RESEARCH PAPER

Analytical Approximate Solution for Large Post-Buckling Behavior of a Fixed-Pinned Beam Subjected to Terminal Force with Shear Force Effect

Yongping Yu[1] · Lihui Chen[1] · Ping Yu[2] · Weipeng Sun[3]

Abstract

Based on Euler–Bernoulli beam theory, this paper investigates large post-buckling deformation of a slender elastic beam with fixed-pinned end. Owing to the asymmetric boundary conditions, it is difficult to establish analytic solution. Based on the Maclaurin series expansion and orthogonal Chebyshev polynomials, the governing differential equation with both sinusoidal nonlinearity and cosinusoidal nonlinearity can be reduced to a polynomial equation, and the geometry condition with sinusoidal nonlinearity can also be simplified to be a cubic polynomial integral equation. The admissible lateral displacement function to satisfy the fixed-pinned boundary conditions is derived in an elegant way. The analytical approximations are obtained with the harmonic balance method. Two approximate formulae for axial load and lateral load are established for small as well as large angle of rotation at the pinned end. These approximate solutions show excellent agreement with those of the shooting method for a large range of the rotation angle at the pinned end. Moreover, due to brevity of expressions, the present analytical approximate solutions are convenient to investigate effects of various parameters on the large post-buckling response of fixed-pinned beams.

Keywords Analytical approximation · Large post-buckling · Fixed-pinned beam

1 Introduction

There are many practical cases where post-buckling of structures may occur, such as satellite tethers, marine cables, robotic arms and slender rod in building or bridge. When the slenderness ratio is very large, there is a large displacement within the elastic range. The stability of slender rods has drawn the attention of many scientists since Euler classical contributions in the eighteenth century (Lee 1990; Lee and Oh 2000; Nayfeh and Emam 2008; Vaz and Mascaro 2005; Wang 1981). Studies show that the shear force in the equilibrium state is negligible for the Euler beams with common supporting conditions, such as simply supported, fixed–fixed and cantilever. However, for the asymmetric boundary and the statically indeterminate characteristics of the fixed-pinned beam, the shear force cannot be ignored, which will result in its post-buckling problem in a strong nonlinear problem.

Kocatürk and Akbaş (2010) established a total Lagrangian finite element model of the simply supported beam to study geometrically nonlinear static problem. The solution was obtained numerically by the Newton–Raphson procedure. Then, they studied thermal post-buckling analysis of Timoshenko beams with various boundary conditions (Kocaturk and Akbas 2011). Akbas (2014) investigated the large post-buckling of Timoshenko beams subjected to non-follower axial compression loads. Wang (1996) proposed the perturbation solution of the large post-buckling of an elastic rod with fixed-pinned ends and suffered axial compression. The solutions for buckling, initial post-buckling (perturbation), large loads (asymptotic) and numerical integration of the fixed-pinned rod were developed by Vaz and Silva (2003). Li et al. (2015), Li and Zhou (2005), Song and Li (2007) considered various post-buckling

✉ Weipeng Sun
  sunwp@jlu.edu.cn

1 College of Construction Engineering, Jilin University, Changchun 130026, People's Republic of China

2 School of Mechanical and Aerospace Engineering, Jilin University, Changchun 130025, People's Republic of China

3 School of Mathematics, Jilin University, Changchun 130012, People's Republic of China

problems for the slender beam with fixed-pinned end by employing shooting method.

Employing the elliptic integration, many scholars obtained the solution of the post-buckling problem of fixed-pinned beams. Yan et al. (2019) derived explicit analytical expressions of buckled beams with fixed-pinned ends. Humer (2013) studied the solution in terms of elliptic integrals for the buckling and post-buckling problem of beams with the influence of axial compressibility and shear deformation. Applying a geometrically exact theory of the elastica, Prechtl et al. (2012) presented a geometrically exact solution to the aforementioned Euler buckling problems. The governing equilibrium equation and Jacobi equation of the elastic rod with various boundary conditions were derived from variational principles by Batista (2015), and their solutions were presented in terms of the Jacobi elliptical functions. Mikata et al. (2007), Singh and Goss (2018) studied the solutions in terms of the Jacobi elliptical functions for a clamped-pinned beam. Sano and Wada (2018) studied snap buckling of an elastic strip with fixed-pinned ends. However, the exact solution was expressed by the elliptic integral of the second kind, which needed to be solved numerically, and was inconvenient for application.

To the authors' knowledge, the previous researches on the post-buckling problem of a fixed-pinned beam are limited to the solution in terms of numerical form or elliptic integral form. No one has derived a simple and convenient explicit analytical formula for the post-buckling of a fixed-pinned beam. In this paper, the analytical solution for the large post-buckling deformation of a slender elastic beam with fixed-pinned ends is investigated. The governing differential equation is established with Euler–Bernoulli beam theory. Employing the Maclaurin series expansion and orthogonal Chebyshev polynomials, the nonlinear governing equation included sinusoidal function and cosinusoidal function can be reduced to a polynomial equation. A reasonable initial approximation satisfying the boundary conditions is given, and the analytical solutions for the large post-buckling deformation of a fixed-pinned beam are obtained by the harmonic balance method.

2 Mathematical Model and Solution Methodology

The sketch depicted in Fig. 1 shows undeformed and deformed configurations of a slender elastic beam subjected to an axial load, fixed at one end and pinned at the other end. A classic Euler–Bernoulli beam model investigating large post-buckling deflection is written as (Humer 2013; Vaz and Silva 2003; Wang 1996):
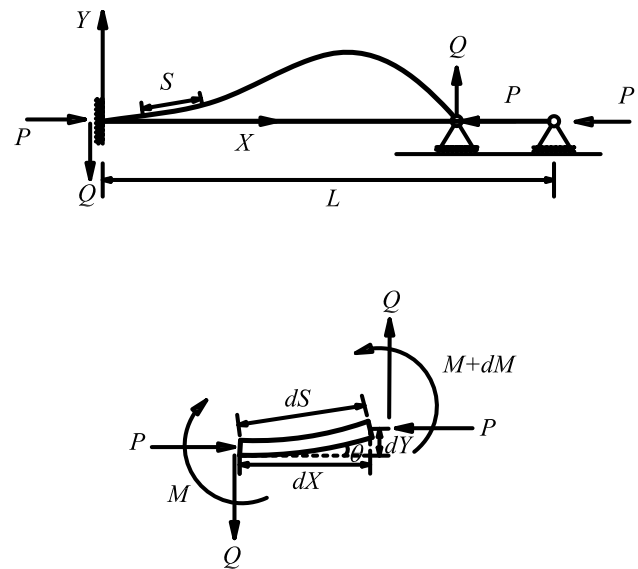

Fig. 1 Schematic of the slender elastic beam

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}s^2} + p\sin\theta + q\cos\theta = 0, \tag{1}$$

and the boundary conditions are

$$\theta(0) = 0,\ \theta(1) = a,\ \frac{\mathrm{d}\theta}{\mathrm{d}s}(1) = 0. \tag{2}$$

Geometric requirements are

$$\frac{\mathrm{d}y}{\mathrm{d}s} = \sin\theta,\ y(0) = y(1) = 0, \tag{3}$$

$$\frac{\mathrm{d}x}{\mathrm{d}s} = \cos\theta,\ x(0) = 0. \tag{4}$$

Equation (3) can also be written as

$$\int_0^1 \sin\theta\,\mathrm{d}s = 0, \tag{5}$$

where

$$s = \frac{S}{L},\ x = \frac{X}{L},\ y = \frac{Y}{L},\ p = \frac{PL^2}{\mathrm{EI}},\ q = \frac{QL^2}{\mathrm{EI}}, \tag{6}$$

and $S \in [0, L]$ is the actual curve length, L is the length of the beam, (X, Y) constitutes Cartesian coordinates of the beam, P and Q are the axial load and lateral load and EI is the bending stiffness of the beam, respectively. The analysis of the asymmetrically supported Euler buckling beam with one end clamped and the other end pinned is complicated.

A new independent variable $\tau = \pi/2 - \pi/2s$ is introduced. Then, Eq. (1) can be rewritten in the following dimensionless form:

$$\frac{d^2\theta}{d\tau^2} + \beta \sin\theta + \gamma \cos\theta = 0, \tag{7}$$

where

$$\beta = 4p/\pi^2, \gamma = 4q/\pi^2, \tag{8}$$

and the boundary conditions are

$$\theta(0) = a, \theta\left(\frac{\pi}{2}\right) = 0, \frac{d\theta}{d\tau}(0) = 0, \tag{9}$$

and the geometry conditions are rewritten as

$$\int_0^{\frac{\pi}{2}} \sin\theta \, d\tau = 0. \tag{10}$$

In the present work, along with the Maclaurin series expansion and the Chebyshev polynomials, and let $\theta = au$, a new nonlinear equation with no trigonometric function is constructed (Yu et al. 2013):

$$\frac{dy}{ds} = B_1 u + B_2 u^3 \tag{11}$$

$$\frac{dx}{ds} = D_0 + D_1 u^2 + D_2 u^4 \tag{12}$$

where

$$\begin{aligned}
B_1 &= a - \frac{a^5}{384} + \frac{a^7}{11520} - \frac{a^9}{737280}, \\
B_2 &= -\frac{a^3}{6} + \frac{a^5}{96} - \frac{a^7}{3840} + \frac{a^9}{276480}, \\
D_0 &= 1 - \frac{a^6}{23040} + \frac{a^8}{737280} - \frac{a^{10}}{51609600}, \\
D_1 &= -\frac{a^2}{2} + \frac{a^6}{1280} - \frac{a^8}{46080} + \frac{a^{10}}{3440640}, \\
D_2 &= \frac{a^4}{24} - \frac{a^6}{480} + \frac{a^8}{23040} - \frac{a^{10}}{1935360}.
\end{aligned} \tag{13}$$

Substituting Eqs. (11) and (12) in Eq. (7), the nonlinear equation could be expressed as

$$\frac{d^2u}{d\tau^2} + \frac{\beta\left(B_1 u + B_2 u^3\right)}{a} + \frac{\gamma\left(D_0 + D_1 u^2 + D_2 u^4\right)}{a} = 0. \tag{14}$$

Using Eqs. (10) and (11), the extra condition could be obtained

$$\int_0^{\frac{\pi}{2}} \left(B_1 u + B_2 u^3\right) d\tau = 0. \tag{15}$$

Following the harmonic balance approximation, a reasonable initial approximation satisfying the boundary conditions could be taken as

$$u(\tau) = b\cos\tau + (1-b)\cos 3\tau, \tag{16}$$

where b is the root of Eq. (15). Note that Eq. (15) which is a 3rd algebraic equation could be analytically solved.

In the following step, harmonic balancing process will be performed: substituting Eq. (16) in Eq. (14), and letting the resulting coefficients of the items $\cos\tau$ and $\cos 3\tau$ vanish (Sun et al. 2009; Wu et al. 2006; Yu et al. 2012), give

$$\begin{aligned}
\alpha_1 \times \gamma_0 + \alpha_2 \times \beta_0 + \alpha_3 &= 0, \\
\omega_1 \times \gamma_0 + \omega_2 \times \beta_0 + \omega_3 &= 0,
\end{aligned} \tag{17}$$

where

$$\begin{aligned}
\alpha_1 = 16\{&45045D_0 + 8[2187 \\
&+16b(-407 + 1098b - 1200b^2 + 560b^3)]D_2 \\
&+858[27 + 8b(-5 + 6b)]D_1\},
\end{aligned}$$

$$\alpha_2 = 45045b\pi\{4B_1 + 3[2 + b(-3 + 2b)]B_2\},$$

$$\alpha_3 = -180180ab\pi,$$

$$\begin{aligned}
\omega_1 = -16\{&15015D_0 + 8[1001 \\
&+16b(-797 + 1806b - 1936b^2 + 784b^3)]D_2 \\
&+286[35 + 8b(-29 + 22b)]D_1\},
\end{aligned}$$

$$\omega_2 = -45045\{4(-1 + b)B_1 + [-3 + b(9 - 15b + 8b^2)]B_2\}\pi,$$

$$\omega_3 = 1621620a(-1 + b)\pi..$$

From Eq. (17), the analytical approximations for β and γ can be solved and expressed as function of a, as

$$\begin{aligned}
\beta_0 &= -\frac{\alpha_3\omega_1 - \alpha_1\omega_3}{\alpha_2\omega_1 - \alpha_1\omega_2}, \\
\gamma_0 &= \frac{\alpha_3\omega_2 - \alpha_2\omega_3}{\alpha_2\omega_1 - \alpha_1\omega_2},
\end{aligned} \tag{18}$$

and the analytical approximate solution for θ can be obtained as

$$\begin{aligned}
\theta_0(\tau) = a\big[&b\cos(\pi/2 - \pi/2s) + (1-b) \\
&\cos(3\pi/2 - 3\pi/2s)\big], \quad s \in [0,1]
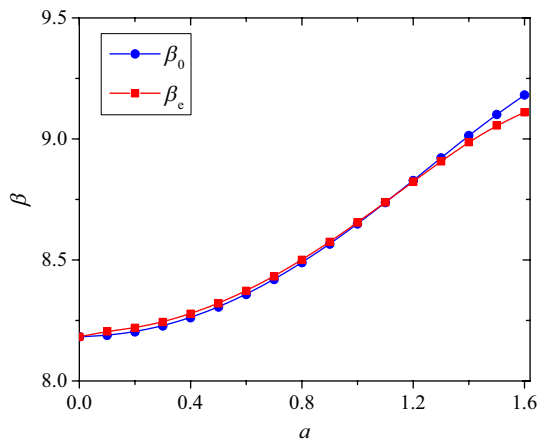\end{aligned} \tag{19}$$

Fig. 2 Comparison of the exact and approximate values of β with respect to the rotation angle a
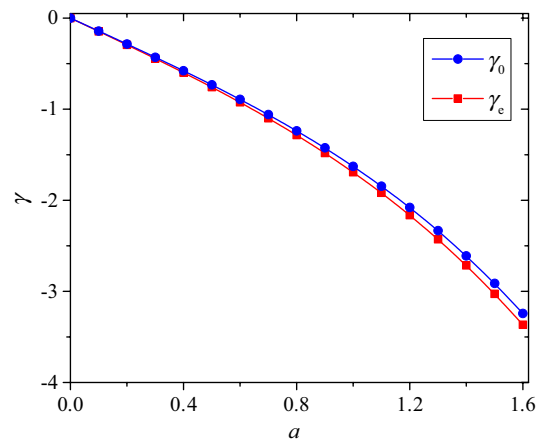
Fig. 3 Comparison of the exact and approximate values of γ with respect to the rotation angle a

3 Results and Discussion

The exact (numeric) solutions β_e, γ_e, θ_e could be obtained by solving Eqs. (3), (4), (7) and (9) with a shooting method. The approximate solutions β_0, γ_0 could be calculated, respectively, by Eq. (18). For comparison, exact and approximate values of β and γ with respect to the angle of rotation a are shown in Figs. 2 and 3, respectively.
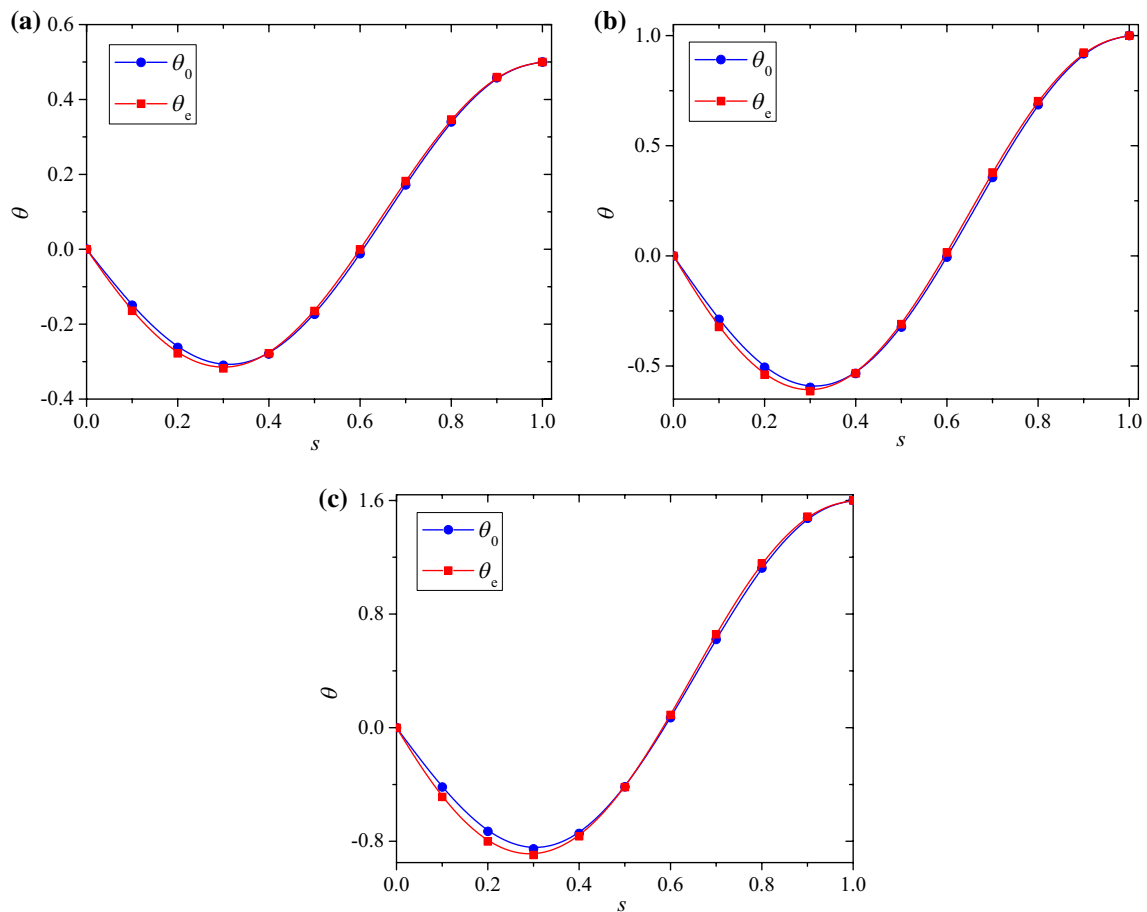

Fig. 4 Comparison of the exact and approximate values of $\theta(s)$: **a** for $a = 0.5$, **b** for $a = 1.0$ and **c** for $a = 1.6$
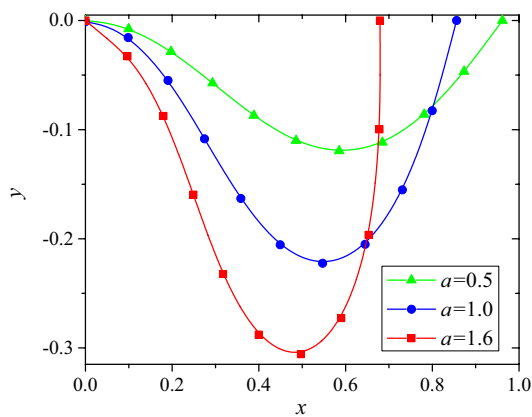
Fig. 5 Variations of the approximate values of lateral displacement y with axial displacement x

Figures 2 and 3 indicate that Eq. (18) could give excellent approximations to parameters β and γ for small as well as large rotation angle a.

The exact (numeric) solutions θ_e and approximate solution θ_0 given in Eq. (19) are displayed in Fig. 4: (a) for $a = 0.5$, (b) for $a = 1.0$ and (c) for $a = 1.6$. From Fig. 4, one could find that Eq. (19) could present excellent approximate accuracy of θ for a large range of rotation angle a. Note that $a = 1.6$ means the angle of rotation in pinned end exceeding $\pi/2$ and it is quite a large deformation. Furthermore, if one would like to obtain solutions with higher approximate accuracy, further approximations need to be constructed with Newton harmonic method (Wu et al. 2006; Yu et al. 2012) or the improved Galerkin method (Sun et al. 2015). The length of the formulae, however, must be longer.

Finally, approximate solutions of lateral displacement y and axial displacement x analytically obtained from Eqs. (11), (12) and (16) are plotted in Fig. 5.

4 Conclusions

This paper has focused on large post-buckling deformation of a slender elastic column with fixed-pinned ends, based on the Euler–Bernoulli beam theory. An accurate and brief approach is provided to solve the problem mentioned-above. The approximate solutions have been established. Results for lateral load, axial load, lateral displacement and axial displacement are presented in dimensionless format. Compared with the solution calculated by applying the shooting method, the results obtained by the new method show excellent accuracy for a large range of the rotation angle in the pinned end. The new approach presented here offers analytical approximate solutions which help in analytically

investigating the effect of the physical parameters on the post-buckling response of the beam with shear force subjected to terminal forces with fixed-pinned ends. The method and accurate approximations derived here are very useful for robotic arms and slender rod in building or bridge engineering applications. By applying the explicit and analytical formulas, the applied force and deflection can be explicitly analyzed with ease and without numerical integration. However, there is a gap between the model and actual structure for the sake of many hypotheses. Therefore, the more practical and precise models should be constructed by reducing assumptions. An improved analytical approximate method will be investigated to solve other structures with shear deformation, such as beams, bars or plates supported by nonlinear elastic foundation.

Compliance with Ethical Standards

Conflict of interest The authors declare that they have no conflict of interest.

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