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Novel hybrid decision-making methods based on *m*F rough information

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Abstract
The soft and rough set theories have a great importance in literature to deal vagueness and uncertainty, but both the theories are unable to deal the vagueness and uncertainty with multipolar information in multi-criteria decision-making. Hybrid models always provide more precise and accurate results in multi-criteria decision making. Thus in this research article, we present a novel frame work for handling multi-criteria decision-making by combining the theory of *m*F sets with rough sets and soft sets to introduce the novel models called *m*F rough set model and soft *m*F rough set model, which approximate the data under multipolar information. Further, we explain the fundamental operations of *m*F rough sets including union, intersection, composition, and investigate some of their properties. We also study some operations of soft *m*F rough sets. Moreover, we explore potential applications of *m*F rough sets and soft *m*F rough sets in multi-criteria decision-making. We also develop algorithms of our proposed hybrid models to solve multi-criteria decision-making problems.

Keywords *m*F Rough sets · *m*F Approximation space · Pseudo *m*F soft sets · Soft *m*F rough sets · Multi-criteria · Decision-making

1 Introduction

Data related to most of our practical life problems including medical science, engineering, economics, and environmental science are imprecise and its corresponding solutions contain the use of mathematical conventions based on imprecision and uncertainty. We cannot use traditional mathematical tools to overcome uncertainties existing in these problems, thus to handle such uncertainties, a number of theories have been introduced including, fuzzy set theory (Zadeh 1965) probability and rough set theory (Pawlak 1982). For applications of fuzzy set theory, the readers are referred to (Chen 1998; Chen and Chang 2011; Chen et al. 2001, 2012; Chen and Tanuwijaya 2011; Wang and Chen 2008). All of these theories have their inherit difficulties identified by Molodtsov (1999). Molodtsov introduced a new idea of soft sets as a new mathematical tool to deal these difficulties. Soft set theory has significant use in game theory, smoothness of function, operational research and probability theory (Molodtsov 1999, 2004). Maji et al. (2003) presented some basic algebraic operations on soft sets and provided an analytical approach to theory of soft sets. Ali et al. (2009) suggested some different operations for soft sets and developed the idea of complement of soft sets. They showed that the certain De Morgan's laws hold in soft sets. Park et al. (2012) considered properties of equivalence soft set relation. Maji et al. (2002) discussed the use of soft sets in decision-making problems. It is observed that fuzzy sets, soft sets, and rough sets are conveniently related notions. Maji et al. (2001) combined soft sets with other mathematical structures and introduced a new hybrid model called fuzzy soft sets, which is the fuzzy generalization of soft sets. Further they investigated many useful results related to it. Majumdar and Samanta (2010) revised the definition of fuzzy soft sets and proposed the concept of generalized fuzzy soft sets based on (Maji et al. 2003). By combining the interval-valued fuzzy sets with soft sets, Yang et al. (2009) introduced a new hybrid model called interval-valued fuzzy soft sets. Garg and Arora have a great contribution in literature of fuzzy soft sets and discussed the novel decision-making methods for solving problems (Arora and Garg 2017, 2018;

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Garg and Arora 2018a, b). For other notations, terminologies and applications, the readers are referred to (Akram et al. 2018, 2019; Akram and Ali 2018; Kreinovich 2016; Liu et al. 2016; Wang et al. 2017).

As a theory of data analysis and processing, the rough set theory proposed by Pawlak (1982, 1991, 1996, 2002) is a new mathematical tool to deal with incomplete, imprecise and uncertain information. In this theory, two precise boundary lines are established to describe the imprecise concepts. Therefore, the rough set theory is a certain mathematical tool to solve uncertain problems. The theory has promoted many applications in certain fields, such as medicine, engineering, and decision support. Pawlak (1996) worked on rough relations and rough functions. Zhang et al. (2015) considered the union and intersection operations on rough sets. Many new concepts were established by relating Pawlak rough sets with other uncertainty theories such as fuzzy set theory and soft set theory (Feng 2009, 2011; Feng et al. 2011, 2014). Dubois and Prade (1990) combined fuzzy set theory and rough set theory to propose the idea of rough fuzzy sets and fuzzy rough sets. Further, rough set approximations have also been combined with intuitionistic fuzzy sets to generate a new concept of intuitionistic fuzzy rough sets and rough intuitionistic fuzzy sets (Cornelis et al. 2003; Zhang et al. 2012; Zhou and Wu 2011). Yang et al. (2012a) studied transformation of bipolar fuzzy rough set models. Feng et al. (2010) related fuzzy sets, rough sets and soft sets all together, which are considered a strong base for certain interesting new models such as soft rough sets, rough soft sets and soft rough fuzzy sets. For the sake of new approach to decision-making problems, Zhang et al. (2014) proposed the notions of soft rough intuitionistic fuzzy sets and intuitionistic fuzzy soft rough sets. Moreover, Sun and Ma (2014) proposed the concept of soft fuzzy rough sets and discussed the application in decision making. For other transformations of rough sets models and applications readers are referred to (Pei and Xu 2007; Wu et al. 2003; Yang et al. 2013; Yao 1998; Zhan et al. 2017; Zhang and Shu 2015; Zhou and Wu 2008).

Chen et al. (2014) introduced the notion of *m*F sets as a generalization of bipolar fuzzy sets (Zhang 1994) and presented that bipolar fuzzy sets and 2-polar fuzzy sets are cryptomorphic mathematical notions. An *m*F set on a set X is a mapping : $X \to [0, 1]^m$. The concept is, multipolar information occurs because data of actual world problems are sometimes from multiple characters and agents. The membership values in *m*F sets are more understanding in obtaining uncertainty of data. *m*F sets concede higher graphical representation of vague data, which promote significantly better investigation in similarity measures, incompleteness and data relationships. Akram et al. introduced several notions based on *m*F sets and graphs including (Akram and Adeel 2016, 2017; Akram and Waseem 2016; Akram et al. 2018, 2017).

The *m*F set theory expresses the vagueness in terms of m possible membership functions of a crisp set and the rough set theory expresses the vagueness in terms of a boundary region of a crisp set. All these theories are unable to approximate the data under *m*F knowledge. Hybrid models always provide more accurate and consistent results when we have to deal with the systems with more than one agreements. Thus in this research article, we present a novel framework by putting together the theory of *m*F sets with rough sets and soft sets to introduce the novel models called *m*F rough sets and soft *m*F rough sets. These models are used to handle the multi-criteria decision-making problems, in which the data come from multipolar information. As compared to previously existing models, an *mF* rough set model is more flexible and practical for real-world problems when we have to approximate the data comes from multipolar information and soft *mF* rough set model shows its flexibility, when we have to evaluate such type of data on basis of its parameters. Proposed models are considered as the generalization of previously existing models because previous models have such a deficiency to handle the uncertain data with multipolar information. To extend the range of number of parameters with multipolar information in many practical life applications, we propose the novel approaches to multi-criteria decision-making based on *m*F rough set model and soft *m*F rough set model. We demonstrate the computational process of the proposed approaches by some practical examples. The complexity to approximate the date under multipolar information is overcome with the proposed approaches.

The organization of this research article is as follows.

In Sect. 2, we introduce our first hybrid model (*m*F rough sets) and investigate some of its basic properties. In Sect. 3, we illustrate this novel concept with real-life examples. Moreover, in Sect. 4, we propose our second hybrid model (soft *m*F rough sets) with its basic operations. In Sect. 5, we describe potential applications of soft *m*F rough sets. We also present our proposed methods as algorithms. In Sect. 6, we study sensitivity and comparison analysis. In Sect. 7, we present the conclusion and future directions.

2 *m*F rough sets and their operations

In this section, we introduce our first novel hybrid model called *m*F rough sets and discuss its properties. The proposed model emerges from the hybridization of *m*F set theory with rough sets. The fundamental and essential concept behind proposed model is the approximation of lower and upper spaces of a set with multipolar information under *m*F relation.

Definition 2.1 (Chen et al. 2014) An *mF set* on a universe S is a function $R = (p_1 \circ R(r), p_2 \circ R(r), \dots, p_m \circ R(r)) : S \to [0,1]^m$, where the $i-th$ projection mapping is defined as $p_i \circ R : [0,1]^m \to [0,1]$. $\mathbf{0} = (0, 0, \dots, 0)$ is the smallest element in $[0,1]^m$ and $\mathbf{1} = (1, 1, \dots, 1)$ is the largest element in $[0,1]^m$.

We first define mF relation, when two crisp universes are given.

Definition 2.2 Let S and T be two nonempty universes, an mF set $\xi \in mF(S \times T)$ of the universe $S \times T$ be called an *mF relation* from S to T. In general, for any $s \in S, t \in T$, the degree of the membership $\xi(s,t) = (p_1 \circ \xi(s,t), p_2 \circ \xi(s,t), \dots, p_m \circ \xi(s,t))$ denotes the degree of the relations of s and t. If $S = T$, then the mF relation $\xi \in mF(S \times T)$ is called an mF relation on S.

Example 2.3 If $S = \{s_1, s_2, s_3\}$ and $T = \{t_1, t_2, t_3\}$ are two universes then a 3−polar fuzzy relation $\xi : S \to T$ of the universe $S \times T$ is given in Table 1.

Definition 2.4 Let S and T be two finite universes of discourses and ξ an mF relation from S to T, the triple (S, T, ξ) be called m*F approximation space*. For any set $X \in mF(T)$, the lower and upper approximations $\underline{\xi}(X)$ and $\overline{\xi}(X)$ w.r.t. approximation space (S, T, ξ) are mF sets of S, whose membership functions for each $s \in S$ are defined as

$$\underline{\xi}(X)(s) = \bigwedge_{t \in T} \Big((\mathbf{1} - \xi(s,t)) \vee X(t) \Big),$$
$$\overline{\xi}(X)(s) = \bigvee_{t \in T} \Big(\xi(s,t) \wedge X(t) \Big).$$

The pair $(\underline{\xi}(X), \overline{\xi}(X))$ is called an mF*rough set* of X w.r.t. (S, T, ξ) and $\underline{\xi}, \overline{\xi} : mF(T) \to mF(S)$ are called lower and upper mF rough approximation operators, respectively. Furthermore, if $\underline{\xi}(X) = \overline{\xi}(X)$, then X is said to be definable.

Remark 1 If $S = T$, then the pair $(\underline{\xi}(X), \overline{\xi}(X))$ is called an mF rough set of X w.r.t. (S, ξ) and $\underline{\xi}, \overline{\xi} : mF(S) \to mF(S)$ are called lower and upper mF rough approximation operators, respectively.

Table 1 3-Polar fuzzy relation

ξ	t_1	t_2	t_3
s_1	(0.6,0.3,0.1)	(0.4,0.7,0.6)	(0.4,0.6,0.2)
s_2	(0.5,0.3,0.2)	(0.5,0.2,0.8)	(0.6,0.9,0.6)
s_3	(0.3,0.2,0.1)	(0.3,0.4,0.8)	(0.7,0.3,0.5)

Example 2.5 Let $S = \{s_1, s_2, s_3, s_4\}$ and $T = \{t_1, t_2, t_3\}$ be two universes of discourses and $X = \Big\{ \Big(\frac{0.5,0.6,0.7,0.2}{t_1} \Big), \Big(\frac{0.3,0.2,0.1,0.8}{t_2} \Big), \Big(\frac{0.5,0.4,0.6,0.1}{t_3} \Big) \Big\}$ be a 4−polar fuzzy set, for these two universes a 4−polar fuzzy relation $\xi : S \to T$ is given in Table 2.

By Definition 2.4, we have

$$\underline{\xi}(X)(s_1) = (0.3, 0.6, 0.4, 0.3), \quad \overline{\xi}(X)(s_1) = (0.3, 0.4, 0.2, 0.5),$$
$$\underline{\xi}(X)(s_2) = (0.8, 0.4, 0.2, 0.4), \quad \overline{\xi}(X)(s_2) = (0.2, 0.6, 0.1, 0.8),$$
$$\underline{\xi}(X)(s_3) = (0.5, 0.4, 0.4, 0.3), \quad \overline{\xi}(X)(s_3) = (0.5, 0.6, 0.7, 0.8),$$
$$\underline{\xi}(X)(s_4) = (0.3, 0.5, 0.6, 0.1), \quad \overline{\xi}(X)(s_4) = (0.5, 0.6, 0.7, 0.2).$$

Thus, $\underline{\xi}(X) = \Big\{ \Big(\frac{0.3,0.6,0.4,0.3}{s_1} \Big), \Big(\frac{0.8,0.4,0.2,0.4}{s_2} \Big), \Big(\frac{0.5,0.4,0.4,0.3}{s_3} \Big), \Big(\frac{0.3,0.5,0.6,0.1}{s_4} \Big) \Big\}$,

$\overline{\xi}(X) = \Big\{ \Big(\frac{0.3,0.4,0.2,0.5}{s_1} \Big), \Big(\frac{0.2,0.6,0.1,0.8}{s_2} \Big), \Big(\frac{0.5,0.6,0.7,0.8}{s_3} \Big), \Big(\frac{0.5,0.6,0.7,0.2}{s_4} \Big) \Big\}$.

Hence, the pair $(\underline{\xi}(X), \overline{\xi}(X))$ is referred as a 4−polar fuzzy rough set.

Theorem 2.6 *Let* (S, T, ξ)*be an mF approximation space, the lower and upper approximations* $\underline{\xi}(X)$*and* $\overline{\xi}(X)$*satisfy the following properties for any* $X, Y \in mF(T)$,

1. $\underline{\xi}(X) = \sim \overline{\xi}(\sim X)$,
2. $X \subseteq Y \Rightarrow \underline{\xi}(X) \subseteq \underline{\xi}(Y)$,
3. $\underline{\xi}(X \cup Y) \supseteq \underline{\xi}(X) \cup \underline{\xi}(Y)$,
4. $\underline{\xi}(X \cap Y) = \underline{\xi}(X) \cap \underline{\xi}(Y)$,
5. $\overline{\xi}(X) = \sim \underline{\xi}(\sim X)$,
6. $X \subseteq Y \Rightarrow \overline{\xi}(X) \subseteq \overline{\xi}(Y)$,
7. $\overline{\xi}(X \cup Y) = \overline{\xi}(X) \cup \overline{\xi}(Y)$,
8. $\overline{\xi}(X \cap Y) \subseteq \overline{\xi}(X) \cap \overline{\xi}(Y)$.

Proof The properties of the lower approximation operator $\underline{\xi}$, for any $X, Y \in mF(T)$ are proved as follows:

Table 2 4-Polar fuzzy relation

ξ	t_1	t_2	t_3
s_1	(0.3,0.4,0.2,0.7)	(0.8,0.2,0.6,0.5)	(0.1,0.3,0.2,0.5)
s_2	(0.2,0.9,0.1,0.5)	(0.1,0.3,0.8,0.9)	(0.2,0.9,0.1,0.6)
s_3	(0.3,0.6,0.8,0.2)	(0.2,0.4,0.6,0.8)	(0.8,0.9,0.5,0.7)
s_4	(0.5,0.7,0.7,0.2)	(0.8,0.9,0.5,0.7)	(0,0.5,1,0.9)

1. For all $s \in S$,

$$\begin{aligned}\underline{\xi}(X) &= \bigwedge_{t\in T}\left[\left(\mathbf{1}-\xi(s,t)\right)\vee X(t)\right]\\ &= \bigwedge_{t\in T}\left[\left(\mathbf{1}-\xi(s,t)\right)\vee \sim X(t)\right]\\ &=\sim \bigwedge_{t\in T}\left[\left(\mathbf{1}-\xi(s,t)\right)\vee \sim X(t)\right]\\ &= \bigvee_{t\in T}\left[\xi(s,t)\wedge X(t)\right].\end{aligned}$$

Thus, $\underline{\xi}(X) =\sim \overline{\xi}(\sim X)$.
2. It can be proved directly using Definition 2.4.
3. For all $s \in S$,

$$\begin{aligned}\underline{\xi}(X\cup Y) &= \bigwedge_{t\in T}\left[\left(\mathbf{1}-\xi(s,t)\right)\vee \left(X\cup Y\right)(t)\right]\\ &\supseteq \bigwedge_{t\in T}\left[\left(\mathbf{1}-\xi(s,t)\right)\vee \left(X(t)\vee Y(t)\right)\right]\\ &= \bigwedge_{t\in T}\left[\left((\mathbf{1}-\xi(s,t))\vee X(t)\right)\vee \left((\mathbf{1}-\xi(s,t))\vee Y(t)\right)\right]\\ &= \left[\bigwedge_{t\in T}\left((\mathbf{1}-\xi(s,t))\vee X(t)\right)\right]\vee\left[\bigwedge_{t\in T}\left((\mathbf{1}-\xi(s,t))\vee Y(t)\right)\right]\\ &= \underline{\xi}(X)\cup\underline{\xi}(Y).\end{aligned}$$

Thus, $\underline{\xi}(X\cup Y) \supseteq \underline{\xi}(X)\cup\underline{\xi}(Y)$.
4. For all $s \in S$,

$$\begin{aligned}\underline{\xi}(X\cap Y) &= \bigwedge_{t\in T}\left[\left(\mathbf{1}-\xi(s,t)\right)\vee \left(X\cap Y\right)(t)\right]\\ &= \bigwedge_{t\in T}\left[\left(\mathbf{1}-\xi(s,t)\right)\vee \left(X(t)\wedge Y(t)\right)\right]\\ &= \bigwedge_{t\in T}\left[\left((\mathbf{1}-\xi(s,t))\vee X(t)\right)\wedge \left((\mathbf{1}-\xi(s,t))\vee Y(t)\right)\right]\\ &= \left[\bigwedge_{t\in T}\left((\mathbf{1}-\xi(s,t))\vee X(t)\right)\right]\wedge\left[\bigwedge_{t\in T}\left((\mathbf{1}-\xi(s,t))\vee Y(t)\right)\right]\\ &= \underline{\xi}(X)\cap\underline{\xi}(Y).\end{aligned}$$

Thus, $\underline{\xi}(X\cap Y) = \underline{\xi}(X)\cap\underline{\xi}(Y)$.

Similarly, the properties $(5-8)$ for the upper approximation operator $\overline{\xi}$, for any $X, Y \in mF(T)$, can also be proved using above arguments. □

Definition 2.7 Let (S, T, ξ_1) and (S, T, ξ_2) be two mF approximation spaces.

- The mF approximation space $(S, T, \xi_1 \cup \xi_2)$ is called the union of (S, T, ξ_1) and (S, T, ξ_2).
- The mF approximation space $(S, T, \xi_1 \cap \xi_2)$ is called the intersection of (S, T, ξ_1) and (S, T, ξ_2).

Theorem 2.8 *Let $(S, T, \xi_1)$and (S, T, ξ_2)be two mF approximation spaces and $\xi = \xi_1 \cup \xi_2$,for any $s \in S$and $X \in mF(T)$,*

1. $\underline{\xi} = \xi_1 \cup \xi_2$,
2. $\overline{\xi}(X) = \overline{\xi}_1(X) \cup \overline{\xi}_2(X)$,
3. $\underline{\xi}(X) = \underline{\xi}_1(X) \cap \underline{\xi}_2(X)$.

Proof

1. For all $s \in S$ and $t \in T$,

$$\begin{aligned}\xi(s) = \xi(s,t) &= (\xi_1\cup\xi_2)(s,t)\\ &= \left((\xi_1)(s,t)\vee(\xi_2)(s,t)\right)\\ &= \left((\xi_1)(s)\vee(\xi_2)(s)\right)\\ &= \left(\xi_1\cup\xi_2\right)(s).\end{aligned}$$

Thus, $\xi = \xi_1 \cup \xi_2$.
2. For all $s \in S$,

$$\begin{aligned}(\overline{\xi}(X))(s) &= \bigvee_{t\in T}\left(\xi(s,t)\wedge X(t)\right)\\ &= \bigvee_{t\in T}\left[\left((\xi_1)(s,t)\vee(\xi_2)(s,t)\right)\wedge X(t)\right]\\ &= \bigvee_{t\in T}\left[\left((\xi_1)(s,t)\wedge X(t)\right)\vee\left((\xi_2)(s,t)\wedge X(t)\right)\right]\\ &= \left[\bigvee_{t\in T}\left((\xi_1)(s,t)\wedge X(t)\right)\right]\vee\left[\bigvee_{t\in T}\left((\xi_2)(s,t)\wedge X(t)\right)\right]\\ &= \left((\overline{\xi}_1(X))(s)\vee(\overline{\xi}_2(X))(s)\right)\\ &= \left(\overline{\xi}_1(X)\cup\overline{\xi}_2(X)\right)(s).\end{aligned}$$

Thus, $\overline{\xi}(X) = \overline{\xi}_1(X) \cup \overline{\xi}_2(X)$.
3. Using the property $\underline{\xi}(X) =\sim \overline{\xi}(\sim X)$, we have

$$\begin{aligned}\underline{\xi}(X) &=\sim \overline{\xi}(\sim X)\\ &=\sim\left(\overline{\xi_1}(\sim X)\cup\overline{\xi_2}(\sim X)\right)\\ &= \left(\sim\overline{\xi_1}(\sim X)\right)\cap\left(\sim\overline{\xi_2}(\sim X)\right)\\ &= \underline{\xi_1}(X)\cap\underline{\xi_2}(X).\end{aligned}$$

Thus, $\underline{\xi}(X) = \underline{\xi_1}(X) \cap \underline{\xi_2}(X)$.

□

Corollary 2.9 *Let (S, T, ξ_1) and (S, T, ξ_2)be two mF approximation spaces. If $\xi_1 \subseteq \xi_2$, then for any $X \in mF(T)$we have*

$$\overline{\xi}_1(X) \subseteq \overline{\xi}_2(X), \quad \underline{\xi}_1(X) \supseteq \underline{\xi}_2(X).$$

Proof It can be proved directly using Definition 2.4. □

For n different mF relations Theorem 2.8 can be generalized as,

Theorem 2.10 *Let* (S, T, ξ_j) be *mF approximation spaces and* $\xi = \cup_{j=1}^{n} \xi_j$. *Then for any* $s \in S$*and* $X \in mF(T)$,

1. $\underline{\xi} = \cup_{j=1}^{n} \xi_j$,
2. $\overline{\xi}(X) = \cup_{j=1}^{n} \overline{\xi}_j(X)$,
3. $\underline{\xi}(X) = \cap_{j=1}^{n} \underline{\xi}_j(X)$.

Proof It is easy to prove using similar arguments, as used in Theorem 2.8. □

Theorem 2.11 *Let* (S, T, ξ_1) *and* (S, T, ξ_2) *be two mF approximation spaces and* $\xi = \xi_1 \cap \xi_2$,*for any* $s \in S$*and* $X \in mF(T)$,

1. $\underline{\xi} = \xi_1 \cap \xi_2$,
2. $\overline{\xi}(X) \subseteq \overline{\xi}_1(X) \cap \overline{\xi}_2(X)$,
3. $\underline{\xi}(X) \supseteq \underline{\xi}_1(X) \cup \underline{\xi}_2(X)$.

Proof

1. For all $s \in S$ and $t \in T$,

$$\begin{aligned}\underline{\xi}(s) = \xi(s,t) &= (\xi_1 \cap \xi_2)(s,t)\\ &= \Big((\xi_1)(s,t) \wedge (\xi_2)(s,t)\Big)\\ &= \Big((\xi_1)(s) \wedge (\xi_2)(s)\Big)\\ &= \Big(\xi_1 \cap \xi_2\Big)(s).\end{aligned}$$

Thus, $\xi = \xi_1 \cap \xi_2$.

2. For all $s \in S$,

$$\begin{aligned}(\overline{\xi}(X))(s) &= \bigvee_{t \in T}\Big(\xi(s,t) \wedge X(t)\Big)\\ &= \bigvee_{t \in T}\Big[\Big((\xi_1)(s,t) \wedge (\xi_2)(s,t)\Big) \wedge X(t)\Big]\\ &= \bigvee_{t \in T}\Big[\Big((\xi_1)(s,t) \wedge X(t)\Big) \wedge \Big((\xi_2)(s,t) \wedge X(t)\Big)\Big]\\ &\leq \Big[\bigvee_{t \in T}\Big((\xi_1)(s,t) \wedge X(t)\Big)\Big] \wedge \Big[\bigvee_{t \in T}\Big((\xi_2)(s,t) \wedge X(t)\Big)\Big]\\ &\leq \Big((\overline{\xi}_1(X))(s) \wedge (\overline{\xi}_2(X))(s)\Big)\\ &= \Big(\overline{\xi}_1(X) \cap \overline{\xi}_2(X)\Big)(s).\end{aligned}$$

Thus, $\overline{\xi}(X) \subseteq \overline{\xi}_1(X) \cap \overline{\xi}_2(X)$.

3. Using the property $\underline{\xi}(X) = \sim \overline{\xi}(\sim X)$, we have

$$\begin{aligned}\underline{\xi}(X) &= \sim \overline{\xi}(\sim X)\\ &\supseteq \sim \Big(\overline{\xi}_1(\sim X) \cap \overline{\xi}_2(\sim X)\Big)\\ &= \Big(\sim \overline{\xi}_1(\sim X)\Big) \cup \Big(\sim \overline{\xi}_2(\sim X)\Big)\\ &= \underline{\xi}_1(X) \cup \underline{\xi}_2(X).\end{aligned}$$

Thus, $\underline{\xi}(X) \supseteq \underline{\xi}_1(X) \cup \underline{\xi}_2(X)$.

□

Example 2.12 Consider (S, T, ξ) is an mF approximation space, where $S = \{s_1, s_2, s_3\}$ and $T = \{t_1, t_2, t_3\}$ are two universes of discourses and ξ_1, ξ_2 are two 3-polar fuzzy relations given in Tables 3 and 4.

Intersection $\xi = \xi_1 \cap \xi_2$ of two 3-polar fuzzy relations ξ_1 and ξ_2 is given in Table 5.

If $X = \left\{\left(\frac{0.2,0.5,0.4}{t_1}\right), \left(\frac{0.3,0.4,0.1}{t_2}\right), \left(\frac{0.7,0.5,0.2}{t_3}\right)\right\}$ is a 3−polar fuzzy set, then using Definition 2.4, we have

$$\begin{aligned}\underline{\xi}_1(X) &= \left\{\left(\frac{0.5, 0.5, 0.4}{s_1}\right), \left(\frac{0.4, 0.5, 0.2}{s_2}\right), \left(\frac{0.4, 0.4, 0.5}{s_3}\right)\right\},\\ \underline{\xi}_2(X) &= \left\{\left(\frac{0.2, 0.4, 0.4}{s_1}\right), \left(\frac{0.3, 0.5, 0.2}{s_2}\right), \left(\frac{0.8, 0.5, 0.3}{s_3}\right)\right\},\\ \underline{\xi}(X) &= \left\{\left(\frac{0.7, 0.7, 0.5}{s_1}\right), \left(\frac{0.7, 0.6, 0.7}{s_2}\right), \left(\frac{0.8, 0.5, 0.8}{s_3}\right)\right\},\\ \overline{\xi}_1(X) &= \left\{\left(\frac{0.5, 0.5, 0.4}{s_1}\right), \left(\frac{0.7, 0.4, 0.2}{s_2}\right), \left(\frac{0.5, 0.5, 0.2}{s_3}\right)\right\},\\ \overline{\xi}_2(X) &= \left\{\left(\frac{0.3, 0.5, 0.4}{s_1}\right), \left(\frac{0.7, 0.5, 0.2}{s_2}\right), \left(\frac{0.2, 0.5, 0.4}{s_3}\right)\right\},\\ \overline{\xi}(X) &= \left\{\left(\frac{0.3, 0.3, 0.4}{s_1}\right), \left(\frac{0.7, 0.4, 0.2}{s_2}\right), \left(\frac{0.2, 0.5, 0.2}{s_3}\right)\right\}.\end{aligned}$$

It is easy to see that

$$\underline{\xi}(X) \neq \underline{\xi}_1(X) \cup \underline{\xi}_2(X), \quad \overline{\xi}(X) \neq \overline{\xi}_1(X) \cap \overline{\xi}_2(X).$$

For n different mF relations Theorem 2.11 can be generalized as,

Theorem 2.13 *Let* (S, T, ξ_j) *be mF approximation spaces and* $\xi = \cap_{j=1}^{n} \xi_j$,*for any* $s \in S$*and* $X \in mF(T)$,

1. $\underline{\xi} = \cap_{j=1}^{n} \xi_j$,
2. $\overline{\xi}(X) \subseteq \cap_{j=1}^{n} \overline{\xi}_j(X)$,
3. $\underline{\xi}(X) \supseteq \cup_{j=1}^{n} \underline{\xi}_j(X)$.

Table 3 3-Polar fuzzy relation ξ_1

ξ_1	t_1	t_2	t_3
s_1	(0.1,0.5,0.6)	(0.5,0.2,0.6)	(0.5,0.3,0.1)
s_2	(0.3,0.4,0.2)	(0.6,0.5,0.3)	(0.7,0.2,0.8)
s_3	(0.5,0.8,0.1)	(0.6,0.7,0.2)	(0.5,0.4,0.2)

Table 4 3-Polar fuzzy relation ξ_2

ξ_2	t_1	t_2	t_3
s_1	(0.8,0.2,0.5)	(0.1,0.6,0.2)	(0.3,0.5,0.6)
s_2	(0.7,0.6,0.1)	(0.4,0.3,0.8)	(0.9,0.7,0.2)
s_3	(0.2,0.9,0.5)	(0.1,0.2,0.7)	(0.1,0.8,0.2)

Table 5 Intersection $\xi = \xi_1 \cap \xi_2$ of two 3-polar fuzzy relations

$\xi = \xi_1 \cap \xi_2$	t_1	t_2	t_3
s_1	(0.1,0.2,0.5)	(0.1,0.2,0.2)	(0.3,0.3,0.1)
s_2	(0.3,0.4,0.1)	(0.4,0.3,0.3)	(0.7,0.2,0.2)
s_3	(0.2,0.8,0.1)	(0.1,0.2,0.2)	(0.1,0.4,0.2)

Proof It is easy to prove using similar arguments as used in Theorem 2.11. □

Definition 2.14 Let (S, ξ_1) and (S, ξ_2) be two mF approximation spaces. The approximation space $(S, \xi_1 \circ \xi_2)$ is called the composition of (S, ξ_1) and (S, ξ_2).

Theorem 2.15 *Let (S, ξ_1) and (S, ξ_2) be two mF approximation spaces and $\xi = \xi_1 \circ \xi_2$, for any $X \in mF(S)$,*

1. $\overline{\xi}(X) = [\overline{\xi}_1 \circ \overline{\xi}_2](X) = \overline{\xi}_1(\overline{\xi}_2(X))$,
2. $\underline{\xi}(X) = [\underline{\xi}_1 \circ \underline{\xi}_2](X) = \underline{\xi}_1(\underline{\xi}_2(X))$.

Proof

1. For all $s \in S$,

$$\begin{aligned}
(\overline{\xi}_1(\overline{\xi}_2(X)))(s) &= \bigvee_{t \in S} \left[(\xi_1)(s,t) \wedge \overline{\xi}_2(X)(t) \right] \\
&= \bigvee_{t \in S} \left[(\xi_1)(s,t) \wedge \left(\bigvee_{r \in S} (\xi_2)(t,r) \wedge X(r) \right) \right] \\
&= \bigvee_{t \in S} \bigvee_{r \in S} \left[(\xi_1)(s,t) \wedge \left((\xi_2)(t,r) \wedge X(r) \right) \right] \\
&= \bigvee_{r \in S} \left[\bigvee_{t \in S} \left((\xi_1)(s,t) \wedge (\xi_2)(t,r) \right) \wedge X(r) \right] \\
&= \bigvee_{r \in S} \left[(\xi)(s,r) \wedge X(r) \right] = (\overline{\xi}(X))(s).
\end{aligned}$$

Thus, $\overline{\xi}(X) = [\overline{\xi}_1 \circ \overline{\xi}_2](X) = \overline{\xi}_1(\overline{\xi}_2(X))$.

2. For all $s \in S$,

$$\begin{aligned}
(\underline{\xi}_1(\underline{\xi}_2(X)))(s) &= \bigwedge_{t \in S} \left[(\mathbf{1} - \xi_1(s,t)) \vee \underline{\xi}_2(X)(t) \right] \\
&= \bigwedge_{t \in S} \left[(\mathbf{1} - \xi_1(s,t)) \vee \left(\bigwedge_{r \in S} (\mathbf{1} - \xi_2(t,r)) \vee X(r) \right) \right] \\
&= \bigwedge_{t \in S} \bigwedge_{r \in S} \left[(\mathbf{1} - \xi_1(s,t)) \vee \left((\mathbf{1} - \xi_2(t,r)) \vee X(r) \right) \right] \\
&= \bigwedge_{r \in S} \left[\bigwedge_{t \in S} \left((\mathbf{1} - \xi_1(s,t)) \vee (\mathbf{1} - \xi_2(t,r)) \right) \vee X(r) \right] \\
&= \bigwedge_{r \in S} \left[(\mathbf{1} - \xi(s,r)) \vee X(r) \right] = (\underline{\xi}(X))(s).
\end{aligned}$$

Thus, $\underline{\xi}(X) = [\underline{\xi}_1 \circ \underline{\xi}_2](X) = \underline{\xi}_1(\underline{\xi}_2(X))$.

□

Theorem 2.16 *Let (Q, S, ξ_1) and (S, T, ξ_2) be two mF approximation spaces and $\xi = \xi_1 \circ \xi_2$, for any $X \in mF(T)$,*

1. $\overline{\xi}(X) = [\overline{\xi}_1 \circ \overline{\xi}_2](X) = \overline{\xi}_1(\overline{\xi}_2(X))$,
2. $\underline{\xi}(X) = [\underline{\xi}_1 \circ \underline{\xi}_2](X) = \underline{\xi}_1(\underline{\xi}_2(X))$.

Proof It can be proved easily using similar arguments as used in Theorem 2.15. □

3 Applications of *m*F rough sets

Multi-criteria decision-making includes the situations where the information about the alternatives comes from different sources. In this section, we present the decision-making approach of our proposed model called mF rough sets in an Algorithm 1 and discuss its applications in multi-criteria decision-making.

Algorithm 1

Step 1. **Input**

S and T, universes of discourses.
$\widetilde{F}$, different features of universe S.
X, an mF set such that $X \in mF(T)$.
$\widetilde{X}$, different types of set X.

Step 2. Compute mF relation $\xi : S \to T$.
Step 3. Compute lower and upper approximations $\underline{\xi}(X)$ and $\overline{\xi}(X)$, for any set $X \in mF(T)$ w.r.t. approximation space (S, T, ξ) as

$$\underline{\xi}(S)(s) = \bigwedge_{t \in T} \Big((\mathbf{1} - \xi(s,t)) \vee X(t) \Big), \quad s \in S,$$

$$\overline{\xi}(S)(s) = \bigvee_{t \in T} \Big(\xi(s,t) \wedge X(t) \Big), \quad s \in S.$$

Step 4. Compute choice value ρ_s as

$$\rho_s = \max(p_i \circ \underline{\xi}(X), p_i \circ \overline{\xi}(X)), i \in m.$$

Step 5. Repeat this process for different varieties and features.

Step 6. **Output**

S_m, the alternative for which ρ_s is maximum.

Working of Algorithm 1 is described as follows:

In algorithm 1, step 1 is taking values as inputs. Step 2 computes the mF relation between S and T. Step 3 computes the lower and upper approximations $\underline{\xi}(X)$ and $\overline{\xi}(X)$, for any set $X \in mF(T)$. Step 4 computes the choice value ρ_s. Step 5 repeats the process. Step 6 shows the best alternative as output.

3.1 Selection of prints and shades for variety of fabrics

Nowadays, the selection of suitable patterns, colors and shades for fabrics is difficult task for designers. To handle this difficult situation, we present the concept of mF rough set model, which provides us information about the selection of stuffs and variety of colors combination with different patterns and shades. It also provides us information about the variety of fabrics.

Suppose a textile designing company wants to manufacture different types of fabrics with suitable patterns, colors, and shades. Company also wants to prepare such a kind of fabric with same design and different variety of materials. So, company hands over this task to a designer for such types of fabrics.

Let P and T be the two universes of discourses, with P $= \{pattern, color, shade\}$ the set of prints and shades of fabrics, T $= \{cotton, wool, silk, linen\}$ the set of type of material, used in manufacturing of fabrics and $\xi : P \to T$ be a 4−polar fuzzy relation. The universe P is further classified as

- The "Patterns" of fabrics include tartan, cross tee, polka dotted and chevron.
- The "Colors" of fabrics include pink, yellow, grey, and peach.
- The "Shades" of fabrics include light, dark, dull and bright.

A 4−polar fuzzy relation is given in Table 6 as follows:

A 4−polar fuzzy relation $\xi \in (P \times T)$ provides us information about the patterns, colors and shades of different fabrics. For example, if we consider

- "Patterns in cotton" then cotton has 60% tartan, 80% cross tee, 20% polka dotted and 10% chevron pattern.
- "Colors in cotton" are classified as 50% pink, 30% yellow, 10% grey and 50% peach.
- "Shades in cotton" are 40% light, 80% dark, 20% dull and 30% bright.

Similarly, for all other fabrics we can select patterns, colors, and shades.

For different variety of fabrics we take a set M as $M = \left\{ \left(\frac{0.7,0.8,0.1,0.5}{cotton}\right), \left(\frac{0.6,0.2,0.3,0.7}{wool}\right), \left(\frac{0.6,0.4,0.3,0.1}{silk}\right), \left(\frac{0.3,0.9,0.1,0.7}{linen}\right) \right\}$, which describes the further types of fabrics, classified as

- The "types of cotton" include drill cotton, dutch cotton, gauze cotton and flannel cotton.
- The "types of wool" include merino wool, alpaca wool, mohair wool and lama wool.
- The "types of silk" include charmeuse silk, filament silk, georgette silk and habutai silk.
- The "type of linen" include damask linen, blended linen, bird's eye linen, cambric linen.

Now to decide the fabrics of different variety with same patterns, colors and shades we apply lower and upper mF rough approximation operators on a 4−polar fuzzy set M using Definition 2.4. Further, for final decision we define choice value as

$$\rho_s = \max(p_i \circ \underline{\xi}(M), p_i \circ \overline{\xi}(M)), i \in m.$$

Table 6 4−Polar fuzzy relation

ξ	Cotton	Wool	Silk	Linen
Pattern	(0.6,0.8,0.2,0.1)	(0.4,0.2,0.9,0.5)	(0.6,0.5,0.2,0.2)	(0.9,0.7,0.3,0.6)
Color	(0.5,0.3,0.1,0.5)	(0.6,0.6,0.2,0.1)	(0.8,0.1,0.7,0.6)	(0.8,0.2,0.6,0.1)
Shade	(0.4,0.8,0.2,0.3)	(0.4,0.9,0.2,0.6)	(0.1,0.1,0.3,0.4)	(0.6,0.7,0.3,0.2)

Thus,

Thus, from choice value $\rho_p = (0.7, 0.8, 0.3, 0.7)$ calculated in Table 7, we conclude

- 70% "Tartan Pattern" is suitable for drill cotton, merino wool, charmeuse silk and damask linen.
- 80% "Cross Tee Pattern" is suitable for dutch cotton, alpaca wool, filament silk and blended linen.
- 30% "Polka Dotted Pattern" is suitable for gauze cotton, mohair wool, georgette silk and birds eye's linen.
- 70% "Chevron Pattern" is suitable for flament cotton, lama wool, habutai silk and cambric linen.

Similarly, from other choice values as calculated in Table 7, we can easily find the suitable colors and shades for different types of fabrics.

3.2 Selection of features for different models of mobiles

With the advent of new technology, the way of communication is also changed. Today is the era of wireless communication which gives rise to mobile phones. Mobiles are the latest invention and common way to communicate now-a-days. Mobile phones are now inexpensive, easy to use, comfortable and equipped with almost every latest features we desire. Feature specifications of different types of mobiles is a complicated task for a company. For this multi-criteria decision-making we use the concept of mF rough sets. Suppose a mobile company wants to launch a mobile phone with different features and specifications. Let (F, T, ξ) be an mF approximation space, where F and T be two universes of discourses and $\xi : F \to T$ be a 5−polar fuzzy relation . Let $\mathrm{F} = \{os, size, battery, processor, memory, network, displays, sensors, camera\}$ be the set of features of mobiles and $\mathrm{T} = \{classic, flip, slider, qwerty, touch\}$ be the set of types of mobiles. The universe F is further classified into five different features as

- The "Os" includes Android, Blackberry, Java, Symbion, and Window.
- The "Size" includes 3.5 inch, 4 inch, 4.5 inch, 5 inch and 5.5 inch.

Table 7 Choice value

	$\underline{\xi}(M)$	$\overline{\xi}(M)$	Choice value ρ_s
Pattern	(0.3,0.5,0.3,0.7)	(0.7,0.8,0.3,0.6)	ρ_p =(0.7,0.8,0.3,0.7)
Color	(0.3,0.4,0.3,0.4)	(0.6,0.3,0.3,0.5)	ρ_c =(0.6,0.4,0.3,0.5)
Shade	(0.4,0.2,0.7,0.6)	(0.4,0.8,0.3,0.6)	ρ_s =(0.4,0.8,0.7,0.6)

- The "Battery" includes lithium polymer, nickel cadmium, nickel metal hydride, lithium ion and new lithium technology.
- The "Processor" includes dual core, quad core, octa core, intel and any other.
- The "Memory" includes drum, floating body, MRAM, NAND and ReRAM.
- The "Network" includes wifi, G, 2G, 3G and 4G.
- The "Displays" includes LCD, amoled, OLED, IPS LCD and retina.
- The "Sensor" includes vibrations, motions, contact switch, ambient light and sound.
- The "Camera" includes ultrawide angle, wide angle, normal, telephoto and super telephoto.

A 5−polar fuzzy relation $\xi \in (F \times T)$ is given in Table 8 as follows:

For different models of mobiles, we take 5−polar fuzzy set M as

$$M = \left\{ \left(\frac{0.3, 0.4, 0.5, 0.8, 0.6}{classic} \right), \left(\frac{0.8, 0.2, 0.6, 0.5, 0.7}{flip} \right), \left(\frac{0.5, 0.9, 0.2, 0.6, 0.7}{slider} \right), \left(\frac{0.6, 0.8, 0.6, 0.2, 0.3}{qwerty} \right), \left(\frac{0.6, 0.2, 0.3, 0.5, 0.9}{touch} \right) \right\}.$$

The models of mobiles are classified as

- The "models of classic mobile" include Samsung S, Motorala raza $V3$, Nokia 3310, Motorala 8000 dyna TAC and Nokia 1110.
- The "models of flip mobile" include Samsung convoy, Blackberry style, LG 450, Samsung gusto and Nokia 6350.
- The "models of slider mobile" include Samsung $G600$, $C205$, LG cosmos slide, Motorola milestone and Nokia $N95$.
- The "models of qwerty mobile" include Blackberry classic, Blackberry bold 9790, Nokia asha 210, Blackberry $Q10$ and Nokia $C3$.
- The "models of touch mobile" include Samsung galaxy $E7$, Blackberry DTE $K50$, LG$K10$, Lenovo $A6000$ and HTC tough HD.

Now to decide the mobiles of different models with same set of features we apply lower and upper mF rough approximation operators on a 5−polar fuzzy set M, using Definition 2.4. Further, for final decision we define choice value as

$$\rho_s = \max(p_i \circ \underline{\xi}(M), p_i \circ \overline{\xi}(M)), i \in m.$$

Table 8 5−Polar fuzzy relation

ξ	Classic	Flip	Slider
OS	(0.1,0.3,0.6,0.7,0.1)	(0.3,0.1,0.5,0.8,0.2)	(0.4,0.2,0.3,0.3,0.4)
Size	(0.8,0.7,0.5,0.5,0.2)	(0.7,0.8,0.6,0.3,0.6)	(0.1,0.3,0.4,0.5,0.2)
Battery	(0.3,0.4,0.6,0.7,0.5)	(0.6,0.2,0.3,0.5,0.8)	(0.6,0.2,0.5,0.9,0.1)
Processor	(0.3,0.1,0.1,0.2,0.6)	(0.5,0.3,0.4,0.6,0.1)	(0.4,0.8,0.2,0.1,0.3)
Memory	(0.3,0.5,0.3,0.2,0.6)	(0.2,0.5,0.6,0.1,0.3)	(0.8,0.2,0.3,0.4,0.6)
Network	(0.1,0.2,0.1,0.3,0.1)	(0.3,0.4,0.5,0.2,0.1)	(0.5,0.2,0.6,0.3,0.2)
Displays	(0.5,0.6,0.5,0.2,0.1)	(0.8,0.9,0.7,0.3,0.1)	(0.8,0.5,0.2,0.9,0.7)
Sensors	(0.3,0.5,0.2,0.8,0.7)	(0.6,0.7,0.8,0.9,0.2)	(0.8,0.2,0.3,0.7,0.5)
Camera	(0.2,0.3,0.5,0.4,0.9)	(0.6,0.8,0.9,0.5,0.4)	(0.3,0.5,0.4,0.3,0.2)

ξ	Qwerty	Touch
OS	(0.5,0.7,0.5,0.3,0.2)	(0.9,0.7,0.6,0.5,0.8)
Size	(0.1,0.3,0.8,0.7,0.5)	(0.2,0.3,0.5,0.7,0.9)
Battery	(0.2,0.5,0.9,0.3,0.5)	(0.2,0.6,0.7,0.2,0.7)
Processor	(0.5,0.9,0.7,0.2,0.1)	(0.5,0.4,0.7,0.3,0.1)
Memory	(0.7,0.2,0.1,0.6,0.2)	(0.3,0.4,0.8,0.1,0.2)
Network	(0.8,0.3,0.9,0.7,0.6)	(0.5,0.6,0.8,0.3,0.1)
Displays	(0.6,0.7,0.8,0.9,0.2)	(0.3,0.5,0.8,0.9,0.7)
Sensors	(0.8,0.3,0.5,0.6,0.7)	(0.3,0.2,0.7,0.8,0.7)
Camera	(0.8,0.9,0.5,0.2,0.3)	(0.7,0.6,0.5,0.2,0.3)

Thus,

Thus, from choice value $\rho_{os} = (0.6, 0.7, 0.5, 0.7, 0.8)$ calculated in Table 9, we conclude

- 60% "OS Android" is suitable for Samsung s, Samsung convoy, Samsung $G600$, Blackberry classic and Samsung galaxy $E7$.
- 70% "OS Blackberry" is suitable for Motorola raza $V3$, Blackberry style, $C205$, Blackberry bold 9790 and Blackberry $DTEK50$.
- 50% "OS Java" is suitable for Nokia 3310, LG450, LG Cosmos slide, Nokia asha 210 and LGK10.
- 70% "OS Symbion" is suitable for Motorola 8000 dyna TAC, Samsung gusto, Motorola milestone, Blackberry $Q10$ and Lenovo $A6000$.
- 80% "OS Window" is suitable for Nokia 1110, Nokia 6350, Nokia $N95$, Nokia $C3$ and HTC tough HD.

Similarly, from other choice values as calculated in Table 9, we can easily find the other suitable features for different models of mobiles.

4 Soft *m*F rough sets

In this section, we introduce the concept of pseudo mF soft sets, which provide the information about the features of alternatives with multipolar information. Further, we propose a new model called soft mF rough sets by combing soft sets with mF rough sets, which is the generalization of previously defined models.

Definition 4.1 Let S be a universe and A be a set of parameters, where $N \subseteq A$. Define $\zeta : N \to mF(S)$, where $mF(S)$ is the collection of all mF subsets of S, then (ζ, N) is called an *mF soft set* over a universe S, which is defined by,

$$(\zeta, N) = \{(s, p_i \circ N_\epsilon(s)) \,:\, s \in S \text{ and } \epsilon \in N\}.$$

Definition 4.2 Let S be a universe and A be a set of parameters. A pair $(\tilde{\zeta}^{-1}, A)$ is called a *pseudo mF soft set* over the universe S if and only if $\tilde{\zeta}^{-1} : S \to mF(A)$ is a mapping of S into all mF subsets of the set A, where $mF(A)$ expresses all mF subsets of parameter set A., i.e., $\tilde{\zeta}^{-1}(s, a) \in [0, 1]^m$, $\forall s \in S, a \in A$.

Remark 2 From Definition of pseudo mF soft set we know that the pseudo mF mapping $\tilde{\zeta}^{-1} : S \to mF(A)$ is a binary mF relation defined between the universe S and parameter set A. i.e., for any $s_j \in S, a_k \in A, \tilde{\zeta}^{-1}(s_j, a_k) \in mF(S \times A)$.

In general, reflexive, symmetric and transitive properties do not hold in $\tilde{\zeta}^{-1}(s_j, a_k)$. Therefore, $\tilde{\zeta}^{-1}(s_j, a_k)$ is an arbitrary mF binary relation.

Table 9 Choice value

	$\underline{\xi}(M)$	$\overline{\xi}(M)$	Choice value ρ_s
OS	(0.6,0.3,0.4,0.5,0.7)	(0.6,0.7,0.5,0.7,0.8)	(0.6,0.7,0.5,0.7,0.8)
Size	(0.3,0.2,0.5,0.3,0.5)	(0.7,0.7,0.6,0.5,0.8)	(0.7,0.7,0.6,0.5,0.8)
Battery	(0.5,0.4,0.3,0.5,0.5)	(0.7,0.4,0.6,0.7,0.7)	(0.7,0.4,0.6,0.7,0.7)
Processor	(0.6,0.6,0.3,0.5,0.6)	(0.5,0.8,0.6,0.5,0.6)	(0.6,0.8,0.6,0.5,0.6)
Memory	(0.5,0.5,0.3,0.6,0.6)	(0.6,0.4,0.6,0.4,0.6)	(0.6,0.5,0.6,0.6,0.6)
Network	(0.5,0.4,0.3,0.3,0.4)	(0.6,0.3,0.6,0.3,0.3)	(0.6,0.4,0.6,0.3,0.4)
Displays	(0.5,0.2,0.3,0.2,0.7)	(0.8,0.7,0.6,0.6,0.7)	(0.8,0.7,0.6,0.6,0.7)
Sensors	(0.5,0.3,0.3,0.4,0.3)	(0.6,0.4,0.6,0.8,0.7)	(0.6,0.4,0.6,0.8,0.7)
Camera	(0.6,0.2,0.2,0.5,0.6)	(0.6,0.8,0.6,0.5,0.7)	(0.6,0.8,0.6,0.5,0.7)

Example 4.3 Let $C = \{c_1, c_2, c_3, c_4, c_5\}$ be a universe of five cars under observation, and $A = \{a_1, a_2, a_3, a_4\}$ be the set of parameters, where the parameter,

"a_1" represents the price of car,
"a_2" represents the color of car,
"a_3"represents the body types of car,
"a_4" represents the attractiveness of car.

We give further characteristics of these parameters.

- The "price of car" may be cheap, costly, very costly.
- The "color of car" may be the combination of white, grey, silver.
- The "body type of car" may be sedan, coupe, hatchback.
- The "attractiveness of car" may include flexibility, comfort, speed.

To define mF soft set, we means to specify the characteristics of price, color, body type and attractiveness of a car. The mF soft set (ζ, N) expresses the characteristics of car that Mr. Z(say) wants to buy. It is shown in Table 10 as follows:

The pseudo mF soft set specifies the features of each car including price, color, body type and attractiveness with their different characteristics. By Definition 4.2, we have the following results:

Table 10 3-Polar fuzzy soft relation

C/A	a_1	a_2	a_3	a_4
c_1	(0.2,0.3,0.7)	(0.3,0.8,0.9)	(0.8,0.3,0.7)	(0.4,0.3,0.2)
c_2	(0.8,0.3,0.5)	(0.6,0.4,0.3)	(0.2,0.1,0.3)	(0.5,0.4,0.6)
c_3	(0.6,0.7,0.8)	(0.8,0.7,0.5)	(0.5,0.3,0.2)	(0.6,0.8,0.9)
c_4	(0.9,0.3,0.5)	(0.9,0.2,0.3)	(0.8,0.9,0.1)	(0.2,0.5,0.4)
c_5	(0.7,0.5,0.4)	(0.5,0.4,0.2)	(0.7,0.8,0.4)	(0.8,0.2,0.3)

$$\tilde{\zeta}^{-1}(c_1) = \left\{ \left(\frac{0.2, 0.3, 0.7}{a_1} \right), \left(\frac{0.3, 0.8, 0.7}{a_2} \right), \left(\frac{0.8, 0.3, 0.7}{a_3} \right), \left(\frac{0.4, 0.3, 0.2}{a_4} \right) \right\},$$

$$\tilde{\zeta}^{-1}(c_2) = \left\{ \left(\frac{0.8, 0.3, 0.5}{a_1} \right), \left(\frac{0.6, 0.4, 0.3}{a_2} \right), \left(\frac{0.2, 0.1, 0.3}{a_3} \right), \left(\frac{0.5, 0.4, 0.6}{a_4} \right) \right\},$$

$$\tilde{\zeta}^{-1}(c_3) = \left\{ \left(\frac{0.6, 0.7, 0.8}{a_1} \right), \left(\frac{0.8, 0.7, 0.5}{a_2} \right), \left(\frac{0.5, 0.3, 0.2}{a_3} \right), \left(\frac{0.6, 0.8, 0.9}{a_4} \right) \right\},$$

$$\tilde{\zeta}^{-1}(c_4) = \left\{ \left(\frac{0.9, 0.3, 0.5}{a_1} \right), \left(\frac{0.9, 0.2, 0.3}{a_2} \right), \left(\frac{0.8, 0.9, 0.1}{a_3} \right), \left(\frac{0.2, 0.5, 0.4}{a_4} \right) \right\},$$

$$\tilde{\zeta}^{-1}(c_5) = \left\{ \left(\frac{0.7, 0.5, 0.4}{a_1} \right), \left(\frac{0.5, 0.4, 0.2}{a_2} \right), \left(\frac{0.7, 0.8, 0.4}{a_3} \right), \left(\frac{0.4, 0.2, 0.3}{a_4} \right) \right\}.$$

This means that the car $\}\}c_1''$ has different features as follows:

- The "price of car" shows, it is 20% cheap, 30% costly and 70% very costly for most of the customers.
- The "color of car" shows, it may have combination of colors as 30% white, 80% grey and 70% silver.
- The "body type of car" shows, its shape is 80% sedan, 30% coupe and 70% hatchback.
- The "attractiveness of car" shows, it has 40% flexibility , 30% comfort and 20% speed.

Definition 4.4 Let S be a universe and $(\tilde{\zeta}^{-1}, A)$ be a pseudo mF soft set over universe S, where $\tilde{\zeta}^{-1}$ be a mapping defined as, $\tilde{\zeta}^{-1} : S \to mF(A)$. The triple $(S, A, \tilde{\zeta}^{-1})$ is called *soft mF* *approximation space*. For any set $X \in mF(A)$, the lower and upper approximations of X, $\underline{\zeta}(X)$ and $\overline{\zeta}(X)$ w.r.t. soft mF approximation space $(S, A, \tilde{\zeta}^{-1})$ are the mF sets of S whose membership functions for each $s \in S$ are defined respectively, as follows:

$$\underline{\zeta}(X)(s) = \bigwedge_{a \in A} \left((\mathbf{1} - \tilde{\zeta}^{-1}(s,a)) \vee X(a) \right),$$

$$\overline{\zeta}(X)(s) = \bigvee_{a \in A} \left(\tilde{\zeta}^{-1}(s,a) \wedge X(a) \right).$$

The pair $(\underline{\zeta}(X), \overline{\zeta}(X))$ is called *soft mF rough set* of X w.r.t. $(S, A, \tilde{\zeta}^{-1})$ and $\underline{\zeta}, \overline{\zeta} : mF(A) \to mF(S)$ are called lower and upper soft mF rough approximation operators respectively. Furthermore, if $\underline{\zeta}(X) = \overline{\zeta}(X)$, then X is said to be definable.

Example 4.5 Re-consider the Example 4.3 and define a 3−polar fuzzy set of attributes, X as

$$X = \left\{ (\frac{0.3, 0.4, 0.5}{a_1}), (\frac{0.6, 0.5, 0.8}{a_2}), (\frac{0.9, 0.7, 0.2}{a_3}), (\frac{0.8, 0.2, 0.3}{a_4}) \right\}.$$

From Definition 4.4, lower and upper approximations of X can be calculated respectively, as follows:

$$\underline{\zeta}(X)(c_1) = (0.7, 0.5, 0.3), \qquad \overline{\zeta}(X)(c_1) = (0.8, 0.5, 0.8),$$
$$\underline{\zeta}(X)(c_2) = (0.3, 0.6, 0.4), \qquad \overline{\zeta}(X)(c_2) = (0.6, 0.4, 0.5),$$
$$\underline{\zeta}(X)(c_3) = (0.4, 0.2, 0.3), \qquad \overline{\zeta}(X)(c_3) = (0.6, 0.5, 0.5),$$
$$\underline{\zeta}(X)(c_4) = (0.3, 0.5, 0.5), \qquad \overline{\zeta}(X)(c_4) = (0.8, 0.7, 0.5),$$
$$\underline{\zeta}(X)(c_5) = (0.3, 0.5, 0.6), \qquad \overline{\zeta}(X)(c_5) = (0.8, 0.7, 0.4).$$

Thus,

$$\underline{\zeta}(X) = \left\{ \left(\frac{0.7, 0.5, 0.3}{c_1}\right), \left(\frac{0.3, 0.6, 0.4}{c_2}\right), \left(\frac{0.4, 0.2, 0.3}{c_3}\right), \left(\frac{0.3, 0.5, 0.5}{c_4}\right), \left(\frac{0.3, 0.5, 0.6}{c_5}\right) \right\},$$

$$\overline{\zeta}(X) = \left\{ \left(\frac{0.8, 0.5, 0.8}{c_1}\right), \left(\frac{0.6, 0.4, 0.5}{c_2}\right), \left(\frac{0.6, 0.5, 0.5}{c_3}\right), \left(\frac{0.8, 0.7, 0.5}{c_4}\right), \left(\frac{0.8, 0.7, 0.4}{c_5}\right) \right\}.$$

Hence, $\underline{\zeta}(X)$ and $\overline{\zeta}(X)$ are the lower and upper approximations of 3−polar fuzzy subset X in parameter set A, and the pair $(\underline{\zeta}(X), \overline{\zeta}(X))$ specifies the soft 4−polar fuzzy rough set.

Theorem 4.6 *Let $(S, A, \tilde{\zeta}^{-1})$be the soft mF approximation space. The lower and upper approximations $\underline{\zeta}(X)$and $\overline{\zeta}(X)$ satisfy the following properties for any $X, Y \in mF(A)$,*

1. $\underline{\zeta}(X) = \sim \overline{\zeta}(\sim X)$,
2. $X \subseteq Y \Rightarrow \underline{\zeta}(X) \subseteq \underline{\zeta}(Y)$,
3. $\underline{\zeta}(X \cup Y) \supseteq \underline{\zeta}(X) \cup \underline{\zeta}(Y)$,
4. $\underline{\zeta}(X \cap Y) = \underline{\zeta}(X) \cap \underline{\zeta}(Y)$,
5. $\overline{\zeta}(X) = \sim \underline{\zeta}(\sim X)$,
6. $X \subseteq Y \Rightarrow \overline{\zeta}(X) \subseteq \overline{\zeta}(Y)$,
7. $\overline{\zeta}(X \cup Y) = \overline{\zeta}(X) \cup \overline{\zeta}(Y)$,
8. $\overline{\zeta}(X \cap Y) \subseteq \overline{\zeta}(X) \cap \overline{\zeta}(Y)$.

Proof It can easily be proved using Definition 4.4. □

5 Applications of soft *m*F rough sets

In this section, we apply the concept of our second proposed model called soft mF rough sets in real-life examples and present its decision-making in an Algorithm 2, which shows its importance in multi-criteria decision-making.

Algorithm 2

Step 1. **Input**

S , as a universe.
A, as a set of parameters.
$\widetilde{C}$, different characteristics of parameters set A.

Step 2. Compute the ideally conventional decision object X

$X = \max\{p_i \circ \tilde{\zeta}^{-1}(s_j, a_k) | s_j \in S\}$, where $i \in m$, $j = 1, 2, 3, \ldots, n, k = 1, 2, 3, \ldots, l$.

Step 3. Compute the lower and upper approximations $\underline{\zeta}(X)$ and $\overline{\zeta}(X)$ for any $X \in mF(A)$ w.r.t. approximation space $(S, A, \tilde{\zeta}^{-1})$ as

$$underline\zeta(X)(s) = \bigwedge_{a \in A} \left((\mathbf{1} - \tilde{\zeta}^{-1}(s,a)) \vee X(a) \right), \quad s \in S,$$

$$\overline{\zeta}(X)(s) = \bigvee_{a \in A} \left(\tilde{\zeta}^{-1}(s,a) \wedge X(a) \right), \quad s \in S.$$

Step 4. Compute the choice value ρ_{s_j} as

$$\rho_{s_j} = p_i \circ \underline{\zeta}(X)(s_j) + p_i \circ \overline{\zeta}(X)(s_j), \quad s_j \in S.$$

Step 5. Compute the maximum choice value ρ_{s_k} as $\rho_{s_k} = \max_j \rho_{s_j}, j = 1, 2, \ldots, |S|$.

Step 6. **Output**

S_M, the alternative for which ρ_{s_k} is maximum.

Step 7. If no choice value is maximum, compute
$P_{sum} = \sum_{i=1}^{m} p_i \circ \rho_{sj}$, where $j = 1, 2, 3, \dots, n$.

Step 8. **Output**
$\widetilde{S_M}$, the alternative for which P_{sum} is maximum.

Working of Algorithm 2 is described as follows:

In algorithm 2, step 1 is taking values as inputs. Step 2 computes the ideally conventional decision object X. Step 3 computes the lower and upper approximations $\underline{\zeta}(X)$ and $\overline{\zeta}(X)$ for any $X \in mF(A)$. Step 4 computes the choice value ρ_{s_j}. Step 5 computes the maximum value for alternatives. Step 6 shows the best alternative as output. If step 6 does not show any output than compute the maximum value in step 7. Step 8 shows the best alternative as output according to decision of step 7.

5.1 Comparison of popular mobile phones for selection

Mobile phones are essential part of our daily communications. All mobile phones have range for voice and simple text messaging services. Recently, mobiles with many more features and functions have become available. So, in this age of competition it has become difficult to compare the features of mobiles for selection.

An Apple's iPhone company launches a new mobile phone with different features and specifications. Company wants to compare its new launched mobile phone with latest mobile phones of other companies. For this purpose, we propose the idea of soft mF rough sets.

Let $(M, A, \tilde{\zeta}^{-1})$ be a soft mF approximation space, where $M = \{m_1, m_2, m_3, m_4, m_5\}$ be a universe of five mobile phones specified as

- m_1 = Apple's iPhone 6,
- m_2 = Amazons Fire Phone,
- m_3 = Samsung Galaxy $S6$,
- m_4 = Motorala Moto X ($2nd$ gen.),
- m_5 = HTC One (MB).

Let $A = \{a_1, a_2, a_3, a_4\}$ be the set of parameters related to the mobile phones in M, where

"a_1" represents the Measurements,
"a_2" represents the Key Facts,
"a_3"represents the Visual Effects,
"a_4" represents the Price.
We give further characteristics of these parameters as

- The "Measurements" include dimensions, weights, slimness.
- The "Key Facts" include operating system, processor, memory.
- The "Visual Effects" include camera, display, sensor.
- The "Price" include cheap, costly, very costly.

However, for such a multi-criteria decision-making problem, one wishes to determine the decision substitute in universe with the estimation value as greater as possible on the whole estimated index. Thus, we construct an ideally conventional decision object X on the mF set of parameters A as follows:

$$X = \max\{p_i \circ \tilde{\zeta}^{-1}(m_j, a_k) | m_j \in M\}, \quad i \in m.$$

Now, the soft mF rough lower approximation $\underline{\zeta}(X)$ and upper approximation $\overline{\zeta}(X)$ of the ideally conventional decision object X are calculated in Table 11, using the Definition 4.4. Moreover, the rough lower and upper approximations are relatively close values to the approximated set of universe of mobiles. Thus, we attain relatively close values $\underline{\zeta}(X)(m_j)$ and $\overline{\zeta}(X)(m_j)$ to the decision substitute $m_j \in M$, by the soft mF rough lower and upper approximations of the mF subset X. Thus, we enumerate the choice value ρ_{m_j}, for the decision substitute m_j on the universe of sites M as follows:

$$\rho_{m_j} = p_i \circ \underline{\zeta}(X)(m_j) + p_i \circ \overline{\zeta}(X)(m_j), \quad i \in m, \quad m_j \in M.$$

Finally choosing the mobile phone $m_j \in M$, which has the maximum choice value ρ_{m_j} as the most favorable decision for the given multi-criteria decision-making problem. From Table 12. it is easy to compare the choice values of all the mobile phones. m_1 =Apple's iPhone 6, has the maximum choice value as compared to all other mobile phones. So, Apple's iPhone 6 is best for selection as compared to all other mobile phones.

Generally , if there occurs two or more items $m_j \in M$ with the same maximum choice value ρ_{m_j} , then take one of them according to your choice as the ideal decision for the given multi-criteria decision-making problems.

5.2 Selection of a site for construction of a grid station

An electricity grid is an interdependent chain for providing the electricity from source to user. It is an amenity project aiming to provide relief to citizens, rather than a commercial activity. Selection of site for construction of grid station is the early and significant process. This requires accurate planning, skillful investigation and administration so that the

Table 11 3−Polar fuzzy soft relation

$\tilde{\zeta}^{-1}$	Measurements	Key Facts	Visual Effects	Price
m_1	(0.2,0.6,0.8)	(0.8,0.5,0.6)	(0.9,0.6,0.2)	(0,0.9,0.1)
m_2	(0.1,0.5,0.7)	(0.2,0.9,0.1)	(0,0.2,0.6)	(0.1,0.1,0.2)
m_3	(0.6,0.2,0.1)	(0.4,0.3,0.6)	(0.2,0.3,0.3)	(0.2,0.8,0.1)
m_4	(0.8,0.1,0.7)	(0.3,0.5,0.4)	(0.5,0.2,0.5)	(0.6,0.4,0.4)
m_5	(0.2,0.3,0.2)	(0.8,0.7,0.6)	(0.6,0.7,0.1)	(0.2,0.3,0.4)
X	(0.8,0.6,0.8)	(0.8,0.9,0.6)	(0.9,0.7,0.6)	(0.6,0.9,0.4)

selected site is mechanically, economically, environmentally, and socially perfect for requirements.

For this purpose, we use the concept of soft mF rough set theory. Let $(S, A, \tilde{\zeta}^{-1})$ be a soft mF approximation space, and $S = \{S_s, S_n, S_e, S_w, S_c\}$ be a set of sites for grid station specified as

- S_s = Site in "south" of city,
- S_n = Site in "north" of city,
- S_e = Site in "east" of city,
- S_w = Site in "west" of city,
- S_c = Site in "center" of city.

Let $A = \{a_1, a_2, a_3, a_4\}$ be the set of parameters related to the sites of grid station in S, where,

"a_1" represents the Energy Sources,
"a_2" represents the Transportation,
"a_3"represents the Area Attributes,
"a_4" represents the Energy Storage Stations.

Further characteristics of parameters are given in Figure 1, that explains four different parameters with the deep classification of characteristics, each parameter is further classified in three different characteristics.

An ideally conventional decision object X on the mF set of parameters A is calculated in Table 13.

The choice value ρ_{s_j}, for the decision substitute s_j on the universe of sites S is calculated in Table 14.

From Table 14, it is easy to compare the choice values of all the required sites. Site in "west" of city, has the maximum choice value as compared to all other sites. So, it is the best site for construction as compared to all other sites.

Table 12 Choice values

	$\underline{\zeta}(X)$	$\overline{\zeta}(X)$	Choice value (ρ_{m_j})
m_1	(0.8,0.6,0.6)	(0.9,0.9,0.8)	ρ_{m_1} =(1.7,1.6,1.4)
m_2	(0.8,0.6,0.6)	(0.2,0.9,0.7)	ρ_{m_2} =(1.0,1.5,1.3)
m_3	(0.8,0.7,0.6)	(0.6,0.8,0.6)	ρ_{m_3} =(1.4,1.5,1.2)
m_4	(0.6,0.8,0.6)	(0.8,0.5,0.7)	ρ_{m_4} =(1.4,1.3,1.3)
m_5	(0.8,0.7,0.6)	(0.8,0.7,0.6)	ρ_{m_5} =(1.6,1.4,1.2)

5.3 Comparison of patients for recovery of heart disease

Traditionally, health plans, medicare, and medicaid pay providers for whatever services they deliver, regardless of whether the services truly benefit the patient. How long some one takes to recover after an episode in intensive care depends on many things, including their age, prevention, health care and medication etc. For such a comparison in patients that whose patient will recover soon with prevention and medication we use the approach of soft mF rough sets.

Let $(P, A_p, \tilde{\zeta}^{-1})$ be a soft mF approximation space and $P = \{p_1, p_2, p_3, p_4, p_5, p_6\}$ be a set of six patients. Let $A_p = \{a_1, a_2, a_3, a_4\}$ be the set of parameters of prevention and treatment of heart disease related to the patients in P, where

"a_1" represents the Vaccination,
"a_2" represents the Health care,
"a_3"represents the Medication,
"a_4" represents the Surgery.
We give further characteristics of these parameters as

- The "Vaccination" includes live-attended vaccines, inactivated vaccines, live non-pathogenic vaccines and live active vaccines.
- The "Health Care" includes exercise, balanced diet, rest and plenty of liquid.
- The "Medication" includes statins, beta blockers, antiplatelet and ACE inhibitors.
- The "Surgery" includes coronary artery bypass grafting, heart valve repair or replacement, aneurysm repair and heart transplant.

An ideally conventional decision object X on the mF set of parameters A_p is calculated in Table 15.

The choice value ρ_{p_j}, for the decision substitute p_j on the universe of patients P is calculated in Table 16.

From Table 16, it is easy to see that no choice value is maximum, so it is difficult for someone to take a decision that whose patient will recover soon. For taking such a decision we calculate

$$\sum_{i=1}^{4} p_i \circ \rho_{pj}, \quad j = 1, 2, \ldots, 6.$$

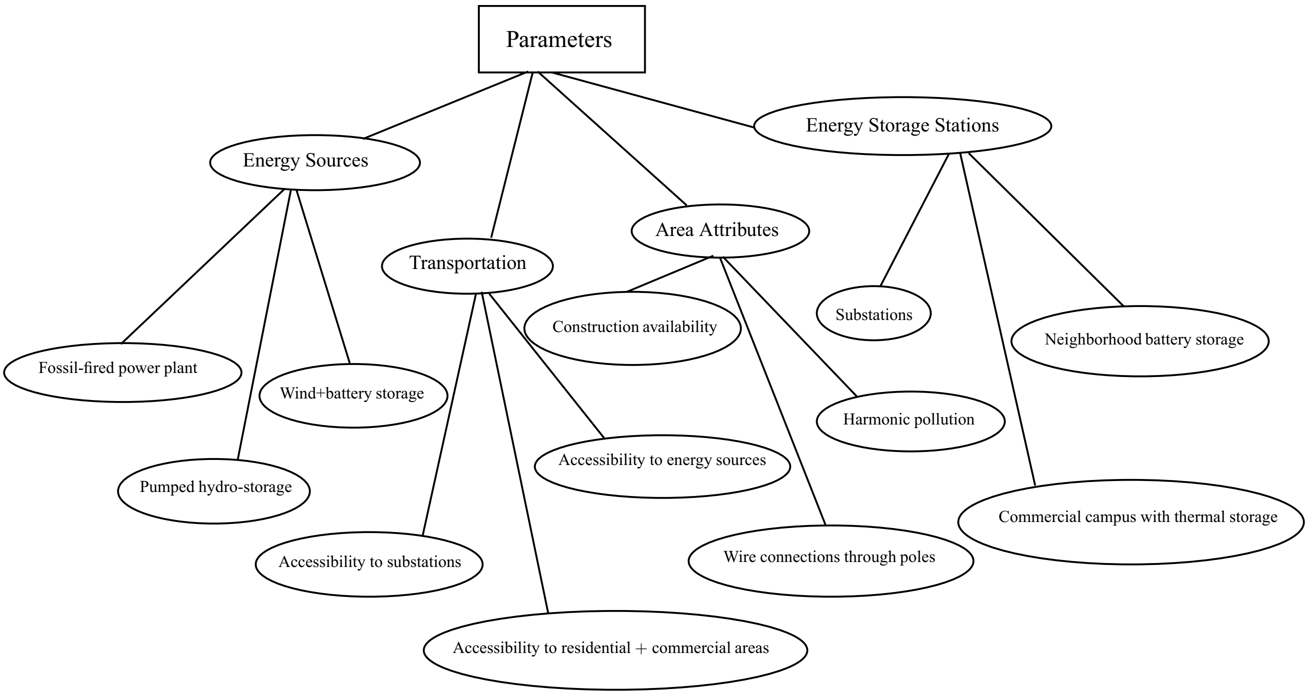

Fig. 1 Characteristics of parameters

Table 13 3−Polar fuzzy soft relation

$\tilde{\zeta}^{-1}$	Energy Sources	Transportation	Area Attributes	Energy Storage Stations
S_s	(0.45,0.32,0.81)	(0.52,0.58,0.23)	(0.35,0.76,0.89)	(0.35,0.81,0.48)
S_n	(0.51,0.68,0.77)	(0.12,0.76,0.77)	(0.22,0.78,0.70)	(0.72,0.76,0.59)
S_e	(0.23,0.82,0.55)	(0.52,0.80,0.63)	(0.57,0.65,0.27)	(0.67,0.39,0.23)
S_w	(0.53,0.82,0.29)	(0.92,0.36,0.27)	(0.20,0.72,0.88)	(0.73,0.58,0.55)
S_c	(0.88,0.21,0.66)	(0.89,0.76,0.54)	(0.70,0.20,0.73)	(0.55,0.46,0.60)
X	(0.88,0.82,0.81)	(0.92,0.80,0.77)	(0.70,0.78,0.89)	(0.73,0.81,0.60)

Table 14 Choice value

	$\underline{\zeta}(X)$	$\overline{\zeta}(X)$	Choice value (ρ_{s_j})
S_s	(0.70,0.78,0.60)	(0.52,0.81,0.81)	ρ_{s_s} =(1.22,1.59,1.41)
S_n	(0.73,0.78,0.60)	(0.72,0.78,0.77)	ρ_{s_n} =(1.45,1.56,1.37)
S_e	(0.70,0.78,0.77)	(0.67,0.82,0.63)	ρ_{s_e} =(1.37,1.60,1.40)
S_w	(0.73,0.78,0.60)	(0.92,0.82,0.88)	ρ_{s_w} =(1.65,1.60,1.48)
S_c	(0.70,0.78,0.60)	(0.89,0.76,0.73)	ρ_{s_c} =(1.59,1.54,1.33)

Finally, taking the patients $p_j \in P$ with the maximum sum of poles of choice value ρ_{p_j} as the ideal decision for the given multi-criteria decision-making problem. From Table 16, it is easy to compare the choice values of all the required patients. Fourth patient has the maximum sum of poles of choice value as compared to all other patients. So, he will recover soon from heart disease with prevention and treatment as compared to others.

6 Sensitivity and comparison analysis

Generally, the real-world decision-making problems occur in complex environment under imprecise, uncertain, and multipolar information, which is difficult to handle. Proposed models are very suitable for the situations when the information is complex, multipolar, imprecise, and uncertain. The fundamental concept behind the proposed models is to approximate the data under multipolar information. In proposed models the lower and upper approximations of a set are used to approximate the date under imprecise and multipolar information. The subset generated by lower approximations is characterized by objects that definitely

Table 15 4−Polar fuzzy soft relation

$\tilde{\zeta}^{-1}$	Vaccination	Healthcare	Medication	Surgery
p_1	(0.3,0,0.8,0.4)	(0.7,0.2,0.6,0.8)	(0.5,0.3,0.2,0.7)	(0.6,0.2,0.8,0.7)
p_2	(0.4,0.7,0.1,0.3)	(0.1,0.1,0.5,0.7)	(0.4,0.6,0.3,0.2)	(0.5,0.4,0.6,0.7)
p_3	(0.4,0.4,0.1,0.6)	(0.7,0.2,0.5,0.8)	(0.7,0.5,0.6,0.2)	(0.6,0.7,0.5,0.6)
p_4	(0.7,0.5,0.2,0.4)	(0.6,0.8,0.5,0.5)	(0.5,0.8,0.5,0.3)	(0.7,0.3,0.8,0.8)
p_5	(0.1,0.6,0.1,0.5)	(0.2,0.7,0.5,0.7)	(0.2,0.8,0.3,0.2)	(0.6,0.8,0.7,0.5)
p_6	(0.5,0.6,0.1,0.3)	(0.5,0.6,0.7,0.8)	(0.5,0.6,0.4,0.2)	(0.5,0.8,0.6,0.6)
X	(0.7,0.7,0.8,0.6)	(0.7,0.8,0.7,0.8)	(0.7,0.8,0.6,0.7)	(0.7,0.8,0.8,0.8)

Table 16 Choice value

	$\underline{\zeta}(X)$	$\overline{\zeta}(X)$	Choice value ρ_{p_j}	$\sum_{i=1}^{4} p_i \circ \rho_{pj}$
p_1	(0.7,0.8,0.7,0.6)	(0.7,0.3,0.8,0.7)	ρ_{p_1} =(1.4,1.1,1.5,1.3)	ρ_{p_1} =5.2
p_2	(0.7,0.7,0.7,0.6)	(0.5,0.7,0.6,0.7)	ρ_{p_2} =(1.2,1.4,1.3,1.3)	ρ_{p_2} =5.2
p_3	(0.7,0.7,0.6,0.6)	(0.7,0.6,0.6,0.8)	ρ_{p_3} =(1.4,1.3,1.2,1.4)	ρ_{p_3} =5.3
p_4	(0.7,0.7,0.6,0.6)	(0.7,0.8,0.8,0.8)	ρ_{p_4} =(1.4,1.5,1.4,1.4)	ρ_{p_4} =5.7
p_5	(0.7,0.7,0.7,0.6)	(0.6,0.8,0.7,0.7)	ρ_{p_5} =(1.3,1.5,1.4,1.3)	ρ_{p_5} =5.5
p_6	(0.7,0.7,0.6,0.7)	(0.5,0.8,0.7,0.8)	ρ_{p_6} =(1.2,1.5,1.3,1.5)	ρ_{p_6} =5.5

form a part of an interest subset, whereas the subset generated by the upper approximations is characterized by objects that possibly form a part of an interest subset. Both the models provide more compatibility and flexibility as compared to previously defined models, which deal multipolar information with enough number of parameters to handle the uncertain facts. All others models are unable to deal with such a kind of multipolar information. In literature, previously defined models based on mF set theory also deal the multipolar information but in those models, the data used to relate the universes are crisp which are unable to handle the uncertain and mF information about the relation of different universes. In short, we can say such type of models approximate an mF input under crisp knowledge, whereas in decision-making our proposed models are able to provide the complete information about universes under mF knowledge.

7 Conclusion

The mF set theory has a significant use in various fields and attracted a number of researchers. The combination of mF set theory with other mathematical theories is a useful tool for dealing with various types of uncertainties with multipolar information, because hybrid models always provide more accurate and consistent results when we have to deal with the systems with more than one agreements. In this research article, we have presented new hybrid models by combining the theory of mF sets with rough sets and soft sets to introduce the idea of mF rough sets and soft mF rough sets, which have their own importance in multi-criteria decision-making. We have introduced the basic operations of our proposed models and investigated some of their properties. To extend the range of number of parameters with multipolar information in many practical life applications, we also have proposed the novel approaches to multi-criteria decision-making based on proposed models and demonstrated the computational process by some practical examples. The complexity to approximate the data under multipolar information is overcome with our proposed approaches. We also have developed algorithms for multi-criteria decision-making problems. We will extend our research work on hybrid models of mF rough sets such as, (1) soft mF rough hyper graphs, (2) soft mF rough graphs, (3) soft rough mF graphs. Our proposed models may be extended to new directions including multi-criteria decision-making and aggregation operators based on (Garg 2018; Garg and Singh 2018; Garg and Kumar 2018).

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Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest regarding the publication of the research article.

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