Iranian Journal of Science
https://doi.org/10.1007/s40995-024-01677-x

RESEARCH PAPER

Optimal Convergence of Thermoelastic Contact Problem Involving Nonlinear Hencky-Type Materials with Friction Conditions

H. El Khalfi[1] · Z. Faiz[1] · O. Baiz[2] · H. Benaissa[1]

Received: 18 March 2024 / Accepted: 1 July 2024

Abstract
We study the linear finite element approximation of thermoelastic frictional contact problem. The unilateral contact condition is weakly imposed by the penalty method. Our analysis yields error estimates that are contingent upon the penalty parameter ε and the mesh size h. Furthermore, provided the solution maintains regularity, we establish a convergence result.

Keywords Thermoelastic contact · Hencky-type materials · Priori error estimates · Penalty method

Mathematics Subject Classification 47J20 · 49J40 · 74F05 · 74G30 · 74M10 · 74M15 · 74S05

1 Introduction

Contact-related challenges pervade various domains within mechanics, physics, and engineering applications. Instances within the automotive sector include the interaction between brake pads and rotors, as well as that between pistons and cylinders. The thermal aspects of contact processes have significant implications, exerting an influence on both the composition and rigidity of surfaces in contact, as well as triggering thermal stresses within the interacting bodies, see Shillor et al. (2004) for more details. Reciprocally, the prevailing temperature can impact the elastic material response. Numerous studies in the literature, illustrated for instance by Chouly et al. (2014); Benkhira et al. (2019a, 2019b); Benaissa et al. (2016); khalfi et al. (2023); Faiz et al. (2024, 2023) and the references therein (Benaissa et al. 2015), have delved into diverse thermomechanical frictional problems. In these works, not only were rigorous mathematical models of contact incorporating thermal effects established, but their unique weak solvability was also demonstrated through the application of variational and hemi-variational inequalities. Furthermore, other contributions in the literature have explored different aspects related to mechanical contact phenomena.

In recent literature, there has been an emergence of a new theoretical framework for modeling frictionless contact in thermoelastic materials, as discussed in Liu et al. (2021). This model introduces two sets of unilateral constraints: one governing normal displacement through the Signorini condition on a specified boundary portion, and the other imposing a unilateral restriction on temperature within a defined domain. Unlike the model presented in Liu et al. (2021), our study focuses on numerically investigating a frictionless contact scenario involving a thermoelastic body and a thermally conductive foundation. Notably, the determination of the heat exchange coefficient in this scenario relies on a function of the contact pressure, as detailed in Ramaniraka (1997). The novelty of our work lies in the numerical methodology employed, which employs two distinct contact algorithms based on penalty and augmented Lagrangian approaches. These algorithms are extensively discussed and effectively applied to simulate the considered system. Our research aims to explore

✉ Z. Faiz
zakaria.faiz@usms.ma

H. El Khalfi
e.hamidos20@gmail.com

O. Baiz
othman.baiz@gmail.com

H. Benaissa
hi.benaissa@gmail.com

[1] Lab. LMRI, FP of Khouribga, Sultan Moulay Slimane University, Beni-Mellal, Morocco

[2] Lab. LSIE., FP of Ouarzazate, Ibno Zohr University, Agadir, Morocco

Published online: 06 August 2024

the implications of incorporating a temperature field in the contact process between a thermoelastic body and a rigid foundation. Specifically, we consider a model describing a static frictional contact problem between a thermoelastic body and a thermally conductive foundation, assuming small deformations. The constitutive law governing material behavior incorporates Hencky's nonlinear law and considers the interplay between mechanical and thermal properties.

The paper's organization is as follows: In Sect. 2, we introduce relevant notations and preliminaries, and present a model depicting the process of frictional contact between a thermoelastic body and a rigid foundation. Section 3 focuses on deriving the penalized weak formulation of the model for both frictionless and Tresca's friction scenarios. This section also addresses issues of existence and uniqueness, and discusses the finite element approximation of the penalized weak problems in detail. In Sect. 4, we establish error estimates for the numerical approximation, considering the dependence on both the penalty parameter ε and the mesh size h. Moreover, provided specific regularity assumptions for the solution of contact problems and stipulated requirements on parameters ε and h, we offer results regarding the convergence rate of the finite element approximation of the penalized solution.

2 Physical Statement of Problem ($\mathcal{P}$)

Problem ($\mathcal{P}$) : we consider a thermoelastic body whose material particles occupy a polygonal or polyhedral domain Ω of $\mathbb{R}^d$ $(d = 2, 3)$. The body's equilibrium equations are characterized by

$$\sigma = \mathcal{A}\varepsilon(u) - \mathcal{M}\theta \quad \text{in} \quad \Omega, \tag{2.1}$$

$$q_T = -\mathcal{K}\nabla\theta \quad \text{in} \quad \Omega, \tag{2.2}$$

$$Div\,\sigma + f_0 = 0 \quad \text{in} \quad \Omega, \tag{2.3}$$

$$div\,q_T - q_0 = 0 \quad \text{in} \quad \Omega. \tag{2.4}$$

Here u is the displacement field, and θ is the temperature field. The symbols σ and q_T stand for the stress tensor and the heat flux vector field, respectively. The operator $\mathcal{A} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d$ is the nonlinear elasticity operator that describes the behavior of Hencky's materials, given by (see Han (2005); Haslinger and Mäkinen (1992); Chouly et al. (2014); Benkhira et al. (2019a, 2019b) for more details)

$$\mathcal{A}\varepsilon(u) = k_0\,tr(\varepsilon(u))\,I + 2g(\|\bar{\varepsilon}(u)\|^2)\,\bar{\varepsilon}(u) \quad \text{in } \Omega, \tag{2.5}$$

where $k_0 > 0$ is a material coefficient, I is the second-order identity tensor, $tr(\varepsilon) = \varepsilon_{ii}$ denotes the trace of ε, and $\bar{\varepsilon}$ represents its deviatoric part defined as follows

$$\bar{\varepsilon} = \varepsilon - \frac{1}{d}tr(\varepsilon)I.$$

The operators $\mathcal{M} = (\mathcal{M}_{ij})$ and $\mathcal{K} = (\mathcal{K}_{ij})$ describe respectively, magentathe purely elastic, the thermal expansion and thermal conductivity properties of the material. The linearized strain $\varepsilon(u)$ is given by $\varepsilon(u) = \frac{1}{2}(\nabla u + (\nabla u)^T)$, where $(\nabla u)^T$ is the transpose of (∇u). We recall that Div and div denote the divergence operator for tensors and vector valued functions. The densities $f_0 \in L^2(\Omega)^d$ and $q_0 \in L^2(\Omega)$ represent the body force and the volume of heat source on the body, respectively.

We assume that the boundary $\Gamma = \partial\Omega$ of Ω is smooth and made of three mutually disjoined parts Γ_1, Γ_2, Γ_3. On part Γ_1, we assume that the body is clamped and a given temperature is described, we choose it equal to zero. On part Γ_2, we prescribe a surface forces and a heat flux of densities $q_2 \in L^2(\Gamma_2)$ and $f_2 \in L^2(\Gamma_2)^d$, respectively. Finally, on Γ_3, the contact is unilateral, resulting in Signorini boundary conditions for mechanical effects, and thermal conditions dictate zero heat flux at points without contact and a prescribed temperature at points with contact for thermal effects. The boundary conditions on Γ_1 and Γ_2 are then specified as follows

$$u = 0 \text{ on } \Gamma_1, \quad \theta = 0 \quad \text{on} \quad \Gamma_1 \cup \Gamma_2, \tag{2.6}$$

$$\sigma\nu = f_2 \text{ on } \Gamma_2, \quad q_T \cdot \nu = q_2 \text{ on } \Gamma_2, \tag{2.7}$$

where the vector ν is the unit outward normal on Γ. We adopt the following decomposition: if v is a given vector field on Γ_3, we split it into its normal component v_ν and its tangential component v_τ by

$$v_\nu = v \cdot \nu, \quad v_\tau = v - v_\nu \nu.$$

Similarly, if σ is a given tensor field on Γ, its normal and tangential components on Γ are defined by

$$\sigma_\nu = (\sigma\nu) \cdot \nu, \quad \sigma_\tau = \sigma\nu - \sigma_\nu \nu.$$

According to this notation, u_ν and u_τ are the normal and tangential components of the displacement vector u, and σ_ν and σ_τ are the normal and tangential components of the stress tensor σ. Furthermore, to describe the unilateral contact on part Γ_3, we consider the following nonlinear boundary conditions

$$u_\nu \le g, \quad \sigma_\nu \le 0, \quad (u_\nu - g)\sigma_\nu = 0 \quad \text{on} \quad \Gamma_3, \tag{2.8}$$

$$\left.\begin{array}{r}\|\sigma_\tau\| \le S,\\ \|\sigma_\tau\| < S \Rightarrow u_\tau = 0,\\ \|\sigma_\tau\| = -S\,\dfrac{u_\tau}{\|u_\tau\|} \Rightarrow \exists\,\lambda \in \mathbb{R}^+;\quad \sigma_\tau = -\lambda\, u_\tau\end{array}\right\} \quad \text{on} \quad \Gamma_3, \tag{2.9}$$

$$q_T \cdot \nu = k_T(u_\nu - g)\,\varphi_L(\theta - \theta_F) \quad \text{on } \Gamma_3. \tag{2.10}$$

In conditions (2.8), the function g denotes the maximum penetration between the body and the foundation. These equations (2.8) correspond to the classical Signorini conditions. The relations (2.9) describe the Tresca-type friction law, where $S \in L^2(\Gamma_3)$ is a given nonnegative function. Equation (2.10) specifies the heat flux condition, with θ_F representing the foundation's temperature and k_T denoting the heat exchange function between the foundation and the body [see Duvaut (1981)]. Additionally, φ_L is a truncation function defined for a large constant $L > 0$ as follows

$$\varphi_L(s) = \begin{cases} -L & \text{if } s < -L,\\ s & \text{if } -L \le s \le L,\\ L & \text{if } s < L.\end{cases}$$

Our aim is a finite element analysis of Problem $(\mathcal{P})$, given by (2.1)–(2.10), using penalty method. Todo that, consider $H^m(\Omega)$ where $m \ge 0$, and let $L^2(\Omega)$ denote the conventional Sobolev space $H^0(\Omega)$ equipped with its customary norms $\|\cdot\|_{m,\Omega}$. Our initial step involves introducing the subsequent

$$H = L^2(\Omega)^d,\ H_1 = H^1(\Omega)^d,\ \mathcal{H} = \{\tau = (\tau_{ij}) \in H : \tau_{ij} = \tau_{ji}\},$$
$$\mathcal{H}_1 = \{\sigma \in \mathcal{H} : \sigma_{ij,j} \in H\}.$$

These are real Hilbert subspaces for the Euclidean associated norms to the following inner products

$$(u,v)_H = \int_\Omega u_i v_i\,dx,\quad (u,v)_{H_1} = (u,v)_H + (\varepsilon(u),\varepsilon(v))_{\mathcal{H}},$$
$$(\sigma,\tau)_{\mathcal{H}} = \int_\Omega \sigma_{ij}\tau_{ij}\,dx,\quad (\sigma,\tau)_{\mathcal{H}_1} = (\sigma,\tau)_{\mathcal{H}} + (Div\sigma, Div\tau)_{\mathcal{H}}.$$

According to the mechanical part of a condition (2.6), we consider the subspace

$$V := \left\{v \in \left(H^1(\Omega)\right)^d : v = 0 \text{ on } \Gamma_1\right\},$$

and the set K of admissible displacements satisfying the non-interpenetration condition, i.e.,

$$K := \{v \in V : v_\nu \le g \text{ on } \Gamma_3\}.$$

Since $meas(\Gamma_1) > 0$, the following Korn's inequality holds, i.e.,

$$\|\varepsilon(v)\|_{\mathcal{H}} \ge c_k\,\|v\|_{H_1}, \quad \forall\, v \in V, \tag{2.11}$$

for a constant $c_k > 0$ that depends only on Ω and Γ_1. Over the subspace V, let us consider the inner product and its associated Euclidean, defined as below

$$(u,v)_V = (\varepsilon(u),\varepsilon(v))_{\mathcal{H}},\quad \|u\|_V = (u,u)_V^{\frac{1}{2}}, \tag{2.12}$$

Thus $(V, \|\cdot\|_V)$ is a real Hilbert space. Moreover, by Sobolev trace theorem, relations (2.11) and (2.12), there exists a constant $c_0 > 0$ which depends only on Ω, Γ_1 and Γ_3 such that

$$\|v\|_{L^2(\Gamma)^d} \le c_0\|v\|_V, \quad \forall\, v \in V. \tag{2.13}$$

Now, according to the thermal part of a condition (2.6), we introduce the subspace

$$Q = \{\xi \in H^1(\Omega)\,|\,\xi = 0 \text{ on } \Gamma_1 \cup \Gamma_2\}.$$

The spaces $(Q, \|\cdot\|_Q)$ is a real Hilbert space for the associated norm of the following scalar product

$$(\xi,\psi)_Q = (\nabla\xi, \nabla\psi)_{L^2(\Omega)}.$$

It is worth noting that, given $meas(\Gamma_1) > 0$, the Friedrichs-Poincaré inequality is applicable, implying the existence of a constant $c_F > 0$ dependent solely on Ω and Γ_a, as follows

$$\|\xi\|_Q \ge c_F\,\|\xi\|_{H^1(\Omega)}, \quad \forall\, \psi \in Q. \tag{2.14}$$

Moreover, by Sobolev trace theorem, there exists $c_1 > 0$, depending only on Ω, Γ_1 and Γ_3, such that

$$\|\xi\|_{L^2(\Gamma_3)} \le c_1\|\xi\|_Q, \quad \forall\, \xi \in Q. \tag{2.15}$$

Also there exists a constant $c_T > 0$, depending only on Ω, Γ_1, Γ_2 and Γ_3, such that

$$\|\nabla\xi\|_H \le c_T\|\xi\|_{H^1(\Omega)}, \quad \forall\, \xi \in Q.$$

For any real Banach space $(X, \|\cdot\|)$, we denote by $\langle\cdot,\cdot\rangle$ the duality pairing between X and its dual X'. Next, in the study of Problem $(\mathcal{P})$, we need the following hypotheses.

$(\mathcal{H}_1)$ The function g is continuously differentiable in $[0,\infty)$ and satisfies

$$0 < g_0 \le g(t) \le \frac{1}{2}\,d\,k_0, \tag{2.16}$$

$$0 < \alpha_1 \le g(t) + 2g'(t)t \le \alpha_2, \tag{2.17}$$

where g_0, α_1 and α_2 are a given positive constants.

($\mathcal{H}_2$) The thermal conductivity tensor $\mathcal{K} = (k_{ij}) : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ satisfies the standard properties:

$$k_{ij} = k_{ji} \in L^\infty(\Omega),$$

and there exists a nonnegative constant $m_\mathcal{K}$ such that

$$k_{ij}(x)\zeta_i\zeta_j \geq m_\mathcal{K}\|\zeta\|^2, \quad \forall \zeta = (\zeta_i) \in \mathbb{R}^d \text{ a.e. } x \in \Omega.$$

Let $M_\mathcal{F} = \sup_{i,j} \|k_{ij}\|_{L^\infty(\Omega)}$ denote the norm of $\mathcal{K}$.

($\mathcal{H}_3$) The tensor $\mathcal{M} = (m_{ij}) : \Omega \times \mathbb{S}^d \to \mathbb{R}^d$ satisfies the properties

$$m_{ij} = m_{ji} \in L^\infty(\Omega).$$

Let $\|\mathcal{M}\| = \sup_{i,j} \|m_{ij}\|_{L^\infty(\Omega)}$ be the norm of the thermal expansion tensor $\mathcal{M}$.

($\mathcal{H}_4$) The thermal conductance function $\psi = k_T : \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+$, satisfy the properties

(a) there exists $M_\varphi > 0$ such that $|k_T(x,u)| \leq M_\varphi$ for all $u \in \mathbb{R}$, a.e. $x \in \Gamma_3$,

(b) the mapping $x \mapsto \psi(x,u)$ is measurable on Γ_3 for all $u \in \mathbb{R}$,

(c) there exists $L_\varphi > 0$ such that, for all $u_1, u_2 \in \mathbb{R}$, one has

$$|\psi(x,u_1) - \psi(x,u_2)| \leq L_\varphi|u_1 - u_2| \text{ a.e. } x \in \Gamma_3.$$

($\mathcal{H}_5$) The body forces, traction and heat source densities satisfy the following properties

$$f_0 \in L^2(\Omega)^d, f_2 \in L^2(\Gamma_2)^d, q_0 \in L^2(\Omega), q_2 \in L^2(\Gamma_2).$$

($\mathcal{H}_6$) The friction bound, the gap function and the foundation's temperature satisfy

$$S \geq 0 \text{ a.e. } x \in \Gamma_3, \quad S \in L^2(\Gamma_3),$$
$$g \geq 0 \text{ a.e. } x \in \Gamma_3, \quad g \in L^2(\Gamma_3),$$
$$\theta_F \in L^2(\Gamma_3).$$

In the other hand, to write the weak formulation, we define the following operators

$$\langle \mathcal{A}u, v\rangle_V = \int_\Omega \mathcal{A}\varepsilon(u)\varepsilon(v)\,dx, \quad \forall\, u,\, v \in V, \tag{2.18}$$

$$\langle \mathcal{P}\theta, v\rangle_V = \int_\Omega \mathcal{P}\theta\,\varepsilon(v)\,dx, \quad \forall\, \theta \in Q, \forall\, v \in V, \tag{2.19}$$

$$\langle K\theta, \eta\rangle_Q = \int_\Omega \mathcal{K}\nabla\theta\,\nabla\eta\,dx, \quad \forall\, \theta,\, \eta \in Q. \tag{2.20}$$

Next, we consider the elements $f \in V'$ and $q \in Q'$ given by

$$\langle f, v\rangle_V = \int_\Omega f_0 \cdot v\,dx + \int_{\Gamma_2} f_2 \cdot v\,da, \quad \forall\, v \in V, \tag{2.21}$$

$$\langle q, \eta\rangle_Q = \int_\Omega q_0\,\eta\,dx - \int_{\Gamma_2} q_2\eta\,da, \quad \forall\, \eta \in Q. \tag{2.22}$$

3 Existence and Uniqueness Results

3.1 Penalty Formulation of the Frictionless Problem

Consider (P') as the frictionless counterpart of Problem (P), derived by substituting (2.9) with the following condition

$$\sigma_\tau(u,\theta) = 0. \tag{3.1}$$

Subsequently, the weak formulation of the frictionless unilateral problem defined by (2.1)–(2.10) is as follows

Problem $(\mathcal{PV})$. *Find a displacement field $u \in K$ and a temperature field $\theta \in Q$ such that*

$$(\mathcal{A}\varepsilon(u), \varepsilon(v) - \varepsilon(u))_\mathcal{H} - (\mathcal{M}\theta, \varepsilon(v) - \varepsilon(u))_\mathcal{H} \geq (f, v-u)_V, \quad \forall\, v \in K, \tag{3.2}$$

$$(\mathcal{K}\nabla\theta, \nabla\xi)_{L^2(\Omega)} + \ell(u,\theta,\xi) = (q,\xi)_Q, \quad \forall\, \xi \in Q, \tag{3.3}$$

where the functional $\ell : V \times Q \times Q \to \mathbb{R}$ defined as follows

$$\ell(u,\theta,\xi) = \int_{\Gamma_3} k_T(u_\nu - g)\,\varphi_L(\theta - \theta_F)\,\xi\,da.$$

The existence of a unique solution to Problem $(\mathcal{PV})$ relies on elliptic variational inequalities and fixed point arguments, as discussed in, for instance, Duvaut (1981). We then examine the product space $X = V \times Q$, which forms a Hilbert space with the corresponding norm defined by the inner product below

$$(x,y)_X = (u,v)_V + (\theta,\xi)_Q, \quad \forall\, x = (u,\theta),\, y = (v,\xi) \in X. \tag{3.4}$$

We introduce the operator $B : X \to X$, defined as follows

$$(Bx,y)_X = (\mathcal{A}\varepsilon(u), \varepsilon(v))_\mathcal{H} - (\mathcal{M}\theta, \varepsilon(v))_\mathcal{H} + (\mathcal{K}\nabla\theta, \nabla\xi)_{L^2(\Omega)} \tag{3.5}$$

We also introduce the functional $j : X \to \mathbb{R}$ and the element $f^e \in X$ given by

$$j(x,y) = \ell(u,\theta,\xi), \tag{3.6}$$

$$f^e = (f,q) \in X. \tag{3.7}$$

Let $U = K \times Q$ be non-empty closed convex of X. Then, we get the following equivalent problem **Problem** $(\mathcal{PV})$. *Find* $x = (u,\theta) \in U$ *such that*

$$(Bx, y-x)_X + j(x,y) - j(x,x) \geq (f^e, y-x)_X, \quad \forall y = (v,\xi) \in U. \tag{3.8}$$

Lemma 3.1 *The operator B is strongly monotone and Lipschitz continuous.*

Proof To establish this, we need to ensure that the nonlinear elasticity operator $\mathcal{A}$, as defined in (2.5), is both strongly monotone and Lipschitz continuous. By performing algebraic manipulations akin to those in references (Benkhira et al. 2019a, b), we obtain:

$$(\mathcal{A}\varepsilon(u) - \mathcal{A}\varepsilon(v), \varepsilon(u) - \varepsilon(v))_{\mathcal{H}} \geq m_{\mathcal{A}}\|u-v\|_V^2, \quad \forall u, v \in V \quad \text{with} \quad m_{\mathcal{A}} = 2\alpha_1, \tag{3.9}$$

$$\|\mathcal{A}\varepsilon(u) - \mathcal{A}\varepsilon(v)\|_{\mathcal{H}} \leq M_{\mathcal{A}}\|u-v\|_V, \quad \forall u, v \in V \quad \text{with} \quad M_{\mathcal{A}} = 2d^2k_0. \tag{3.10}$$

With (3.9)–(3.10) in mind, and employing similar algebraic manipulations as in Benkhira et al. (2019a, 2019b), we can readily establish the existence of positive constants m_B and M_B, depending solely on $\mathcal{A}$, $\mathcal{M}$, $\mathcal{K}$, and Ω, such that:

$$(Bx - By, x-y)_X \geq m_B\|x-y\|_X^2, \quad \forall x, y \in X, \tag{3.11}$$

$$\|Bx - By\|_X \leq M_B\|x-y\|_X, \quad \forall x, y \in X. \tag{3.12}$$

Therefore, we conclude the proof of the Lemma 3.1. □

Let's define the notation $[\cdot]^+$ to represent the positive part of each scalar $a \in \mathbb{R}$ as follows:

$$[a]^+ = \begin{cases} a & \text{if} \quad a \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Throughout the remainder of this document, we will frequently utilize the following common properties:

$$a \leq [a]^+, \quad a \cdot [a]^+ = [a]^{+2}, \quad \forall a \in \mathbb{R}. \tag{3.13}$$

The monotonicity property can be derived from the aforementioned properties, as detailed in Chouly and Hild (2013a); Chouly et al. (2014); Chouly and Hild (2013b).

$$\left([a]^+ - [b]^+\right)(a-b) \geq \left([a]^+ - [b]^+\right)^2. \tag{3.14}$$

The variational inequality (3.8) poses challenges for solution using various methods, particularly due to the constraint subspace K which is not conducive for computations. Therefore, alternative techniques are needed, and one of the classical and widely used methods to address this inequality constraint is the penalty method. The penalty technique is a well-established approach for numerically handling constrained problems (see, for example, Li (1998); Kikuchi and Oden (1988)). Unlike the Lagrange multiplier technique, the penalty method does not require the introduction of a new variable. Furthermore, it is more readily implementable in many numerical algorithms. However, it's important to note that this method still represents an approximation, as the solution of the penalized problem is expected to converge to the solution of the original problem only as the penalty parameter tends to zero.

Problem $(\mathcal{PV}_\varepsilon)$. *Find a displacement field* $u_\varepsilon \in V$ *and a temperature field* $\theta_\varepsilon \in Q$ *such that*

$$(Bx_\varepsilon, y)_X + j(x_\varepsilon, y) + \frac{1}{\varepsilon}\int_{\Gamma_3} [u_{\varepsilon,\nu}]^+ v_\nu \, da = (f^e, y)_X, \quad \forall y = (v,\xi) \in X = V \times Q. \tag{3.15}$$

Note that this formulation is obtained by taking

$$\sigma_\nu(u_\epsilon, \theta_\epsilon) = \frac{1}{\varepsilon}[u_{\varepsilon,\nu}]^+ \quad \text{where} \quad u_{\varepsilon,\nu} = u_\varepsilon \cdot \nu.$$

We have the following theorem, the proof of which can be found in (Bourichi et al. 2016, Theorem 3.1):

Theorem 3.2 *Problem* (PV_ϵ) *has a unique solution* $x_\epsilon = (u_\epsilon, \theta_\epsilon) \in X = V \times Q$.

3.2 Penalty Formulation of Tresca's Friction Problem

Initially, employing conventional methods rooted in Green's formula, we establish that when (u, σ, θ, q) denote regular functions satisfying (2.1)-(2.10), the weak formulation of the Tresca's friction problem can be articulated as follows.

Problem $(\overline{\mathcal{PV}})$. *Find* $(u,\theta) \in V \times Q$ *such that*

$$(\mathcal{A}\varepsilon(u), \varepsilon(v) - \varepsilon(u))_{\mathcal{H}} - (\mathcal{M}\theta, \varepsilon(v) - \varepsilon(u))_{\mathcal{H}} + j_S(v) - j_S(u) \geq (f, v-u)_V, \quad \forall v \in V, \tag{3.16}$$

$$(\mathcal{K}\nabla\theta, \nabla\xi)_{L^2(\Omega)} + \ell(u,\theta,\xi) = (q,\xi)_Q, \quad \forall \xi \in W, \tag{3.17}$$

where the functionals j_S and ℓ are defined as following

$$j_S = \int_{\Gamma_3} S\|u_\tau\| \, ds,$$

$$\ell(u,\theta,\xi) = \int_{\Gamma_3} k_T(u_\nu - g)\varphi_L(\theta - \theta_F)\,\xi \, da.$$

The following theorem can be found in Duvaut (1981).

Theorem 3.3 *With assumptions* $(\mathcal{H}_1)$–$(\mathcal{H}_6)$, *Problem* $(\overline{\mathcal{PV}})$ *possesses at least one solution. Furthermore, if the function k is given for some* $\alpha \in \mathbb{R}$, *by*

$$k(\xi) = \alpha k_0(\xi), \tag{3.18}$$

Then, there exists $\alpha_1 > 0$ such that if

$$0 \le \alpha \le \alpha_1, \tag{3.19}$$

the solution of Problem $(\overline{\mathcal{PV}})$, is unique.

Lemma 3.4 *The couple $x = (u, \theta) \in X$ is a solution to Problem $(\overline{\mathcal{PV}})$ if and only if*

$$(Bx, y - x)_X + j(x, y) - j(x, x) + \int_{\Gamma_3} S(\|v_\tau\| - \|u_\tau\|) ds \ge (f^e, y - x)_X, \ \forall y = (v, \xi) \in U. \tag{3.20}$$

By employing the penalty method on Problem $(\overline{\mathcal{PV}}_\varepsilon)$, we derive the subsequent penalized weak formulation.

Problem $(\overline{\mathcal{PV}}_\varepsilon)$. *Find a displacement field $u_\varepsilon \in V$ and a temperature field $\theta_\varepsilon \in Q$ such that*

$$(Bx_\varepsilon, y)_X + j(x_\varepsilon, y) + \frac{1}{\varepsilon} \int_{\Gamma_3} [u_{\varepsilon,\nu}]^+ v_\nu da + \frac{1}{\varepsilon} \int_{\Gamma_3} [u_{\varepsilon,\tau}]_{\varepsilon g} v_\tau da = (f^e, y)_X, \quad \forall y = (v, \xi) \in X, \tag{3.21}$$

where the notation $[\cdot]_\alpha$ $(\alpha \in \mathbb{R}^+)$ stands for the truncation of a scalar $x \in \mathbb{R}$, i.e.;

$$[x]_\alpha = \begin{cases} x & \text{if } \|x\| \le \alpha, \\ \alpha \dfrac{x}{\|x\|} & \text{otherwise.} \end{cases}$$

The following theorem can be found in (Chouly and Hild 2013b, Theorem 4.1) and (Bourichi et al. 2016, Theorem 3.1).

Theorem 3.5 *Under the assumptions $(\mathcal{H}_1)$–$(\mathcal{H}_6)$, $(\overline{\mathcal{PV}}_\varepsilon)$ has a unique solution.*

3.3 Approximation of the Penalty Weak Formulations

For a given discretization parameter $h > 0$, we denote by $\mathcal{T}^h$ a coherent set of triangular finite element partitions of the closed domain $\overline{\Omega}$, which are compatible with the boundary partitions $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$. We contemplate two finite-dimensional subspaces $V^h \subset V$ and $Q^h \subset Q$, which approximate the spaces V and Q, respectively, as follows

$$V^h = \left\{ v^h \in C(\overline{\Omega})^d : \ v^h_{|T} \in \mathbb{P}_1(T)^d, \ \forall T \in \mathcal{T}^h \text{ and } v^h = 0 \text{ on } \Gamma_1 \right\},$$

$$Q^h = \left\{ \xi^h \in C(\overline{\Omega}) : \ \xi^h_{|T} \in \mathbb{P}_1(T), \ \forall T \in \mathcal{T}^h \text{ and } \xi^h = 0 \text{ on } \Gamma_1 \cup \Gamma_2 \right\}$$

where $\mathbb{P}_1(T)$ represents the space of polynomial functions with a global degree less than or equal to 1 within an arbitrary element $T \in \mathcal{T}^h$. Additionally, we examine the space $\mathcal{X}^h(\Gamma_3)$ comprising the normal traces on Γ_3 for discrete functions in V^h, namely

$$\mathcal{X}^h(\Gamma_3) = \left\{ \mu_h \in C(\overline{\Gamma_3}) : \ \exists v^h \in V^h, \ \forall T \in \mathcal{T}^h; \ v^h \cdot \nu = \mu_h \right\}.$$

The following lemmas outline pertinent properties of the $L^2(\Gamma_3)$-projection operator $\mathcal{P}^h : L^2(\Gamma_3) \to \mathcal{X}^h(\Gamma_3)$, with further details available in Bernardi et al. (1994), Bramble et al. (2001), Bramble and Xu (1991).

Lemma 3.6 *Suppose the mesh linked with $\mathcal{X}^h(\Gamma_3)$ exhibits local quasi-uniformity, indicating that the ratio of the diameter of a simplex to the diameter of the largest ball enclosed within the simplex remains bounded irrespective of h for all simplices across all triangulations, as elaborated in Bramble et al. (2001). Under these conditions, for any $r \in [0, 1]$ and every $v \in H^r(\Gamma_3)$, the subsequent stability and interpolation estimates apply*

$$\|\mathcal{P}^h(v)\|_{r,\Gamma_3} \le c, \|v\|_{r,\Gamma_3}, \tag{3.22}$$

$$\|v - \mathcal{P}^h(v)\|_{0,\Gamma_3} \le ch^r, \|v\|_{r,\Gamma_3}, \tag{3.23}$$

where the constant $c > 0$ in the two inequalities remains independent of v and the discretization size h.

Lemma 3.7 *Suppose the mesh on Γ_3 is quasi-uniform. Then, there exists an extension operator $\mathcal{R}^h : \mathcal{X}^h(\Gamma_3) \to V^h(\Gamma_3)$ and a constant $c > 0$, independent of v and h, satisfying*:

$$\mathcal{R}^h(\mu_h)| \ \Gamma 3 = \mu_h, \tag{3.24}$$

$$\|\mathcal{R}^h(\mu_h)\|_{1,\Omega} \le c, \|\mu_h\|_{\frac{1}{2},\Gamma_3}, \quad \forall, \mu_h \in \mathcal{X}^h(\Gamma_3). \tag{3.25}$$

Subsequently, consider $\mathbf{P}^h$ and $\mathbf{R}^h$ as the vector representations of the operators $\mathcal{P}^h$ and $\mathcal{R}^h$, respectively, which are defined as follows

$$\mathbf{P}^h(\mathbf{w}) := (\mathcal{P}^h(w_i))_{1 \le i \le d}, \quad \forall \ \mathbf{w} := (w_i)_{1 \le i \le d} \in L^2(\Gamma_3), \tag{3.26}$$

$$\mathbf{R}^h(\mathbf{w}) := (\mathcal{R}^h(w_i))_{1 \le i \le d}, \quad \forall \ \mathbf{w} := (w_i)_{1 \le i \le d} \in \mathcal{X}^h(\Gamma_3). \tag{3.27}$$

It's worth noting that $\mathbf{P}^h$ and $\mathbf{R}^h$ adhere to the stability and interpolation properties stated earlier in equations (3.22) through (3.25).

Remark 3.8 Consider a sequence of triangulations $\mathcal{T} = (\mathcal{T}^h)_{h>0}$. We define it as quasi-uniform if the ratio h_τ/ρ_τ, where h_τ denotes the diameter of an element $\tau \in \mathcal{T}^h$ and ρ_τ represents the diameter of its inscribed circle, is bounded by a constant $\hat{\sigma}$ independent of both τ and h. Furthermore, there exists a constant $c > 0$ that is invariant with respect to h and satisfies the following condition:

$$h_\tau \geq ch, \quad \forall \tau \in \mathcal{T}^h.$$

It's important to highlight that the quasi-uniformity of the mesh on Γ_3 (the mesh associated with $\mathcal{X}^h(\Gamma_3)$) implies its local quasi-uniformity.

3.3.1 Approximation of the Frictionless Problem

The finite element discretization $(\mathcal{PV}_\varepsilon^h)$ of the penalized problem $(\mathcal{PV}_\varepsilon)$ is outlined as follows.

Problem $(\mathcal{PV}_\varepsilon^h)$. *Find a displacement field* $u_\varepsilon^h \in V^h$ *and a temperature field* $\theta_\varepsilon^h \in Q^h$ *such that*

$$\begin{aligned}(Bx_\varepsilon^h, y^h)_X + j(x_\varepsilon^h, y^h) + \frac{1}{\varepsilon}\int_{\Gamma_3} [u_{\varepsilon,\nu}^h]^+ v_\nu^h \, da \\ = (f^e, y^h)_X, \quad \forall y^h = (v^h, \xi^h) \in X^h = V^h \times Q^h.\end{aligned} \tag{3.28}$$

Proof By applying Theorem 3.2, replacing V and Q with V^h and Q^h respectively, we deduce that Problem $(\mathcal{PV}_\varepsilon^h)$ has a unique solution $(u_\varepsilon^h, \theta_\varepsilon^h) \in X^h = V^h \times Q^h$. □

3.3.2 Approximation of the Tresca's Friction Problem

The numerical approximation $(\overline{\mathcal{PV}}_\varepsilon^h)$ for the penalized problem $(\overline{\mathcal{PV}}_\varepsilon)$, is described as follows.

Problem $(\overline{\mathcal{PV}}_\varepsilon^h)$. *Find* $u_\varepsilon^h \in V^h$ *and* $\theta_\varepsilon^h \in Q^h$ *such that*

$$\begin{aligned}(Bx_\varepsilon^h, y^h)_X + j(x_\varepsilon^h, y^h) + \frac{1}{\varepsilon}\int_{\Gamma_3} [u_{\varepsilon,\nu}^h]^+ v_\nu^h da \\ + \frac{1}{\varepsilon}\int_{\Gamma_3} [u_{\varepsilon,\tau}^h]_{\varepsilon g} v_\tau^h da = (f^e, y^h)_X, \\ \text{for all } y^h = (v^h, \xi^h) \in X^h = V^h \times Q^h.\end{aligned} \tag{3.29}$$

Remark 3.9 The uniqueness of the solution to problem $(\overline{\mathcal{PV}}_\varepsilon^h)$ can be established using similar reasoning as in (Chouly and Hild 2013b, Theorem 4.1) and (Bourichi et al. 2016, Theorem 3.1), with the respective spaces V^h and Q^h replacing V and Q.

The penalized problem $(\overline{\mathcal{PV}}_\varepsilon)$ is consistent with the finite element penalized problem $(\overline{\mathcal{PV}}_\varepsilon^h)$ in such a way that the solution $x_\varepsilon = (u_\varepsilon, \theta_\varepsilon)$ of problem $(\overline{\mathcal{PV}}_\varepsilon)$ satisfies equation (3.29) for all test functions $y^h = (v^h, \xi^h) \in X^h = V^h \times Q^h \subset X = V \times Q$. This implies that

$$\begin{aligned}(Bx_\varepsilon, y^h)_X + j(x_\varepsilon, y^h) + \frac{1}{\varepsilon}\int_{\Gamma_3} [u_{\varepsilon,\nu}]^+ v_\nu^h da \\ + \frac{1}{\varepsilon}\int_{\Gamma_3} [u_{\varepsilon,\tau}]_{\varepsilon g} v_\tau^h da = (f^e, y^h)_X.\end{aligned} \tag{3.30}$$

We revisit the fundamental characteristics of projections as follows

$$\begin{aligned}&\big([x]_\alpha - [y]_\alpha\big) \cdot (x - y) \geq 0 \\ &\text{and} \quad \big|[x]_\alpha - [y]_\alpha\big| \leq |x - y|, \quad \forall x, y \in \mathbb{R}^{d-1},\end{aligned} \tag{3.31}$$

expressed for any v and w belonging to V, we can represent this as

$$\begin{aligned}&\big([v_\tau]_{\epsilon g} - [w_\tau]_{\epsilon g}\big) \cdot (v_\tau - w_\tau) \geq 0 \\ &\text{and} \quad \big|[v_\tau]_{\epsilon g} - [w_\tau]_{\epsilon g}\big| \leq |v_\tau - w_\tau| \quad \text{on} \quad \Gamma_3.\end{aligned} \tag{3.32}$$

4 Approximation and a Priori Estimate Results

4.1 A Priori Estimate of the Frictionless Problem

Lemma 4.1 *Let* $x = (u, \theta) \in [H^{\frac{3}{2}+r}(\Omega)]^d \times H^{\frac{3}{2}+r}(\Omega)$ *with* $r \in (0, \frac{1}{2}]$, *respectively,* $x_\varepsilon^h = (u_\varepsilon^h, \theta_\varepsilon^h)$ *and* $x_\varepsilon = (u_\varepsilon, \theta_\varepsilon)$ *be solutions of the problems* $(\mathcal{PV})$, $(\mathcal{PV}_\varepsilon^h)$ *and* $(\mathcal{PV}_\varepsilon)$. *Then, we have*

$$\begin{aligned}&\left\|\sigma_\nu(u,\theta) + \frac{1}{\varepsilon}[u_{\varepsilon,\nu}^h]^+\right\|_{-r,\Gamma_3} \\ &\leq C\left[h^r\left\|\sigma_\nu(u,\theta) + \frac{1}{\varepsilon}[u_{\varepsilon,\nu}^h]^+\right\|_{0,\Gamma_3} \right. \\ &\left. + h^{r-\frac{1}{2}}\Big(\|u - u_\varepsilon^h\|_{1,\Omega} + \|\theta - \theta_\varepsilon^h\|_{1,\Omega}\Big)\right]\end{aligned} \tag{4.1}$$

and

$$\begin{aligned}&\left\|\sigma_\nu(u,\theta) + \frac{1}{\varepsilon}[u_{\varepsilon,\nu}]^+\right\|_{-r,\Gamma_3} \\ &\leq C\left[\varepsilon^r\left\|\sigma_\nu(u,\theta) + \frac{1}{\varepsilon}[u_{\varepsilon,\nu}]^+\right\|_{0,\Gamma_3} \right. \\ &\left. + \varepsilon^{r-\frac{1}{2}}\Big(\|u - u_\varepsilon\|_{1,\Omega} + \|\theta - \theta_\varepsilon\|_{1,\Omega}\Big)\right],\end{aligned} \tag{4.2}$$

where the non-negative constant C is independent of ε, u , u_ε^h and h.

Proof First, we have [see Chouly and Hild (2013b) for more details]

$$\left\|\sigma_\nu(u,\theta)+\frac{1}{\varepsilon}[u^h_{\varepsilon,\nu}]^+\right\|_{-r,\Gamma_3} = \sup_{v\in H^r(\Gamma_3)} \frac{\left\langle \sigma_\nu(u,\theta)+\frac{1}{\varepsilon}[u^h_{\varepsilon,\nu}]^+, v\right\rangle_{\Gamma_3}}{\|v\|_{r,\Gamma_3}}.$$

Hence, by using the relations (3.22)–(3.25), we get that

$$\begin{aligned}&\left\|\sigma_\nu(u,\theta)+\frac{1}{\varepsilon}[u^h_{\varepsilon,\nu}]^+\right\|_{-r,\Gamma_3}\\ &\leq \sup_{v\in H^r(\Gamma_3)} \frac{\left\langle \sigma_\nu(u,\theta)+\frac{1}{\varepsilon}[u^h_{\varepsilon,\nu}]^+, v-\mathcal{P}^h v\right\rangle_{\Gamma_3}}{\|v\|_{r,\Gamma_3}}\\ &\quad+\sup_{v\in H^r(\Gamma_3)} \frac{\left\langle \sigma_\nu(u,\theta)+\frac{1}{\varepsilon}[u^h_{\varepsilon,\nu}]^+, \mathcal{P}^h v\right\rangle_{\Gamma_3}}{\|v\|_{r,\Gamma_3}}\\ &\leq \left\|\sigma_\nu(u,\theta)+\frac{1}{\varepsilon}[u^h_{\varepsilon,\nu}]^+\right\|_{0,\Gamma_3}\sup_{v\in H^r(\Gamma_3)}\frac{\|v-\mathcal{P}^h v\|_{0,\Gamma_3}}{\|\mathcal{P}^h v\|_{r,\Gamma_3}}\\ &\quad+C\sup_{v\in H^r(\Gamma_3)} \frac{\left\langle \sigma_\nu(u,\theta)+\frac{1}{\varepsilon}[u^h_{\varepsilon,\nu}]^+, \mathcal{P}^h v\right\rangle_{\Gamma_3}}{\|\mathcal{P}^h v\|_{r,\Gamma_3}}\\ &\leq Ch^r\left\|\sigma_\nu(u,\theta)+\frac{1}{\varepsilon}[u^h_{\varepsilon,\nu}]^+\right\|_{0,\Gamma_3}\\ &\quad+\sup_{v\in H^r(\Gamma_3)} \frac{\left\langle \sigma_\nu(u,\theta)+\frac{1}{\varepsilon}[u^h_{\varepsilon,\nu}]^+, \mathcal{P}^h v\right\rangle_{\Gamma_3}}{\|\mathcal{P}^h v\|_{r,\Gamma_3}}.\end{aligned}$$

Furthermore, we know that for all $v\in V$, we have

$$(\mathcal{A}\varepsilon(u),\varepsilon(v))_{\mathcal{H}}-(\mathcal{M}\theta,\varepsilon(v))_{\mathcal{H}}=(f,v)_V+\langle\sigma_\nu(u,\theta),v_\nu\rangle_{\Gamma_3},$$
$$(\mathcal{A}\varepsilon(u_\varepsilon),\varepsilon(v))_{\mathcal{H}}-(\mathcal{M}\theta_\varepsilon,\varepsilon(v))_{\mathcal{H}}=(f,v)_V+\left\langle-\frac{1}{\varepsilon}[u_{\varepsilon,\nu}]^+,v_\nu\right\rangle_{\Gamma_3}.$$

Then, we find that

$$\begin{aligned}\left\langle\sigma_\nu(u,\theta)+\frac{1}{\varepsilon}[u_{\varepsilon,\nu}]^+,v_\nu\right\rangle_{\Gamma_3}&=(\mathcal{A}\varepsilon(u-u_\varepsilon),\varepsilon(v))_{\mathcal{H}}\\ &\quad+(\mathcal{M}(\theta_\varepsilon-\theta),\varepsilon(v))_{\mathcal{H}},\quad\forall v\in V.\end{aligned}\tag{4.3}$$

Similarly, by using V^h instead of V, we deduce that

$$\begin{aligned}\left\langle\sigma_\nu(u,\theta)+\frac{1}{\varepsilon}[u^h_{\varepsilon,\nu}]^+,v^h_\nu\right\rangle_{\Gamma_3}&=(\mathcal{A}\varepsilon(u-u^h_\varepsilon),\varepsilon(v^h))_{\mathcal{H}}\\ &\quad+(\mathcal{M}(\theta^h_\varepsilon-\theta),\varepsilon(v^h))_{\mathcal{H}}.\end{aligned}$$

On another side, the continuity of $(u,v)\mapsto(\mathcal{A}\varepsilon(u),\varepsilon(v))$ and $(\theta,v)\mapsto(\mathcal{M}\theta,\varepsilon(v))$ lead to

$$\begin{aligned}&\sup_{v\in H^r(\Gamma_3)} \frac{\left\langle \sigma_\nu(u,\theta)+\frac{1}{\varepsilon}[u^h_{\varepsilon,\nu}]^+, \mathcal{P}^h v\right\rangle_{\Gamma_3}}{\|\mathcal{P}^h v\|_{r,\Gamma_3}}\\ &\leq\sup_{v\in H^r(\Gamma_3)} \frac{\left\langle \sigma_\nu(u,\theta)+\frac{1}{\varepsilon}[u^h_{\varepsilon,\nu}]^+, \mathcal{R}^h(\mathcal{P}^h v)_{|\Gamma_3}\cdot\nu\right\rangle_{\Gamma_3}}{\|\mathcal{P}^h v\|_{r,\Gamma_3}}\\ &\leq\sup_{v\in H^r(\Gamma_3)}\frac{(\mathcal{A}\varepsilon(u-u^h_\varepsilon),\varepsilon\mathcal{R}^h(\mathcal{P}^h v))_{\mathcal{H}}+(\mathcal{M}(\theta^h_\varepsilon-\theta),\varepsilon\mathcal{R}^h(\mathcal{P}^h v))_{\mathcal{H}}}{\|\mathcal{P}^h v\|_{r,\Gamma_3}}\\ &\leq C\left(\|u-u^h_\varepsilon\|_{1,\Omega}+\|\theta-\theta^h_\varepsilon\|_{1,\Omega}\right)\sup_{v\in H^r(\Gamma_3)}\frac{\|\mathcal{R}^h(\mathcal{P}^h v)\|_{1,\Omega}}{\|\mathcal{P}^h v\|_{r,\Gamma_3}}\\ &\leq C\left(\|u-u^h_\varepsilon\|_{1,\Omega}+\|\theta-\theta^h_\varepsilon\|_{1,\Omega}\right)\sup_{v\in H^r(\Gamma_3)}\frac{\|\mathcal{R}^h(\mathcal{P}^h v)\|_{\frac{1}{2},\Gamma_3}}{\|\mathcal{P}^h v\|_{r,\Gamma_3}}.\end{aligned}$$

Using next the following inverse inequality

$$\|\mathcal{P}^h v\|_{\frac{1}{2},\Gamma_3}\leq Ch^{r-\frac{1}{2}}\|\mathcal{P}^h v\|_{r,\Gamma_3}$$

to find

$$\begin{aligned}&\sup_{v\in H^r(\Gamma_3)} \frac{\left\langle \sigma_\nu(u,\theta)+\frac{1}{\varepsilon}[u^h_{\varepsilon,\nu}]^+, \mathcal{P}^h v\right\rangle_{\Gamma_3}}{\|\mathcal{P}^h v\|_{r,\Gamma_3}}\\ &\leq Ch^{r-\frac{1}{2}}\left(\|u-u^h_\varepsilon\|_{1,\Omega}+\|\theta-\theta^h_\varepsilon\|_{1,\Omega}\right).\end{aligned}$$

Finally, we the following estimate

$$\begin{aligned}&\left\|\sigma_\nu(u,\theta)+\frac{1}{\varepsilon}[u^h_{\varepsilon,\nu}]^+\right\|_{-r,\Gamma_3}\\ &\leq C\left[h^{r-\frac{1}{2}}(\|u-u^h_\varepsilon\|_{1,\Omega}+\|\theta-\theta^h_\varepsilon\|_{1,\Omega})\right.\\ &\quad\left.+h^r\left\|\sigma_\nu(u,\theta)+\frac{1}{\varepsilon}[u^h_{\varepsilon,\nu}]^+\right\|_{0,\Gamma_3}\right].\end{aligned}$$

We now introduce V^ε a fictitious finite element space, defined identically as V^h and with the choice of mesh size $h=\varepsilon$. We note $\mathcal{P}^\varepsilon: L^2(\Gamma_3)\to\mathcal{X}^\varepsilon(\Gamma_3)$ the $L^2(\Gamma_3)$-projection operator onto $\mathcal{X}^\varepsilon(\Gamma_3)$. Therefore, we can write

$$\begin{aligned}&\left\|\sigma_\nu(u,\theta)+\frac{1}{\varepsilon}[u^h_{\varepsilon,\nu}]^+\right\|_{-r,\Gamma_3}\\ &\leq C\left[\varepsilon^{r-\frac{1}{2}}(\|u-u^h_\varepsilon\|_{1,\Omega}+\|\theta-\theta^h_\varepsilon\|_{1,\Omega})\right.\\ &\quad\left.+\varepsilon^r\left\|\sigma_\nu(u,\theta)+\frac{1}{\varepsilon}[u^h_{\varepsilon,\nu}]^+\right\|_{0,\Gamma_3}\right],\end{aligned}$$

which is the desired result (4.2). □

Theorem 4.2 *Suppose $\Omega\subset\mathbb{R}^d$ is a bounded polygonal domain. Let $x=(u,\theta)$ and $x_\varepsilon=(u_\varepsilon,\theta_\varepsilon)$ denote the solutions of Problems $(\mathcal{P})$ and $(\mathcal{PV}_\varepsilon)$, respectively. Assuming that $(u,\theta)\in[H^{\frac{3}{2}+r}(\Omega)]^d\times H^{\frac{3}{2}+r}(\Omega)$ (where $r\in(0,1/2]$), the subsequent prior estimate holds*

$$\|u-u_\varepsilon\|_{1,\Omega}+\|\theta-\theta_\varepsilon\|_{1,\Omega}+\varepsilon^{\frac{1}{2}}\left\|\sigma_\nu(u,\theta)+\frac{1}{\varepsilon}[u_{\varepsilon,\nu}]^+\right\|_{0,\Gamma_3}$$
$$\le C\,\varepsilon^{\frac{1}{2}+r}\left(\|u\|_{\frac{3}{2}+r,\Omega}+\|\theta\|_{\frac{3}{2}+r,\Omega}\right) \tag{4.4}$$

for a constant $C>0$ which is independent of a parameter ε and of a solution x of Problem $(\mathcal{P})$.

Proof By taking test functions $v\in(H^1(\Omega)^d)$ and $\psi\in H^1(\Omega)$ and multiplying Eq. (2.1) and (2.2) by them, we subsequently employ Green's formula and incorporate the boundary conditions (2.6), (2.7), and (2.10) to derive

$$(Bx,y)_X+\ell(u,\theta,\psi)+\int_{\Gamma_3}\sigma_\nu(u,\theta)\,v_\nu\,da$$
$$=(f^e,y)_X,\quad \forall y=(v,\psi)\in X. \tag{4.5}$$

It's worth mentioning that the Eq. (4.5) can be interpreted meaningfully if its integral term is viewed as a duality pairing between $H_{\Gamma_3}=H^{1/2}(\Gamma_3)$ and its dual space $H^*_{\Gamma_3}=H^{-1/2}(\Gamma_3)$. Hence, assuming $u\in[H^{\frac{3}{2}+r}(\Omega)]^d$ and $\psi\in H^{\frac{3}{2}+r}(\Omega)$ provides justification for this relationship by ensuring

$\sigma_\nu\in H^r(\Gamma_3)$.

Considering the ellipticity relation (3.11) of the operator A, along with the relations (3.15) and (4.5), we infer that

$$\begin{aligned}
m_B\|x-x_\varepsilon\|_X^2&\le(Bx-Bx_\varepsilon,x-x_\varepsilon)_X\\
&\le(Bx,x-x_\varepsilon)_X-(Bx_\varepsilon,x-x_\varepsilon)_X\\
&\le\int_{\Gamma_3}\left(\sigma_\nu(u,\theta)+\frac{1}{\varepsilon}[u_{\varepsilon,\nu}]^+\right)(u_\nu-u_{\varepsilon,\nu})\,da\\
&\quad+\ell(u_\varepsilon,\theta_\varepsilon,\theta-\theta_\varepsilon)-\ell(u,\theta,\theta-\theta_\varepsilon)\\
&\le\int_{\Gamma_3}\sigma_\nu(u,\theta)\,u_\nu\,da+\int_{\Gamma_3}\frac{1}{\varepsilon}[u_{\varepsilon,\nu}]^+\,u_\nu\,da+\mathcal{T}\\
&\quad-\int_{\Gamma_3}\sigma_\nu(u,\theta)\,u_{\varepsilon,\nu}\,da-\int_{\Gamma_3}\frac{1}{\epsilon}[u_{\varepsilon,\nu}]^+\,u_{\varepsilon,\nu}\,da,
\end{aligned} \tag{4.6}$$

where

$\mathcal{T}=\ell(u_\varepsilon,\theta_\varepsilon,\theta-\theta_\varepsilon)-\ell(u,\theta,\theta-\theta_\varepsilon)$.

Due to the contact conditions (2.8) on Γ_3, we observe that

$$\int_{\Gamma_3}\sigma_\nu(u,\theta)\,u_\nu\,da=0, \tag{4.7}$$

$$\int_{\Gamma_3}\frac{1}{\epsilon}[u_{\varepsilon,\nu}]^+\,u_\nu\,da\le0. \tag{4.8}$$

Additionally, recalling the beneficial property (3.13), the identical condition (2.8) results in

$$-\int_{\Gamma_3}\sigma_\nu(u,\theta)\,u_{\varepsilon,\nu}\,da\le-\int_{\Gamma_3}\sigma_\nu(u,\theta)\,[u_{\varepsilon,\nu}]^+\,da, \tag{4.9}$$

$$-\int_{\Gamma_3}\frac{1}{\varepsilon}[u_{\varepsilon,\nu}]^+\,u_{\varepsilon,\nu}\,da=-\int_{\Gamma_3}\frac{1}{\varepsilon}[u_{\varepsilon,\nu}]^+\,[u_{\varepsilon,\nu}]^+\,da. \tag{4.10}$$

Then, using the relations (4.7)–(4.10) and the well-known Young inequality, (4.6) becomes

$$\begin{aligned}
&m_B\|x-x_\varepsilon\|_X^2\\
&\le-\int_{\Gamma_3}\left(\sigma_\nu(u,\theta)+\frac{1}{\varepsilon}[u_{\varepsilon,\nu}]^+\right)[u_{\varepsilon,\nu}]^+\,da+\mathcal{T}\\
&\le-\varepsilon\int_{\Gamma_3}\left(\sigma_\nu(u,\theta)+\frac{1}{\varepsilon}[u_{\varepsilon,\nu}]^+\right)\\
&\left(\sigma_\nu(u,\theta)-\sigma_\nu(u,\theta)+\frac{1}{\varepsilon}[u_{\varepsilon,\nu}]^+\right)da+\mathcal{T}\\
&\le-\varepsilon\left\|\sigma_\nu(u,\theta)+\frac{1}{\varepsilon}[u_{\varepsilon,\nu}]^+\right\|_{0,\Gamma_3}^2\\
&\quad+\varepsilon\int_{\Gamma_3}\left(\sigma_\nu(u,\theta)+\frac{1}{\varepsilon}[u_{\varepsilon,\nu}]^+\right)\sigma_\nu(u,\theta)\,da+\mathcal{T}\\
&\le-\varepsilon\left\|\sigma_\nu(u,\theta)+\frac{1}{\varepsilon}[u_{\varepsilon,\nu}]^+\right\|_{0,\Gamma_3}^2+\varepsilon^\delta\|\sigma_\nu(u,\theta)\\
&\quad+\frac{1}{\varepsilon}[u_{\varepsilon,\nu}]^+\Big\|_{-r,\Gamma_3}\varepsilon^{1-\delta}\|\sigma_\nu(u,\theta)\|_{r,\Gamma_3}+\mathcal{T}\\
&\le-\varepsilon\|\sigma_\nu(u,\theta)\\
&\quad+\frac{1}{\varepsilon}[u_{\varepsilon,\nu}]^+\Big\|_{0,\Gamma_3}^2+\frac{\varepsilon^{2\delta}}{2\beta}\left\|\sigma_\nu(u,\theta)+\frac{1}{\varepsilon}[u_{\varepsilon,\nu}]^+\right\|_{-r,\Gamma_3}^2\\
&\quad+\frac{\beta\varepsilon^{2-2\delta}}{2}\|\sigma_\nu(u,\theta)\|_{r,\Gamma_3}^2+\mathcal{T},
\end{aligned} \tag{4.11}$$

where $\delta\in[0,1]$ and $\beta>0$. Considering the two estimates (4.1)–(4.2), we can infer that

$$\begin{aligned}
&m_B\|x-x_\varepsilon\|_X^2\\
&\le-\varepsilon\left(1-C\,\frac{\varepsilon^{2(\delta+r)-1}}{\beta}\right)\left\|\sigma_\nu(u,\theta)+\frac{1}{\varepsilon}[u_{\varepsilon,\nu}]^+\right\|_{0,\Gamma_3}^2\\
&\quad+C\,\frac{\varepsilon^{2(\delta+r)-1}}{\beta}\left[\|u-u_\varepsilon\|_{1,\Omega}^2+\|\theta-\theta_\varepsilon\|_{1,\Omega}^2\right]\\
&\quad+\frac{\beta\varepsilon^{2(1-\delta)}}{2}\|\sigma_\nu(u,\theta)\|_{r,\Gamma_3}^2+\mathcal{T}.
\end{aligned} \tag{4.12}$$

On the other hand, we have

$$\begin{aligned}\mathcal{T} &= \int_{\Gamma_3} k_T(u_{\varepsilon,\nu}-g)\,\varphi_L(\theta_\varepsilon-\theta_F)(\theta-\theta_\varepsilon)\,da\\ &\quad-\int_{\Gamma_3} k_T(u_\nu-g)\,\varphi_L(\theta-\theta_F)(\theta-\theta_\varepsilon)\,da\\ &=\int_{\Gamma_3} k_T(u_{\varepsilon,\nu}-g)\big(\varphi_L(\theta_\varepsilon-\theta_F)-\varphi_L(\theta-\theta_F)\big)(\theta-\theta_\varepsilon)\,da\\ &\quad+\int_{\Gamma_3}\big(k_T(u_{\varepsilon,\nu}-g)-k_T(u_\nu-g)\big)\,\varphi_L(\theta-\theta_F)(\theta-\theta_\varepsilon)\,da.\end{aligned}$$

Then, we deduce for some non-negative constant $C > L_\varphi L$ that

$$\begin{aligned}|\mathcal{T}| &\le M_\varphi \|\theta-\theta_\varepsilon\|^2_{0,\Gamma_3} + L_\varphi L\,\|u_{\varepsilon,\nu}-u_\nu\|_{0,\Gamma_3}\,\|\theta-\theta_\varepsilon\|_{0,\Gamma_3}\\ &\le \left(M_\varphi+\frac{L_\varphi L}{4\alpha}\right)\|\theta-\theta_\varepsilon\|^2_{0,\Gamma_3}+\alpha L_\varphi L\,\|u_{\varepsilon,\nu}-u_\nu\|^2_{0,\Gamma_3}\\ &\le \left(M_\varphi+\frac{C}{4\alpha}\right)\|\theta-\theta_\varepsilon\|^2_{1,\Omega}+C\,\alpha\,\|u_\varepsilon-u\|^2_{1,\Omega}.\end{aligned} \tag{4.13}$$

We next use the two estimates (4.13) and (4.12) to deduce

$$\begin{aligned}&\left(m_B - C\,\frac{\varepsilon^{2(\delta+r)-1}}{\beta}-C\,\alpha\right)\|u-u_\varepsilon\|^2_{1,\Omega}\\ &\quad+\left(m_B-C\,\frac{\varepsilon^{2(\delta+r)-1}}{\beta}-M_\varphi-\frac{C}{4\alpha}\right)\|\theta-\theta_\varepsilon\|^2_{1,\Omega}\\ &\quad+\varepsilon\left(1-C\,\frac{\varepsilon^{2(\delta+r)-1}}{\beta}\right)\left\|\sigma_\nu(u,\theta)+\frac{1}{\varepsilon}[u_{\varepsilon,\nu}]^+\right\|^2_{0,\Gamma_3}\\ &\le \beta\,\frac{\varepsilon^{2(1-\delta)}}{2}\,\|\sigma_\nu(u,\theta)\|^2_{r,\Gamma_3}.\end{aligned} \tag{4.14}$$

We then choose $\delta = \frac{1}{2}-r$ (which give $2(\delta+r)-1=0$), and

$$\beta = C\left(1+\frac{1}{m_B}+\frac{1}{m_B+M_\varphi}\right),$$
$$\alpha = \frac{1}{C}\left(m_B-\frac{C}{\beta}\right)+\frac{C}{2\left(m_B+M_\varphi-\frac{C}{\beta}\right)}.$$

This choice of δ, α and β was made in order to guarantee that

$$m_B - C\,\frac{\varepsilon^{2(\delta+r)-1}}{\beta}-C\,\alpha>0,$$
$$m_B - C\,\frac{\varepsilon^{2(\delta+r)-1}}{\beta}-M_\varphi-\frac{C}{4\alpha}>0,$$
$$1-C\,\frac{\varepsilon^{2(\delta+r)-1}}{\beta}>0.$$

Thus, the desired bound (4.4) follows from the estimate

$$\|\sigma_\nu(u,\theta)\|_{r,\Gamma_3}\le C\left(\|u\|_{\frac{3}{2}+r,\Omega}+\|\theta\|_{\frac{3}{2}+r,\Omega}\right).$$

□

4.2 A Priori Estimate of of Tresca's Friction Problem

Theorem 4.3 *Suppose $x=(u,\theta)$ and $x_\varepsilon=(u_\varepsilon,\theta_\varepsilon)$ represent the solutions of Problem $(\overline{\mathcal{PV}})$ and Problem $(\overline{\mathcal{PV}}_\varepsilon)$, respectively. If we assume the regularities $u\in(H^{\frac{3}{2}+r}(\Omega))^d$ and $\theta\in H^{\frac{3}{2}+r}(\Omega)$ with $0<r\le 1/2$, then the following a priori estimate holds*

$$\begin{aligned}&\|u-u_\varepsilon\|_{1,\Omega}+\|\theta-\theta_\varepsilon\|_{1,\Omega}+\varepsilon^{\frac{1}{2}}\left\|\sigma_\nu(u,\theta)+\frac{1}{\varepsilon}[u_{\varepsilon,\nu}]^+\right\|_{0,\Gamma_3}\\ &\quad+\varepsilon^{\frac{1}{2}}\left\|\sigma_\tau(u,\theta)+\frac{1}{\varepsilon}[u_{\varepsilon,\tau}]_{\varepsilon g}\right\|_{0,\Gamma_3}\\ &\quad\le C\,\varepsilon^{\frac{1}{2}+r}\left(\|u\|_{\frac{3}{2}+r,\Omega}+\|\theta\|_{\frac{3}{2}+r,\Omega}\|\right),\end{aligned} \tag{4.15}$$

where $C>0$ does not depend on a penalized parameter ε, nor on $x=(u,\theta)$ solution of Problem $(\overline{\mathcal{PV}})$.

Proof It's notable to observe that the friction conditions (2.9) and the definition of $[\cdot]_{\varepsilon,g}$ entail

$$\begin{aligned}&\int_{\Gamma_3}\left(\sigma_\tau(u,\theta)+\frac{1}{\epsilon}\,[u_{\varepsilon,\tau}]_{\varepsilon g}\right)(u_\tau-u_{\varepsilon,\tau})\,da\\ &\quad\le -\int_{\Gamma_3}\left(\sigma_\tau(u,\theta)+\frac{1}{\epsilon}[u_{\varepsilon,\tau}]_{\varepsilon g}\right)u_{\varepsilon,\tau}\,da\\ &\quad\le -\int_{\Gamma_3}\left(\sigma_\tau(u,\theta)+\frac{1}{\varepsilon}[u_{\varepsilon,\tau}]_{\varepsilon g}\right)[u_{\varepsilon,\tau}]_{\varepsilon g}\,da.\end{aligned}$$

The estimate (4.2) obtained in Lemma 4.1 still holds, by replacing $\|\sigma_\nu(u,\theta)+\frac{1}{\varepsilon}[u_{\varepsilon,\nu}]^+\|_{s,\Gamma_3}$ by

$$\left\|\sigma_\nu(u,\theta)+\frac{1}{\varepsilon}[u_{\varepsilon,\nu}]^+\right\|_{s,\Gamma_3}+\left\|\sigma_\tau(u,\theta)+\frac{1}{\varepsilon}[u_{\varepsilon,\tau}]_{\varepsilon g}\right\|_{s,\Gamma_3},$$

for $s=-r$ or $s=0$, see (Chouly and Hild 2013b, Theorem 4.1). Subsequently, employing similar methodologies as presented in (Chouly and Hild 2013b, Theorem 4.1), we can derive the estimate (4.15), thereby establishing Theorem 4.2. □

4.3 A Priori Estimate of the Approximation Frictionless Problem

Theorem 4.4 *Consider $x^h_\varepsilon=(u^h_\varepsilon,\theta^h_\varepsilon)$ and $x_\varepsilon=(u_\varepsilon,\theta_\varepsilon)$ as the solutions of problems $(\mathcal{PV}^h_\varepsilon)$ and $(\mathcal{PV}_\varepsilon)$, respectively. Then, for any $\varepsilon>0$ and any $h>0$, the subsequent a priori error estimate is valid*

$$\|u_\varepsilon - u_\varepsilon^h\|_{1,\Omega} + \|\theta_\varepsilon - \theta_\varepsilon^h\|_{1,\Omega} + (\varepsilon^{\frac{1}{2}} + Ch^{\frac{1}{2}}) \left\| \frac{1}{\varepsilon}[u_{\varepsilon,\nu}]^+ - \frac{1}{\varepsilon}[u_{\varepsilon,\nu}^h]^+ \right\|_{0,\Gamma_3} \leq C\Big\{ \|u - u_\varepsilon\|_{1,\Omega} + \|u - v^h\|_{1,\Omega} + \|\theta - \theta_\varepsilon\|_{1,\Omega} + \|\theta - \xi^h\|_{1,\Omega} \Big\}, \quad \forall y^h = (v^h, \xi^h) \in X^h, \tag{4.16}$$

where $C > 0$ is independent of the constrained solution $x = (u, \theta)$ and the parameters h and ε.

Proof Taking $y^h - x_\epsilon^h$ as test function in Problem $(\mathcal{PV}_\epsilon^h)$, we obtain

$$(Bx_\varepsilon^h, y^h - x_\varepsilon^h)_X + \ell(u_\varepsilon^h, \theta_\varepsilon^h, \xi^h - \theta_\varepsilon^h) + \frac{1}{\varepsilon}\int_{\Gamma_3} [u_{\varepsilon,\nu}^h]^+ (v_n^h - u_{\varepsilon,\nu}^h)\, da = (f^e, y^h - x_\varepsilon^h)_X, \quad \forall y^h = (v^h, \xi^h) \in X^h. \tag{4.17}$$

The penalized problem $(\mathcal{PV}_\varepsilon)$ aligns with the finite element problem $(\mathcal{PV}_\epsilon^h)$ in such a way that the solution $x_\varepsilon = (u_\varepsilon, \theta_\varepsilon)$ of Problem $(\mathcal{PV}_\varepsilon)$ also satisfies, for every $y^h = (v^h, \xi^h) \in X^h$, the equation

$$(Bx_\varepsilon, y^h)_X + \ell(u_\varepsilon, \theta_\varepsilon, \xi^h) + \frac{1}{\varepsilon}\int_{\Gamma_3} [u_{\varepsilon,\nu}]^+ v_\nu^h\, da = (f^e, y^h)_X. \tag{4.18}$$

We take the test function $x_\varepsilon^h = (u_\varepsilon^h, \theta_\varepsilon^h) \in X^h$ in equation (4.18) to deduce

$$(Bx_\varepsilon, y^h - x_\varepsilon^h)_X + \ell(u_\varepsilon, \theta_\varepsilon, \xi^h - \theta_\varepsilon^h) + \frac{1}{\varepsilon}\int_{\Gamma_3} [u_{\varepsilon,\nu}]^+ (v_n^h - u_{\varepsilon,\nu}^h)\, da = (f^e, y^h - x_\varepsilon^h)_X, \quad \forall y^h = (v^h, \xi^h) \in X^h. \tag{4.19}$$

Given that the operator B exhibits strong monotonicity and Lipschitz continuity (see (3.11)–(3.12)), we have

$$\begin{aligned} m_B \|x_\varepsilon - x_\varepsilon^h\|_X^2 &\leq (Bx_\varepsilon - Bx_\varepsilon^h, x_\varepsilon - x_\varepsilon^h)_X \\ &\leq (Bx_\varepsilon - Bx_\varepsilon^h, x_\varepsilon - y^h)_X + (Bx_\varepsilon - Bx_\varepsilon^h, y^h - x_\varepsilon^h)_X \\ &\leq M_B \|x_\varepsilon - x_\varepsilon^h\|_X \|x_\varepsilon - y^h\|_X \\ &\quad + (Bx_\varepsilon, y^h - x_\varepsilon^h)_X - (Bx_\varepsilon^h, y^h - x_\varepsilon^h)_X, \\ &\qquad \forall y^h = (v^h, \xi^h) \in X^h. \end{aligned} \tag{4.20}$$

Employing Young's inequality and the triangle inequality, the preceding estimate (4.20) is transformed into

$$\begin{aligned} m_B \|x_\varepsilon - x_\varepsilon^h\|_X^2 &\leq \frac{1}{2\alpha} \|x_\varepsilon - x_\varepsilon^h\|_X^2 + \frac{\alpha M_B^2}{2} \|x_\varepsilon - y^h\|_X^2 \\ &\quad + (Bx_\varepsilon, y^h - x_\varepsilon^h)_X - (Bx_\varepsilon^h, y^h - x_\varepsilon^h)_X \\ &\leq \frac{1}{2\alpha} \|x_\varepsilon - x_\varepsilon^h\|_X^2 + \alpha M_B^2 \left(\|x_\varepsilon - x\|_X^2 + \|x - y^h\|_X^2 \right) \\ &\quad + (Bx_\varepsilon, y^h - x_\varepsilon^h)_X - (Bx_\varepsilon^h, y^h - x_\varepsilon^h)_X, \quad \forall y^h = (v^h, \xi^h) \in X^h. \end{aligned} \tag{4.21}$$

To assess the last two terms of the previous inequality, we utilize equations (4.17) and (4.19) to derive

$$\begin{aligned} &(Bx_\varepsilon, y^h - x_\varepsilon^h)_X - (Bx_\varepsilon^h, y^h - x_\varepsilon^h)_X \\ &= \int_{\Gamma_3} \left(\frac{1}{\varepsilon}[u_{\varepsilon,\nu}^h]^+ - \frac{1}{\varepsilon}[u_{\varepsilon,\nu}]^+ \right) (v_\nu^h - u_{\varepsilon,\nu}^h)\, da \\ &\quad + \ell(u_\varepsilon^h, \theta_\varepsilon^h, \xi^h - \theta_\varepsilon^h) - \ell(u_\varepsilon, \theta_\varepsilon, \xi^h - \theta_\varepsilon^h) \\ &= \int_{\Gamma_3} \left(\frac{1}{\varepsilon}[u_{\varepsilon,\nu}]^+ - \frac{1}{\varepsilon}[u_{\varepsilon,\nu}^h]^+ \right) (u_{\varepsilon,\nu}^h - u_{\varepsilon,\nu})\, da \\ &\quad + \int_{\Gamma_3} \left(\frac{1}{\varepsilon}[u_{\varepsilon,\nu}]^+ - \frac{1}{\varepsilon}[u_{\varepsilon,\nu}^h]^+ \right) (u_{\varepsilon,\nu} - v_\nu^h)\, da \\ &\quad + \ell(u_\varepsilon^h, \theta_\varepsilon^h, \xi^h - \theta_\varepsilon^h) - \ell(u_\varepsilon, \theta_\varepsilon, \xi^h - \theta_\varepsilon^h). \end{aligned} \tag{4.22}$$

We estimate the first term of (4.22) by

$$\begin{aligned} &\int_{\Gamma_3} \left(\frac{1}{\varepsilon}[u_{\varepsilon,\nu}]^+ - \frac{1}{\varepsilon}[u_{\varepsilon,\nu}^h]^+ \right) (u_{\varepsilon,\nu}^h - u_{\varepsilon,\nu})\, da \\ &= -\varepsilon \int_{\Gamma_3} \left(\frac{1}{\varepsilon}[u_{\varepsilon,\nu}]^+ - \frac{1}{\varepsilon}[u_{\varepsilon,\nu}^h]^+ \right) \left(\frac{1}{\varepsilon} u_{\varepsilon,\nu} - \frac{1}{\varepsilon} u_{\varepsilon,\nu}^h \right) da \\ &\leq -\varepsilon \left\| \frac{1}{\varepsilon}[u_{\varepsilon,\nu}]^+ - \frac{1}{\varepsilon}[u_{\varepsilon,\nu}^h]^+ \right\|_{0,\Gamma_3}^2. \end{aligned} \tag{4.23}$$

For the second term of (4.22), we (3.26), (3.27) and Cauchy–Schwartz inequality to obtain

$$\begin{aligned} &\int_{\Gamma_3} \left(\frac{1}{\varepsilon}[u_{\varepsilon,\nu}]^+ - \frac{1}{\varepsilon}[u_{\varepsilon,\nu}^h]^+ \right) (u_{\varepsilon,\nu} - v_\nu^h)\, da \\ &= \int_{\Gamma_3} \left(\frac{1}{\varepsilon}[u_{\varepsilon,\nu}]^+ - \frac{1}{\varepsilon}[u_{\varepsilon,\nu}^h]^+ \right) \left((u_\varepsilon - v^h)\cdot\nu - \mathbf{P}^h(u_\varepsilon - v^h)\cdot\nu \right) da \\ &\quad + \int_{\Gamma_3} \left(\frac{1}{\varepsilon}[u_{\varepsilon,\nu}]^+ - \frac{1}{\varepsilon}[u_{\varepsilon,\nu}^h]^+ \right) \mathbf{P}^h(u_\varepsilon - v^h)\cdot\nu\, da \\ &\leq \left\| \frac{1}{\varepsilon}[u_{\varepsilon,\nu}]^+ - \frac{1}{\varepsilon}[u_{\varepsilon,\nu}^h]^+ \right\|_{0,\Gamma_3} \|(u_\varepsilon - v^h)\cdot\nu - \mathbf{P}^h(u_\varepsilon - v^h)\cdot\nu\|_{0,\Gamma_3} \\ &\quad + \int_{\Gamma_3} \left(\frac{1}{\varepsilon}[u_{\varepsilon,\nu}]^+ - \frac{1}{\varepsilon}[u_{\varepsilon,\nu}^h]^+ \right) \mathbf{R}^h(\mathbf{P}^h(u_\varepsilon - v^h)\cdot\nu)\, da. \end{aligned} \tag{4.24}$$

Through the utilization of interpolation (3.23), in conjunction with the continuity of the trace operator and Young's inequality, we obtain

$$\begin{aligned}&\left\|\frac{1}{\varepsilon}\,[u_{\varepsilon,\nu}]^{+}-\frac{1}{\varepsilon}\,[u^h_{\varepsilon,\nu}]^{+}\right\|_{0,\Gamma_3}\|(u_\varepsilon-v^h)\cdot\nu-\mathbf{P}^h(u_\epsilon-v^h)\cdot\nu\|_{0,\Gamma_3}\\&\quad\le ch^{\frac12}\left\|\frac{1}{\varepsilon}\,[u_{\epsilon,\nu}]^{+}-\frac{1}{\varepsilon}\,[u^h_{\varepsilon,\nu}]^{+}\right\|_{0,\Gamma_3}\|u_\varepsilon-v^h\|_{\frac12,\Gamma_3}\\&\quad\le ch\left\|\frac{1}{\varepsilon}\,[u_{\varepsilon,\nu}]^{+}-\frac{1}{\varepsilon}\,[u^h_{\varepsilon,\nu}]^{+}\right\|^2_{0,\Gamma_3}+c\,\|u_\varepsilon-v^h\|^2_{1,\Omega}\\&\quad\le Ch\left\|\frac{1}{\varepsilon}\,[u_{\varepsilon,\nu}]^{+}-\frac{1}{\varepsilon}\,[u^h_{\varepsilon,\nu}]^{+}\right\|^2_{0,\Gamma_3}\\&\qquad+C\left(\|u_\varepsilon-u\|^2_{1,\Omega}+\|u-v^h\|^2_{1,\Omega}\right).\end{aligned}\tag{4.25}$$

Using condition (4.18) and stability properties in (3.22) and (3.25), and choosing

$$y^*=\left(\mathbf{R}^h(\mathbf{P}^h(u_\epsilon-v^h)),\theta_\epsilon-\xi^h\right),$$
$$\mathcal{K}=\ell(u_\varepsilon,\theta_\varepsilon,\theta_\epsilon-\xi^h)-\ell(u^h_\varepsilon,\theta^h_\varepsilon,\theta_\epsilon-\xi^h),$$

we can deduce from Problem $(\mathcal{PV}^h_\varepsilon)$ that

$$\begin{aligned}&\int_{\Gamma_3}\left(\frac{1}{\varepsilon}\,[u_{\varepsilon,\nu}]^{+}-\frac{1}{\varepsilon}\,[u^h_{\varepsilon,\nu}]^{+}\right)\mathbf{R}^h\left(\mathbf{P}^h(u_\varepsilon-v^h)\cdot\nu\right)da\\&\quad\le(Bx_\varepsilon-Bx^h_\varepsilon,y^*)_X+\ell(u_\varepsilon,\theta_\varepsilon,\theta_\epsilon-\xi^h)-\ell(u^h_\varepsilon,\theta^h_\varepsilon,\theta_\epsilon-\xi^h)\\&\quad\le M_B\,\|x_\varepsilon-x^h_\varepsilon\|_X\,\|y^*\|_X+\mathcal{K}\\&\quad\le M_B\,\|x_\varepsilon-x^h_\varepsilon\|_X\,\{\|\mathbf{R}^h(\mathbf{P}^h(u_\varepsilon-v^h))\|_{1,\Omega}\\&\qquad+\|\theta_\varepsilon-\xi^h\|_{1,\Omega}\}+\mathcal{K}\\&\quad\le CM_B\,\|x_\varepsilon-x^h_\varepsilon\|_X\,\{\|\mathbf{P}^h(u_\varepsilon-v^h)\|_{\frac12,\Gamma_3}+\|\theta_\varepsilon-\xi^h\|_{1,\Omega}\}+\mathcal{K}\\&\quad\le CM_B\|x_\varepsilon-x^h_\varepsilon\|_X\,\{\|u_\varepsilon-u\|_{1,\Omega}\\&\qquad+\|u-v^h\|_{1,\Omega}+\|\theta_\varepsilon-\theta\|_{1,\Omega}+\|\theta-\xi^h\|_{1,\Omega}\}+\mathcal{K}\\&\quad\le\frac{1}{\alpha}\|x_\varepsilon-x^h_\varepsilon\|^2_X\\&\quad+\frac{\alpha\,(CM_B)^2}{2}\,\{\|u_\varepsilon-u\|^2_{1,\Omega}+\|u-v^h\|^2_{1,\Omega}\\&\qquad+\|\theta_\varepsilon-\theta\|^2_{1,\Omega}+\|\theta-\xi^h\|^2_{1,\Omega}\}+\mathcal{K},\end{aligned}\tag{4.26}$$

where the term $\mathcal{K}$ is as follows

$$\begin{aligned}\mathcal{K}&=\int_{\Gamma_3}k_T(u_{\varepsilon,\nu}-g)\,\varphi_L(\theta_\varepsilon-\theta_F)(\theta_\varepsilon-\xi^h)\,da\\&\quad-\int_{\Gamma_3}k_T(u^h_{\varepsilon,\nu}-g)\,\varphi_L(\theta^h_\varepsilon-\theta_F)(\theta_\varepsilon-\xi^h)\,da\\&=\int_{\Gamma_3}k_T(u_{\varepsilon,\nu}-g)\,\left(\varphi_L(\theta_\varepsilon-\theta_F)-\varphi_L(\theta^h_\varepsilon-\theta_F)\right)(\theta_\varepsilon-\xi^h)\,da\\&\quad+\int_{\Gamma_3}\left(k_T(u_{\varepsilon,\nu}-g)-k_T(u^h_{\varepsilon,\nu}-g)\right)\varphi_L(\theta^h_\varepsilon-\theta_F)\,(\theta_\varepsilon-\xi^h)\,da.\end{aligned}$$

Then, we finally find

$$\begin{aligned}&|\mathcal{K}|\\&\quad\le M_\varphi\|\theta_\varepsilon-\theta^h_\varepsilon\|_{0,\Gamma_3}\,\|\theta_\varepsilon-\xi^h\|_{0,\Gamma_3}\\&\qquad+L_\varphi L\,\|u_{\varepsilon,\nu}-u^h_{\varepsilon,\nu}\|_{0,\Gamma_3}\,\|\theta_\varepsilon-\xi^h\|_{0,\Gamma_3}\\&\quad\le\frac{M_\varphi}{4\alpha}\,\|\theta_\varepsilon-\theta^h_\varepsilon\|^2_{0,\Gamma_3}+\left(\alpha M_\varphi+\frac{L_\varphi L}{4\alpha}\right)\|\theta_\varepsilon-\xi^h\|^2_{0,\Gamma_3}\\&\qquad+\alpha L_\varphi L\,\|u_{\varepsilon,\nu}-u^h_{\varepsilon,\nu}\|^2_{0,\Gamma_3}\\&\quad\le\frac{M_\varphi}{4\alpha}\,\|\theta_\varepsilon-\theta^h_\varepsilon\|^2_{1,\Omega}\\&\qquad+\left(\alpha M_\varphi+\frac{C}{4\alpha}\right)\|\theta_\varepsilon-\xi^h\|^2_{1,\Omega}+\alpha C\,\|u_\varepsilon-u^h_\varepsilon\|^2_{1,\Omega}\\&\quad\le\frac{M_\varphi}{4\alpha}\,\|\theta_\varepsilon-\theta^h_\varepsilon\|^2_{1,\Omega}+\left(\alpha M_\varphi+\frac{C}{4\alpha}\right)\Big(\|\theta_\varepsilon-\theta\|^2_{1,\Omega}\\&\qquad+\|\theta-\xi^h\|^2_{1,\Omega}\Big)+\alpha C\,\|u_\varepsilon-u^h_\varepsilon\|^2_{1,\Omega}.\end{aligned}\tag{4.27}$$

We now combine (4.25), (4.26) and (4.27) to rewrite the estimate (4.24) as follows

$$\begin{aligned}&\int_{\Gamma_3}\left(\frac{1}{\varepsilon}\,[u_{\varepsilon,\nu}]^{+}-\frac{1}{\varepsilon}\,[u^h_{\varepsilon,\nu}]^{+}\right)(u_{\varepsilon,\nu}-v^h_\nu)\,da\\&\quad\le\frac{1}{\alpha}\,\|x_\varepsilon-x^h_\varepsilon\|^2_X+ch\left\|\frac{1}{\varepsilon}\,[u_{\varepsilon,\nu}]^{+}-\frac{1}{\varepsilon}\,[u^h_{\varepsilon,\nu}]^{+}\right\|^2_{0,\Gamma_3}\\&\qquad+C\left(1+\frac{\alpha M_B^2}{2}\right)\left(\|u_\varepsilon-u\|^2_{1,\Omega}+\|u-v^h\|^2_{1,\Omega}\right)\\&\qquad+\left(\alpha M_\varphi+\frac{C}{4\alpha}+\frac{\alpha CM_B^2}{2}\right)\left(\|\theta_\varepsilon-\theta\|^2_{1,\Omega}+\|\theta-\xi^h\|^2_{1,\Omega}\right)\\&\qquad+\alpha C\,\|u_\varepsilon-u^h_\varepsilon\|^2_{1,\Omega}+\frac{M_\varphi}{4\alpha}\,\|\theta_\varepsilon-\theta^h_\varepsilon\|^2_{1,\Omega}.\end{aligned}\tag{4.28}$$

For the third term of (4.22), wee know that

$$\begin{aligned}&\ell(u^h_\varepsilon,\theta^h_\varepsilon,\xi^h-\theta^h_\varepsilon)-\ell(u_\varepsilon,\theta_\varepsilon,\xi^h-\theta^h_\varepsilon)\\&\quad=\int_{\Gamma_3}k_T(u^h_{\varepsilon,\nu}-g)\varphi_L(\theta^h_\varepsilon-\theta_F)(\xi^h-\theta^h_\varepsilon)\,da\\&\qquad-\int_{\Gamma_3}k_T(u_{\varepsilon,\nu}-g)\varphi_L(\theta_\varepsilon-\theta_F)(\xi^h-\theta^h_\varepsilon)\,da\\&\quad=\int_{\Gamma_3}k_T(u^h_{\varepsilon,\nu}-g)\left(\varphi_L(\theta^h_\varepsilon-\theta_F)-\varphi_L(\theta_\varepsilon-\theta_F)\right)(\xi^h-\theta^h_\varepsilon)\,da\\&\qquad+\int_{\Gamma_3}\left(k_T(u^h_{\varepsilon,\nu}-g)-k_T(u_{\varepsilon,\nu}-g)\right)\varphi_L(\theta_\varepsilon-\theta_F)(\xi^h-\theta^h_\varepsilon)\,da.\end{aligned}$$

Then, the following majoration inequality holds.

$$
\begin{aligned}
&\left|\ell(u^h_\varepsilon,\theta^h_\varepsilon,\xi^h-\theta^h_\varepsilon)-\ell(u_\varepsilon,\theta_\varepsilon,\xi^h-\theta^h_\varepsilon)\right|\\
&\quad\le M_\varphi\|\theta^h_\varepsilon-\theta_\varepsilon\|_{0,\Gamma_3}\,\|\xi^h-\theta^h_\varepsilon\|_{0,\Gamma_3}\\
&\qquad+L_\varphi L\,\|u^h_{\varepsilon,v}-u_{\varepsilon,v}\|_{0,\Gamma_3}\,\|\xi^h-\theta^h_\varepsilon\|_{0,\Gamma_3}\\
&\quad\le\frac{M_\varphi}{4\alpha}\,\|\theta^h_\varepsilon-\theta_\varepsilon\|^2_{0,\Gamma_3}+\left(\alpha M_\varphi+\frac{L_\varphi L}{4\alpha}\right)\\
&\|\theta^h_\varepsilon-\xi^h\|^2_{0,\Gamma_3}+\alpha L_\varphi L\,\|u_{\varepsilon,v}-u^h_{\varepsilon,v}\|^2_{0,\Gamma_3}\\
&\quad\le\frac{M_\varphi}{4\alpha}\,\|\theta_\varepsilon-\theta^h_\varepsilon\|^2_{1,\Omega}\\
&\qquad+\left(\alpha M_\varphi+\frac{C}{4\alpha}\right)\|\theta^h_\varepsilon-\xi^h\|^2_{1,\Omega}+\alpha C\,\|u_\varepsilon-u^h_\varepsilon\|^2_{1,\Omega}\\
&\quad\le\frac{M_\varphi}{4\alpha}\,\|\theta_\varepsilon-\theta^h_\varepsilon\|^2_{1,\Omega}+\left(\alpha M_\varphi+\frac{C}{4\alpha}\right)\\
&\left(\|\theta_\varepsilon-\theta^h_\varepsilon\|^2_{1,\Omega}+\|\theta_\varepsilon-\theta\|^2_{1,\Omega}+\|\theta-\xi^h\|^2_{1,\Omega}\right)\\
&\qquad+\alpha C\,\|u_\varepsilon-u^h_\varepsilon\|^2_{1,\Omega}.
\end{aligned}
\tag{4.29}
$$

Next, we combine (4.23), (4.28) and (4.29) to reformulate the estimate (4.22), we express it in the following manner

$$
\begin{aligned}
&(Bx_\varepsilon,y^h-x^h_\varepsilon)_X-(Bx^h_\varepsilon,y^h-x^h_\varepsilon)_X\\
&\quad\le\frac{1}{\alpha}\,\|x_\varepsilon-x^h_\varepsilon\|^2_X+(Ch-\varepsilon)\left\|\frac{1}{\varepsilon}\,[u_{\varepsilon,v}]^+-\frac{1}{\varepsilon}\,[u^h_{\varepsilon,v}]^+\right\|^2_{0,\Gamma_3}\\
&\qquad+C\left[1+\frac{\alpha M_B^2}{2}\right]\left(\|u_\varepsilon-u\|^2_{1,\Omega}+\|u-v^h\|^2_{1,\Omega}\right)\\
&\qquad+\left[2\alpha M_\varphi+\frac{C}{2\alpha}+\frac{\alpha C M_B^2}{2}\right]\left(\|\theta_\varepsilon-\theta\|^2_{1,\Omega}+\|\theta-\xi^h\|^2_{1,\Omega}\right)\\
&\qquad+2\alpha C\,\|u_\varepsilon-u^h_\varepsilon\|^2_{1,\Omega}+\left[\frac{M_\varphi}{2\alpha}+\alpha M_\varphi+\frac{C}{4\alpha}\right]\|\theta_\varepsilon-\theta^h_\varepsilon\|^2_{1,\Omega}.
\end{aligned}
\tag{4.30}
$$

Recalling $\|\cdot\|^2_X=\|\cdot\|^2_V+\|\cdot\|^2_Q$, $\|\cdot\|_{H_1}\underset{V}{\approx}\|\cdot\|_V$ and $\|\cdot\|_Q=\|\cdot\|_{1,\Omega}$, we use the inequality (4.30) to rewrite the estimate (4.21) as follows

$$
\begin{aligned}
&\left[m_B-\frac{3}{2\alpha}-2\alpha C\right]\|u_\varepsilon-u^h_\varepsilon\|^2_{1,\Omega}\\
&\qquad+\left[m_B-\frac{3}{2\alpha}-\frac{M_\varphi}{2\alpha}-\alpha M_\varphi-\frac{C}{4\alpha}\right]\|\theta_\varepsilon-\theta^h_\varepsilon\|^2_{1,\Omega}\\
&\qquad+\varepsilon\left\|\frac{1}{\varepsilon}[u_{\varepsilon,v}]^+-\frac{1}{\varepsilon}[u^h_{\varepsilon,v}]^+\right\|^2_{0,\Gamma_3}\\
&\quad\le Ch\left\|\frac{1}{\varepsilon}[u_{\varepsilon,v}]^+-\frac{1}{\varepsilon}[u^h_{\varepsilon,v}]^+\right\|^2_{0,\Gamma_3}\\
&\qquad+C\left[1+\frac{3\alpha M_B^2}{2}\right]\left(\|u_\varepsilon-u\|^2_{1,\Omega}+\|u-v^h\|^2_{1,\Omega}\right)\\
&\qquad+\left[2\alpha M_\varphi+\frac{C}{2\alpha}+\frac{3C\alpha M_B^2}{2}\right]\left(\|\theta_\varepsilon-\theta\|^2_{1,\Omega}+\|\theta-\xi^h\|^2_{1,\Omega}\right).
\end{aligned}
\tag{4.31}
$$

Under appropriate mathematical condition on $\alpha>0$ involving m_B, M_φ and C, the terms $m_B-\frac{3}{2\alpha}-2\alpha C$ and $m_B-\frac{3}{2\alpha}-\frac{M_\varphi}{2\alpha}-\alpha M_\varphi-\frac{C}{4\alpha}$ are non-negative. So, (4.31) implies the desired estimation (4.16). □

Theorem 4.5 *Given the conditions stated in Theorem 4.4, if $x=(u,\theta)$ represents the solution of Problem (3.8) such that*

$$u\in H^{\frac{3}{2}+r}(\Omega)^d,\quad \theta\in H^{\frac{3}{2}+r}(\Omega)\quad\text{with}\quad 0<r\le 2,$$

then the ensuing estimate holds true

$$
\begin{aligned}
&\left\|u-u^h_\varepsilon\right\|_{1,\Omega}+\left\|\theta-\theta^h_\varepsilon\right\|_{1,\Omega}+(\varepsilon^{\frac{1}{2}}-Ch^{\frac{1}{2}})\,\Big\|\sigma_v(u,\theta)\\
&\qquad+\frac{1}{\varepsilon}\,[u^h_{\varepsilon,v}]^+\Big\|_{0,\Gamma_3}\\
&\quad\le C\left(h^{\frac{1}{2}+r}+\varepsilon^{\frac{1}{2}+r}\right)\left(\|u\|_{\frac{3}{2}+r,\Omega}+\|\theta\|_{\frac{3}{2}+r,\Omega}\right)
\end{aligned}
\tag{4.32}
$$

where $C>0$ is independent of $x=(u,\theta)$, ε and h.

Proof Given that $x^h=(u^h,\theta^h)\in X^h$, we select

$$x^h=(u^h,\theta^h)=(\mathcal{I}^1_h(u),\mathcal{I}^1_h(\theta)),$$

where $\mathcal{I}^1_h$ denotes Lagrange's interpolation operator associated with $X^h=V^h\times Q^h$. The conventional Lagrange interpolation approximations in the $H^1(\Omega)$ norm are provided for $r\in(-\frac{1}{2},\frac{1}{2}]$ as detailed in prior works (see khalfi et al. (2023); Bourichi et al. (2016); Dione (2019); Ern and Guermond (2004))

$$
\begin{aligned}
&\left\|u-\mathcal{I}^1_h(u)\right\|_{1,\Omega}\le ch^{\frac{1}{2}+r}\|u\|_{\frac{3}{2}+r,\Omega},\\
&\left\|\theta-\mathcal{I}^1_h(\theta)\right\|_{1,\Omega}\le ch^{\frac{1}{2}+r}\|\theta\|_{\frac{3}{2}+r,\Omega}.
\end{aligned}
\tag{4.33}
$$

We opt for penalty and mesh parameters to ensure $\varepsilon>h$. Subsequently, utilizing the triangle inequality, Theorem 4.5 is derived from Theorems 4.4, 4.2, and the interpolation estimate (4.33). □

Remark 4.6 Ultimately, to establish a convergence rate for the approximation (4.32), we may select $\varepsilon(h):=ch^\gamma$, where c and γ are fixed positive constants; hence, the penalty parameter ε becomes a function of the mesh size h. Consequently, we obtain

1. When ε scales in accordance with h, meaning $\varepsilon(h):=(C+1)^2h$, we derive the following a priori estimate from Theorem 4.4:

$$
\begin{aligned}
&\|u-u^h_\varepsilon\|_{1,\Omega}+\|\theta-\theta^h_\varepsilon\|_{1,\Omega}+h^{\frac{1}{2}}\|\sigma_v(u,\theta)\\
&\quad+\frac{1}{\varepsilon}\,[u^h_{\varepsilon,v}]^+\Big\|_{0,\Gamma_3}\le Ch^{\frac{1}{2}+r}\left(\|u\|_{\frac{3}{2}+r,\Omega}+\|\theta\|_{\frac{3}{2}+r,\Omega}\right).
\end{aligned}
$$

2. When $\varepsilon(h):=C^2h^\gamma$ with $0<\gamma<1$, Theorem 4.4 gives us the following a priori estimate:

$$\|u-u_\varepsilon^h\|_{1,\Omega}+\|\theta-\theta_\varepsilon^h\|_{1,\Omega} + Ch^{\frac{\gamma}{2}}\left(1-h^{\frac{1-\gamma}{2}}\right)\left\|\sigma_\nu(u,\theta)+\frac{1}{\varepsilon}[u_{\varepsilon,\nu}^h]^+\right\|_{0,\Gamma_3} \le Ch^{\gamma(\frac{1}{2}+r)}\left(\|u\|_{\frac{3}{2}+r,\Omega}+\|\theta\|_{\frac{3}{2}+r,\Omega}\right).$$

4.4 A Priori Estimate of the Approximation Tresca's Friction Problem

Theorem 4.7 *Let $x_\epsilon=(u_\epsilon,\theta_\epsilon)$ and $x_\varepsilon^h=(u_\varepsilon^h,\theta_\varepsilon^h)$ solutions of problems $(\overline{\mathcal{PV}}_\varepsilon)$ and $(\overline{\mathcal{PV}}_\varepsilon^h)$, respectively. Subsequently, for any $\varepsilon>0$ and $h>0$, the ensuing a priori estimate is obtained*

$$\|u_\varepsilon-u_\varepsilon^h\|_{1,\Omega}+\|\theta_\varepsilon-\theta_\varepsilon^h\|_{1,\Omega} + \left(\varepsilon^{\frac{1}{2}}-Ch^{\frac{1}{2}}\right)\left[\left\|\frac{1}{\varepsilon}[u_{\varepsilon,\nu}]^+ +\frac{1}{\varepsilon}[u_{\varepsilon,\nu}^h]^+\right\|_{0,\Gamma_3}+\left\|\frac{1}{\varepsilon}[u_{\varepsilon,\tau}]_{\varepsilon g}-\frac{1}{\varepsilon}[u_{\varepsilon,\tau}^h]_{\varepsilon g}\right\|_{0,\Gamma_3}\right] \le C\left(\|u_\varepsilon-u\|_{1,\Omega}+\|u-v^h\|_{1,\Omega} +\|\theta_\varepsilon-\theta\|_{1,\Omega}+\|\theta-\xi^h\|_{1,\Omega}\right),\quad \forall y^h=(v^h,\xi^h)\in X^h, \tag{4.34}$$

where a constant $C>0$ remains independent of $x=(u,\theta),\varepsilon$ and h.

Proof The proof technique employed here resembles that of Theorem 4.4. Initially, as utilized in establishing (4.21), we have

$$\begin{aligned} m_B\|x_\varepsilon-x_\varepsilon^h\|_X^2 &\le \frac{1}{2\alpha}\|x_\varepsilon-x_\varepsilon^h\|_X^2+\frac{\alpha M_B^2}{2}\|x_\varepsilon-y^h\|_X^2 + (Bx_\varepsilon,y^h-x_\varepsilon^h)_X-(Bx_\varepsilon^h,y^h-x_\varepsilon^h)_X \\ &\le \frac{1}{2\alpha}\|x_\varepsilon-x_\varepsilon^h\|_X^2+\alpha M_B^2\left(\|x_\varepsilon-x\|+\|x-y^h\|_X^2\right)_X^2 + (Bx_\varepsilon,y^h-x_\varepsilon^h)_X-(Bx_\varepsilon^h,y^h-x_\varepsilon^h)_X,\ \forall y^h \\ &= (v^h,\xi^h)\in X^h. \end{aligned} \tag{4.35}$$

Recalling (3.29) and (3.30), where the terms $u_{\varepsilon,\nu}$ and $u_{\varepsilon,\tau}$ are introduced, we derive

$$\begin{aligned} &(Bx_\varepsilon,y^h-x_\varepsilon^h)_X-(Bx_\varepsilon^h,y^h-x_\varepsilon^h)_X \\ &=\int_{\Gamma_3}\left(\frac{1}{\varepsilon}[u_{\varepsilon,\nu}^h]^+-\frac{1}{\varepsilon}[u_{\varepsilon,\nu}]^+\right)(v_\nu^h-u_{\varepsilon,\nu}^h)\,da +\int_{\Gamma_3}\left(\frac{1}{\varepsilon}[u_{\varepsilon,\tau}^h]_{\varepsilon g}-\frac{1}{\varepsilon}[u_{\varepsilon,\tau}]_{\varepsilon g}\right)(v_\tau^h-u_{\varepsilon,\tau}^h)\,da \\ &\quad+\ell(u_\varepsilon^h,\theta_\varepsilon^h,\xi^h-\theta_\varepsilon^h)-\ell(u_\varepsilon,\theta_\varepsilon,\xi^h-\theta_\varepsilon^h) \\ &=\int_{\Gamma_3}\left(\frac{1}{\varepsilon}[u_{\varepsilon,\nu}]^+-\frac{1}{\varepsilon}[u_{\varepsilon,\nu}^h]^+\right)(u_{\varepsilon,\nu}^h-u_{\varepsilon,\nu})\,da +\int_{\Gamma_3}\left(\frac{1}{\varepsilon}[u_{\varepsilon,\nu}]^+-\frac{1}{\varepsilon}[u_{\varepsilon,\nu}^h]^+\right)(u_{\varepsilon,\nu}-v_\nu^h)\,da \\ &\quad+\int_{\Gamma_3}\left(\frac{1}{\varepsilon}[u_{\varepsilon,\tau}]_{\varepsilon g}-\frac{1}{\varepsilon}[u_{\varepsilon,\tau}^h]_{\varepsilon g}\right)(u_{\varepsilon,\tau}^h-u_{\varepsilon,\tau})\,da +\int_{\Gamma_3}\left(\frac{1}{\varepsilon}[u_{\varepsilon,\tau}]_{\varepsilon g}-\frac{1}{\varepsilon}[u_{\varepsilon,\tau}^h]_{\varepsilon g}\right)(u_{\varepsilon,\tau}-v_\tau^h)\,da \\ &\quad+\ell(u_\varepsilon^h,\theta_\varepsilon^h,\xi^h-\theta_\varepsilon^h)-\ell(u_\varepsilon,\theta_\varepsilon,\xi^h-\theta_\varepsilon^h) \\ &=S_1+S_2+S_3+S_4+S_5, \end{aligned} \tag{4.36}$$

where the quantities S_1, S_2, S_3 , S_4 and S_5 are defined as follows.

$$S_1=\int_{\Gamma_3}\left(\frac{1}{\varepsilon}[u_{\varepsilon,\nu}]^+-\frac{1}{\varepsilon}[u_{\varepsilon,\nu}^h]^+\right)(u_{\varepsilon,\nu}^h-u_{\varepsilon,\nu})\,da, \tag{4.37}$$

$$S_2=\int_{\Gamma_3}\left(\frac{1}{\varepsilon}[u_{\varepsilon,\nu}]^+-\frac{1}{\varepsilon}[u_{\varepsilon,\nu}^h]^+\right)(u_{\varepsilon,\nu}-v_\nu^h)\,da, \tag{4.38}$$

$$S_3=\int_{\Gamma_3}\left(\frac{1}{\varepsilon}[u_{\varepsilon,\tau}]_{\varepsilon g}-\frac{1}{\varepsilon}[u_{\varepsilon,\tau}^h]_{\varepsilon g}\right)(u_{\varepsilon,\tau}^h-u_{\varepsilon,\tau})\,da, \tag{4.39}$$

$$S_4=\int_{\Gamma_3}\left(\frac{1}{\varepsilon}[u_{\varepsilon,\tau}]_{\varepsilon g}-\frac{1}{\varepsilon}[u_{\varepsilon,\tau}^h]_{\varepsilon g}\right)(u_{\varepsilon,\tau}-v_\tau^h)\,da, \tag{4.40}$$

$$S_5=\ell(u_\varepsilon^h,\theta_\varepsilon^h,\xi^h-\theta_\varepsilon^h)-\ell(u_\varepsilon,\theta_\varepsilon,\xi^h-\theta_\varepsilon^h). \tag{4.41}$$

We already have estimated the term S_1 in (4.23), and we get

$$S_1\le-\varepsilon\left\|\frac{1}{\varepsilon}[u_{\varepsilon,\nu}]^+-\frac{1}{\varepsilon}[u_{\varepsilon,\nu}^h]^+\right\|_{0,\Gamma_3}^2. \tag{4.42}$$

Using the properties (3.31)–(3.32), we estimate the term S_3 as follows

$$S_3=-\varepsilon\int_{\Gamma_3}\left(\frac{1}{\varepsilon}[u_{\varepsilon,\tau}]_{\varepsilon g}-\frac{1}{\varepsilon}[u_{\varepsilon,\tau}^h]_{\varepsilon g}\right)\left(\frac{1}{\varepsilon}u_{\varepsilon,\tau}-\frac{1}{\varepsilon}u_{\varepsilon,\tau}^h\right)da \le-\varepsilon\left\|\frac{1}{\varepsilon}[u_{\varepsilon,\tau}]_{\varepsilon g}-\frac{1}{\varepsilon}[u_{\varepsilon,\tau}^h]_{\varepsilon g}\right\|_{0,\Gamma_3}^2. \tag{4.43}$$

For the remaining terms S_2 and S_4, we use (3.24)–(3.27) and Cauchy-Schwartz inequality to get

$$
\begin{aligned}
&S_2+S_4\\
&=\int_{\Gamma_3}\left(\frac{1}{\varepsilon}\,[u_{\varepsilon,\nu}]^+-\frac{1}{\varepsilon}\,[u^h_{\varepsilon,\nu}]^+\right)\left((u_\varepsilon-v^h)\cdot\nu-\mathbf{P}^h(u_\varepsilon-v^h)\cdot\nu\right)da\\
&\quad+\int_{\Gamma_3}\left(\frac{1}{\varepsilon}\,[u_{\varepsilon,\nu}]^+-\frac{1}{\varepsilon}\,[u^h_{\varepsilon,\nu}]^+\right)\mathbf{P}^h(u_\varepsilon-v^h)\cdot\nu\,da\\
&\quad+\int_{\Gamma_3}\left(\frac{1}{\varepsilon}\,[u_{\varepsilon,\tau}]_{\varepsilon g}-\frac{1}{\varepsilon}\,[u^h_{\varepsilon,\tau}]_{\varepsilon g}\right)\left((u_\varepsilon-v^h)_\tau-\mathbf{P}^h(u_\varepsilon-v^h)_\tau\right)da\\
&\quad+\int_{\Gamma_3}\left(\frac{1}{\varepsilon}\,[u_{\varepsilon,\tau}]_{\varepsilon g}-\frac{1}{\varepsilon}\,[u^h_{\varepsilon,\tau}]_{\varepsilon g}\right)\mathbf{P}^h(u_\varepsilon-v^h)_\tau\,da.
\end{aligned}
$$

Hence, we conclude that

$$
\begin{aligned}
S_2+S_4\le&\left\|\frac{1}{\varepsilon}\,[u_{\varepsilon,\nu}]^+-\frac{1}{\varepsilon}\,[u^h_{\varepsilon,\nu}]^+\right\|_{0,\Gamma_3}\|(u_\varepsilon-v^h)\\
&-\mathbf{P}^h(u_\varepsilon-v^h)\cdot\nu\|_{0,\Gamma_3}\\
&+\int_{\Gamma_3}\left(\frac{1}{\varepsilon}\,[u_{\varepsilon,\nu}]^+-\frac{1}{\varepsilon}\,[u^h_{\varepsilon,\nu}]^+\right)\mathbf{R}^h(\mathbf{P}^h(u_\varepsilon-v^h))\cdot\nu\,da\\
&+\left\|\frac{1}{\varepsilon}\,[u_{\varepsilon,\tau}]_{\varepsilon g}-\frac{1}{\varepsilon}\,[u^h_{\varepsilon,\tau}]_{\varepsilon g}\right\|_{0,\Gamma_3}\|\left((u_\varepsilon-v^h)-\mathbf{P}^h(u_\varepsilon-v^h)\right)_\tau\|_{0,\Gamma_3}\\
&+\int_{\Gamma_3}\left(\frac{1}{\varepsilon}\,[u_{\varepsilon,\tau}]_{\varepsilon g}-\frac{1}{\varepsilon}\,[u^h_{\varepsilon,\tau}]_{\varepsilon g}\right)\mathbf{R}^h(\mathbf{P}^h(u_\varepsilon-v^h))_\tau\,da.
\end{aligned}
\tag{4.44}
$$

Moreover, by the same arguments as those used in the estimation (4.25)–(4.26)–(4.27), we obtain

$$
\begin{aligned}
&\left\|\frac{1}{\varepsilon}\,[u_{\varepsilon,\nu}]^+-\frac{1}{\varepsilon}\,[u^h_{\varepsilon,\nu}]^+\right\|_{0,\Gamma_3}\|\left((u_\varepsilon-v^h)\right.\\
&\quad\left.-\mathbf{P}^h(u_\varepsilon-v^h)\right)\cdot\nu\|_{0,\Gamma_3}\\
&\quad+\left\|\frac{1}{\varepsilon}\,[u_{\varepsilon,\tau}]_{\varepsilon g}-\frac{1}{\varepsilon}\,[u^h_{\varepsilon,\tau}]_{\varepsilon g}\right\|_{0,\Gamma_3}\|\left((u_\varepsilon-v^h)\right.\\
&\quad\left.-\mathbf{P}^h(u_\varepsilon-v^h)\right)_\tau\|_{0,\Gamma_3}\\
&\le Ch\left\{\left\|\frac{1}{\varepsilon}\,[u_{\varepsilon,\nu}]^+-\frac{1}{\varepsilon}\,[u^h_{\varepsilon,\nu}]^+\right\|^2_{0,\Gamma_3}\right.\\
&\quad\left.+\left\|\frac{1}{\varepsilon}\,[u_{\varepsilon,\tau}]_{\varepsilon g}-\frac{1}{\varepsilon}\,[u^h_{\varepsilon,\tau}]_{\varepsilon g}\right\|^2_{0,\Gamma_3}\right\}\\
&\quad+C\left\{\|u_\varepsilon-u\|^2_{1,\Omega}+\|u-v^h\|^2_{1,\Omega}\right\}
\end{aligned}
\tag{4.45}
$$

and

$$
\begin{aligned}
&\int_{\Gamma_3}\left(\frac{1}{\varepsilon}\,[u_{\varepsilon,\nu}]^+-\frac{1}{\varepsilon}\,[u^h_{\varepsilon,\nu}]^+\right)\mathbf{R}^h(\mathbf{P}^h(u_\varepsilon-v^h))\cdot\nu\,da\\
&\quad+\int_{\Gamma_3}\left(\frac{1}{\varepsilon}\,[u_{\varepsilon,\tau}]_{\varepsilon g}-\frac{1}{\varepsilon}\,[u^h_{\varepsilon,\tau}]_{\varepsilon g}\right)\mathbf{R}^h(\mathbf{P}^h(u_\varepsilon-v^h))_\tau\,da\\
&\le\frac{1}{\alpha}\,\|x_\varepsilon-x^h_\varepsilon\|^2_X+\frac{\alpha\,CM_B^2}{2}\left(\|u_\varepsilon-u\|^2_{1,\Omega}+\|u-v^h\|^2_{1,\Omega}\right)\\
&\quad+\left(\alpha M_\varphi+\frac{C}{4\alpha}+\frac{\alpha CM_B^2}{2}\right)\left(\|\theta_\varepsilon-\theta\|^2_{1,\Omega}+\|\theta-\xi^h\|^2_{1,\Omega}\right)\\
&\quad+\alpha C\,\|u_\varepsilon-u^h_\varepsilon\|^2_{1,\Omega}+\frac{M_\varphi}{4\alpha}\,\|\theta_\varepsilon-\theta^h_\varepsilon\|^2_{1,\Omega}.
\end{aligned}
\tag{4.46}
$$

Thus, the sum S_2+S_4 can be estimated as follows

$$
\begin{aligned}
&S_2+S_4\\
&\le Ch\left\{\left\|\frac{1}{\varepsilon}\,[u_{\varepsilon,\nu}]^+-\frac{1}{\varepsilon}\,[u^h_{\varepsilon,\nu}]^+\right\|^2_{0,\Gamma_3}+\left\|\frac{1}{\varepsilon}\,[u_{\varepsilon,\tau}]_{\varepsilon g}-\frac{1}{\varepsilon}\,[u^h_{\varepsilon,\tau}]_{\varepsilon g}\right\|^2_{0,\Gamma_3}\right\}\\
&\quad+\frac{1}{\alpha}\,\|x_\varepsilon-x^h_\varepsilon\|^2_X+C\left(1+\frac{\alpha M_B^2}{2}\right)\left(\|u_\varepsilon-u\|^2_{1,\Omega}+\|u-v^h\|^2_{1,\Omega}\right)\\
&\quad+\left(\alpha M_\varphi+\frac{C}{4\alpha}+\frac{\alpha CM_B^2}{2}\right)\left(\|\theta_\varepsilon-\theta\|^2_{1,\Omega}+\|\theta-\xi^h\|^2_{1,\Omega}\right)\\
&\quad+\alpha C\,\|u_\varepsilon-u^h_\varepsilon\|^2_{1,\Omega}+\frac{M_\varphi}{4\alpha}\,\|\theta_\varepsilon-\theta^h_\varepsilon\|^2_{1,\Omega}.
\end{aligned}
\tag{4.47}
$$

Finally, we already have estimated the term S_5 in (4.29), as following

$$
\begin{aligned}
S_5\le&\frac{M_\varphi}{4\alpha}\,\|\theta_\varepsilon-\theta^h_\varepsilon\|^2_{1,\Omega}+\alpha C\,\|u_\varepsilon-u^h_\varepsilon\|^2_{1,\Omega}\\
&+\left(\alpha M_\varphi+\frac{C}{4\alpha}\right)\left(\|\theta_\varepsilon-\theta^h_\varepsilon\|^2_{1,\Omega}+\|\theta_\varepsilon-\theta\|^2_{1,\Omega}+\|\theta-\xi^h\|^2_{1,\Omega}\right).
\end{aligned}
\tag{4.48}
$$

We combine (4.35) with the estimates (4.42), (4.43), (4.47) and (4.48) to get

$$
\begin{aligned}
&\left(m_B-\frac{3}{2\alpha}-2\alpha C\right)\|u_\varepsilon-u^h_\varepsilon\|^2_{1,\Omega}\\
&\quad+\left(m_B-\frac{3}{2\alpha}-\frac{M_\varphi}{2\alpha}-\alpha M_\varphi-\frac{C}{4\alpha}\right)\|\theta_\varepsilon-\theta^h_\varepsilon\|^2_{1,\Omega}\\
&\quad+\varepsilon\left[\left\|\frac{1}{\varepsilon}\,[u_{\varepsilon,\nu}]^+-\frac{1}{\varepsilon}\,[u^h_{\varepsilon,\nu}]^+\right\|^2_{0,\Gamma_3}+\left\|\frac{1}{\varepsilon}\,[u_{\varepsilon,\tau}]_{\varepsilon g}\right.\right.\\
&\quad\left.\left.-\frac{1}{\varepsilon}\,[u^h_{\varepsilon,\tau}]_{\varepsilon g}\right\|^2_{0,\Gamma_3}\right]\\
&\le Ch\left[\left\|\frac{1}{\varepsilon}\,[u_{\varepsilon,\nu}]^+-\frac{1}{\varepsilon}\,[u^h_{\varepsilon,\nu}]^+\right\|^2_{0,\Gamma_3}+\left\|\frac{1}{\varepsilon}\,[u_{\varepsilon,\tau}]_{\varepsilon g}\right.\right.\\
&\quad\left.\left.-\frac{1}{\varepsilon}\,[u^h_{\varepsilon,\tau}]_{\varepsilon g}\right\|^2_{0,\Gamma_3}\right]\\
&\quad+C\left(1+\frac{3\alpha M_B^2}{2}\right)\left(\|u_\varepsilon-u\|^2_{1,\Omega}+\|u-v^h\|^2_{1,\Omega}\right)\\
&\quad+\left(2\alpha M_\varphi+\frac{C}{2\alpha}+\frac{3C\,\alpha M_B^2}{2}\right)\left[\|\theta_\varepsilon-\theta\|^2_{1,\Omega}+\|\theta-\xi^h\|^2_{1,\Omega}\right],
\end{aligned}
\tag{4.49}
$$

where $C>L_\varphi L$ is independent of $x=(u,\theta),\varepsilon$ and h. As we have noted for estimation (4.31), we recall that for appropriate condition on α we have

$$
m_B-\frac{3}{2\alpha}-2\alpha C>0\quad\text{and}\quad m_B-\frac{3}{2\alpha}-\frac{M_\varphi}{2\alpha}-\alpha M_\varphi-\frac{C}{4\alpha}>0,
$$

which let us conclude the desired estimation (4.34). □

Theorem 4.8 *Under hypotheses of Theorem 4.7, if the solution $x=(u,\theta)$ of Problem (3.20) is such that*

$$
u\in H^{\frac{3}{2}+r}(\Omega)^d\quad\text{and}\quad\theta\in H^{\frac{3}{2}+r}(\Omega)\quad\text{with}\quad 0<r\le 2,
$$

there exists $c>0$ independent of $x=(u,\theta)$, ϵ and h such that

$$\left\|u-u_{\varepsilon}^{h}\right\|_{1,\Omega}+\left\|\theta-\theta_{\varepsilon}^{h}\right\|_{1,\Omega}+(\varepsilon^{\frac{1}{2}}-Ch^{\frac{1}{2}})\left[\left\|\sigma_{\nu}(u,\theta)+\frac{1}{\varepsilon}\,[u_{\varepsilon,\nu}^{h}]^{+}\right\|_{0,\Gamma_3}+\left\|\sigma_{\tau}(u,\varphi)+\frac{1}{\varepsilon}\,[u_{\varepsilon,\tau}^{h}]_{\varepsilon g}\right\|_{0,\Gamma_3}\right] \leq C(h^{\frac{1}{2}+r}+\varepsilon^{\frac{1}{2}+r})\left(\|u\|_{\frac{3}{2}+r,\Omega}+\|\theta\|_{\frac{3}{2}+r,\Omega}\right). \tag{4.50}$$

Remark 4.9 Similar to the frictionless scenario, to ascertain the convergence rate outlined in Theorem 4.8, we adjust the penalty parameter as a function of the mesh size. Therefore, if we consider, for instance, $\varepsilon(h) := \left(C+1\right)^{2}h$, signifying that the penalty parameter ε aligns with the mesh size h, we obtain the ensuing a priori estimate

$$\left\|u-u_{\varepsilon}^{h}\right\|_{1,\Omega}+\left\|\theta-\theta_{\varepsilon}^{h}\right\|_{1,\Omega}+h^{\frac{1}{2}}\left[\left\|\sigma_{\nu}(u,\theta)+\frac{1}{\varepsilon}\,[u_{\varepsilon,\nu}^{h}]^{+}\right\|_{0,\Gamma_3}+\left\|\sigma_{\tau}(u,\theta)+\frac{1}{\varepsilon}\,[u_{\varepsilon,\tau}^{h}]_{\varepsilon g}\right\|_{0,\Gamma_3}\right] \leq Ch^{\frac{1}{2}+r}\left(\|u\|_{\frac{3}{2}+r,\Omega}+\|\theta\|_{\frac{3}{2}+r,\Omega}\right).$$

5 Conclusions

A thermoelastic unilateral contact problem in $d=2,3$ dimensional domain Ω with and without Tresca's friction law has been presented in this work. First, the variational formulations and their corresponding linear finite element approximations are provided. The unilateral contact condition is weakly imposed here using the penalty method. Finally, error estimates dependent on the penalty parameter ε and the mesh size h were obtained. Furthermore, assuming the solution maintains regularity, a convergence result was established.

Acknowledgements The authors would like to express their thanks to the Editors and Reviewers for their comments.

Author Contributions All authors are contributed equally to this work.

Funding Not applicable.

Data availability Not applicable.

Declarations

Conflict of interest No conflit of interest to declare.

References

Benaissa H, Essoufi EL-H, Fakhar R (2015) Existence results for unilateral contact problem with friction of thermo-electro-elasticity. Appl Math Mech 36(7):911–926

Benaissa H, Essoufi EL-H, Fakhar R (2016) Analysis of a Signorini problem with nonlocal friction in thermo-piezoelectricity. Glasnik Matematicki 51(71):391–411

Benkhira EL-H, Fakhar R, Mandyly Y (2019) Analysis and numerical approximation of a contact problem involving nonlinear Hencky-Type materials with nonlocal Coulomb's friction law. Numer Funct Anal Optim 40(11):1291–1314

Benkhira EL-H, Fakhar R, Mandyly Y (2019) Numerical approximation of a frictional contact problem in elasto-plasticity based on the penalty approach. Z Angew Math Mech 99(12):e201800300

Bernardi C, Maday Y, Patera AT (1994) A new nonconforming approach to domain decomposition: the mortar element method. In: Nonlinear partial differential equations and their applications, Collège de France Seminar, vol XI, pp 13–51

Bourichi S, Essoufi E-L, Fakhar R (2016) A priori error estimates of a signorini contact problem of electro-elastic materials. Int J Num Anal Mod 13(4):627–647

Bramble JH, Xu J (1991) Some estimates for a weighted L^2 projection. Math Comput 56:463–476

Bramble JH, Pasciak JE, Steinbach O (2001) On the stability of the L^2 projection in $H_1(\Omega)$. Math Comput 71:147–156

Chouly F, Hild P (2013) On convergence of the penalty method for unilateral contact problems. Appl Numer Math 65:27–40

Chouly F, Hild P (2013) A Nitsche-based method for unilateral contact problems: numerical analysis. SIAM J Numer Anal 51:1295–1307

Chouly F, Hild P, Renard Y (2014) Symmetric and non-symmetric variants of Nitsche's method for contact problems in elasticity: theory and numerical experiments. Math Comput 84:1089–1112

Dione I (2019) Optimal convergence analysis of the unilateral contact problem with and without Tresca friction conditions by the penalty method. J Math Anal Appl 472(1):266–284

Duvaut G (1981) Non-linear boundary value problem in thermoelasticity. In: Carlson DE, Shield RT (eds) Proceedings of the IUTAM Symposium on Finite Elasticity. Springer: Dordrecht

Ern A, Guermond JL (2004) Theory and practice of finite elements. Appl Math Sci 159, Springer-Verlag, New York

Faiz Z, Baiz O, Benaissa H, El Moutawkil D (2023) Analysis and approximation of hemi-variational inequality for a frictional thermo-electro-visco-elastic contact problem with damage. Taiw J Math 27(1):81–111

Faiz Z, Benaissa H, Baiz O (2024) Nonlinear inclusion for thermo-electro-elastic: existence, dependence and optimal control, Communications in Combinatorics and Optimization

Han W (2005) A posteriori error analysis via duality theory with applications in modeling and Numerical Approximations. Advances in Mechanics and Mathematics vol 8, Springer, New York

Haslinger J, Mäkinen R (1992) Shape optimization of elasto-plastic bodies under plane strains: Sensitivity analysis and numerical implementation. Struct Optim 4:133–141

Khalfi H El, Baiz O, Benaissa H (2023) Error estimates of piezoelectric Signorinis contact problems. Z Angew Math Mech 103:e202300112. https://doi.org/10.1002/zamm.202300112

Kikuchi N, Oden JT (1988) Contact problems in Elasticity: a study of variational inequalities and finite element methods. Soc Indu Appl Math

Li ZC (1998) Combined methods for elliptic equations with singularities, interfaces and infinities. Kluwer Academic Publishers, Amsterdam

Liu J, Migórski S, Yang X, Zeng S (2021) Existence and convergence results for a nonlinear thermoelastic contact problem. J Nonlin Variational Anal 5(5):647–664

Ramaniraka N (1997) Thermomécanique des contacts entre deux solides déformables. Thèse de doctorat, Ecole Polytechnique Fédérale de Lausanne

Shillor M, Sofonea M, Telega JJ (2004) Models and analysis of quasistatic contact variational methods: variational methods. In: Lecture notes in physics, vol 655, Springer, Berlin