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RESEARCH PAPER

Analysis of the Mechanical and Thermal Buckling of Laminated Beams by New Refined Shear Deformation Theory

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Abstract

Thermo-mechanical buckling analysis of symmetric and antisymmetric laminated composite beams is performed based on a refined simple *n*th higher-order shear deformation theory. The theory accounts for the parabolic distribution of the transverse shear strains and satisfies the zero traction boundary conditions on the surfaces of the beam without using shear correction factors. The governing equations and corresponding simply boundary conditions are obtained with the aid of minimum total potential energy principle. The effects of temperatures on non-dimensional critical buckling loads are investigated. Numerical results due to present theory are compared with data available in the literature to show the accuracy and simplicity of the proposed theory in analyzing the thermo-mechanical buckling of laminated composite beams.

Keywords Laminated · Beams · Buckling · Mechanical · Thermal · Refined simple *n*th higher-order shear deformation theory

1 Introduction

The use of composite materials has been greatly increased in the weight-sensitive applications such as aerospace, marine, civil and mechanical engineering structures because of their superior mechanical properties such as high strength-to-weight ratio and high stiffness-to weight-ratio as well as their directionality property capable of providing the desired elastic couplings through the proper selection of the layup parameters. The laminated composite beams are basic structural components and are widely used in various structures. For the safe design of composite beams, accurate knowledge of their vibration characteristics and buckling behaviors are necessary. The wide use of laminated beams has stimulated considerable interest in their dynamic and buckling analyses. A lot of relevant papers have been published in recent decades, and many mathematical models and solution techniques have been developed, for example, Touratier (1991), Soldatos(1992), Khdeir and Reddy (1997), Khdeir (2001), Karama et al. (2003), Kapuria et al.(2004), Aydogdu (2006, 2009), Reddy (1997), Emam (2011), Xiaohui and Wanji (2009), Kim (2009), Della and Shu(2009), Akbas and Kocaturk (2012), Yu and Sun (2012), Mohri et al. (2012), Vo and Thai (2012a, b), Kim and Choi (2013), Huang et al. (2014), Aktaş and Balcıoğlu (2014).

Recently, Akgöz, and Civalek (2015) developed new non-classical sinusoidal plate model via modified strain gradient theory. This model takes into account the effects of shear deformation without any shear correction factors and also can capture the size effects due to additional material length-scale parameters. Li and Qiao (2015a, b) extended the Reddy's high-order shear deformation beam theory with a von Karman type of kinematic nonlinearity for mechanical and thermal post-buckling analysis of anisotropic laminated beams with different boundary conditions. Kahya(2016) developed the multilayer beam finite element for vibration and buckling of laminated composite and sandwich beams via the first-order shear deformation theory. The multilayered beam element consists of N layers and includes totally $3N+7$ degrees of freedom (DOFs); in addition, slip and delamination between the layers are not allowed. Canales

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and Mantari (2016) derived the Ritz solution for vibration and buckling analysis of composite beams using a generalized higher-order shear deformation theory. Ergun et al. (2016) studied experimentally the free vibration and buckling behaviors of hybrid composite beams having different span lengths and orientation angles subjected to different impact energy levels.

The analytical solutions for simply supported and clamped boundary conditions of a post-buckling sandwich beam are obtained in the thermal environment presented by Li et al. (2018). The effect of the humidity conditions on thermal buckling analysis of grapheme system containing two layers under different boundary conditions is developed by Sobhy and Zenkour (2018); Dihaj et al.(2018) studied the free vibration analysis of chiral double-walled carbon nanotube embedded in an elastic medium using non-local elasticity theory and Euler–Bernoulli beam model. Akbaş (2018) presented a new method for post-buckling responses of a simply supported laminated composite beam subjected to a non-follower axially compression loads using nonlinear kinematic model of the laminated beam in conjunction with Timoshenko beam theory and total Lagrangian approach. The unified approach for compressive buckling analysis of stiffened composite plates, which takes into account the contribution of stringers' rotational stiffness, achieves a closed-form solution presented by Feng (2018).

In the present paper, the authors combine the displacement field of theory developed by Xiang (2014) and the displacement field of refined shear deformation theory to develop a new refined simple nth higher-order shear deformation theory for thermo-mechanical buckling analysis of laminated beam. The theory satisfies that the transverse shear stresses should be vanished at the top and bottom surfaces of beam, and so there is no need for using a shear correction factor. That is because the present simplified refined nth-order theory is based on the assumption that the in-plane and transverse displacements consist of bending and shear components, in which the bending components do not contribute toward shear forces and, likewise, the shear components do not contribute toward bending moments. The governing differential equations and corresponding simply boundary conditions in buckling are derived with the aid of minimum total potential energy principle. The thermal effects on the critical buckling loads of simply supported laminate beams are investigated. The accuracy of this theory is demonstrated according to some numerical examples and comparisons with the corresponding data in the literature.

2 Theoretical Formulations

The displacement field of the conventional nth-order shear deformation theory is given by Xiang (2014):

$$\begin{aligned} &U(x,y,z) = u_0(x,y) + z\varphi_x(x,y) - \frac{1}{n}\left(\frac{2}{h}\right)^{n-1} z^n \left(\varphi_x(x,y) + \frac{\partial w(x,y)}{\partial x}\right) \\ &n = 3,5,7,9,\ldots \\ &W(x,y,z) = w_0(x,y) \end{aligned} \tag{1}$$

where w_0 and φ_x are two unknown displacement functions of the mid-plane of the beam; and h is the thickness of the beam. By dividing the transverse displacement into bending and shear parts (i.e., $w_0 = w_b + w_s$) and making further assumptions given by $\varphi_x = -\partial w_b / \partial x$, the displacement field of the new refined theory can be rewritten in a simpler form as:

where

$$\begin{aligned} &U(x,y,z) = u_0(x,y) - z\frac{\partial w_b}{\partial x} - \frac{1}{n}\left(\frac{2}{h}\right)^{n-1} z^n \left(\frac{\partial w_s}{\partial x}\right) \\ &n = 3,5,7,9,\ldots \\ &W(x,y,z) = w_b(x,y) + w_s(x,y) \end{aligned} \tag{2}$$

Clearly, the displacement field in Eq. (2) contains only two unknowns, w_b and w_s. The nonzero strains associated with the displacement field in Eq. (2) are:

$$\begin{aligned} &\varepsilon_x = \varepsilon_x^0 + zk_x^b + f(z)k_x^s \\ &\gamma_{xz} = g(z)\gamma_{xz}^s \end{aligned} \tag{3}$$

where

$$\begin{aligned} &\varepsilon_x^0 = \frac{\partial u_0}{\partial x}, \quad k_x^b = -\frac{\partial^2 w_b}{\partial x^2}, \quad k_x^s = -\frac{\partial^2 w_s}{\partial x^2}, \\ &\gamma_{xz}^s = \frac{\partial w_s}{\partial x} g(z) = 1 - f'(z), \quad f'(z) = \frac{\mathrm{d}f(z)}{\mathrm{d}z}, \\ &f(z) = \frac{1}{n}\left(\frac{2}{h}\right)^{n-1} z^n, \quad g(z) = 1 - \left(\frac{2z}{h}\right)^{n-1} \end{aligned} \tag{4}$$

Constitutive relation can be written in matrix form as follows:

$$\begin{Bmatrix} \sigma_x \\ \sigma_{xz} \end{Bmatrix}^{(k)} = \begin{bmatrix} \bar{Q}_{11} & 0 \\ 0 & \bar{Q}_{55} \end{bmatrix} \left(\begin{Bmatrix} \varepsilon_x \\ \gamma_{xz} \end{Bmatrix} - \alpha_x \Delta T \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \right)^{(k)} \tag{5}$$

where usual notations for normal and shear stress components are adopted. The relationship of the global reduced stiffness matrix $\bar{Q}_{ij}$ and transformed coefficient of thermal expansion can be referred to any standard texts such as (Reddy (1997)).

After substituting Eq. (3) in Eq. (5), the resulting equation is integrated through the thickness of the laminate. Then, the laminated constitutive equations take the form

$$\begin{Bmatrix} N_x \\ M_x^b \\ M_x^s \end{Bmatrix} = \begin{bmatrix} A_{11} & B_{11} & B_{11}^s \\ B_{11} & D_{11} & D_{11}^s \\ B_{11}^s & D_{11}^s & H_{11}^s \end{bmatrix} \begin{Bmatrix} \varepsilon_x^0 \\ k_x^b \\ k_x^s \end{Bmatrix} - \begin{Bmatrix} N_x^T \\ M_x^{bT} \\ M_x^{sT} \end{Bmatrix} \tag{6}$$

$$Q_{xz} = A_{55}^s \frac{\partial w_s}{\partial x}$$

where

$$\begin{aligned} &\left\{N_x, M_x^b, M_x^s\right\} = b \int_{-h/2}^{h/2} (1, z, f(z))\sigma_x \mathrm{d}z = b \sum_{k=1}^{N} \int_{Z_k}^{Z_{k+1}} (1, z, f(z))\sigma_x \mathrm{d}z \\ &Q_{xz} = b \int_{-h/2}^{h/2} \sigma_{xz} g(z) \mathrm{d}z = b \sum_{k=1}^{N} \int_{Z_k}^{Z_{k+1}} \sigma_{xz} g(z) \mathrm{d}z \\ &\left\{N_x^T, M_x^{bT}, M_x^{sT}\right\} = b \sum_{k=1}^{N} \int_{Z_k}^{Z_{k+1}} (1, z, f(z)) \bar{Q}_{11} \alpha_x \Delta T \mathrm{d}z \\ &(A_{11}, B_{11}, D_{11}, B_{11}^s, D_{11}^s, H_{11}^s) = b \sum_{k=1}^{N} \int_{Z_k}^{Z_{k+1}} \bar{Q}_{11} (1, z, z^2, f(z), zf(z), (f(z))^2) \mathrm{d}z \\ &A_{55}^s = b \sum_{k=1}^{N} \int_{Z_k}^{Z_{k+1}} \bar{Q}_{55} (g(z))^2 \mathrm{d}z \end{aligned} \tag{7}$$

where $\bar{Q}_{11}$ is the reduced stiffness matrix, α_x the transformed thermal coefficient of expansion and ΔT the constant temperature rise or drop through the thickness.

The strain energy of the beam can be written as

$$U = \frac{1}{2} \int_V \sigma_{ij} \varepsilon_{ij} \mathrm{d}V = \frac{1}{2} \int_V \left(\sigma_x \varepsilon_x + \sigma_{xz} \gamma_{xz}\right) \mathrm{d}V \tag{8}$$

Substituting Eqs. (3) and (5) into Eq. (8) and integrating through the thickness of the beam, the strain energy of the beam due to the normal force, shear force, bending moment and shear moment can be rewritten as

$$U = \frac{1}{2} \int_0^L \left[N_x \delta\varepsilon_x^0 + M_x^b \delta k_x^b + M_x^s k_x^s + Q_{xz}^s \gamma_{xz}^s\right] \mathrm{d}x \tag{9}$$

The work of the beam done by applied forces (mechanical force and forces due to variation of temperature ΔT) can be written as

$$V = \frac{1}{2} \int_0^L \left[\left(P + N_x^T\right) \frac{\partial^2 (w_b + w_s)}{\partial x^2}\right] \mathrm{d}x \tag{10}$$

where P is a mechanical force and N_x^T are applied forces due to variation of temperature.

The principle of minimum total potential energy (Reddy (1997)) is used here to derive the equations governing the displacement field and the constitutive equation. The principle can be stated in analytical form as

$$\delta(U + V) = 0 \tag{11}$$

where δ indicates a variation in relation to x.

Substituting Eqs. (9) and (10) into Eq. (11) and integrating the equation by parts, collecting the coefficients of δu_0, δw_b and δw_s, the governing equations can be obtained as follows:

$$\frac{\partial N_x}{\partial x} = 0 \tag{12a}$$

$$\frac{\partial^2 M_x^b}{\partial x^2} + \bar{N} \frac{\partial^2 (w_b + w_s)}{\partial x^2} = 0 \tag{12b}$$

$$\frac{\partial^2 M_x^s}{\partial x^2} + \frac{\partial Q_{xz}^s}{\partial x} + \bar{N} \frac{\partial^2 (w_b + w_s)}{\partial x^2} = 0 \tag{12c}$$

where

$$\bar{N} = P + N_x^T \tag{13}$$

Equations (12a–12c) can be expressed in terms of displacements $\left(u_0, w_b, w_s\right)$ by substituting for the force and moment stress resultants from Eq. (7). For laminated beam, the governing Eqs. (12a–12c) take the form

$$A_{11} \frac{\partial^2 u}{\partial x^2} - B_{11} \frac{\partial^3 w_b}{\partial x^3} - B_{11}^s \frac{\partial^3 w_s}{\partial x^3} = 0 \tag{14a}$$

$$B_{11} \frac{\partial^3 u}{\partial x^3} - D_{11} \frac{\partial^4 w_b}{\partial x^4} - D_{11}^s \frac{\partial^4 w_s}{\partial x^4} + \bar{N} \frac{\partial^2 (w_b + w_s)}{\partial x^2} = 0 \tag{14b}$$

$$B_{11}^s \frac{\partial^3 u}{\partial x^3} - D_{11}^s \frac{\partial^4 w_b}{\partial x^4} - H_{11}^s \frac{\partial^4 w_s}{\partial x^4} + A_{55}^s \frac{\partial^2 w_s}{\partial x^2} + \bar{N} \frac{\partial^2 (w_b + w_s)}{\partial x^2} = 0 \tag{14c}$$

By following the Navier solution procedure, the solutions to the problem are assumed to take the following forms:

$$u(x) = \sum_{m=1}^{\infty} U_{mn} \cos \lambda x \tag{15a}$$

$$w_b(x, y) = \sum_{m=1}^{\infty} W_{bmn} \sin \lambda x \tag{15b}$$

$$w_s(x, y) = \sum_{m=1}^{\infty} W_{smn} \sin \lambda x \tag{15c}$$

whereU_{mn}, W_{bmn}, W_{smn} are arbitrary parameters to be determined, $\lambda = m\pi / L$.

Substituting the expansions of (u_0, w_b, w_s) from Eqs. (15a)–(15c) into the stability Eqs. (14a)–(14c), the analytical solutions can be obtained from the following equations:

$$\left(\begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{12} & k_{22} & k_{23} \\ k_{13} & k_{23} & k_{33} \end{bmatrix} + \xi \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \right) \begin{Bmatrix} U_{mn} \\ W_{bmn} \\ W_{smn} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{16}$$

where

$$\begin{aligned} & k_{11} = A_{11}\lambda^2, \quad k_{12} = -B_{11}\lambda^3, \quad k_{13} = -B^s_{11}\lambda^3 \\ & k_{22} = D_{11}\lambda^4, \quad k_{23} = D^s_{11}\lambda^4, \quad k_{33} = H^s_{11}\lambda^4 + A^s_{55}\lambda^2, \quad \xi = -\left(P + N^T\right)\lambda^2 \end{aligned} \tag{17}$$

For nontrivial solution, the determinant of the coefficient matrix in Eq. (16) must be zero. This gives the following expression for buckling load:

$$P = -N^T + \frac{\det\left[k_{ij}\right]}{k_{11}\lambda^2\left(k_{22} + k_{33} - 2k_{23}\right) - k_{12}\lambda^2\left(k_{12} - 2k_{13}\right) - k_{13}^2\lambda^2} \tag{18}$$

The equation below shows that temperature reduces the critical buckling load.

On the other hand, if the beam is subjected to temperature change only with no mechanical loads, it is possible to define the critical temperature change that causes buckling. In light of Eq. (16), the critical temperature change is given as follows:

$$\Delta T_{cr} = \frac{1}{T_{11}} \left\{ \frac{\det\left[k_{ij}\right]}{k_{11}\lambda^2\left(k_{22} + k_{33} - 2k_{23}\right) - k_{12}\lambda^2\left(k_{12} - 2k_{13}\right) - k_{13}^2\lambda^2} \right\} \tag{19}$$

where

$$T_{11} = \int_{-h/2}^{h/2} \bar{Q}_{11}\alpha_x \mathrm{d}A \tag{20}$$

3 Results and discussion

In this section, a number of numerical examples are presented and analyzed to verify the accuracy of the present theory and to investigate the critical buckling of symmetric and antisymmetric laminated simply supported shear-deformable composite beam. All laminates are of equal thickness and made of the same orthotropic material, whose properties are as follows (Khdeir and Reddy(1997), Khdeir(2001), Aydogdu(2006,009), Reddy (1997), Vo and Thai (2012a, 2012b), Li and Qiao(2015a, b), Kahya(2016), Canales and Mantari (2016)):

Material I: $E_1/E_2 = \text{Open}, \quad G_{12} = G_{13} = 0.6E_2,$
$G_{23} = 0.5E_2, \quad \nu_{12} = 0.25, \quad \alpha_1/\alpha_2 = \text{Open}$
Material II: $E_1/E_2 = \text{open}, \quad G_{12} = G_{13} = 0.5E_2,$
$G_{23} = 0.2E_2, \quad \nu_{12} = 0.25$

We use the model developed in the present study to determine the non-dimensional first critical buckling load for laminated beams. The results are compared with those available in the literature. Firstly, mechanical buckling analysis of simply supported composite beams with symmetric cross-ply (0/90/0) is performed. Materials I and II with $E_1/E_2 = 10$ and 40 are used. The critical buckling loads for different length-to-thickness ratios are compared with analytical solutions (Khdeir and Reddy 1997; Aydogdu 2006) and the finite elements method (Vo and Thai 2012a; Kahya 2016) in Tables 1 and 2. The comparisons are well justified.

The second example concerns mechanical buckling behavior of simply supported cross-ply laminated beams. The non-dimensional critical buckling loads $\left(PL^2 / E_2bh^3\right)$ have been obtained for two-layer cross-ply beams with various values of length-to-thickness ratio L/h, which are presented in Table 3. The Material I is used with $E_1/E_2 = 40$. The present results are compared with those due to third-order beam theory (TOBT), the results given in Khdeir and Reddy (1997), the parabolic shear deformation beam theory (PSDBT) given in Aydogdu (2006), the refined shear deformation theory (RSDT) given in Vo and Thai (2012a, b), higher-order shear deformation beam theory (HOSDBT) given in Li and Qiao (2015a) and the finite element method

Table 1 Comparison of the non-dimensional critical buckling loads $\left(PL^2/E_2bh^3\right)$ of symmetric cross-ply composite laminated beams (0/90/0) with simply supported boundary condition (Materials I and II with $E_1/E_2=10$)

Materials	Theory	L/h			
		5	10	20	50
Material I	FSDBT[a]	4.752	6.805	7.630	7.897
	HOBT[b]	4.726	–	7.666	–
	HOBT[a]	4.709	6.778	7.620	7.896
	Present $n=3$	4.7268	6.8141	7.6664	7.9451
	Present $n=5$	4.7804	6.8453	7.6765	7.9468
	Present $n=7$	4.8357	6.8746	7.6857	7.9484
	Present $n=9$	4.8775	6.8961	7.6925	7.9496
Material II	FSDBT[a]	4.069	6.420	7.503	7.875
	HOBT[b]	3.728	–	7.459	–
	HOBT[a]	3.717	6.176	7.416	7.860
	Present $n=3$	3.7281	6.2060	7.4600	7.9088
	Present $n=5$	3.9340	6.3534	7.5132	7.9183
	Present $n=7$	4.0477	6.4287	7.5395	7.9230
	Present $n=9$	4.1190	6.4741	7.5551	7.9258

[a]Aydogdu (2006)

[b]Vo and Thai (2012a)

Table 2 Comparison of the non-dimensional critical buckling loads $\left(PL^2/E_2bh^3\right)$ of symmetric cross-ply composite laminated beams (0/90/0) with simply supported boundary condition (Materials I and II with $E_1/E_2=40$)

Materials	Theory	L/h			
		5	10	20	50
Material I	FSDBT[a]	8.606	18.989	–	–
	FSDBT[b]	8.604	18.974	27.154	30.882
	HOBT[a]	8.613	18.832	–	–
	HOBT[d]	8.613	–	27.084	–
	HOBT[b]	8.609	18.814	27.050	30.859
	FEM[c]	8.6132	18.8319	27.0860	30.9056
	Present $n=3$	8.58499	18.8846	27.0982	–
	Present $n=5$	8.6995	19.0389	27.2063	30.9314
	Present $n=7$	8.8533	19.2562	27.3208	30.9553
	Present $n=9$	8.9819	19.4212	27.4051	30.9726
Material II	FSDBT[b]	6.600	16.253	25.620	30.549
	HOBT[d]	5.896	–	24.685	–
	HOBT[b]	5.895	14.857	24.655	30.319
	Present $n=3$	5.8968	14.8682	24.6851	30.3643
	Present $n=5$	6.2890	15.6814	25.2637	30.5041
	Present $n=7$	6.5440	16.1311	25.5601	30.5732
	Present $n=9$	6.7164	16.4129	25.7387	30.6141

[a]Khdeir and Reddy (1997)

[b]Vo and Thai (2012a)

[c]Kahya (2016)

[d]Aydogdu (2006)

Table 3 Comparison of the non-dimensional critical buckling loads $\left(PL^2/E_2bh^3\right)$ of antisymmetric cross-ply composite laminated beams (0/90) with simply supported boundary condition (Material I $E_1/E_2=40$)

Theories	(0/90)		
	$L/h=5$	$L/h=10$	$L/h=20$
TOBT[a]	–	–	–
PSDBT[b]	3.906	–	5.296
RSDT[c]	3.903	4.936	5.290
HOSDBT[d]	3.9054	4.9399	5.2945
FEM[e]	3.28557	4.64637	5.20132
Present $n=3$	3.9066	4.9420	5.2969
Present $n=5$	3.8166	4.9076	5.2871
Present $n=7$	3.8008	4.9019	5.2856
Present $n=9$	3.8016	4.9026	5.2858

[a]Khdeir and Reddy (1997)

[b]Aydogdu (2006)

[c]Vo and Thai (2012b)

[d]Li and Qiao (2015a)

[e]Kahya (2016)

(FEM) given in Kahya (2016). The differences between non-dimensional critical buckling loads obtained by the present formulation and those using different higher-order beam theories and the finite element method are very small.

The next comparison example is presented in Table 4 that reports the non-dimensional critical buckling load for orthotropic the unidirectional composite beams ($\theta=0$ and 90) and symmetric cross-ply laminated beams four-layer (0/90)s a simply supported for different length-to-thickness ratios ($L/h=10$, 20, 100). The Material II is used with $E_1/E_2=25$. The obtained are compared with those of Reddy (1997) based on the Timoshenko beam theory (TBT) and the results of Kahya (2016) based on finite elements methods via first-order shear deformation theory (FSDT). It is seen that the present results are in excellent agreement with the literature values using shear deformation theory as seen from the validation checks.

In order to discuss the applicability of the present refined simple nth higher-order shear deformation beam theory to other laminate schemes, the mechanical non-dimensional critical buckling load for laminated beams with a variety of stacking sequences, the results are reported in Table 5. The Material I is used with $E_1/E_2=40$. The present results have been compared with those reported in Mantari (2016). Once again, the present theory is in good agreement with the Ritz solution buckling analysis of composite beams via a generalized higher-order shear deformation theory given by Canales and Mantari (2016).

The effect of length-to-thickness ratios of the beam on non-dimensional critical buckling loads is shown in Fig. 1,

Table 4 Comparison of the non-dimensional critical buckling loads $\left(PL^2/E_2bh^3\right)$ of composite laminated beams with simply supported boundary condition (Material II with $E_1/E_2=25$)

L/h	Layup	TBT[a]	FEM[b]	Present			
				$n=3$	$n=5$	$n=7$	$n=9$
10	0	13.768	13.7679	13.8175	13.8167	14.0020	14.0884
	90	0.784	0.8066	0.7857	0.7864	0.7873	0.7880
	(0/90)s	11.179	12.3115	10.1716	10.6356	10.9386	11.1313
20	0	18.304	18.3041	18.3473	18.3473	18.4355	18.4735
	90	0.812	0.8185	0.8145	0.8147	0.8149	0.8151
	(0/90)s	15.689	16.1905	15.1566	15.4152	15.5740	15.6716
100	0	20.461	20.4614	20.5117	20.5117	20.5162	20.5181
	90	0.822	0.8223	0.8241	0.8241	0.8241	0.8241
	(0/90)s	18.015	18.0101	17.9977	18.0122	18.0208	18.0261

[a]Reddy (1997)

[b]Kahya (2016)

Table 5 Comparison of the non-dimensional critical buckling loads $\left(PL^2/E_2bh^3\right)$ for simply supported composite laminated beams with a variety of stacking sequences (Material I with $E_1/E_2=40$)

L/h	Theories	Layup					
		(0/30/0)	(0/45/0)	(0/60/0)	(0/90/0)	(0/45/–45/0)	(0/60/–60/0)
5	3D-HSDT[a]	9.0658	8.8846	8.7340	8.5561	8.7382	8.5092
	Present $n=3$	9.1024	8.9391	8.7762	8.6132	8.7744	8.5432
	Present $n=5$	9.1191	8.9735	8.8338	8.6995	8.7512	8.5273
	Present $n=7$	9.2498	9.1093	8.9773	8.8533	8.8598	8.6377
	Present $n=9$	9.3681	9.2296	9.1011	8.9819	8.9654	8.7439
10	3D-HSDT[a]	19.6135	19.3191	19.0597	18.8294	18.5976	18.1286
	Present $n=3$	19.5961	19.3066	19.0527	18.8319	18.5607	18.0910
	Present $n=5$	19.7244	19.4506	19.2226	19.0389	18.6066	18.1345
	Present $n=7$	19.9110	19.6429	19.4252	19.2562	18.7550	18.2829
	Present $n=9$	20.0608	19.7955	19.5827	19.4212	18.8839	18.4118

[a]Canales and Mantari (2016)

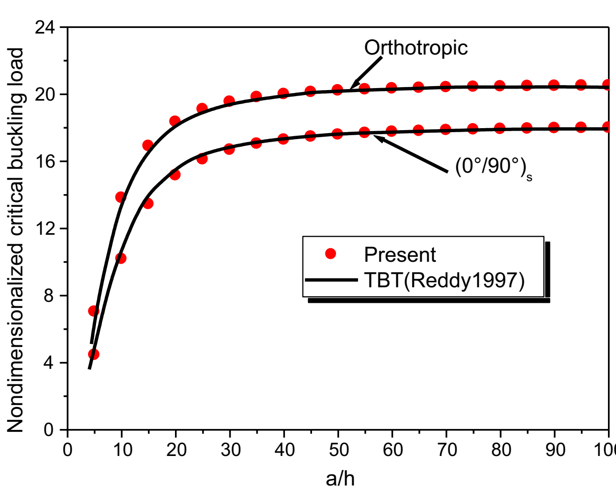

Fig. 1 Effect of length-to-thickness ratio (a/h) on the non-dimensional critical buckling loads $\left(PL^2/E_2bh^3\right)$, with simply boundary conditions, ($n=3$, Material II with $E_1/E_2=25$)

for simply supported orthotropic composite beams and symmetric cross-ply laminated beams with four-layer (0/90)s and the material properties II used with $E_1/E_2=25$. The obtained results based on the refined simple nth higher-order shear deformation theory are compared with those of Reddy (1997) based on the Timoshenko beam theory. It can be seen that the results of the present theory are in excellent agreement with those of Timoshenko beam theory for all values of length-to-thickness ratios. Also, it can be seen that non-dimensional critical buckling load increases by the increase in length-to-thickness ratio.

Finally, in this part, we still discuss the evaluation of the present refined simple nth higher-order shear deformation theory in the study of thermal buckling behavior. Tables 6, 7 and 8 present the non-dimensional critical temperature for different length-to-thickness ratios, different modulus ratios and different thermal expansions, respectively. The Material I is used. On the other hand, the critical buckling temperatures

Table 6 Comparison of the non-dimensional critical temperature $\overline{T}_{cr} = T_{cr}\alpha_1(L/h)^2$ results for a three layer (0/90/0) symmetric cross-ply beams, for different length-to-thickness ratios (Material I with $E_1/E_2 = 20,\ \ \alpha_1/\alpha_2 = 3$)

L/h	HOBT[a]	FOBT[a]	CBT[a]	HOSDBT[b]	Present			
					$n=3$	$n=5$	$n=7$	$n=9$
5	0.4678	0.4715	–	0.44908	0.4678	0.4739	0.4813	0.4871
10	0.8229	0.8281	–	0.78912	0.8229	0.8291	0.8352	0.8397
20	1.0190	1.0212	–	0.97666	1.0190	1.0215	1.0238	1.0255
50	1.0921	1.0925	1.1072	1.04656	1.0921	1.0926	1.0930	1.0933

[a]Khdeir (2001)

[b]Li and Qiao (2015b)

Table 7 Comparison of the non-dimensional critical temperature $\overline{T}_{cr} = T_{cr}\alpha_1(L/h)^2$ results for three-layer (0/90/0) symmetric cross-ply beams, for different modulus ratios, (Material I, $L/h = 10$, $\alpha_1/\alpha_2 = 3$)

E_1/E_2	HOBT[a]	FOBT[a]	CBT[a]	HOSDBT[b]	Present			
					$n=3$	$n=5$	$n=7$	$n=9$
3	0.7612	0.7625	0.8022	0.62508	0.7612	0.7625	0.7637	0.7645
10	0.8832	0.8868	1.0370	0.81683	0.8832	0.8873	0.8911	0.8939
20	0.8229	0.8281	1.1072	0.78912	0.8229	0.8291	0.8352	0.8397
30	0.7471	0.7528	1.1329	0.72608	0.7471	0.7543	0.7616	0.7670
40	0.6796	0.6853	1.1462	0.66506	0.6796	0.6871	0.6949	0.7009

[a]Khdeir (2001)

[b]Li and Qiao (2015b)

Table 8 Comparison of the non-dimensional critical temperature $\overline{T}_{cr} = T_{cr}\alpha_1(L/h)^2$ results for three-layer (0/90/0) symmetric cross-ply beams, for different thermal expansion coefficients ratio (Material I, with $E_1/E_2 = 20$, $L/h = 10$)

α_2/α_1	HOBT[a]	FOBT[a]	CBT[a]	HOSDBT[b]	Present			
					$n=3$	$n=5$	$n=7$	$n=9$
3	0.8229	0.8281	1.1072	0.7891	0.8229	0.8291	0.8352	0.8397
10	0.7077	0.7121	0.9522	0.6392	0.7077	0.7130	0.7183	0.7222
20	0.5898	0.5935	0.7935	0.5027	0.5898	0.5942	0.5986	0.6018
50	0.3932	0.3956	0.5290	0.30640	0.3932	0.3961	0.3990	0.4012
100	0.2528	0.2543	0.3401	0.1856	0.2528	0.2547	0.2565	0.2579

[a]Khdeir (2001)

[b]Li and Qiao (2015b)

were compared with the theoretical results of the Euler–Bernoulli classical beam theory (CBT), the first-order beam theory (FOBT), the third-order beam theory (HOBT) developed by Khdeir (2001) and the higher-order shear deformation beam theory (HOSDBT) developed by Li and Qiao (2015a, b). The critical buckling temperature of the present theory is in excellent agreement with the results of the other theories of shear deformation. Also, it is observed that the Euler–Bernoulli classical beam theory overestimates the thermal critical buckling of laminated beams. Hence, in order to obtain accurate results for laminated beam, it is necessary to consider the transverse shear deformation effects by using shear deformation theories.

The effects of temperatures on non-dimensional critical buckling loads $\bar{P}$ are presented in Tables 9 and 10 for two types of stacking sequences of symmetric cross-ply laminated (0/90/0) and (0/90/90/0), respectively. The beams are subjected to the uniform temperature rises. The mechanical properties of each layer Shen (2001) are assumed to be.

$$E_1/E_2 = 40, \quad G_{12} = G_{13} = 0.6E_2, \quad G_{23} = 0.5E_2,$$
$$v_{12} = 0.25, \quad \alpha_1 = 1.14 \times 10^{-6}, \quad \alpha_2 = 11.4 \times 10^{-6}$$

It is seen from Tables 9 and 10 that difference between one-dimensional buckling loads evaluated by EBT and RPT is more considerable for lower slenderness ratios, while this difference almost disappears for higher slenderness ratios. In other words, an increase in slenderness ratio leads to a decline on effects of shear deformation and difference between the results of EBT and SBT. On the contrary, it can be emphasized that the thermal effects on dimensionless buckling loads become more significant for higher slenderness ratios. These numerical results are useful for numerical benchmarking by others.

Table 9 Effects of temperatures on-dimensional critical buckling loads $\left(PL^2/E_2bh^3\right)$ of simply supported symmetric (0/90/0) beam under three sets of thermal loading conditions

ΔT (°C)	Theories	L/h		
		5	10	20
0	EBT	31.7603	31.7603	31.7603
	RBT $n=3$	8.6132	18.8319	27.0860
	RBT $n=5$	8.6995	19.0389	27.2063
	RBT $n=7$	8.8533	19.2562	27.3208
	RBT $n=9$	8.9819	19.4212	27.4051
100	EBT	31.6747	31.4178	30.3902
	RBT $n=3$	8.5276	18.4893	25.7159
	RBT $n=5$	8.6138	18.6964	25.8361
	RBT $n=7$	8.7676	18.9137	25.9506
	RBT $n=9$	8.8963	19.0787	26.0350
200	EBT	31.5890	31.0752	29.0200
	RBT $n=3$	8.4419	18.1468	24.3457
	RBT $n=5$	8.5282	18.3538	24.4660
	RBT $n=7$	8.6820	18.5711	24.5805
	RBT $n=9$	8.8106	18.7361	24.6648

Table 10 Effects of temperatures on non-dimensional critical buckling loads $\left(PL^2/E_2bh^3\right)$ of simply supported symmetric (0/90/90/0) beam under three sets of thermal loading conditions

ΔT (°C)	Theories	L/h		
		5	10	20
0	EBT	28.9344	28.9344	28.9344
	RBT $n=3$	8.3191	17.7593	24.9880
	RBT $n=5$	8.3262	17.8269	25.0289
	RBT $n=7$	8.4471	17.9888	25.1112
	RBT $n=9$	8.5588	18.1254	25.1787
100	EBT	28.8630	28.6489	27.7926
	RBT $n=3$	8.2478	17.4739	23.8462
	RBT $n=5$	8.2549	17.5415	23.8871
	RBT $n=7$	8.3757	17.7033	23.9694
	RBT $n=9$	8.4874	17.8400	24.0369
200	EBT	28.7916	28.3635	26.6508
	RBT $n=3$	8.1764	17.1885	22.7044
	RBT $n=5$	8.1835	17.2560	22.7453
	RBT $n=7$	8.3043	17.4179	22.8276
	RBT $n=9$	8.4160	17.5545	22.8951

4 Conclusion

Thermo-mechanical buckling response of simply supported laminated beams is investigated on the basis of a refined simple nth higher-order shear deformation beam theory. The governing differential equations are derived by implementing minimum total potential energy principle. Thermal effects on the critical buckling loads of simply supported laminated beams are investigated. The obtained results are compared with other available results in the published references. Significant observations from the results can be summarized as follows:

1. In the present paper, the authors combine the nth-order shear deformation theory developed by Xiang (2014) with the idea of the refined beam theory. The axial displacement field uses parabolic function in terms of thickness ordinate to include the effect of transverse shear deformation. The transverse displacement consists of bending and shear components. These ideas are used for developing the new nth-order shear deformation theory with modified displacement field to its optimization. Closed-form solutions for thermo-mechanical buckling behavior of composite beam are obtained.
2. This theory is seen to behave well, and the results of sample examples show good agreement with those in the literature as seen from the validation checks.
3. Effect of temperature change on buckling characteristic of laminated beams becomes more pronounced for larger values of length-to-thickness ratio.
4. The transverse shear deformation has the effect of decreasing both buckling loads. Thus, the classical laminate theory overpredicts buckling loads. This is primarily due to the assumed infinite rigidity of the transverse normals in the classical laminate theory. Note that the assumption does not yield a conservative result; i.e., if one designs a beam for buckling load based on the classical laminate theory and if no safety factor is used, it will fail for a working load smaller than the critical buckling load.

Appendix

(1) Consider a laminate beam made of n plies. Each ply has a thickness of t_k. Then, the thickness of the laminate h is

$$h = \sum_{k=1}^{n} t_k$$

Then, the location of the mid-plane is $h/2$ from the top or the bottom surface of the laminate. The z-coordinate of each ply k surface (top and bottom) is given by

Ply 1:

$$h_0 = -\frac{h}{2} \quad \text{(top surface)}$$

$$h_1 = -\frac{h}{2} + t_1 \quad \text{(bottom surface)}$$

Ply k: $(k = 2, 3, \ldots n-2, n-1)$

$$h_{k-1} = -\frac{h}{2} + \sum_{=1}^{k-1} t \quad \text{(top surface)}$$

$$h_k = -\frac{h}{2} + \sum_{=1}^{k} t \quad \text{(bottom surface)}$$

(2) Find the value of the reduced stiffness matrix $[Q]$ for each ply using its six elastic moduli, E_1, E_2 , G_{12}, G_{13} , G_{23} , ν_{12} in constants $Q_{11}, Q_{12}, Q_{22}, Q_{66}, Q_{44}$ and Q_{55}.

(3) Find the value of the transformed reduced stiffness matrix for each ply using the $[\bar{Q}]$ matrix calculated in step 2, and the angle of the ply and transformed coefficient of thermal expansion can be referred to any standard texts such as (Reddy (1997)).

(4) Knowing the thickness, t_k, of each ply, find the coordinate of the top and bottom surface, h_i, $\quad i = 1 \ldots, n$, of each ply, using the following equation:

Ply n:

$$h_{n-1} = \frac{h}{2} - t_n \text{(top surface)}$$

$$h_n = \frac{h}{2} \text{(bottom surface)}$$

(5) Use the $[\bar{Q}]$ matrices from step 3 and the location of each ply from step 4 to find the six beam stiffness (A_{11}, $B_{11}, D_{11}, B^s_{11}, D^s_{11}, H^s_{11}$ and A^s_{55}) from Eq. (7).

(6) Substitute the stiffness matrix values found in step 5 and the applied forces and moments in Eq. (6).

(7) Solve the three simultaneous Eqs. (14a–14c). Closed-form solutions are obtained using the Navier solution for simply supported laminated composite beams Eqs. (15a–15c), and the eigenvalue problem is solved to get the corresponding eigenvalues for buckling load equation with the effect temperature reduces the critical buckling load (18) and the critical temperature Eq. (19).

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