among the classes. Hence the fuzzy rough soft set forms chain by the order relation $\asymp$. □

The following example illustrates the Proposition 3.6.

Example 3.3 Let $U = \{p_1, p_2, p_3, p_4, p_5, p_6\}$ be the set of people in a social gathering. Let the parameter set A described the shirts of three colors namely red, white and blue, i.e., $A = \{r, w, b\}$, where r, w and b stand for red, white and blue, respectively. Let us consider the soft set (F, A) with $F(r) = \{p_1, p_3, p_4\}$, $F(w) = \{p_1, p_2, p_4, p_5, p_6\}$ and $F(b) = \{p_1, p_2, p_3, p_6\}$. Then from definition of MSR set $\psi : U \to P(A)$ is constructed as: $\psi(p_1) = \{r, w, b\}$, $\psi(p_2) = \{w, b\}$, $\psi(p_3) = \{r, w\}$, $\psi(p_4) = \{r, w\}$, $\psi(p_5) = \{w\}$, $\psi(p_6) = \{w, b\}$. Let $B = \{$ doctor, teacher, engineer $\} = \{d, t, e\}$, where d, t and e denote doctor, teacher and engineer, respectively. Let (G, B) be another soft set over U is defined as:

Fig. 1 Chain of fuzzy rough soft set



$G(d) = \{p_1, p_2, p_4\}$, $G(t) = \{p_1, p_2\}$, $G(e) = \{p_1, p_3, p_5, p_6\}$. Then for the MSR approximation space (U, ψ), we can write $\underline{G(d)}_{\psi} = \{p_1, p_4\}$, $\overline{G(d)}_{\psi} = \{p_1, p_2, p_4, p_6\}$, $\underline{G(t)}_{\psi} = \{p_1\}$, $\overline{G(t)}_{\psi} = \{p_1, p_2, p_6\}$, $\underline{G(e)}_{\psi} = \{p_1, p_3, p_5\}$, $\overline{G(e)}_{\psi} = \{p_1, p_2, p_3, p_5, p_6\}$. Now, $R_{G(d)} = \frac{|\underline{G(d)}_{\psi}|}{|\overline{G(d)}_{\psi}|} = \frac{2}{4} = \frac{1}{2}$, $R_{G(t)} = \frac{|\underline{G(t)}_{\psi}|}{|\overline{G(t)}_{\psi}|} = \frac{3}{5}$, $R_{G(e)} = \frac{|\underline{G(e)}_{\psi}|}{|\overline{G(e)}_{\psi}|} = \frac{1}{3}$. Therefore, the fuzzy rough soft set of (G, B) is given by $\{(G(d), \frac{1}{2}), (G(t), \frac{1}{3}), (G(e), \frac{3}{5})\}$. Then equivalence classes by the relation $\} \asymp'$ are $[(G(d), \frac{1}{2})]_{\asymp} = \{(G(d), \frac{1}{2})\}$, $[(G(t), \frac{1}{3})]_{\asymp} = \{(G(t), \frac{1}{3})\}$, $[(G(e), \frac{3}{5})]_{\asymp} = \{(G(e), \frac{3}{5})\}$.

Clearly, fuzzy rough soft set of (G, B) forms a chain by the order relation '$\asymp$'.

The Hasse diagram for this chain is depicted in Fig. 1.

4 Conclusion

Soft set theory, rough set theory and fuzzy set theory are three remarkable theories and all are dealing with uncertainty for variety of problems. In this paper, an attempt has been made to combine these theories and as a result fuzzy rough soft set is introduced in the modified soft rough approximation space. Some basic properties of fuzzy rough soft set are investigated. By defining fuzzy rough soft set in a MSR approximation space, flavour of theories of soft sets and rough sets and fuzzy sets are retained altogether. Also lattice theory is studied in the proposed fuzzy rough soft set. In addition to the above, we have concluded that the concept of the paper has opened a new platform for algebraic study.

There are many avenues for further study in this paper. One main avenue is that one can extend the theme of the paper for different types of lattice such as distributive lattice, modular lattice, etc. Another avenue of the paper is that researcher may define fuzzy rough soft relation, congruence relation and lattice ideal under fuzzy rough soft set.

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