



# A Variational Formulation to Find Finite Element Bending, Buckling and Vibration Equations of Nonlocal Timoshenko Beams

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## Abstract

A variational approach is developed to obtain bending, buckling and vibration finite element equations of nonlocal Timoshenko beams in this study. The reason for using the finite element method in this research is to investigate the behavior of nano-beams with complex geometry, material property and different boundary conditions. Weak forms of governing equations are derived, and the nonlocal differential elasticity theory is used to find the finite element formulation of nonlocal Timoshenko beams. In deriving the weak formulations, it is seen that it is impossible to construct the quadratic functional form due to non-symmetric bilinear property. Using the developed concepts and formulations, the bending and buckling of nonlocal Timoshenko beams with four classical boundary conditions are analyzed and the obtained results are compared with those reported in the literature. In order to show the capabilities of the proposed formulation in comparison with exact methods, the simply supported stepped nonlocal Timoshenko beam is selected and bending and buckling analyses are performed as well.

**Keywords** Variational approach · Finite element · Nonlocal Timoshenko beam · Bending · Buckling · Vibration

## 1 Introduction

Nano-beams and nano-plates have become one of the most important structures used widely in NEMS devices due to outstanding mechanical and physical properties. There are many research works about using such structures that can be referred to the works done by Wang et al. (2006, 2008), Ghannadpour and Mohammadi (2010, 2011), Ghannadpour et al. (2013), Wang and Wang (2007), Dinckal (2016) and Ebrahimi and Barati (2016). In the literature, it is observed from both experimental and theoretical simulation that the size effect has an important role on static and dynamic deformation behavior of materials and cannot be negligible in nano-sized structures (Farrokhhabadi and Tavakolian 2017, Tavakolian and Farrokhhabadi 2017 and Tavakolian et al. 2017). Therefore, by applying the size-independent theories, inaccurate results will be obtained.

Due to the difficulties in experimental work at the nanoscale and due to their being time-consuming, numerical simulations of nano-structures have been presented extensively by the researchers and they became interested to develop the size-dependent continuum theories. Therefore, to model the small-sized structures, the various continuum theories that are dependent on size have attracted a lot of attention. They can be pointed out to micro-morphic theory by Eringen and Suhubi (1964), micro-polar theory by Eringen and Suhubi (1964) and Chen et al. (2004), couple stress theory by Toupin (1962) and nonlocal elasticity theory by Eringen (1983).

Micro-morphic theory models a material as a continuous set of deformable point particles. The deformation of a micro-morphic continuum includes the displacements of the center of particles and the microscopic internal motion within the microstructure of a particle. By considering the particle as rigid, micro-morphic theory becomes micro-polar theory. Therefore, micro-polar theory yields only translational and rotational modes of rigid units. In the simplest form of micro-polar theory, the so-called couple stress theory, the rotation vector is dependent on the displacement vector. This theory, which is elaborated by

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Toupin (1962) and Koiter (1964), in addition to the classical constants, contains two material length scale coefficients for an isotropic elastic material. Recently, a modified couple stress theory was proposed and introduced by Yang et al. (2002) which contains one additional material length scale coefficient.

The history of nonlocal elasticity theory goes back to the works introduced by Eringen (1972, 1983). This theory expresses that the stress at any point in the body depends on the strain at this point and also on strains at all points in the body. The nonlocal effect is present due to the introduction of a nonlocal nanoscale which depends on the material and an internal characteristic length, and this parameter takes the zero value at macroscale.

Among the above-mentioned theories, the theory of nonlocal elasticity has been adopted by many researchers, as can be seen in the literature. This theory is used in two general versions, nonlocal differential and integral elasticity (Hu et al. 2008), but the former is more popular due to its simplicity. Many scientists have used the differential type of nonlocal elasticity for static and dynamic analyses of nano-sized structures.

The study of Peddieson et al. (2003) can be considered to be a pioneering work which first applied the nonlocal elasticity theory of Eringen. In this work, a nonlocal Euler beam was selected and its flexural behavior was studied using nonlocal differential elasticity. For integral type of nonlocal elasticity, Polizzotto (2001) developed the total potential energy, the complementary energy and the mixed Hu–Washizu principles.

It is worth noting that most of the scientific community's attention in the literature has been attracted to deriving the governing equations and the corresponding boundary conditions of nano-structures by the well-known technique of the calculus of variation and with nonlocal differential elasticity approach. Accordingly, the exact solution of most analyses on nano-beams with simple domains and classical boundary conditions can be found in the literature such as works carried out by Wang et al. (2006, 2007), Challamel and Wang (2008) and Wang (2005). For example, Wang et al. (2006) have analyzed the elastic buckling behavior of micro- and nanotubes based on differential type of nonlocal elasticity and the Timoshenko beam theory. The governing equations and the corresponding boundary conditions have been developed by the principle of virtual work. Such exact solutions are not generally available for nano-sized structures with complicated geometries. It is quite obvious that the finite element method can efficiently analyze the structures with arbitrary boundary conditions and also complicated geometries. So this has made this technique an efficient alternative to the previous solutions for nano-sized beams depending on beam model. Use of the finite element method in the framework of nonlocal integral elasticity has

been formulated for the first time by Pisano et al. (2009a, b). A new research activity on nonlocal structures with differential type of nonlocal elasticity theory and with Galerkin finite element method is the research carried out by Phadikar and Pradhan (2010). In their study, the Galerkin finite element technique in conjunction with differential type of nonlocal elasticity using both Euler beam theory and classical plate theory has been presented. More recently, Ghannadpour et al. (2013) analyzed the bending, buckling and vibration behaviors of nonlocal Euler beams. Weak form of the governing equation of nonlocal beams was outlined in their work.

With the above descriptions, finite element equations and weak formulations related to the bending, buckling and vibration of Timoshenko beams in the framework of nonlocal differential elasticity approach are presented in this paper. Thus, the first aim of this study is to present the final expressions for the weak form of the weighted residuals based on the differential type of nonlocal elasticity. The second aim is to drive the element matrices of nonlocal Timoshenko beams and analyze the bending and buckling behaviors of such beams.

## 2 Derivation of Governing Equations

In this section, for completeness of formulations, bending, buckling and vibration governing equations for nonlocal Timoshenko beams are derived. It is known that in a shear deformation beam theory, the so-called Timoshenko beam theory (TBT), transverse shear stress and strain are invariant throughout the thickness of the beam. This is due to ignoring the normality assumption in Euler beam theory. Therefore, in Timoshenko beam theory, plane sections remain plane after deformation but not necessarily normal to the longitudinal axis.

By adopting that the  $x$ - and  $z$ -axes are assumed along the length and thickness of the beam, respectively, the relations between strains and displacements according to the Timoshenko beam theory can be written as

$$\varepsilon_{xx} = z \frac{d\phi}{dx} \quad (1)$$

$$\gamma_{xz} = \phi + \frac{dw}{dx} \quad (2)$$

where  $z$  is measured from the mid-plane of the beam,  $\phi$  the rotation due to bending,  $w$  the transverse displacement,  $\varepsilon_{xx}$  the normal strain and  $\gamma_{xz}$  the transverse shear strain.

The virtual strain energy  $\delta U$ , the virtual potential energy  $\delta V$  of axial load  $P$  and transverse distributed load  $q = q(x)$  and the virtual kinetic energy  $\delta T$  of a Timoshenko beam by assuming free harmonic motion and including the effect of rotary inertia can be written as follows:

$$\delta U = \int_0^L \int_A (\sigma_{xx} \delta \varepsilon_{xx} + \sigma_{xz} \delta \gamma_{xz}) dA dx \quad (3)$$

$$\delta V = - \int_0^L \left( P \frac{dw}{dx} \frac{d\delta w}{dx} + q \delta w \right) dx \quad (4)$$

$$\delta T = \int_0^L (\rho A \omega^2 w \delta w + \rho I \omega^2 \phi \delta \phi) dx \quad (5)$$

In the above equations,  $\sigma_{xx}$  is the normal stress,  $\sigma_{xz}$  the transverse shear stress,  $L$  the length of the beam,  $A$  the cross-sectional area of the beam,  $I$  the second moment of area,  $\omega$  the circular frequency of vibration and  $\rho$  the mass density of the beam material.

By substituting Eqs. (1) and (2) into Eq. (3), the final form of the virtual strain energy by considering the shear correction factor  $K_s$  can be expressed as Eq. (6).

$$\begin{aligned} \delta U &= \int_0^L \int_A \left( \sigma_{xx} \delta \left( z \frac{d\phi}{dx} \right) + \sigma_{xz} \delta \left( \phi + \frac{dw}{dx} \right) \right) dA dx \\ &= \int_0^L \int_A \left( \sigma_{xxz} \frac{d\delta \phi}{dx} + \sigma_{xz} \left( \delta \phi + \frac{d\delta w}{dx} \right) \right) dA dx \\ &= \int_0^L \left( M \frac{d\delta \phi}{dx} + Q \left( \delta \phi + \frac{d\delta w}{dx} \right) \right) dx \end{aligned} \quad (6)$$

where  $M = \int_A \sigma_{xxz} dA$  and  $Q = K_s \int_A \sigma_{xz} dA$  are the bending moment and shear force, respectively. The Hamilton principle for the Timoshenko beam theory has the following form:

$$\begin{aligned} \delta T - \delta U - \delta V &= 0 \\ &= \int_0^L \left( \rho A \omega^2 w \delta w + \rho I \omega^2 \phi \delta \phi - M \frac{d\delta \phi}{dx} \right. \\ &\quad \left. - Q \delta \phi - Q \frac{d\delta w}{dx} + P \frac{dw}{dx} \frac{d\delta w}{dx} + q \delta w \right) dx \end{aligned} \quad (7)$$

The Euler differential equations can be derived using the well-known technique of variational calculus. After applying this technique, the following relations are obtained which are known as bending, buckling and vibration governing equations of Timoshenko beams.

$$\frac{dM}{dx} = Q - \rho I \omega^2 \phi \quad (8)$$

$$\frac{dQ}{dx} = P \frac{d^2 w}{dx^2} - \rho A \omega^2 w - q \quad (9)$$

The associated boundary conditions of the beam are also obtained as

$$w = 0 \quad \text{or} \quad V = Q - P \frac{dw}{dx} = 0 \quad (10)$$

$$\phi = 0 \quad \text{or} \quad M = 0 \quad (11)$$

As can be observed, the above equations are the same as the governing equations for local Timoshenko beam theory. As mentioned before, in this study the nonlocal differential elasticity theory is used. The essence of nonlocal theory is that the size effects are captured by considering that the stress at a point in a body is a function of strains at all other points. Therefore, due to the nonlocal constitutive relations, the shear force and bending moment expressions for the nonlocal beams are quite different. The constitutive equations in nonlocal elasticity and in one-dimensional problems can be written as (Eringen 1964 and 1983):

$$\sigma_{xx} - \eta^2 \frac{d^2 \sigma_{xx}}{dx^2} = E \varepsilon_{xx} \quad (12)$$

$$\sigma_{xz} = G \gamma_{xz} \quad (13)$$

where  $E$  is the Young's modulus,  $G$  the shear modulus and  $\eta$  the scale factor incorporating the size effect. The nonlocal effect in this theory is present due to the introduction of a nonlocal scale factor  $\eta = e_0 a$  which depends on the material  $e_0$  and an internal characteristic length  $a$  (e.g., granular distance and C–C bond length), and at macroscale, this parameter takes the zero value. More details on how to find this parameter can be found in the work studied by Wang et al. (2007).

As it can be seen in Eqs. (12) and (13), in the nonlocal differential elasticity the constitutive relation is presented in the differential equations form and also the constitutive relation for the shear stress and strain was assumed the same as in the local beam theory in this study. Multiplying Eq. (12) by  $z dA$  and integrating the result over the area  $A$  yields

$$M - \eta^2 \frac{d^2 M}{dx^2} = EI \frac{d\phi}{dx} \quad (14)$$

By multiplying Eq. (13) by shear correction factor  $K_s$  and integrating over the area  $A$  and also substituting Eqs. (8) and (9) into Eq. (14), the shear force and bending moment expressions for the nonlocal beam theory can be written in the following forms:

$$Q = K_s G A \left( \phi + \frac{dw}{dx} \right) \quad (15)$$

$$M = EI \frac{d\phi}{dx} + \eta^2 \left( P \frac{d^2 w}{dx^2} - \rho A \omega^2 w - \rho I \omega^2 \frac{d\phi}{dx} - q \right) \quad (16)$$

Finally, the governing equations of the nonlocal Timoshenko beam in terms of the displacement components  $w$  and  $\phi$  can be obtained by substituting Eqs. (15) and (16) into Eqs. (8) and (9).

$$-P \frac{d^2 w}{dx^2} + K_s GA \left( \frac{d\phi}{dx} + \frac{d^2 w}{dx^2} \right) + \rho A \omega^2 w + q = 0 \quad (17)$$

$$\begin{aligned} \eta^2 P \frac{d^3 w}{dx^3} + (EI - \eta^2 \rho I \omega^2) \frac{d^2 \phi}{dx^2} - \eta^2 \rho A \omega^2 \frac{dw}{dx} - \eta^2 \frac{dq}{dx} \\ - K_s GA \left( \frac{dw}{dx} + \phi \right) + \rho I \omega^2 \phi \\ = 0 \end{aligned} \quad (18)$$

It is noted that by setting  $\eta = 0$  in the above equations, the governing equations of the local Timoshenko beam can be retrieved.

### 3 Derivation of Weak Form

In this section, derivation of weak form of the obtained governing equations of the nonlocal Timoshenko beams is outlined. As it is known, the weak form for a differential equation can be obtained based on the weighted-integral form that is equivalent to the governing differential equation as well as the associated natural boundary conditions.

Based on the above descriptions, the weak form of the governing Eqs. (17) and (18) is developed using the inverse of variational calculus, i.e., each equation must be multiplied by a weight function. To do this, Eq. (17) is multiplied by a weight function  $-\psi_1$  and Eq. (18) by a weight function  $-\psi_2$ , and then, they are integrated over the beam length.

$$\int_0^L -\psi_1 \left( -P \frac{d^2 w}{dx^2} + K_s GA \left( \frac{d\phi}{dx} + \frac{d^2 w}{dx^2} \right) + \rho A \omega^2 w + q \right) dx = 0 \quad (19)$$

$$\begin{aligned} \int_0^L -\psi_2 \left( \eta^2 P \frac{d^3 w}{dx^3} + (EI - \eta^2 \rho I \omega^2) \frac{d^2 \phi}{dx^2} - \eta^2 \rho A \omega^2 \frac{dw}{dx} \right. \\ \left. - \eta^2 \frac{dq}{dx} - K_s GA \left( \frac{dw}{dx} + \phi \right) + \rho I \omega^2 \phi \right) dx = 0 \end{aligned} \quad (20)$$

After performing integration once by parts on the first two terms of Eq. (19) and on the first four terms of Eq. (20), and using Eqs. (10), (11), (15) and (16), the weak statements can be written in the following final form.

$$\begin{aligned} \int_0^L \left( -P \frac{dw}{dx} \frac{d\psi_1}{dx} + K_s GA \left( \frac{dw}{dx} + \phi \right) \right. \\ \left. \frac{d\psi_1}{dx} - \rho A \omega^2 w \psi_1 - q \psi_1 \right) dx - [\psi_1 V]_0^L = 0 \end{aligned} \quad (21)$$

$$\begin{aligned} \int_0^L \left( \eta^2 P \frac{d^2 w}{dx^2} \frac{d\psi_2}{dx} + (EI - \eta^2 \rho I \omega^2) \frac{d\phi}{dx} \frac{d\psi_2}{dx} \right. \\ \left. - \eta^2 \rho A \omega^2 w \frac{d\psi_2}{dx} - \eta^2 q \frac{d\psi_2}{dx} \right. \\ \left. + K_s GA \left( \frac{dw}{dx} + \phi \right) \psi_2 - \rho I \omega^2 \phi \psi_2 \right) dx - [\psi_2 M]_0^L = 0 \end{aligned} \quad (22)$$

The coefficients of the weight functions in the boundary integrals are called secondary variables, and their specifications constitute the natural boundary conditions. The weight functions  $\psi_1$  and  $\psi_2$  must have the physical interpretations that give  $\psi_1 V$  and  $\psi_2 M$  units of work. Clearly,  $\psi_1$  must be equivalent to (the variation of) the transverse deflection  $w$ , and  $\psi_2$  must be equivalent to (the variation of) the rotation function  $\phi$ .

In the above equations, it can be observed that all the terms are either bilinear or linear. However, the first and the third terms in Eq. (22) are non-symmetric bilinear. Therefore, it is impossible to construct the associated quadratic functional form.

It must be remembered that, as in the case of variational and weighted-residual methods, the aim is to satisfy the governing differential equations in a weighted-integral sense. The type of finite element model depends on the weighted-integral form used to generate the algebraic equations. Thus, if one uses the weak form, the resulting model will be different from those obtained with a weighted-residual statement in which the weight function can be any of several choices. In the remaining of the paper, the weak form finite element models are concerned.

### 4 Weak Form Finite Element Formulation

To construct the finite element model, the global domain  $(0, L)$  of the problem should be divided into a set of sub-domains. Each isolated interval, which is called finite element, is of length  $l$  with domain  $(x_e, x_{e+1})$ . Therefore, the weak form must be applied to one of the isolated elements.

$$\int_0^l \left( -P \frac{dw^e}{dx} \frac{d\psi_1^e}{dx} + K_s GA \left( \frac{dw^e}{dx} + \phi^e \right) \frac{d\psi_1^e}{dx} \right. \\ \left. - \rho A \omega^2 w^e \psi_1^e - q \psi_1^e \right) dx - [\psi_1^e V]_0^l = 0 \quad (23)$$

$$\begin{aligned} \int_0^l \left( \eta^2 P \frac{d^2 w^e}{dx^2} \frac{d\psi_2^e}{dx} + (EI - \eta^2 \rho I \omega^2) \frac{d\phi^e}{dx} \frac{d\psi_2^e}{dx} \right. \\ \left. - \eta^2 \rho A \omega^2 w^e \frac{d\psi_2^e}{dx} - \eta^2 q \frac{d\psi_2^e}{dx} \right. \\ \left. + K_s GA \left( \frac{dw^e}{dx} + \phi^e \right) \psi_2^e - \rho I \omega^2 \phi^e \psi_2^e \right) dx - [\psi_2^e M]_0^l = 0 \end{aligned} \quad (24)$$

A close examination of the terms in Eqs. (23) and (24) shows that the transverse deflection  $w$  is differentiated twice with respect to  $x$  and the rotation function  $\phi$  is differentiated only once. In the present work, the following displacement fields in a local coordinate system  $\bar{x}$  have been defined for  $w$  and  $\phi$ .

$$w^e = \sum_{j=1}^4 N_j(\bar{x}) w_j \quad (25)$$

$$\phi^e = \sum_{j=1}^4 N_j(\bar{x}) \phi_j$$

where  $N_j(\bar{x})$ ,  $j = 1, 2, 3, 4$  are Lagrange interpolation functions which are given as follows:

$$N_1(\bar{x}) = \frac{-1}{2} \left( \frac{\bar{x}}{l} - 1 \right) \left( 3 \frac{\bar{x}}{l} - 1 \right) \left( 3 \frac{\bar{x}}{l} - 2 \right)$$

$$N_2(\bar{x}) = \frac{9\bar{x}}{2l} \left( \frac{\bar{x}}{l} - 1 \right) \left( 3 \frac{\bar{x}}{l} - 2 \right)$$

$$N_3(\bar{x}) = \frac{-9\bar{x}}{2l} \left( \frac{\bar{x}}{l} - 1 \right) \left( 3 \frac{\bar{x}}{l} - 1 \right)$$

$$N_4(\bar{x}) = \frac{1\bar{x}}{2l} \left( 3 \frac{\bar{x}}{l} - 1 \right) \left( 3 \frac{\bar{x}}{l} - 2 \right) \quad (26)$$

and  $w_j$  and  $\phi_j$ ,  $j = 1, 2, 3, 4$  denote the nodal degrees of freedom of the element. By substituting the displacement fields into the weak form Eqs. (23) and (24) and assuming  $q = q_0$  (constant), the following finite element equations are obtained.

$$\sum_{j=1}^4 \left( K_{ij}^{11} - \beta B_{ij}^{11} - \omega^2 M_{ij}^{11} \right) w_j + \sum_{j=1}^4 K_{ij}^{12} \phi_j - Q_i^1 - F_i^1 = 0;$$

$$(i = 1, 2, 3, 4) \quad (27)$$

$$\sum_{j=1}^4 \left( K_{ij}^{21} - \beta B_{ij}^{21} - \omega^2 M_{ij}^{21} \right) w_j + \sum_{j=1}^4 \left( K_{ij}^{22} - \omega^2 \left( M_{ij}^{22} + M_{ij}^{\prime 22} \right) \right) \phi_j - Q_i^2 - F_i^2 = 0;$$

$$(i = 1, 2, 3, 4) \quad (28)$$

where

$$K_{ij}^{11} = \int_0^l K_s GA \frac{dN_i}{d\bar{x}} \frac{dN_j}{d\bar{x}} d\bar{x}$$

$$B_{ij}^{11} = \int_0^l \bar{P} \frac{dN_i}{d\bar{x}} \frac{dN_j}{d\bar{x}} d\bar{x}$$

$$M_{ij}^{11} = \int_0^l \rho A N_i N_j d\bar{x}$$

$$K_{ij}^{12} = \int_0^l K_s GA \frac{dN_i}{d\bar{x}} N_j d\bar{x}$$

$$K_{ij}^{21} = \int_0^l K_s G A N_i \frac{dN_j}{d\bar{x}} d\bar{x}$$

$$B_{ij}^{21} = \int_0^l -\eta^2 \bar{P} \frac{dN_i}{d\bar{x}} \frac{d^2 N_j}{d\bar{x}^2} d\bar{x}$$

$$M_{ij}^{21} = \int_0^l \eta^2 \rho A \frac{dN_i}{d\bar{x}} N_j d\bar{x}$$

$$K_{ij}^{22} = \int_0^l \left( EI \frac{dN_i}{d\bar{x}} \frac{dN_j}{d\bar{x}} + K_s G A N_i N_j \right) d\bar{x}$$

$$M_{ij}^{22} = \int_0^l \rho I N_i N_j d\bar{x}$$

$$M_{ij}^{\prime 22} = \int_0^l \eta^2 \rho I \frac{dN_i}{d\bar{x}} \frac{dN_j}{d\bar{x}} d\bar{x}$$

$$Q_i^1 = \int_0^l q_0 N_i d\bar{x}; \quad Q_i^2 = \int_0^l \eta^2 q_0 \frac{dN_i}{d\bar{x}} d\bar{x}$$

$$F_i^1 = N_i(l) V|_{\bar{x}=l} - N_i(0) V|_{\bar{x}=0}; \quad F_i^2 = N_i(l) M|_{\bar{x}=l} - N_i(0) M|_{\bar{x}=0} \quad (29)$$

and the eigenvalue  $\beta = P/\bar{P}$  represents the ratio of actual buckling load and applied in-plane load. Equations (27) and (28) can be written in matrix form as

$$\left( \begin{bmatrix} K^{11} & K^{12} \\ K^{21} & K^{22} \end{bmatrix} - \beta \begin{bmatrix} B^{11} & 0 \\ B^{21} & 0 \end{bmatrix} \right) \begin{Bmatrix} w \\ \phi \end{Bmatrix} = \begin{Bmatrix} Q^1 \\ Q^2 \end{Bmatrix} + \begin{Bmatrix} F^1 \\ F^2 \end{Bmatrix} - \omega^2 \begin{bmatrix} M^{11} & 0 \\ M^{21} & M^{22} + M^{\prime 22} \end{bmatrix} \begin{Bmatrix} w \\ \phi \end{Bmatrix} \quad (30)$$

or

$$([K] - \beta[B] - \omega^2[M])\{U\} = \{Q\} + \{F\} \quad (31)$$

where  $\{U\}$ ,  $[K]$ ,  $[B]$ ,  $[M]$ ,  $\{Q\}$  and  $\{F\}$  are defined in ‘‘Appendix’’.

It is emphasized that the corresponding equations of the local Timoshenko beam element can be achieved by setting  $[B^{21}] = [M^{21}] = [M^{\prime 22}] = 0$  in Eq. (30). The submatrices  $[B^{21}]$  and  $[M^{21}]$  have non-symmetric effect on the finite element equations of the nonlocal Timoshenko beam element.

Finally, by applying these expressions to obtain the matrices of individual elements, the overall matrices for the whole beam can be assembled by using the conventional routines of finite element method.

## 5 Results and Discussions

In the above sections, the finite element formulations for bending, buckling and vibration of nonlocal Timoshenko beams were presented. However, the results for bending and buckling of nonlocal Timoshenko beams are obtained

and presented in this section. In order to conduct convergence study and also validation of the results, the following data are adopted in generating the bending and buckling results.

Nano-rod with diameter  $d = 1$  nm, Young’s modulus  $E = 1$  TPa, Poisson’s ratio  $\nu = 0.19$ , shear correction factor  $K_s = 0.9$ , applied in-plane load  $\bar{P} = 1$  nN and uniform distributed load  $q_0 = 1$  nN/nm.

The convergence studies carried out for the maximum deflection  $w$  and critical buckling load  $P_{cr}$  of a simply supported nano-rod with the scale coefficient  $\eta$  of 1 and length-to-diameter ratio  $L/d$  of 10 are tabulated in Table 1.

In this table, the results of maximum deflection and critical buckling load obtained by Wang et al. (2006 and 2008) are also included. In the mentioned references, the governing equations and the boundary conditions have been derived using the principle of virtual work and explicit expressions for the transverse deflections and critical buckling loads of nano-beams with various end conditions have been presented.

The convergence study shows that over 20 elements are needed to obtain accurate results for the critical buckling load  $P_{cr}$ . But the data related to the bending analysis converge with fewer numbers of elements. However, for the sake of confidence, all of the results presented in this study have been calculated using 30 elements.

In the next step and in order to compare the results obtained by the proposed FEM, the maximum deflections  $w$  and critical buckling loads  $P_{cr}$  are tabulated and compared with those of obtained by Wang et al. (2006 and 2008) in Tables 2 and 3, respectively. As mentioned before, the results of Wang were obtained by solving the governing equations and so they are named exact solutions.

It can be seen that there is an excellent agreement between the FEM results and exact solutions.

**Table 1** Convergence study of maximum deflection and critical buckling load with  $\eta = 1$  and  $L/d = 10$

$N$	$w$ (nm)	$P_{cr}$ (nN)
4	2.949	4.34289
8	2.949	4.34420
12	2.949	4.34463
16	2.949	4.34480
20	2.949	4.34488
24	2.949	4.34492
28	2.949	4.34494
32	2.949	4.3450
36	2.949	4.3450
	2.949 <sup>b</sup>	4.345 <sup>a</sup>

<sup>a</sup>Exact results obtained by Wang et al. (2006)

<sup>b</sup>Exact results obtained by Wang et al. (2008)

Table 2 shows that for a simply supported nonlocal Timoshenko beam subjected to a uniformly distributed load, the deflection is affected by the small-scale coefficient, whereas in the clamped nonlocal Timoshenko beam example, the deflection is the same as that of local Timoshenko beam. In other words, in the clamped nonlocal Timoshenko beam, the deflection is not affected by the scaling factor (Challamel and Wang 2008).

It can be inferred from Table 3 that the application of the local elasticity models for nano-sized structures analysis would lead to an overprediction of the buckling loads if the effect of small scale is neglected between the atoms.

In order to show the ability of the finite element method, the simply supported stepped nonlocal Timoshenko beam shown in Fig. 1 has been chosen and bending and buckling analyses have been performed.

The following data are adopted in generating the results of stepped beam with circular cross section.

Nano-rod with diameters  $d_1 = 4$  nm,  $d_2 = 3$  nm,  $d_3 = 2$  nm, lengths  $l_1 = l_2 = l_3 = 10$  nm, Young’s modulus  $E = 1$  TPa, Poisson’s ratio  $\nu = 0.19$ , shear correction factor  $K_s = 0.9$ , applied in-plane load  $\bar{P} = 1$  nN and uniform distributed load  $q_0 = 1$  nN/nm.

All the analyses for this beam have been obtained by choosing 45 finite elements of equal length. The maximum deflections for various nonlocal parameters and also critical buckling loads are tabulated in Table 4.

As can be seen from the above table, the deflections and the critical buckling loads of the stepped nonlocal Timoshenko beam are affected by the scale coefficient. This table demonstrates the significant effects of the scale coefficient on higher modes of buckling. This can also be seen from the buckling deflection modes represented in Fig. 2.

**Table 2** Maximum deflections  $w$  for simply supported and clamped rods with various scale coefficients

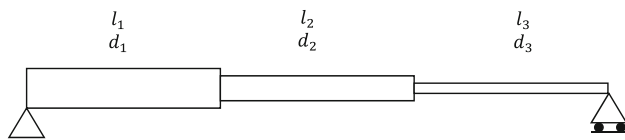
$\eta$	$w$ (nm)			
	Simply supported rod		Clamped rod	
	Exact <sup>a</sup>	FEM	Exact <sup>a</sup>	FEM
0	2.6947	2.6947	0.5726	0.5726
0.5	2.7583	2.7583	0.5726	0.5726
1	2.9493	2.9493	0.5726	0.5726
1.5	3.2676	3.2676	0.5726	0.5726
2	3.7133	3.7133	0.5726	0.5726

<sup>a</sup>Exact results obtained by Wang et al. (2008)

**Table 3** Critical buckling loads  $P_{cr}$  (nN) for rods with various boundary conditions and various length-to-diameter ratios  $L/d$

$L/d$	$\eta$									
	0		0.5		1		1.5		2	
	Exact <sup>a</sup>	FEM	Exact <sup>a</sup>	FEM	Exact <sup>a</sup>	FEM	Exact <sup>a</sup>	FEM	Exact <sup>a</sup>	FEM
<i>Simply supported rod</i>										
10	4.767	4.7670	4.654	4.6540	4.345	4.3450	3.9121	3.9121	3.4333	3.4333
12	3.3267	3.3267	3.2713	3.2713	3.1156	3.1156	2.8865	2.8865	2.6172	2.6172
14	2.4514	2.4514	2.4212	2.4212	2.3348	2.3348	2.2038	2.2038	2.0432	2.0432
16	1.8805	1.8805	1.8626	1.8627	1.8111	1.8111	1.7313	1.7313	1.6306	1.6306
18	1.4878	1.4878	1.4766	1.4766	1.444	1.4440	1.3928	1.3928	1.3269	1.3269
20	1.2063	1.2063	1.1989	1.1989	1.1773	1.1773	1.1431	1.1431	1.0983	1.0983
<i>Clamped-simply supported rod</i>										
10	9.5605	9.5605	9.1179	9.1179	8.0055	8.0054	6.652	6.6518	5.3782	5.3780
12	6.7118	6.7118	6.4904	6.4904	5.9059	5.9058	5.1348	5.1347	4.341	4.3409
14	4.9638	4.9638	4.8416	4.8416	4.5086	4.5085	4.0448	4.0447	3.5355	3.5354
16	3.8168	3.8168	3.7441	3.7441	3.5418	3.5417	3.249	3.2490	2.912	2.9120
18	3.0248	3.0247	2.9789	2.9789	2.8493	2.8493	2.6567	2.6567	2.427	2.4270
20	2.4553	2.4553	2.425	2.4250	2.3384	2.3384	2.2071	2.2070	2.0462	2.0461
<i>Clamped rod</i>										
10	18.192	18.1919	16.649	16.6490	13.273	13.2723	9.92	9.9194	7.3283	7.3279
12	12.874	12.8742	12.082	12.0819	10.199	10.1988	8.0964	8.0959	6.2829	6.2825
14	9.5687	9.5687	9.124	9.1239	8.0077	8.0074	6.6514	6.6510	5.3765	5.3761
16	7.3818	7.3817	7.1143	7.1142	6.4168	6.4166	5.5155	5.5153	4.6092	4.6089
18	5.8631	5.8631	5.6931	5.6930	5.2375	5.2374	4.6212	4.6210	3.9675	3.9673
20	4.767	4.7670	4.654	4.6540	4.345	4.3449	3.9122	3.9120	3.4333	3.4331
<i>Cantilever rod</i>										
10	1.2063	1.2063	1.1989	1.1989	1.1773	1.1773	1.1431	1.1431	1.0983	1.0983
12	0.8387	0.8387	0.8352	0.8352	0.8246	0.8246	0.8077	0.8077	0.7851	0.7851
14	0.6167	0.6167	0.6147	0.6147	0.609	0.6090	0.5997	0.5997	0.5872	0.5872
16	0.4724	0.4724	0.4712	0.4712	0.4679	0.4679	0.4623	0.4624	0.4549	0.4549
18	0.3734	0.3734	0.3726	0.3726	0.3705	0.3705	0.367	0.3671	0.3623	0.3623
20	0.3025	0.3025	0.302	0.3020	0.3006	0.3006	0.2984	0.2984	0.2952	0.2952

<sup>a</sup>Exact results obtained by Wang et al. (2006)



**Fig. 1** Simply supported stepped nonlocal Timoshenko beam

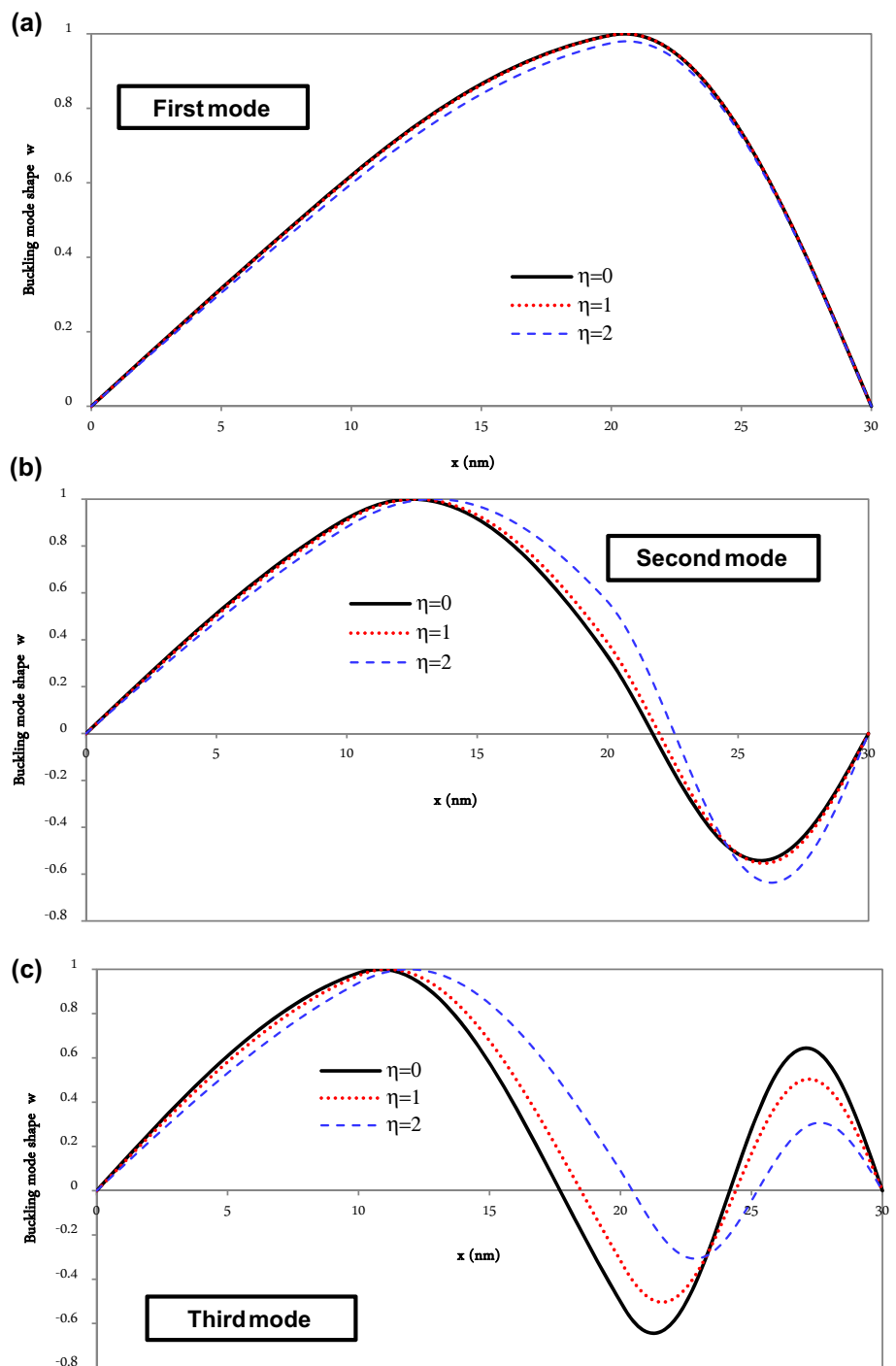
### 6 Conclusion

In this paper, nonlocal Timoshenko beam element matrices in bending, buckling and vibration formulations were derived. The final expressions for the weak form of the weighted residuals using the differential type of nonlocal elasticity were also derived. In the development process, it was found that there are two terms in the weak statements of the governing equations which have non-symmetric bilinear form. Therefore, it was concluded that it is

**Table 4** Maximum deflections and critical buckling loads for the stepped beam

$\eta$	Max. deflection (nm)	Critical buckling load nN (three modes)
0	4.7085	21.077
		103.342
		191.488
1	4.7649	20.621
		96.010
		162.117
2	4.9341	19.345
		76.524
		112.690

**Fig. 2** **a** First mode shape of simply supported stepped nonlocal Timoshenko beam, **b** second mode shape of simply supported stepped nonlocal Timoshenko beam, **c** third mode shape of simply supported stepped nonlocal Timoshenko beam



impossible to construct the quadratic functional form. In this regard and after deriving the finite element formulation, some nano-beams with four classical boundary conditions were analyzed in bending and buckling problems. It was observed that the results of the developed method are in excellent agreement with exact solutions. Also, in order to show the capabilities of the FEM method in comparison

with exact methods, the simply supported stepped nonlocal Timoshenko beam was selected and bending and buckling analyses were performed. The present study will be helpful in the analyses and design of nano-sized structures with complicated geometry, material property and boundary conditions.



## Appendix: Nonlocal Timoshenko Beam element matrices

The element matrices for nonlocal Timoshenko beam are presented as

$$\begin{aligned} \{U\} &= \{w_1 \ w_2 \ w_3 \ w_4 \ \phi_1 \ \phi_2 \ \phi_3 \ \phi_4\}^T \\ [K] &= \frac{K_s GA}{1680l} \begin{bmatrix} 42[C_1] & 21l[C_2] \\ 21l[C_2]^T & l^2[C_3] \end{bmatrix} + \frac{EI}{40l} \begin{bmatrix} 0 & 0 \\ 0 & [C_1] \end{bmatrix} \\ [M] &= \frac{\rho l}{1680} \begin{bmatrix} A[C_3] & 0 \\ 0 & l[C_3] \end{bmatrix} + \frac{\eta^2 \rho}{80l} \begin{bmatrix} 0 & 0 \\ Al[C_2] & 2l[C_1] \end{bmatrix} \\ [B] &= \frac{\bar{P}}{40l} \begin{bmatrix} [C_1] & 0 \\ 0 & 0 \end{bmatrix} + \frac{9\eta^2 \bar{P}}{8l^2} \begin{bmatrix} 0 & 0 \\ [C_4] & 0 \end{bmatrix} \\ \{Q\} &= \frac{q_0 l}{8} \begin{Bmatrix} 1 \\ 3 \\ 3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} + q_0 \eta^2 \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ 1 \end{Bmatrix}; \quad \{F\} = \begin{Bmatrix} -V|_{\bar{x}=0} \\ 0 \\ 0 \\ V|_{\bar{x}=l} \\ -M|_{\bar{x}=0} \\ 0 \\ 0 \\ M|_{\bar{x}=l} \end{Bmatrix} \end{aligned}$$

where  $[C_1]$ ,  $[C_2]$ ,  $[C_3]$  and  $[C_4]$  are defined as

$$\begin{aligned} [C_1] &= \begin{bmatrix} 148 & -189 & 54 & -13 \\ -189 & 432 & -297 & 54 \\ 54 & -297 & 432 & -189 \\ -13 & 54 & -189 & 148 \end{bmatrix} \\ [C_2] &= \begin{bmatrix} -40 & -57 & 24 & -7 \\ 57 & 0 & -81 & 24 \\ -24 & 81 & 0 & -57 \\ 7 & -24 & 57 & 40 \end{bmatrix} \\ [C_3] &= \begin{bmatrix} 128 & 99 & -36 & 19 \\ 99 & 648 & -81 & -36 \\ -36 & -81 & 648 & 99 \\ 19 & -36 & 99 & 128 \end{bmatrix} \\ [C_4] &= \begin{bmatrix} 13 & -31 & 23 & -5 \\ -9 & 27 & -27 & 9 \\ -9 & 27 & -27 & 9 \\ 5 & -23 & 31 & -13 \end{bmatrix} \end{aligned}$$

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