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ORIGINAL PAPER

BEM for thick plates on unilateral Winkler springs

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Abstract
A new direct boundary element (BEM) technique is established to analyze plates on tensionless elastic foundation. The soil is modeled as Winkler springs. The considered BEM is based on the formulation of shear deformable plate bending theory according to Reissner. The developed technique is based on coupling a developed plate bending software with iterative process to eliminate tensile stresses underneath the considered plate. Tensile zones are redistributed until the final contact zone of plate is reached. Examples are tested and results are compared to analytical and previously published results to verify the proposed technique.

Keywords Boundary element method · Tensionless · Soil-structure interaction · Raft

Introduction

Solving plates on tensionless foundation can be divided into two main categories of solution. First solution is to solve the problem using iterative procedure to consider the miscontact between plate and foundation. Second solution is to embed the contact problem in a system of nonlinear equations and solve it using optimization algorithm. Boundary element method (BEM) is widely used to solve plate on foundation problem. Katsikadelis and Armenakas [1] presented analysis of thin plates on elastic foundation. A BEM formulation based on shear deformable plates according to Reissner [2] was derived by Vander Weeën [3]. Rashed et al. [4–6] extended Vander Weeën formulation to model foundation plates.

Several studies are presented to solve plates on tensionless foundations. Weitsman [7] presented analysis of tensionless beams, or plates, and their supporting Winkler or Reissner subgrade due to concentrated loads. Celep [8] presented the behavior of elastic plates of rectangular shape on a tensionless Winkler foundation using auxiliary function. Galerkin's method is used to reduce the problem to a system of algebraic equations. Li and Dempsey [9] used an iterative procedure to analyze unbonded contact of a square thin plate under centrally symmetric vertical loading on elastic Winkler or EHS foundation. Sapountzakis and Katsikadelis [10] presented boundary element solution for unilateral contact problems of thin elastic plates resting on linear or nonlinear subgrade by solving a system of nonlinear algebraic equations. Kexin et al. [11] presented a BE–LCEM solution for thin free edge plates on elastic half space with unilateral contact. Silva et al. [12] used finite element method to discretize the plate and foundation then used three alternative optimization linear complementary problems to solve plates on tensionless elastic foundations. T-element analysis of plates on unilateral elastic Winkler type foundation using hybrid-Trefftz finite element algorithm was presented by Jirousek et al. [13]. Xiao [14] presented a BE–LCEM solution to solve unilateral free edges thick plates. Nonlinear bending behavior of Reissner–Mindlin plates with free edges resting on tensionless elastic foundations using admissible functions was presented by Shen and Yu [15]. Silveira et al. [16] presented a nonlinear analysis of structural elements under unilateral contact constrains studied by a Ritz approach using a mathematical programming technique. Results of finite element analysis of beam elements on unilateral elastic foundation using special zero thickness element designed for foundation modeling are presented by Torbacki [17]. Buczkowski and Torbacki [18] presented finite element analysis

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of plate on layered tensionless foundation. Kongtong and Sukawat [19] used the method of finite Hankel integral transform techniques for solving the mixed boundary value problem of unilaterally supported rectangular plates loaded by uniformly distributed load.

In this paper, an iterative procedure is developed to solve thick plates on tensionless Winkler foundation. The boundary element formulation is used to extract the stiffness matrix of the plate supported on Winkler springs. The main advantage of the proposed technique is that it merges between the advantage of the boundary element method, modeling plate using integral equation, and the simplicity of finite element method, solving nonlinear equations iteratively in matrix form. One of the advantages of the proposed technique is its practicality. In the formulation of Xiao [14] although solving of thick plates on tensionless foundations, the results were not accurate near corners as it will be shown in the examples of this paper. Also, this formulation is suitable to be extended for solving plates on Winkler elastic–plastic foundations. Numerical examples are presented to verify efficiency and practicality of the proposed technique.

BEM for plate on Winkler foundation

Formulation of direct boundary integral equation of the plate [6] based on Reissner plate bending theory [2] is used. Consider a general plate domain Ω and boundary Γ with internal Winkler cells as shown in Fig. 1. The indexical notation is used in this paper where the Greek indexes vary from 1 to 2 and Roman indexes vary from 1 to 3. The integral equation can be represented as follows:

$$C_{ij}(\xi)u_j(\xi) + \int_{\Gamma(x)} T_{ij}(\xi,x)u_j(x)\,\mathrm{d}\Gamma(x) = \int_{\Gamma(x)} U_{ij}(\xi,x)t_j(x)\,\mathrm{d}\Gamma(x) + \sum_c \int_{\Omega(y)} \left[U_{ik}(\xi,y) - \frac{\nu}{(1-\nu)\lambda^2}U_{i\alpha,\alpha}(\xi,y)\delta_{3k}\right] F_k(y)\mathrm{d}\Omega(y), \tag{1}$$

where $T_{ij}(\xi,x)$, $U_{ij}(\xi,x)$ are the two-point fundamental solution kernels for tractions and displacements, respectively [3]. The two points ξ and x are the source and the field points, respectively. $u_j(x)$ and $t_j(x)$ denote the boundary generalized displacements and tractions. $C_{ij}(\xi)$ is the jump term. The symbols ν and λ denote the plate Poison's ratio and shear factor. c denotes the number of internal Winkler cells having domain Ω, and F_k denotes the Winkler cell two bending moments ($F_1 = M_{xx}$, $F_2 = M_{yy}$) and column vertical force ($F_3 = F$). The field point y denotes the point of the internal Winkler cell center.

After discretizing the boundary of the plate to NE quadratic elements, each node has three unknowns. Due to internal Winkler cells inside the domain, additional three unknowns are added, so another collocation scheme is carried out at each internal Winkler cell center to add additional equations. The collocation scheme is carried out as follows:

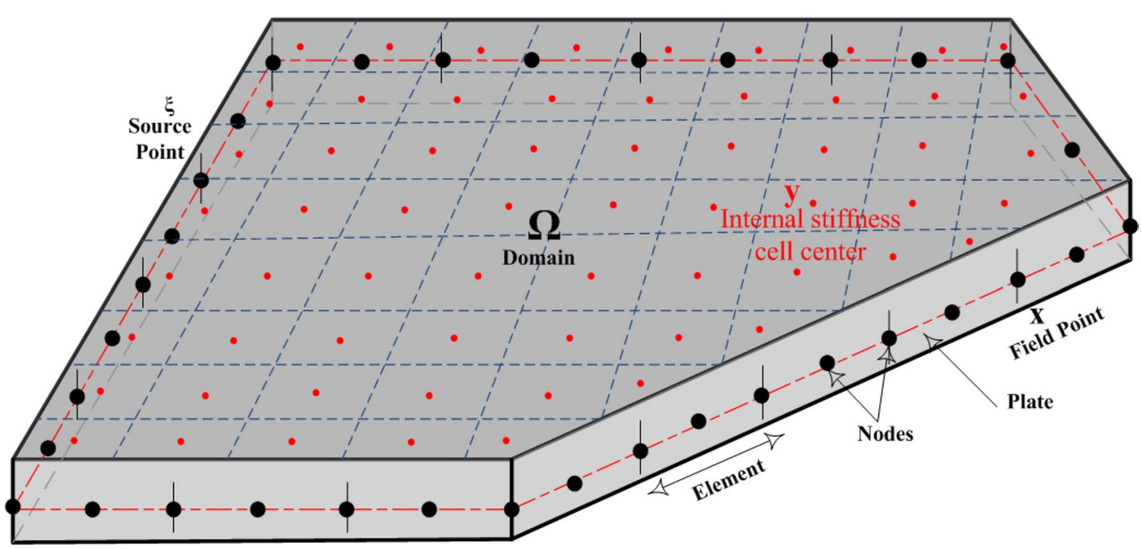


Fig. 1 General plate for BEM formulation of the plate

$$u_i(Y) + \int_{\Gamma(x)} T_{ij}(Y,x)u_j(x)\,\mathrm{d}\Gamma(x) = \int_{\Gamma(x)} U_{ij}(Y,x)t_j(x)\,\mathrm{d}\Gamma(x) + \sum_c \int_{\Omega(y)} \left[U_{ik}(Y,y) - \frac{\nu}{(1-\nu)\lambda^2} U_{i\alpha,\alpha}(Y,y)\delta_{3k}\right] F_k(y)\,\mathrm{d}\Omega(y), \tag{2}$$

where Y is a new source point located at each cell center.

Equations (1) and (2) after discretization of plate boundary can be re-written in a matrix form as follows:

$$\begin{bmatrix} [A]_{3N\times 3N} & [A_2]_{3N\times 3N_c} & [0]_{3N\times 3Nc} \\ [A_1]_{3N_c\times 3N} & [A_3]_{3N_c\times 3N_c} & [I]_{3N_c\times 3N_c} \end{bmatrix} \begin{Bmatrix} \{u/t\}_{3N\times 1} \\ \{F\}_{3N_c\times 1} \\ \{u_c\}_{3N_c\times 1} \end{Bmatrix} = \begin{Bmatrix} \{0\}_{3N\times 1} \\ \{\mathrm{RHS}\}_{3N_c\times 1} \end{Bmatrix}, \tag{3}$$

where $[A]_{3N\times 3N}$, $[A_1]_{3N_c\times 3N}$, $[A_2]_{3N\times 3N_c}$, $[A_3]_{3N_c\times 3N_c}$ and $\{\mathrm{RHS}\}_{3N_c\times 1}$ contain the coefficients of the integrals presented in Eqs. (1) and (2). $[0]_{3N\times 3N_c}$, $[I]_{3N_c\times 3N_c}$ are the null and identity matrices, respectively. The vector $\{u/t\}_{3N\times 1}$ contains the unknown boundary values displacement or traction, the vector $\{F\}_{3N_c\times 1}$ contains the unknown values of internal Winkler cell forces and $\{U_c\}_{3N_c\times 1}$ contains the unknown values of internal Winkler cell displacement. The relation between $\{U_c\}_{3N_c\times 1}$ and$\{F\}_{3N_c\times 1}$ can be as follows:

$$\{F\}_{3N_c\times 1} = [K^c]_{3N_c\times 3N_c}\{u_c\}_{3N_c\times 1}, \tag{4}$$

where $[K^c]_{3N_c\times 3N_c}$ is the stiffness matrix of the internal Winkler cells.

In order to extract the stiffness matrix of the plate like FEM, the following procedure is presented. Since the stiffness matrix of the plate is independent of the loading, neither the no domain load nor the cells area loading is considered. As a result of this assumption, the vector $\{\mathrm{RHS}\}_{3N_c\times 1}$ will be zeros and, consequently, Eq. (3) can be re-written as follows:

$$\begin{bmatrix} [A]_{3N\times 3N} & [A_2]_{3N\times 3N_c} \\ [A_1]_{3N_c\times 3N} & [A_3]_{3N_c\times 3N_c} \end{bmatrix} \begin{Bmatrix} \{u/t\}_{3N\times 1} \\ \{F_c\}_{3N_c\times 1} \end{Bmatrix} = \begin{Bmatrix} \{0\}_{3N\times 1} \\ -\{u_c\}_{3N_c\times 1} \end{Bmatrix}. \tag{5}$$

In order to get $\{F_c\}_{3N_c\times 1}$ representing a certain case of the stiffness $\{u_c\}_{3N_c\times 1}$ is forced to be unity. Cases of loading equal to the $3N_c$ are considered to extract the plate stiffness matrix as follows:

$$\begin{bmatrix} [A]_{3N\times 3N} & [A_2]_{3N\times 3N_c} \\ [A_1]_{3N_c\times 3N} & [A_3]_{3N_c\times 3N_c} \end{bmatrix} \begin{Bmatrix} \{u/t\}_{3N\times 3N_c} \\ \{K_\mathrm{p}\}_{3N_c\times 3N_c} \end{Bmatrix} = \begin{Bmatrix} \{0\}_{3N\times 3N_c} \\ -\{I\}_{3N_c\times 3N_c} \end{Bmatrix}, \tag{6}$$

where $\{K_\mathrm{p}\}_{3N_c\times 3N_c}$ is the required plate stiffness matrix.

Winkler stiffness values of each cell are added as a diagonal matrix to the plate stiffness matrix $[K_\mathrm{p}]$ to extract $[K_\mathrm{Overall}]_{3N_c\times 3N_c}$ the overall stiffness of the plate rested on elastic foundation. Also condensed load vector of plate is computed $\{P_\mathrm{plate}\}_{3N_c\times 1}$. The overall equilibrium equation can be written as follows:

$$[K_\mathrm{Overall}]_{3N_c\times 3Nc} \times \{u_\mathrm{plate}\}_{3N_c\times 1} = \{P_\mathrm{plate}\}_{3N_c\times 1}, \tag{7}$$

Proposed iterative procedure

In this section, simulation of soil as tensionless material is established using a new iterative procedure to solve plate on tensionless foundation. Figure 2 demonstrates flowchart of the iterative procedure. The system of equations in Eq. (7) is solved and then the internal forces of soil are computed. The internal forces of soil are used to get the most tensile stressed cell. Elimination procedure is used to eliminate the most tensile stressed cell DOF stiffness matrix to simulate the loss of contact between plate and soil. The system of equations is resolved. This iterative procedure is used until reaching the real contact zone, i.e. no tensile stresses appear in the soil reactions. To optimize the solution, the formulation is implemented into a computer code. It is worth to be noted that this procedure is valid also in solving plates on elastic perfectly plastic soil by applying a virtual load equal to the allowable compressive stress of the soil at the eliminated cells DOFs.

Numerical examples

Example 1: clamped supported circular plate on tensionless Winkler foundation

A clamped circular plate with unit radius and loaded by a unit concentrated moment $M = 1$ at its center is considered by Sapountzakis and Katsikadelis [10]. The plate is resting on tensionless Winkler foundation as shown in Fig. 3. The modulus of soil K is 15,576.93 and $\lambda = (a/4\sqrt{D/k}) = 3$, hence $D = 192.3077$. Plate Young's modulus is $E = 2E + 6$. Plate is modeled using 16 clamped boundary elements assuming that ($\nu = 0.3$, $t = 0.102$) as shown in Fig. 4. The proposed technique is used to obtain the unilateral results. FEM model is also presented to verify results and to compare them with the results of [10].

Figure 5 demonstrates the contact area of the plate. Figures 6 and 7 demonstrate deflection and bending moment contours, respectively. It can be seen that the results are in good agreements. In addition, Figs. 8 and 9 demonstrate

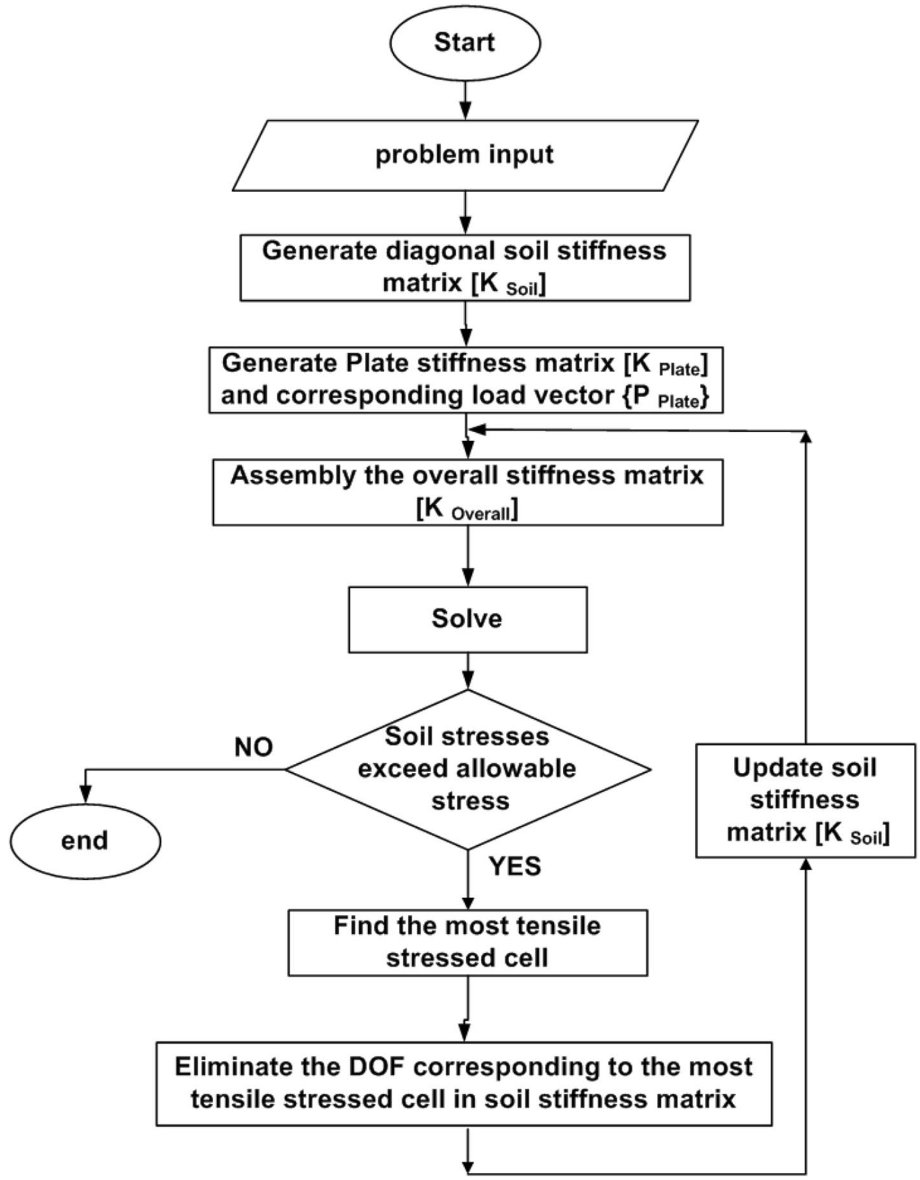


Fig. 2 Flowchart for developed iterative procedure

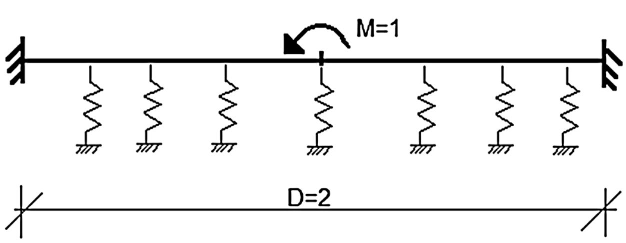


Fig. 3 The clamped circular plate in example 1

bilateral and unilateral results along plate centerline strip, respectively.

Example 2: free edge square plate on tensionless Winkler foundation

This example consists of free edges square plate. The side of plate is a = 400 cm. The Young's modulus is $E = 2.6 \times 10^6$ N/cm^2, Poisson's ratio of the plate material

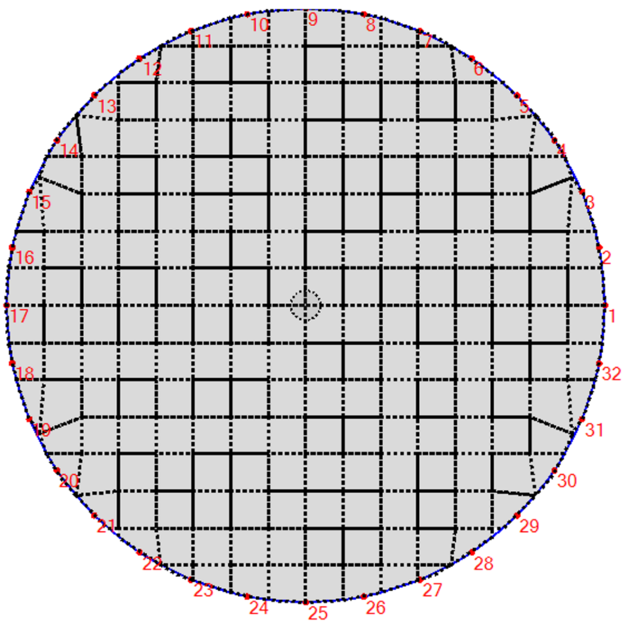


Fig. 4 BEM model for circular plate in example 1

is ($\nu = 0.15$). Different plate thicknesses (t) are employed 30,40,60,80,100 cm. The foundation stiffness parameter K is 500 N/cm^3. Plate is loaded by patch distributed load $q = 1000$ N/cm^2 over area of 20 cm × 20 cm at its center, as shown in Fig. 10. The plate is modeled as thick plate rested on Winkler foundation as shown in Fig. 10 with soil discretization as shown in Fig. 11.

Our results are compared with the results obtained in Xiao [14]. All models are solved with different thicknesses (30, 40, 60, 80,100 cm). Table 1 demonstrates the contact regions for different thicknesses of raft. Figures 12, 13, 14 and 15 demonstrate the deflection along diagonal strip (indicated in Fig. 10) for different thicknesses of raft. Xiao [14] results and FEM for both bilateral results (at left of graph) and unilateral results (at right of graph) are plotted also on these figures for the sake of comparison.

It can be seen from the results that proposed technique obtained the same contact area that is obtained in [14] and in FEM. Deflection is in a good agreement with results of Xiao [14] except at corners, at which some inaccurate results due to singularity problems near boundary elements appear in the

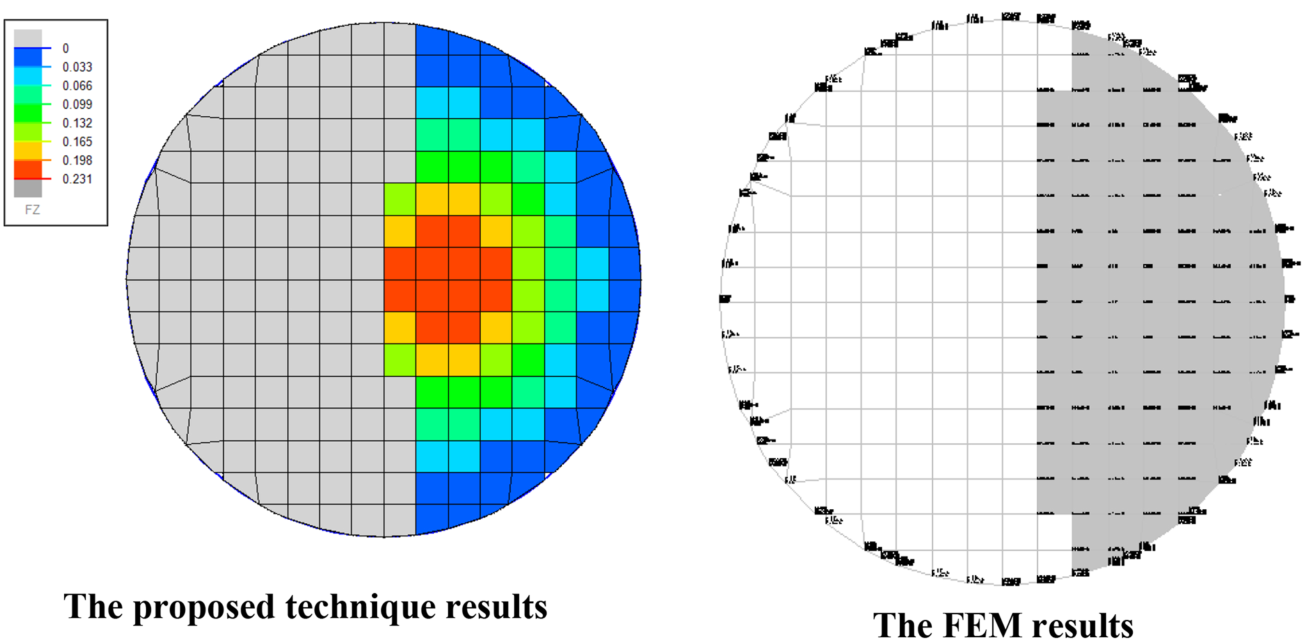


Fig. 5 Contact area of proposed technique and FEM

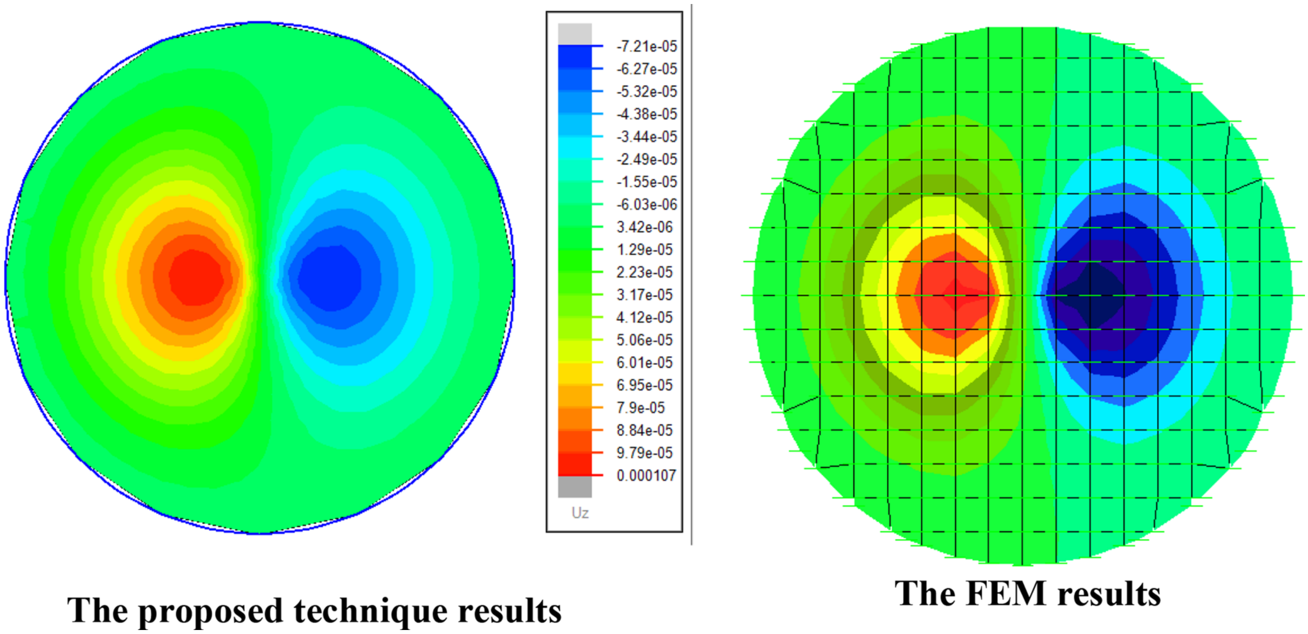


Fig. 6 Deflection contours of proposed technique and FEM

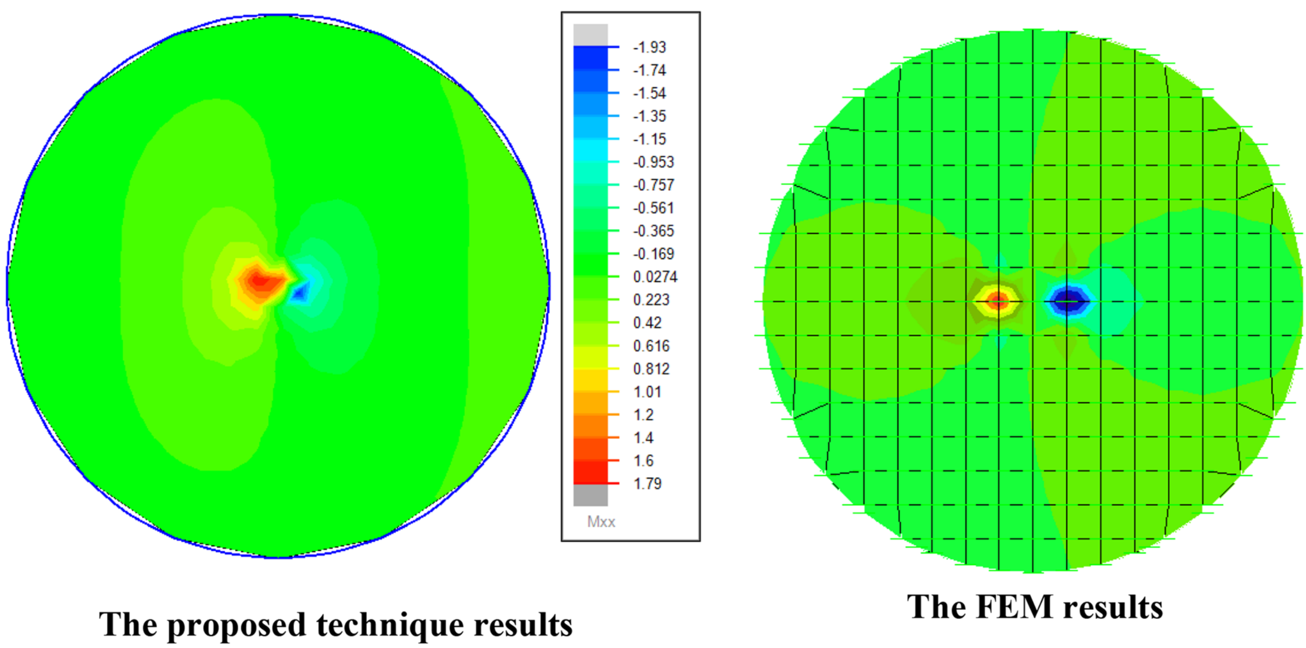


Fig. 7 Moment contours of proposed technique and FEM

formulation of Xiao [14]. It can be seen that the presented formulation results are in good agreement with results from FEM. This demonstrates the advantages of the proposed technique over the formulation of Xiao [14].

Conclusions

In this paper, solving plates on tensionless Winkler soil is presented using an efficient technique. The plate stiffness matrix is extracted using boundary element formulation. An iterative technique is used to eliminate the tensile stresses.

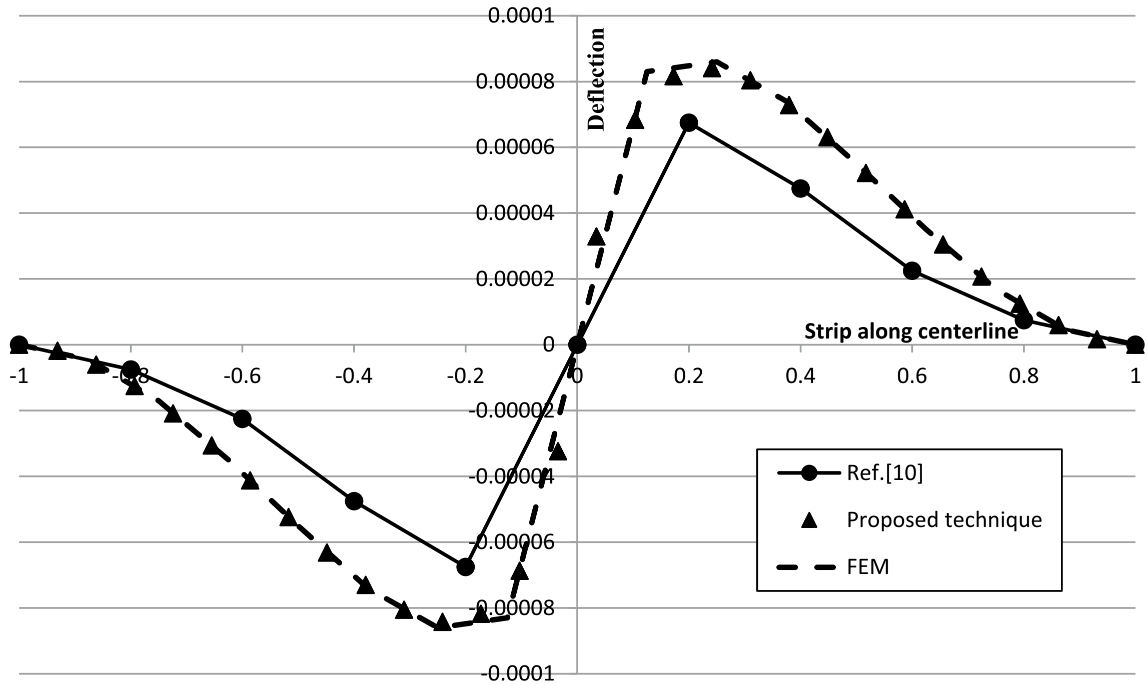


Fig. 8 Bilateral deflection results along plate centerline strip in example 1

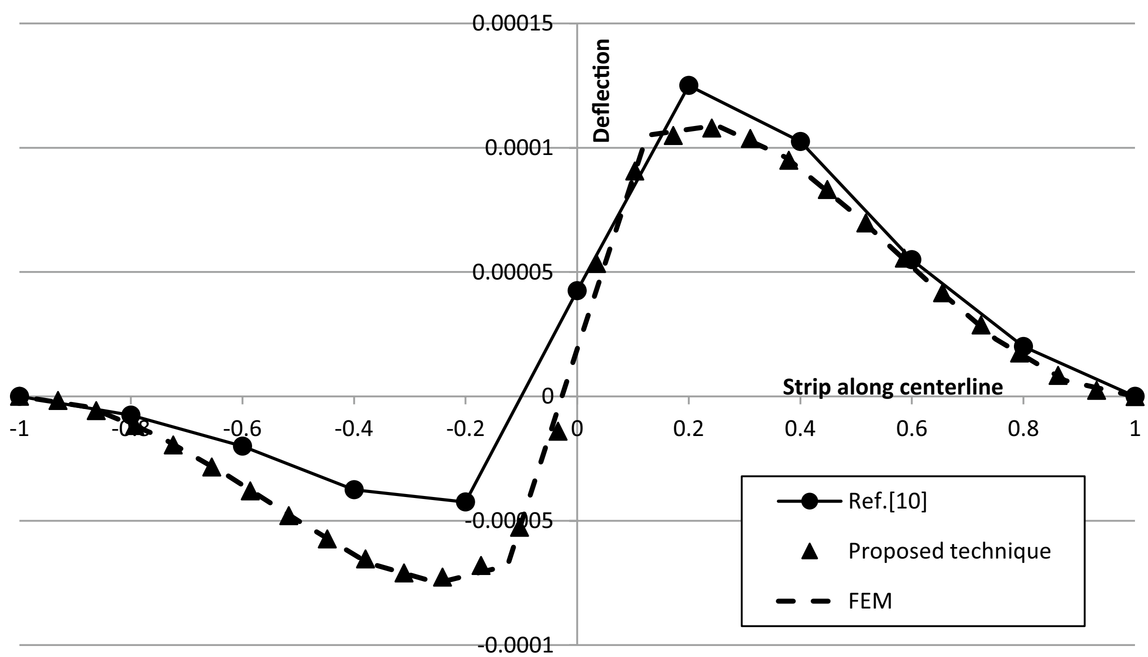
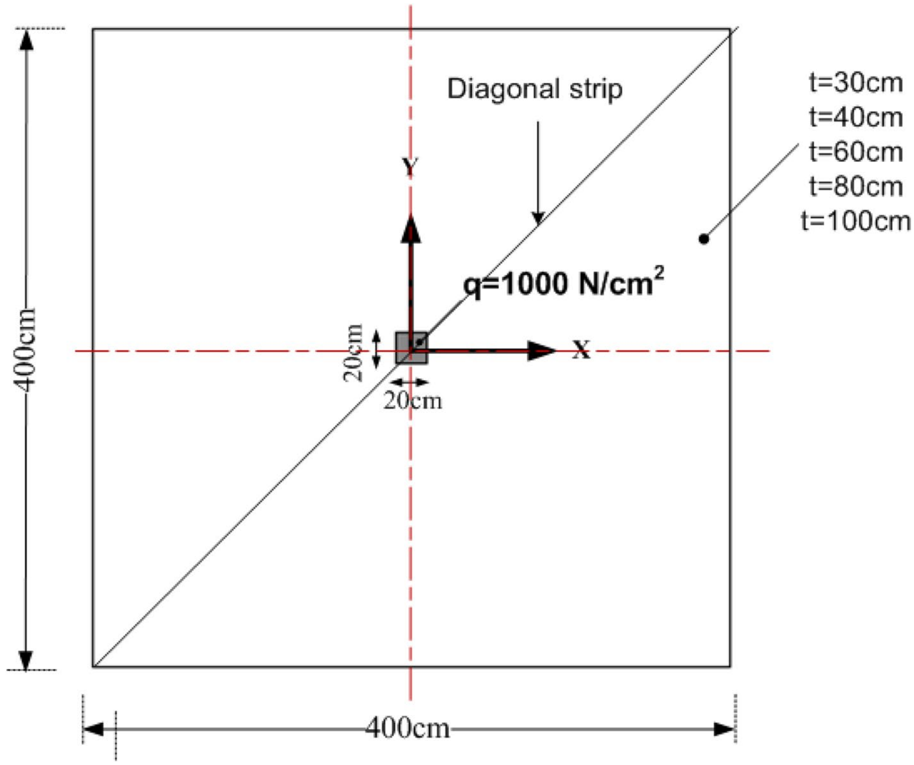


Fig. 9 Unilateral deflection results along plate centerline strip in example 1

Fig. 10 Free edge square plate dimensions under patch load in example 2



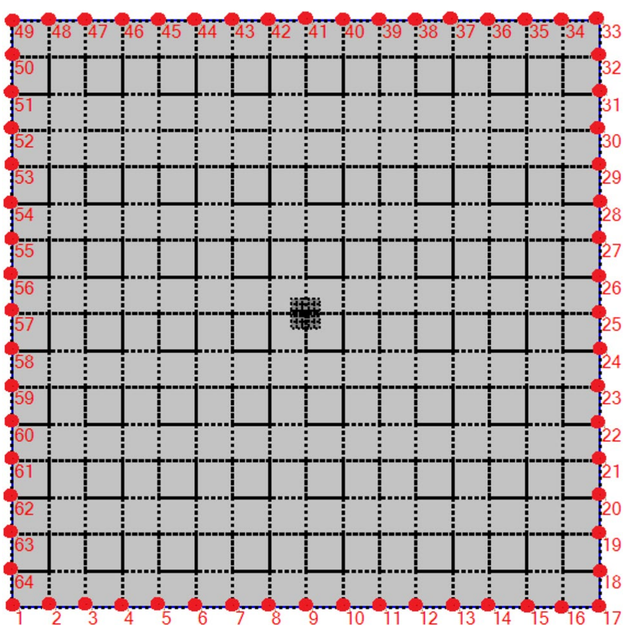
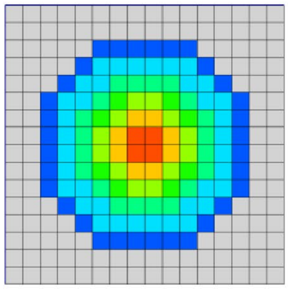
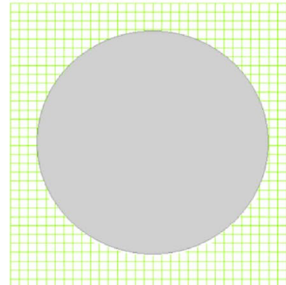
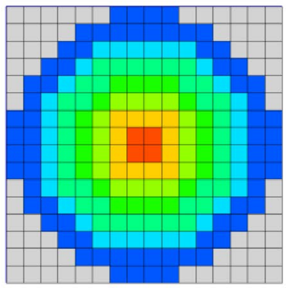
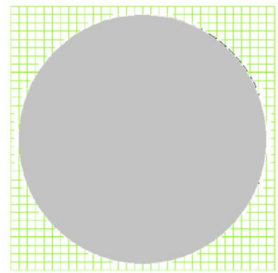
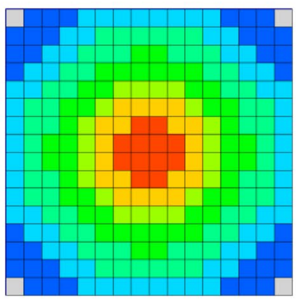
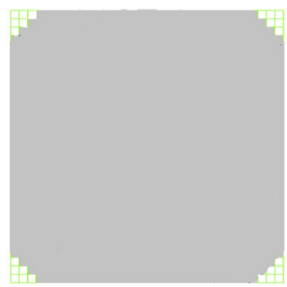
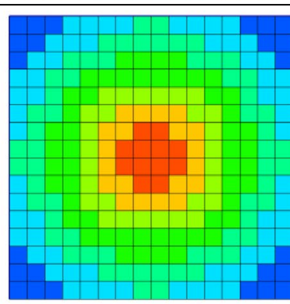
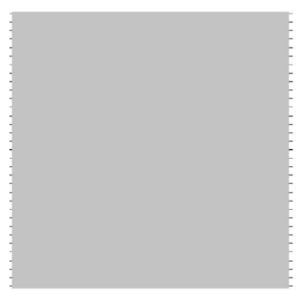
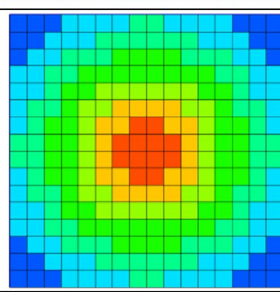


Fig. 11 Proposed technique model of example 2

Several examples are presented and from the results it can be concluded that the main advantages of this technique are:

1. The simplicity of dealing with the nonlinearity nature of the problem.
2. Avoiding the stresses concentration zones appears in finite element solutions using the advantages of boundary element formulation.
3. Suitable to be extended to include solving plates on Winkler elastic–plastic foundations.
4. Accurate results near corners unlike in the formulation of Xiao [14] as shown in the examples of this paper.
5. Solving any geometry and any boundary conditions of the plate as shown in example 1.

Table 1 Contact region for different thicknesses in example 2

t (cm)	Proposed technique results	FEM results
30		
40		
60		
80		
100		

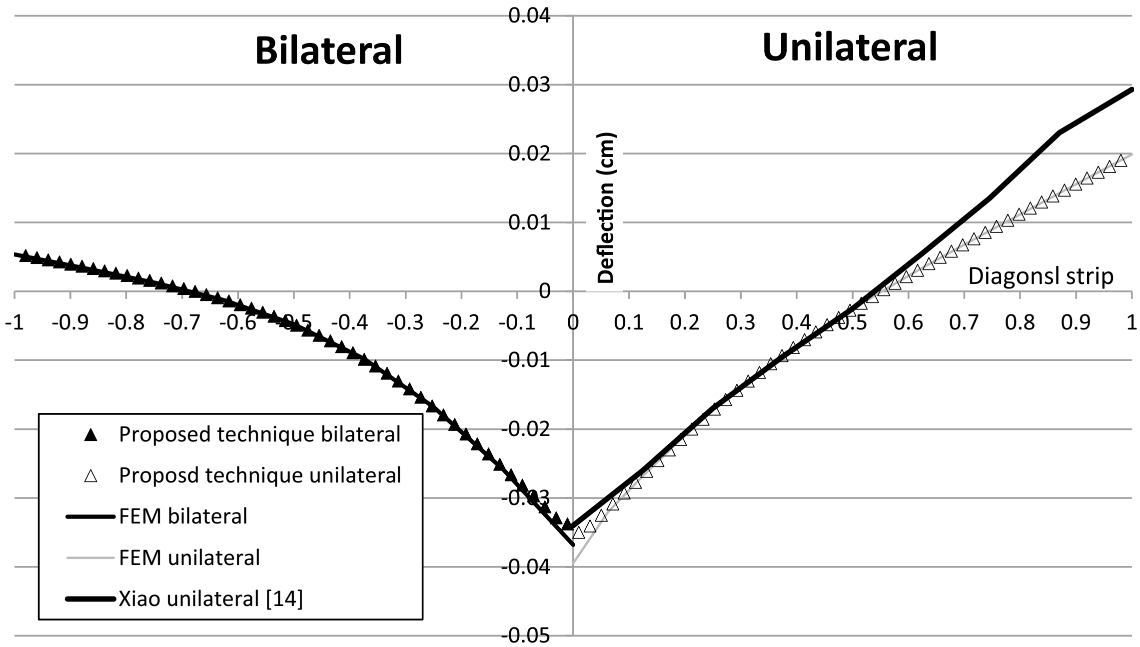


Fig. 12 Deflection along the diagonal strip for plate thickness = 30 cm for all models

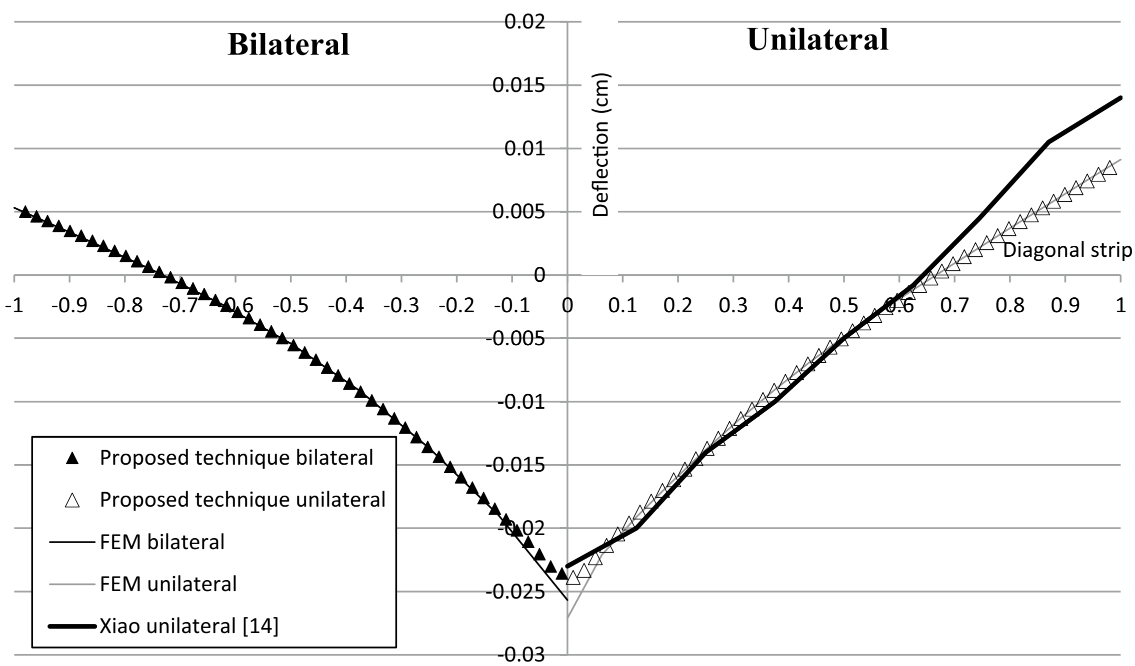


Fig. 13 Deflection along the diagonal strip for plate thickness = 40 cm for all models

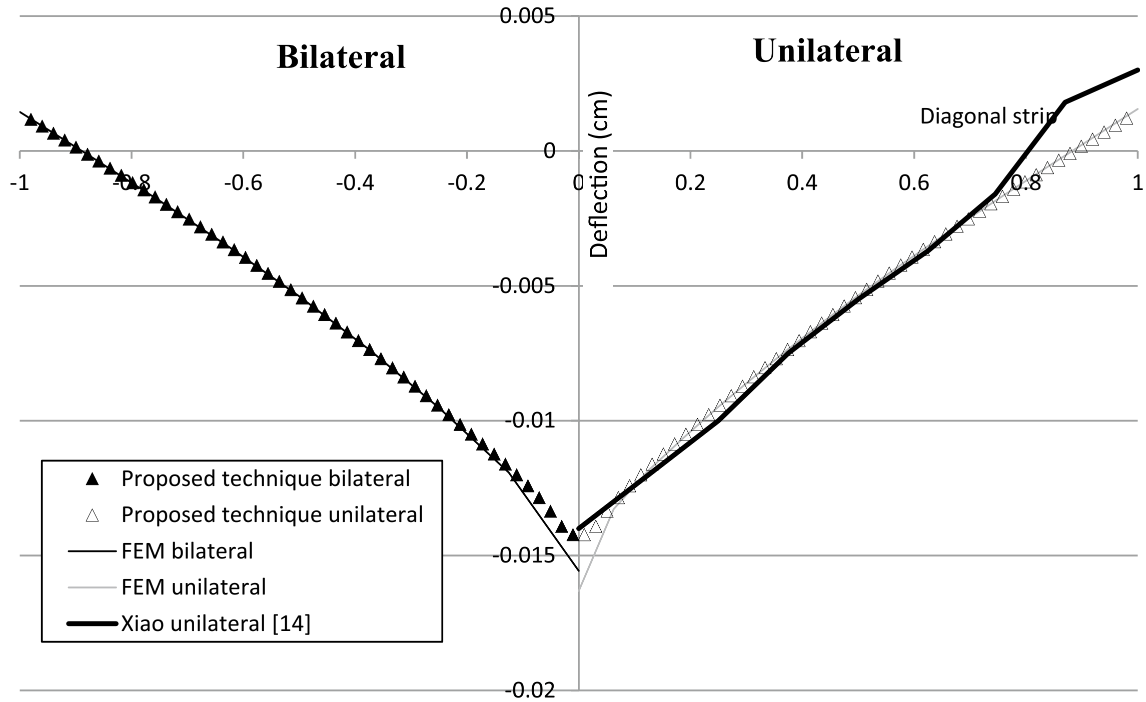


Fig. 14 Deflection along the diagonal strip for plate thickness = 60 cm for all models

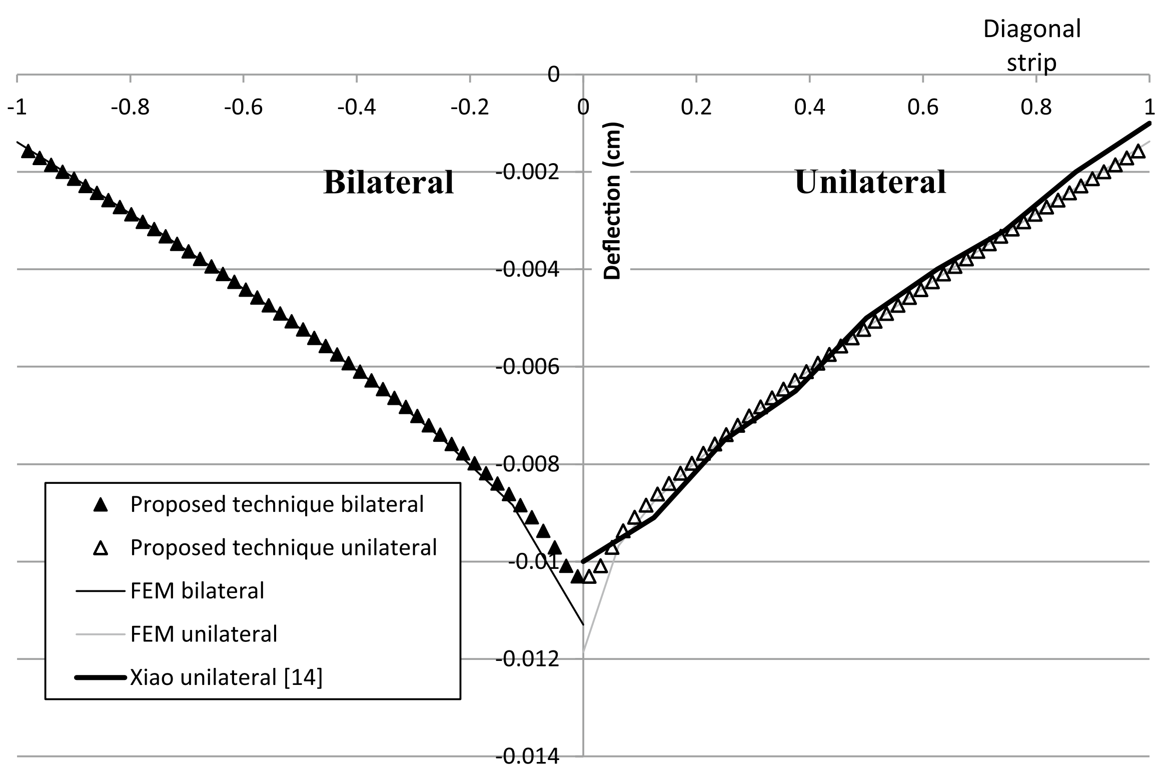


Fig. 15 Deflection along the diagonal strip for plate thickness = 80 cm for all models

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