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RESEARCH PAPER

Direct and Inverse Problems of String Equation by Numerov's Method

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Abstract

It is well known that free vibration of a taut string having mass per unit $m(x)$ and frequency ω is governed by ordinary differential equation $y'' + \omega^2 m(x)y = 0$. In this paper, first we discretize the differential equation by using Numerov's method to obtain a matrix eigenvalue problem of the form $-Au = \Lambda BMu$, where A and B are constant tridiagonal matrices and M is a diagonal matrix related to mass function $m(x)$. In direct problem, for a given $m(x)$, we approximate the first N eigenvalues of the string equation by making a new correction on the eigenvalues of matrix pair $(-A, BM)$. Also we obtain the error order of corrected eigenvalues. For inverse problem, we propose an efficient algorithm for constructing unknown mass function $m(x)$ by using given spectra by solving a nonlinear system. We solve the nonlinear system by using modified Newton's method and a regularization technique. The convergence of Newton's method is proved. Finally, we give some numerical examples to illustrate the efficiency of the proposed algorithm.

Keywords String equation · Sturm-Liouville problem · Numerov's method · Correction term

Mathematics Subject Classification 34A55 · 34B24 · 34L16 · 65F18

1 Introduction

The free vibration of a taut string having mass per unit $m(x)$ and frequency ω is described by ordinary differential equation of the following form

$$y'' + \lambda m(x)y = 0, \quad 0 < x < 1, \tag{1}$$

where $\lambda = \omega^2$, (Gladwell 2004). Equation (1) is called *string equation* which is a special form of Sturm-Liouville equation. Usually, boundary conditions at end points are considered for differential equation (1). The most important boundary conditions are fixed-fixed $(y(0) = y(1) = 0)$, fixed-free $(y(0) = 0, y'(1) = 0)$ and free-free $(y'(0) = 0, y'(1) = 0)$. It is well known that differential equation (1) with one set of the boundary conditions has infinite number of eigenvalues $\{\lambda_n\}_1^\infty$ such that

$$0 \le \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots, \lambda_n \to +\infty.$$

For more details see (Freiling and Yurko 2001; Gladwell 2004; Kirsch 1996). In this paper, we investigate two types of problems related to differential equation (1). First, we consider direct problem i.e. approximating the first N eigenvalues of the string equation with given boundary conditions. Second, we solve the corresponding inverse problem i.e. we construct unknown mass function $m(x)$ using informations of spectral data. By solving inverse problem for a string, we may design a string with a prescribed frequences. It is proved that if $m(x)$ is symmetric with respect to mid point $x = \frac{1}{2}$, i.e. $m(x - \frac{1}{2}) = m(x)$, then one spectrum corresponding to fixed-fixed boundary condition suffices to construct $m(x)$, uniquely. But for non symmetric case of $m(x)$ two spectra corresponding to two boundary conditions e.g. fixed-fixed and fixed-free are required to construct $m(x)$, uniquely (Gladwell 2004; Jiang and Xu 2019). Since $m(x)$ is a positive function, for computational purpose we define $m(x) = \rho^2(x)$. If $\int_0^1 \rho(x)dx = 1$ then by changing of variables

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$$\frac{d\xi}{dx} = \rho(x), \quad z(\xi) = (\rho(x))^{\frac{1}{2}}, \quad v(\xi) = z(\xi)y(x), \tag{2}$$

the string equation (1) can be transformed to the following Sturm-Liouville problem (Gladwell 2004; Jiang and Xu 2019)

$$\ddot{v}(\xi) + (\lambda - q(\xi))v(\xi) = 0, \quad 0<\xi<1, \tag{3}$$

where

$$q(\xi) = \frac{\ddot{z}(\xi)}{z(\xi)}, \quad \cdot \equiv \frac{d}{d\xi}. \tag{4}$$

Direct and inverse Sturm-Liouville problems are well studied problems in the literature. For more details see (Andrew 2005; Ezhak and Telnova 2020; Freiling and Yurko 2001; Gladwell 2004; Jiang et al. 2021; Kirsch 1996; Mirzaei 2017; Perera and Böckmann 2020, 2019; Mosazadeh and Akbarfam 2020; Neamaty and Akbarpoor 2016). Note that construction of $m(x)$ from $q(x)$ using (3) and (4) needs more information and conditions on $m(x)$. That is why the direct and inverse problem corresponding to the string equation have been studied, independently. In papers (Andrew 2003; Andrew and Paine 1986; Andrew 2000; Gao et al. 2015, 2017, 2018; Paine et al. 1981), the direct and inverse problems of Sturm-Liouville equation (3) are studied by using finite difference, finite element and Numerov's methods. They find approximations for the first N eigenvalues as follows

$$\lambda_k = \Lambda_k + \epsilon_{r,k}, \quad k = 1, 2, \ldots, N, \tag{5}$$

where Λ_k is the kth eigenvalue of the matrix obtained by discretization of the Sturm-Liouville equation (3) and $\epsilon_{r,k}$ is the difference between the kth eigenvalue of (3) and the kth eigenvalue of the matrix form of (3) for $q(x) = 0$. It is proved that

$$\epsilon_{r,k} = k^2\pi^2 - \frac{12\sin^2(\frac{k\pi h}{2})}{h^2[3 + (1-r)\sin^2(\frac{k\pi h}{2})]}, \quad r = 1, 2, 3, \quad k = 1, 2, \ldots, N, \tag{6}$$

where r is a parameter that depends on the discretization method. For finite difference method $r = 1$, for Numerov's method $r = 2$ and $r = 3$ for finite element method. In general, solving the direct and inverse problems of the string equation has been less studied comparing to the classical Sturm-Liouville equation. In Jiang and Xu (2019, 2021) construction of the mass function $m(x)$ is considered by using trace formula. In Rundell and Sacks (1992) mass function constructed by an iterative procedure based on Goursat problem. In general, computing eigenvalues and eigenfunctions of the string equation for non constant mass function is impossible, explicitly. In practice, however, finite dimensional numerical methods are used to estimate the spectral data. Using such methods Eq. (1) is then transformed to a matrix eigenvalue problem where the eigenvalues of the resulting matrix become approximations for the first N eigenvalues of the string equation. The eigenvalues of matrix equation can be used to approximate the eigenvalues of lower indices but for eigenvalues of higher indices they generally lead to poor numerical results. In this paper, we discretize the string equation by using Numerov's method to obtain the corresponding matrix eigenvalue problem. In order to make good approximations for the eigenvalues of the string equation we add a new suitable correction term to the eigenvalues of the matrix obtained from Numerov's method. Then we propose an algorithm to solve direct and inverse problems corresponding to the string equation in the cases of symmetric and non symmetric function $m(x)$. Our results show that Numerov's method together with correction technique can be applied successfully to solve direct and inverse problems. To our knowledge, correction idea has not been applied to direct and inverse problems of the string equation.

The rest of the paper is arranged in the following manner. In Sect. 2, we discretize the string equation to obtain a matrix eigenvalue problem. By making new correction on the eigenvalues of resulting matrix eigenvalue problem we approximate the first N eigenvalues of the string equation. Moreover, the error analysis of corrected eigenvalues and some numerical results are presented in this section. In Sect. 3, a method based on correction technique of Sect. 2 is proposed to solve inverse problem of the string equation in symmetric and non symmetric cases. Finally, some different numerical examples are given in Sect. 4 to show the good efficiency of this technique for inverse problem.

2 Direct Problem

In this section, we study direct problem of Eq. (1). First we discretize Eq. (1) by using Numerov's method. For this aim we divide the interval [0, 1] into N subintervals of length h and evaluate Eq. (1) at $x_i = ih$ as follows

$$y_i'' + \lambda m_i y_i = 0, \quad i = 1, 2, \ldots, N-1, \tag{7}$$

where $m_i = m(x_i)$ and $y_i = y(x_i)$. Using central difference formula we have

$$y_i'' = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} - \frac{h^2}{12}y_i^{(4)} + O(h^4). \tag{8}$$

Using Eq. (1) we have $y_i^{(4)} = -\lambda(m(x)y)_i''$. Approximating the second derivative by central difference formula gives the following approximation for $y_i^{(4)}$

$$y_i^{(4)} = -\lambda \frac{m_{i+1}y_{i+1} - 2m_i y_i + m_{i-1}y_{i-1}}{h^2} + O(h^2). \tag{9}$$

Substituting (9) in (8) and approximating y_i'' by (7), we find

$$-y_{i+1} + 2y_i - y_{i-1} \simeq \frac{h^2}{12}\Lambda(m_{i+1}y_{i+1} + 10m_i y_i + m_{i-1}y_{i-1}), \ i = 1, 2, \ldots, N-1. \tag{10}$$

Equation (10) can be written in the following matrix form

$$-A\mathbf{u} = \Lambda BM\mathbf{u}, \tag{11}$$

where Λ and $\mathbf{u} = [u_1, u_2, \ldots, u_{N-1}]^T$ are approximations for λ and $\mathbf{y} = [y_1, y_2, \ldots, y_{N-1}]^T$, respectively. For fixed-fixed boundary conditions the matrices A and B are as follows:

$$A = \frac{1}{h^2}\begin{pmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & & -2 & 1 \\ & & & 1 & -2 \end{pmatrix},$$

$$B = \frac{1}{12}\begin{pmatrix} 10 & 1 & & & \\ 1 & 10 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & & 10 & 1 \\ & & & 1 & 10 \end{pmatrix},$$

and $M = diag(m_1, m_2, \cdots, m_{N-1})$. We have $B = I + \frac{h^2}{12}A$. For fixed-free boundary conditions the condition $y'(1) = 0$ can be discretized by using a first or second order approximation. But in this paper we use Remark 2 to transform the non symmetric string equation on [0, 1] to symmetric string equation on [0, 2] with fixed-fixed boundary conditions. Throughout the paper the eigenvalues of string equation with fixed-fixed boundary conditions are denoted by $\{\lambda_i\}$ and the eigenvalues with fixed-free boundary conditions are denoted by $\{\mu_i\}$, respectively. Similarly, the eigenvalues of the matrix Eq. (11) corresponding to fixed-fixed boundary conditions are denoted by $\{\Lambda_i^1\}$ and the eigenvalues of the matrix equation (11) corresponding to fixed-free boundary conditions are denoted by $\{\Lambda_i^2\}$. Note that Λ_i^1 and Λ_i^2 are approximations for $\{\lambda_i\}$ and $\{\mu_i\}$, respectively which are good approximations for lower eigenvalues but we have poor results for higher indices, see Tables 1 and 2. In order to improve the results, we extend the correction technique (Paine et al. 1981) to the string equation.

If $\int_0^1 \rho(x)dx = 1$, then Eqs. (1) and (3) are isospectral, i.e. they have the same eigenvalues. Therefore it is natural to expect that the correction (5) may be used to approximate the eigenvalues of Eq. (1). If $\int_0^1 \rho(x)dx = c \neq 1$ then we may modify Eq. (1) as follows

$$y'' + \lambda^* m^*(x)y(x) = 0, \tag{12}$$

where $\lambda^* = c^2\lambda$ and $m^* = \frac{1}{c^2}m(x)$. Making correction (5) for Eq. (12) we find

$$\lambda_k^* \simeq \Lambda_k^* + \epsilon_{2,k} \Longrightarrow \lambda_k \simeq \tilde{\Lambda}_k := \Lambda_k + \frac{1}{c^2}\epsilon_{2,k}. \tag{13}$$

Thus, if $c \neq 1$, selecting the correction term as $\epsilon_{2,k}$ does not make a good approximation. In this case we must choose the correction term as $\frac{1}{c^2}\epsilon_{2,k}$.

Table 1 Errors of uncorrected and corrected eigenvalues for $m(x) = 1 - 0.3e^{-20(x-0.5)^2}$ with $n = 32$

i	λ_i	$\lvert\lambda_i - \Lambda_i^1\rvert$	$\lvert\lambda_i - \tilde{\Lambda}_i^1\rvert$
1	12.1880	$3.1e-7$	$5.1e-8$
4	101.8304	$1.1e-3$	$3.8e-5$
8	550.8877	$7.0e-2$	$2.5e-3$
12	910.5795	$8.0e-1$	$2.8e-2$
16	1360.2085	$4.5e+0$	$1.6e-1$
20	1899.7650	$1.8e+1$	$6.4e-1$
24	2529.2479	$5.3e+1$	$2.0e+0$
28	3248.6568	$1.36e+2$	$5.3e+0$
32	3248.6568	$3.07e+2$	$1.3e+1$

2.1 Error Analysis of Corrected Eigenvalues

In this part, using some lemmas and theorems we find the error order for corrected eigenvalues $\tilde{\Lambda}_k$. We prove the results for the case $c = 1$ and fixed-fixed boundary condition. The results can be extend to the case $c \neq 1$ and other boundary conditions, similarly. It is obvious that the eigenvalues of problem (1) with $m(x) = 1$ and fixed-fixed boundary condition, are $\lambda_k^\circ = k^2\pi^2$ and corresponding eigenfunctions are $y_k^\circ(x) = \sin k\pi x$. We have the following lemma for the matrix pair $(-A, B)$.

Lemma 1 (*Andrew* 2000; *Chawla and Katti* 1980) *For matrix pair* $(-A, B)$ *we have*

(i) $-A\mathbf{s}_k = \Lambda_k^\circ B\mathbf{s}_k, \quad \Lambda_k^\circ = \frac{12}{h^2}\frac{\sin^2(\frac{k\pi}{2N})}{3-\sin^2(\frac{k\pi}{2N})},$
$\mathbf{s}_k = (\sin k\pi x_1, \ldots, \sin k\pi x_{N-1}),$
(ii) $\Lambda_k^\circ = k^2\pi^2 + O(k^6h^4).$

Lemma 2 (*Andrew* 2000; *Chawla and Katti* 1980) *If* $\int_0^1 \rho(x)\mathrm{d}x = 1,$ *then*

(i) $\lambda_k = k^2\pi^2 + O(\frac{1}{k^2}),$

Table 2 Errors of uncorrected and corrected eigenvalues for $m(x) = 2 + \sin(\pi(x-1)^2)$ with $n = 32$

i	λ_i	$\lvert\lambda_i - \Lambda_i^1\rvert$	$\lvert\lambda_i - \tilde{\Lambda}_i^1\rvert$	μ_i	$\lvert\mu_i - \Lambda_i^2\rvert$	$\lvert\mu_i - \tilde{\Lambda}_i^2\rvert$
1	3.723	$3.1e-6$	$1.5e-6$	1.056	$1.5e-7$	$1.3e-7$
2	15.569	$1.3e-4$	$2.7e-5$	8.674	$2.5e-5$	$8.1e-6$
3	35.341	$1.3e-3$	$1.7e-4$	24.461	$4.5e-4$	$7.3e-5$
4	63.048	$7.0e-3$	$6.7e-4$	48.203	$3.2e-3$	$3.5e-4$
5	98.680	$2.6e-2$	$2.1e-3$	79.873	$1.4e-2$	$1.2e-3$
6	142.235	$7.8e-2$	$5.5e-3$	119.467	$4.6e-2$	$3.4e-3$
7	193.710	$2.0e-1$	$1.3e-2$	166.982	$1.3e-1$	$8.5e-3$
8	253.106	$4.4e-1$	$2.7e-2$	222.418	$3.0e-1$	$1.9e-2$
9	320.421	$8.9e-1$	$5.3e-2$	285.774	$6.3e-1$	$3.8e-2$
10	395.657	$1.7e+0$	$9.8e-2$	357.049	$1.2e+0$	$7.3e-2$
11	478.812	$3.0e+0$	$1.7e-1$	436.244	$2.3e+0$	$1.3e-1$
12	569.887	$5.1e+0$	$2.9e-1$	523.359	$3.9e+0$	$2.3e-1$
13	668.881	$8.3e+0$	$4.8e-1$	618.394	$6.5e+0$	$3.7e-1$
14	775.795	$1.3e+1$	$7.6e-1$	721.348	$1.0e+1$	$6.0e-1$
15	890.629	$2.0e+1$	$1.2e+0$	832.222	$1.6e+1$	$9.4e-1$
16	1013.382	$2.9e+1$	$1.8e+0$	951.015	$2.4e+1$	$1.4e+0$
17				1077.728	$3.6e+1$	$2.1e+0$

(ii) $y_k(x) = \sin k\pi x + O(\frac{1}{k})$,

(iii) $|\lambda_k - \Lambda_k| = O(k^6 h^4)$.

Note that for arbitrary function $f(x)$, we use the boldface vector $\mathbf{f}$, for $(f(x_1), f(x_2), \ldots, f(x_n))$.

Lemma 3 *Suppose that* $e(x) = y(x) - \sin k\pi x$, $\mathbf{e} = \mathbf{y} - \mathbf{s}$ *and* $\epsilon = \mathbf{u} - \mathbf{s}$, *then we have*

$$(\Lambda - \lambda)\mathbf{u}^T M\mathbf{y} = (\Lambda_k^\circ - k^2\pi^2)\mathbf{u}^T\mathbf{s} + \mathbf{s}^T(\mathbf{e}'' - B^{-1}A\mathbf{e}) + \epsilon^T(\mathbf{e}'' - B^{-1}A\mathbf{e}). \tag{14}$$

Proof By transposing Eq. (11) then multiplying by $\mathbf{y}$ we obtain

$$-\mathbf{u}^T B^{-1}A\mathbf{y} = \Lambda\mathbf{u}^T M\mathbf{y}. \tag{15}$$

Writing (7) in matrix form and multiplying by $\mathbf{u}^T$ we get

$$-\mathbf{u}^T\mathbf{y}'' = \lambda\mathbf{u}^T M\mathbf{y}. \tag{16}$$

Subtracting (16) from (15) we find

$$(\Lambda - \lambda)\mathbf{u}^T M\mathbf{y} = \mathbf{u}^T(\mathbf{y}'' - B^{-1}A\mathbf{y}), \tag{17}$$

substituting $\mathbf{y} = \mathbf{e} + \mathbf{s}$, $\mathbf{s}'' = -k^2\pi^2\mathbf{s}$, $\mathbf{u} = \epsilon + \mathbf{s}$ in the right hand side of (17) we find

$$\begin{aligned}(\Lambda - \lambda)\mathbf{u}^T M\mathbf{y} &= \mathbf{u}^T(\mathbf{e}'' + \mathbf{s}'' - B^{-1}A(\mathbf{e} + \mathbf{s})) \\ &= \mathbf{u}^T(\mathbf{e}'' - k^2\pi^2\mathbf{s} + \Lambda_k^\circ\mathbf{s} - B^{-1}A\mathbf{e}) \\ &= (\Lambda_k^\circ - k^2\pi^2)\mathbf{u}^T\mathbf{s} + \mathbf{u}^T(\mathbf{e}'' - B^{-1}A\mathbf{e}) \\ &= (\Lambda_k^\circ - k^2\pi^2)\mathbf{u}^T\mathbf{s} + \mathbf{s}^T(\mathbf{e}'' - B^{-1}A\mathbf{e}) \\ &\quad + \epsilon^T(\mathbf{e}'' - B^{-1}A\mathbf{e}).\end{aligned}$$

□

Remark 1 We can rewrite the string equation as $y''(x) + \lambda y(x) = \lambda(1 - m(x))y(x)$. Also, the discrete equation (11) can be written as $-A\mathbf{u} + \Lambda B(I - M)\mathbf{u} = \Lambda B\mathbf{u}$. From Andrew and Paine (1985) by comparing these equations with corresponding equations in canonical Sturm-Liouville problem we conclude that for higher index k, $|\mathbf{y}_k - \mathbf{s}_k|$ and $|\mathbf{u}_k - \mathbf{s}_k|$ are of order $O(\frac{1}{k})$.

Lemma 4 *For the function* $e(x)$ *we have*

$$e(x) = \frac{1}{k\pi}\int_0^x (k^2\pi^2 - \lambda m(t))\sin k\pi(x - t)y(t)\mathrm{d}t, \tag{18}$$

$$e^{(j)}(x) = O(k^{(j-1)}), j = 0, 1, \ldots, \tag{19}$$

$$e(0) = e(1) = e''(0) = e''(1) = 0. \tag{20}$$

Proof Differentiating twice of $e(x) = y(x) - \sin(k\pi x)$ we obtain

$$\begin{aligned}e''(x) &= y''(x) + k^2\pi^2\sin k\pi x \\ &= -\lambda m(x)y(x) + k^2\pi^2[y(x) - e(x)],\end{aligned}$$

thus $e'' + k^2\pi^2 e = [k^2\pi^2 - \lambda m(x)]y(x)$. This equation with boundary conditions $e(0) = 0$ and $e(1) = 0$, has the solution of the form (18). Using Lemma 2, part (*ii*) we find $e(x) = O(\frac{1}{k})$. Differentiating (18), implies (19). Simple calculation conclude (20). □

Theorem 1 *Suppose that*

$$f(x) = (k^2\pi^2 - \lambda m(x))y(x),$$
$$\alpha(x,h) = \int_x^{x+h} f(t)\sin k\pi(x+h-t)\mathrm{d}t, \tag{21}$$
$$E_j = \alpha(x_j, h) - \alpha(x_j, -h), \tag{22}$$

then

$$A\mathbf{e} - B\mathbf{e}'' - (k^2\pi^2 - \Lambda_k^\circ)B\mathbf{e} = \left(1 + \frac{h^2\Lambda_k^\circ}{12}\right)\frac{E}{k\pi h^2} - B\mathbf{f}. \tag{23}$$

Proof We have $\mathbf{e}'' + k^2\pi^2\mathbf{e} = \mathbf{f}$, therefore

$$B\mathbf{e}'' = B\mathbf{f} - k^2\pi^2 B\mathbf{e}. \tag{24}$$

On the other hand by using (18) and part (*i*) of Lemma 1, for the *j*th entry of the vector $A\mathbf{e}$ we have

$$\begin{aligned} k\pi h^2(A\mathbf{e})_j &= k\pi(e_{j+1} - 2e_j + e_{j-1}) \\ &= \int_0^{x_j} f(t)[sink\pi(x_{j+1} - t) - 2sink\pi(x_j - t) \\ &\quad + sink\pi(x_{j-1} - t)] + E_j \\ &= -\frac{h^2\Lambda_k^\circ}{12}\int_0^{x_j} f(t)[sink\pi(x_{j+1} - t) \\ &\quad + 10sink\pi(x_j - t) + sink\pi(x_{j-1} - t)] + E_j \\ &= -\frac{h^2\Lambda_k^\circ k\pi}{12}[e(x_{j+1}) + 10e(x_j) + e(x_{j-1})] \\ &\quad + (1 + \frac{h^2\Lambda_k^\circ}{12})E_j, \end{aligned}$$

dividing both sides to $k\pi h^2$ we obtain

$$A\mathbf{e} = -\Lambda_k^\circ B\mathbf{e} + \frac{1}{k\pi h^2}(1 + \frac{h^2\Lambda_k^\circ}{12})\mathbf{E}. \tag{25}$$

Subtracting (24) from (25) we obtain the required result. □

Theorem 2 *Suppose that* $m(x) \in C^4[0,1]$, *then there exists constant* c_1 *such that*

$$|\epsilon^T(B^{-1}A\mathbf{e} - \mathbf{e}'' + (\Lambda_k^\circ - k^2\pi^2)\mathbf{e}| \le c_1 k^5 h^3$$

Proof By definition of E_j we have

$$\begin{aligned} E_j &= \int_{x_j}^{x_{j+1}} f(t)\sin[k\pi(x_{j+1} - t)]\mathrm{d}t \\ &\quad + \int_{x_j}^{x_{j-1}} f(t)\sin[k\pi(x_{j-1} - t)]\mathrm{d}t, \end{aligned}$$

Applying Taylor's expansion of the function *f*(*x*) around x_j, then using integration by parts we find

$$\begin{aligned} E_j &= \frac{2}{k\pi}[1 - \cos(k\pi h)]f_j + \left\{\left(\frac{h^2}{k\pi}\right)\right. \\ &\quad \left. - \frac{2}{(k\pi)^3}(1 - \cos(k\pi h))\right\}f_j'' + O\left(kh^6\|f^{(4)}\|_\infty\right). \end{aligned}$$

On the other hand we have

$$B\mathbf{f} = \mathbf{f} + \frac{h^2A\mathbf{f}}{12} = \mathbf{f} + \frac{h^2}{12}\mathbf{f}'' + O(h^4\|\mathbf{f}^{(4)}\|_\infty).$$

Thus

$$\begin{aligned} &\frac{1}{k\pi}\left(1 + \frac{h^2\Lambda_k^\circ}{12}\right)E_j - h^2(B\mathbf{f})_j \\ &= \left\{\left(\frac{2}{k^2\pi^2}\right)(1 - \cos(k\pi h))\left(1 + \frac{h^2\Lambda_k^\circ}{12}\right) - h^2\right\}f_j \\ &\quad + h^2\left\{\frac{1}{k^2\pi^2}\left[1 - \left(\frac{2}{k^2\pi^2h^2}\right)(1 - \cos(k\pi h))\right]\left(1 + \frac{h^2\Lambda_k^\circ}{12}\right)\right. \\ &\quad \left. - \frac{h^2}{12}\right\}f_j'' + O\left(h^6\|f^{(4)}\|_\infty\right) \\ &= \frac{h^2}{k^2\pi^2}(\Lambda_k^\circ - k^2\pi^2)f_j \\ &\quad + \frac{h^2}{k^4\pi^4}(k^2\pi^2 - \Lambda_k^\circ)\left(1 - \frac{h^2k^2\pi^2}{12}\right)f_j'' \\ &\quad + O\left(h^6\|f^{(4)}\|_\infty\right). \end{aligned} \tag{26}$$

Since $\Lambda_k^\circ - k^2\pi^2 = O(k^6h^4)$, $1 - \frac{h^2k^2\pi^2}{12} = O(1)$, and $\|f^{(j)}\|_\infty = O(k^{(j+2)})$. Thus all terms in (26) are of order $O(k^6h^6)$. Therefore we find

$$\frac{1}{k\pi}\left(1 + \frac{h^2\Lambda_k^\circ}{12}\right)E_j - h^2(B\mathbf{f})_j = O\left(k^6h^6\right). \tag{27}$$

Using Remark 1 we have $\|\epsilon\|_\infty = O(\frac{1}{k})$. Also $\|B^{-1}\| = O(1)$ and $n = O(\frac{1}{h})$. Thus, using (23) and (27) we obtain

$$\begin{aligned} &\left\|\in^T\left(B^{-1}A\mathbf{e} - \mathbf{e}'' + \left(\Lambda_k^\circ - k^2\pi^2\right)\mathbf{e}\right)\right\| \\ &\le n\|\in\|_\infty\|B\|\left\|A\mathbf{e} - B\mathbf{e}'' + \left(\Lambda_k^\circ - k^2\pi^2\right)B\mathbf{e}\right\|_\infty \\ &\le c_1k^5h^3. \end{aligned}$$

□

Theorem 3 *If* $m(x) \in C^4[0,1]$ *then there exists constant* c_2 *such that*

$$\left\|\mathbf{s}^T\mathbf{f}\right\|_\infty \le c_2k^5h^3. \tag{28}$$

Proof Define $F(x) = f(x)\sin k\pi x$. The function $F(x)$ has the following properties

$$F(0) = F(1) = 0, \quad F'(0) = F'(1) = 0. \tag{29}$$

Substituting $x = 1$ in (18) we find $\int_0^1 F(x)\mathrm{d}x = 0$. Suppose that T_hF denote the trapezoidal integration formula of $F(x)$ with step size h on [0, 1]. Using Euler-Maclurin formula (Davis and Rabinowitz 1975), we have

$$\mathbf{s}^T\mathbf{f} = h^{-1}T_hF = h^{-1}\left\{\int_0^1 F(x)dx + \frac{B_4h^4}{4!}[F'''(1) - F'''(0)] - h^4\int_0^1 p_4\left(\frac{x}{h}\right)F^{(4)}(x)dx\right\}, \tag{30}$$

where B_4 is Bernoulli number and $\{p_i\}$ are piecewise polynomials of period one satisfying

$$p'_{j+1} = p_j, p_{2j+1}(0) = p_{2j+1}(1) = 0, p_1(x) = x - \frac{1}{2}.$$

Simple calculations show that

$$F'''(1) - F'''(0) = 3k\pi((-1)^kf''(1) - f''(0)),$$
$$f''(0) = -2\lambda m'(0)y'(0), f''(1) = -2\lambda m'(1)y'(1).$$

Since $\lambda = O(k^2), y' = O(k), \quad m' = O(1),$ we find $F'''(1) - F'''(0) = O(k^4)$. Substituting this in (30), we get

$$\left|\mathbf{s}^T\mathbf{f}\right| = h^3\left|\int_0^1 p_4\left(\frac{x}{h}\right)F^{(4)}(x)dx\right| + O(k^4h^3). \tag{31}$$

By simple calculations, we obtain

$$F^{(4)}(x) = -8k^4\pi^4g(x)\cos 2k\pi x + O(k^5)$$

where $g(x) = k^2\pi^2 - \lambda m(x)$. By similar procedure in proof of Lemma 6 in Andrew and Paine (1985), we obtain

$$\left\|\int_0^1 P_4\left(\frac{x}{h}\right)F^{(4)}(x)dx\right\| = O(k^5). \tag{32}$$

Combining (31) and (32) we obtain the result (28). □

Theorem 4 *Suppose that* $m(x) \in C^4[0, 1]$, *then we have* $|\tilde{\Lambda}_k - \lambda_k| \le ck^6h^4$.

Proof Using part (*i*) of Lemma 1 then adding and subtracting $k^2\pi^2\mathbf{s}^T\mathbf{e}$ we obtain

$$\begin{aligned}\mathbf{s}^T(\mathbf{e}'' - B^{-1}A\mathbf{e}) &= \mathbf{s}^T(\mathbf{e}'' + \Lambda_k^\circ\mathbf{e} + k^2\pi^2\mathbf{e} - k^2\pi^2\mathbf{e})\\ &= \mathbf{s}^T(\mathbf{e}'' + k^2\pi^2\mathbf{e})\\ &\quad + (\Lambda_k^\circ - k^2\pi^2)\mathbf{s}^T\mathbf{e}.\end{aligned}$$

According to the proof of Lemma 4 we have $\mathbf{e}'' + k^2\pi^2\mathbf{e} = \mathbf{f}$, thus we obtain

$$\mathbf{s}^T(\mathbf{e}'' - B^{-1}A\mathbf{e}) = \mathbf{s}^T\mathbf{f} + (\Lambda_k^\circ - k^2\pi^2)\mathbf{s}^T\mathbf{e}.$$

On the other hand we have $\mathbf{u}^T\mathbf{y} = \mathbf{u}^T\mathbf{s} + \mathbf{s}^T\mathbf{e} + \epsilon^T\mathbf{e}$. Using these relations and Theorems 1, 2 and 3 we find

$$\begin{aligned}|\tilde{\Lambda} - \lambda||\mathbf{u}^TM\mathbf{y}| &= |(\Lambda - \lambda)\mathbf{u}^TM\mathbf{y} - (\Lambda_k^\circ - k^2\pi^2)\mathbf{u}^T\mathbf{y}\\ &\quad + (\Lambda_k^\circ - k^2\pi^2)\mathbf{u}^T(I - M)\mathbf{y}|\\ &= |(\Lambda - \lambda)\mathbf{u}^TM\mathbf{y} - (\Lambda_k^\circ - k^2\pi^2)(\mathbf{u}^T\mathbf{s} + \mathbf{s}^T\mathbf{e} + \epsilon^T\mathbf{e})\\ &\quad + (\Lambda_k^\circ - k^2\pi^2)\mathbf{u}^T(I - M)\mathbf{y}|\\ &= |\mathbf{s}^T(\mathbf{e}'' - B^{-1}A\mathbf{e}) + \epsilon^T(\mathbf{e}'' - B^{-1}A\mathbf{e})\\ &\quad - (\Lambda_k^\circ - k^2\pi^2)[\mathbf{s}^T\mathbf{e} + \epsilon^T\mathbf{e} - \mathbf{u}^T(I - M)\mathbf{y}]|\\ &\le c_1k^5h^3 + c_2k^5h^3 + |(\Lambda_k^\circ - k^2\pi^2)\mathbf{u}^T(I - M)\mathbf{y}|.\end{aligned}$$

Thus we get

$$|\tilde{\Lambda} - \lambda| \le \frac{(c_1k^5h^3 + c_2k^5h^3)}{|\mathbf{u}^TM\mathbf{y}|} + |\Lambda_k^\circ - k^2\pi^2|\frac{|\mathbf{u}^T(I - M)\mathbf{y}|}{|\mathbf{u}^TM\mathbf{y}|}.$$

Using Remark 1 we have $\|\mathbf{y} - \mathbf{s}\|_\infty = O(\frac{1}{k}), \|\mathbf{u} - \mathbf{s}\|_\infty = O(\frac{1}{k})$. Also we have $\frac{1}{\mathbf{s}^TM\mathbf{s}} = O(h)$ (Andrew and Paine 1986, 1985). Thus we obtain $\frac{1}{\mathbf{u}^TM\mathbf{y}} = O(h)$. On the other hand we have $\Lambda_k^\circ - k^2\pi^2 = O(k^6h^4)$, thus we obtain $|\tilde{\Lambda} - \lambda| \le ck^6h^4$.□

In Tables 1 and 2, the eigenvalues of string equation corresponding to mass functions $m(x) = 1 - 0.3e^{-20(x-0.5)^2}$ and $m(x) = 2 + \sin(\pi(x - 1)^2)$ are approximated using new correction term given by (13). The exact eigenvalues are computed by Matslise package (Ledoux et al. 2005). The results for $|\lambda_k - \Lambda_k|$ and $|\lambda_k - \tilde{\Lambda}_k|$ show the efficiency of the correction term $\frac{1}{c^2}\epsilon_{2,k}$ to approximate the eigenvalues of the string equation. We state the following remark to obtain the eigenvalues of non symmetric mass functions.

Remark 2 Let $\{\lambda_i\}_1^n$ and $\{\mu_i\}_1^{n+1}$ be the eigenvalues of the string equation (1) with fixed-fixed and fixed-free boundary conditions, respectively. If $m(x)$ is extended to interval [0, 2] as a symmetric function, then it is well known that $\{\lambda_i^*\}_1^{2n+1}$ defined by

$$\lambda_{2i-1}^* = \mu_i, \quad \lambda_{2i}^* = \lambda_i, \quad i = 1, 2, \cdots, n + 1,$$

Table 3 Scaled errors of Λ_i and $\tilde{\Lambda}_i$ for $m(x) = 1 - 0.3e^{-20(x-0.5)^2}$ and $h = \frac{1}{32}\ \frac{1}{64}\ \frac{1}{128}$

i	$\|\Lambda_i - \lambda_i\|/h^4 i^6$			$\|\tilde{\Lambda}_i - \lambda_i\|/h^4 i^6$		
1	12.57	12.97	13.53	0.054	0.056	0.066
4	10.87	11.19	11.36	0.0098	0.010	0.010
8	10.92	11.21	11.37	0.0099	0.010	0.010
12	10.99	11.23	11.37	0.010	0.010	0.010
16	11.10	11.25	11.38	0.010	0.010	0.010
20	11.24	11.29	11.39	0.010	0.010	0.010
24	11.39	11.34	11.40	0.011	0.010	0.010
28	11.56	11.39	11.41	0.011	0.010	0.010
32	11.73	11.45	11.43	0.012	0.010	0.010
36		11.52	11.45		0.011	0.010
40		11.59	11.47		0.011	0.010
44		11.67	11.50		0.011	0.010

Table 4 Scaled Errors of Λ_i and $\tilde{\Lambda}_i$ for $m(x) = 2 + \sin(\pi(x-1)^2)$ and $h = \frac{1}{32}\ \frac{1}{64}\ \frac{1}{128}$

i	$\|\Lambda_i - \lambda_i\|/h^4 i^6$			$\|\tilde{\Lambda}_i - \lambda_i\|/h^4 i^6$		
1	2.56	2.56	2.54	2.16	2.14	2.14
4	0.515	0.514	0.513	0.0785	0.0779	0.0777
8	0.447	0.445	0.444	0.0383	0.0376	0.0374
12	0.438	0.432	0.432	0.0295	0.0286	0.0283
16	0.438	0.430	0.428	0.0266	0.0252	0.0249
20	0.442	0.430	0.426	0.0257	0.0237	0.0233
24	0.447	0.430	0.426	0.0258	0.0230	0.0224
28	0.453	0.432	0.426	0.0265	0.0227	0.0219
32	0.460	0.434	0.426	0.0280	0.0226	0.0216
36		0.436	0.426		0.0226	0.0214
40		0.439	0.427		0.0228	0.0213
44		0.442	0.428		0.0231	0.0212

are the eigenvalues of the string equation on interval [0, 2] with symmetric mass function and fixed-fixed boundary condition (Gladwell 2004). Thus we conclude that the nonsymmetric case on [0, 1] is equivalent to symmetric case on [0, 2]. Note that by change of variable (2) the coefficient c in correction term (13) for nonsymmetric function $m(x)$ is computed as $c = \frac{1}{2}\int_0^2 \rho(x)dx$.

To confirm the results of Theorem 4, we compute the values of scaled errors $|\lambda_i - \tilde{\Lambda}_i|/h^4 i^6$ in Tables 3 and 4. As expected, the results are bounded and more smaller than the corresponding results for Λ_i. The results show that for string equation uncorrected and corrected Numerov's method are of order $O(h^4 i^6)$. But, correction term reduces the asymptotic error constant, significantly.

3 Inverse Problem

In this section, we want to construct unknown mass function $m(x)$ in the differential equation (1) by using spectral data. If $m(x)$ is symmetric then one spectrum corresponding to fixed-fixed boundary condition will be sufficient to construct it, uniquely. For non symmetric case we need two spectra for unique construction of $m(x)$. Here we use the spectra correspond to fixed-fixed and fixed-free boundary conditions. We state the main inverse problem as follows:

Inverse Problem 1 Let $n \in \mathbb{N}$ and $\{\lambda_i\}_1^{n+1}$ be a given set of positive and distinct real numbers. Construct a symmetric mass function $m(x)$ such that $\{\lambda_i\}_1^{n+1}$ are the first $(n+1)$ eigenvalues of the equation (1) with fixed-fixed boundary condition.

Inverse Problem 2 Let $n \in \mathbb{N}$ and $\{\lambda_i\}_1^n$, $\{\mu_i\}_1^{n+1}$ be two given set of distinct and positive real numbers such that $\mu_i < \lambda_i < \mu_{i+1}$. Construct a mass function $m(x)$ such that $\{\lambda_i\}_1^n$ and $\{\mu_i\}_1^{n+1}$ are the first eigenvalues of the equation (1) with fixed-fixed and fixed-free boundary conditions, respectively.

First we verify the solution for Inverse Problem 1. Let $N = 2n$ (also we can take $N = 2n + 1$). Using Numerov's method the string equation can be written in the matrix eigenvalue problem $-A\mathbf{u} = \Lambda BM\mathbf{u}$. According to symmetric assumption of $m(x)$ we have

$$m_i = m_{N-i}, \quad i = 1, 2, 3, \ldots, n. \tag{33}$$

Therefore the Inverse Problem 1 is equivalent to construct $\{m_i\}_1^n$ by using prescribed eigenvalues $\{\lambda_i\}_1^{n+1}$. The construction procedure is as follows. First we find the eigenvalues Λ_i^1 of the matrix pair $(-A, BM)$ by the procedure given in Sect. 2. Using Eq. (13) we may write

$$\Lambda_i^1 = \lambda_i - \beta\epsilon_{2,i}, \quad i = 1, 2, \ldots, n+1, \tag{34}$$

where $\beta = \frac{1}{c^2} = \frac{1}{(\int_0^1 \rho(x)dx)^2}$. Note that β is an unknown parameter, since $m(x) = \rho^2(x)$ is unknown. Thus we have to find β and $\{m_i\}_1^n$ such that $\{\Lambda_i^1\}_1^{n+1}$ defined by (34) are eigenvalues of the matrix pair $(-A, BM)$. Indeed β and $\{m_i\}_1^n$ must be the solutions of the following nonlinear system of algebraic equations

$$P_i(\mathbf{m}) := \Lambda_i^1(\mathbf{m}) - \lambda_i(m(x)) + \beta\epsilon_{2,i} = 0, \quad i = 1, 2, \ldots, n+1, \tag{35}$$

where $\mathbf{m} = [m_1, m_2, \cdots, m_n, \beta]$. Note that parameter β can be computed as follows:

$$\beta = \lim_{i\to\infty} \frac{\lambda_i}{i^2\pi^2}, \tag{36}$$

See (Chawla and Katti 1980). Here we can approximate $\beta \simeq \frac{\lambda_n}{n^2\pi^2}$. But computing the parameter β by solving nonlinear system (35), leads to efficient results for inverse problem, see Tables 6, 7, 8, and 9. We can solve the system (35) by using modified Newton's method (Stoer and Bulirsch 2002). The sequence of recursive Newton's method is given by:

$$\mathbf{m}_{k+1} = \mathbf{m}_k - G^{-1}\mathbf{P}(\mathbf{m}_k), \ \mathbf{m}_0 = [1, 1, \ldots, 1], \quad k = 1, 2, \ldots, \tag{37}$$

where G is Jacobian matrix of system (35) at point $\mathbf{m} = \mathbf{m}_0$. For this aim, we need the eigenpairs of (11) corresponding to $m(x) \equiv 1$ and Jacobian matrix G. The eigenpairs are as follows:

Lemma 5 (*Yueh 2005*) *The orthonormal eigenvector* $\mathbf{y}_i^\circ$ *of the matrix pair* $(-A, B)$ *is given by*

$$y_{i,j}^\circ = \sqrt{\frac{2}{N}}\sin\left(\frac{ij\pi}{N}\right), \quad j = 1, 2, \ldots, N-1. \tag{38}$$

We find the Jacobian matrix in the following lemma.

Lemma 6 *Jacobian matrix corresponding to nonlinear system* (35) *at* $\mathbf{m} = \mathbf{m}_0$ *is given by*:

$$G(i,j) = -\frac{4}{N}\Lambda_i^\circ \sin^2\left(\frac{ij\pi}{N}\right), \quad i = 1, 2, \ldots, n+1, \quad j = 1, 2, \ldots, n,$$

$$G(i, n+1) = \epsilon_{2,i}, \quad i = 1, 2, \ldots, n+1.$$

Proof Using Eq. (35) we find

Table 5 Condition number of Jacobian matrix G

n	4	8	16	32	64
$Cond(G)$	63.21	179.05	823.21	$4.12e+3$	$2.2e+4$

Table 6 Results of the Example 1 for $m_1(x)$ using $P_i(\mathbf{m}) = 0$

x_i	n			
	5	10	15	20
0.1	$7.4e-4$	$2.0e-5$	$1.0e-5$	$6.2e-6$
0.2	$4.3e-5$	$4.1e-7$	$1.1e-6$	$6.9e-7$
0.3	$3.3e-5$	$2.9e-8$	$3.7e-7$	$2.5e-7$
0.4	$2.3e-5$	$4.9e-8$	$2.2e-7$	$1.5e-7$
0.5	$2.1e-5$	$4.8e-8$	$1.9e-7$	$1.3e-7$
β	0.3210	0.3240	0.3246	0.3248
$\|\tilde{\mathbf{m}} - \mathbf{m}\|_2$	$7.5e-4$	$4.6e-4$	$3.6e-4$	$3.4e-4$
$\|\tilde{\mathbf{m}} - \mathbf{m}\|_{2,n}$	$2.5e-3$	$1.0e-3$	$1.5e-4$	$4.2e-4$

Table 7 Results of the Example 1 for $m_2(x)$ using $P_i(\mathbf{m}) = 0$

x_i	n			
	5	10	15	20
0.1	$3.1e-4$	$1.8e-4$	$7.7e-5$	$3.8e-5$
0.2	$1.7e-4$	$3.3e-5$	$1.3e-5$	$6.7e-6$
0.3	$1.5e-4$	$1.5e-5$	$9.1e-6$	$5.6e-6$
0.4	$7.6e-5$	$1.6e-5$	$1.0e-5$	$9.4e-6$
0.5	$2.8e-5$	$2.4e-5$	$1.1e-5$	$1.3e-5$
β	1.2000	1.1927	1.1914	1.1910
$\|\tilde{\mathbf{m}} - \mathbf{m}\|_2$	$4.0e-4$	$4.4e-4$	$5.9e-4$	$1.1e-4$
$\|\tilde{\mathbf{m}} - \mathbf{m}\|_{2,n}$	$1.5e-3$	$2.5e-3$	$5.0e-3$	$6.7e-3$

$$\frac{\partial P_i}{\partial m_j} = \frac{\partial \Lambda_i^1}{\partial m_j}, \quad \frac{\partial P_i}{\partial \beta} = \epsilon_{2,i}. \tag{39}$$

Taking partial derivative from both sides of $-B^{-1}A\mathbf{u}_i = \Lambda_i^1 M\mathbf{u}_i$ with respect to m_j we have

$$-B^{-1}A\frac{\partial \mathbf{u}_i}{\partial m_j} = \frac{\partial \Lambda_i^1}{\partial m_j}M\mathbf{u}_i + \Lambda_i^1\frac{\partial M}{\partial m_j}\mathbf{u}_i + \Lambda_i^1 M\frac{\partial \mathbf{u}_i}{\partial m_j}.$$

Multiplying both sides of the last equation by $\mathbf{y}_i^T$ we obtain

$$-\mathbf{u}_i^T B^{-1}A\frac{\partial \mathbf{u}_i}{\partial m_j} = \frac{\partial \Lambda_i^1}{\partial m_j}\mathbf{u}_i^T M\mathbf{u}_i + \Lambda_i^1\mathbf{u}_i^T\frac{\partial M}{\partial m_j}\mathbf{u}_i + \Lambda_i^1\mathbf{u}_i^T M\frac{\partial \mathbf{u}_i}{\partial m_j}. \tag{40}$$

Orthogonality of the eigenvectors with respect to M implies that $\mathbf{u}_i^T M\mathbf{u}_i = 1$. On the other hand $-B^{-1}A$ is a symmetric matrix (Chawla and Katti 1980). Therefore $-\mathbf{u}_i^T B^{-1}A\frac{\partial \mathbf{u}_i}{\partial m_j} = \Lambda_i^1\mathbf{u}_i^T M\frac{\partial \mathbf{u}_i}{\partial m_j}$. Considering these properties in (40) we obtain

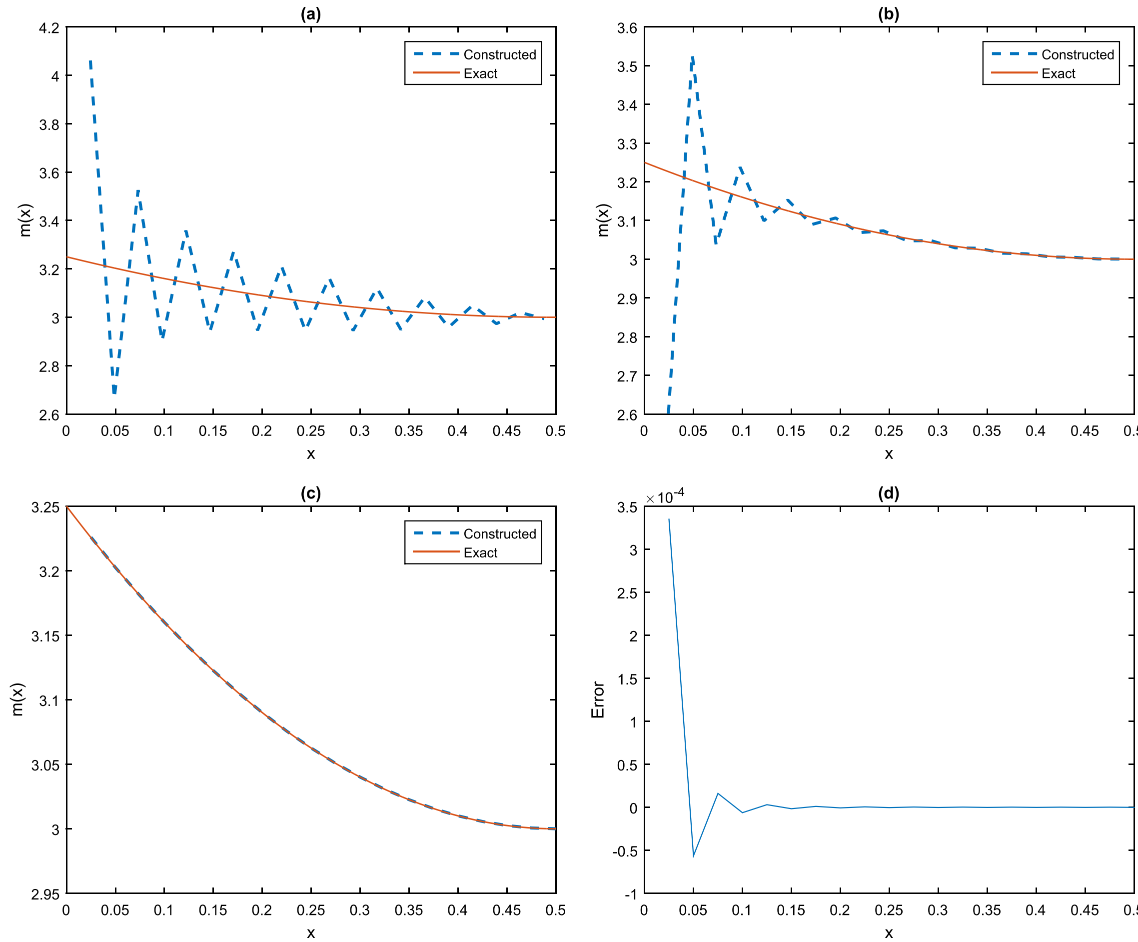

Fig. 1 Results for $m_1(x)$ with $n = 20$. **a** With correction term $\epsilon_{2,i}$, **b** without correction term, **c** with correction term $\beta\epsilon_{2,i}$, **d** the error $(\tilde{m}_i - m_i)$ with correction term $\beta\epsilon_{2,i}$

$$\frac{\partial \Lambda_i^1}{\partial m_j} = -\Lambda_i^1 \mathbf{u}_i^T \frac{\partial M}{\partial m_j} \mathbf{u}_i.$$

Computing $\frac{\partial M}{\partial m_j}$ and substituting $\mathbf{u}_i, \Lambda_i^1$ corresponding to $\mathbf{m} = \mathbf{m}_0$ we find

$$\frac{\partial \Lambda_i^1}{\partial m_j}|_{\mathbf{m}=\mathbf{m}_0} = -\frac{4}{N} \Lambda_i^\circ \sin^2\left(\frac{ij\pi}{N}\right). \tag{41}$$

Combining Eqs. (39) and (41) we find the entries of the required Jacobian matrix. □

Remark 3 The matrix G is a nonsingular constant matrix and independent of mass function $m(x)$. The condition number of G is given in Table 5 for different values of n. For some large values of n, we may need to apply a regularization method for solving nonlinear system (35). Here we apply the quasi-Newton's method as follows

$$\mathbf{m}_{k+1} = \mathbf{m}_k - \alpha_k (G^t G + \sigma I)^{-1} G^t \mathbf{P}(\mathbf{m}_k), \tag{42}$$

where α_k satisfy the Wolf conditions (Gilbert 1997) and $\sigma > 0$ is a regularization parameter.

Remark 4 The equation (1) is a special case of the Sturm-Liouville equation $(py')' + (\lambda w + q)y = 0$ thus, the parameter λ is a differentiable function with respect to $m(x)$ (Zhang and Li 2020).

The following theorem shows the convergence of modified Newton's sequence (37).

Theorem 5 *Let $n \in \mathbb{N}, p \geq 1$ and $\|.\|_p$ denotes the L_p norm on* [0, 1], *there exists a positive number $c_p(n)$ such that if $\|m(x) - 1\|_p < c_p(n)$ and the sequence $\mathbf{m}_k$ obtained from* (37) *be positive, then the recursive sequence* (37) *with initial value $\mathbf{m}_0$ is convergent to the solution of the system* (35).

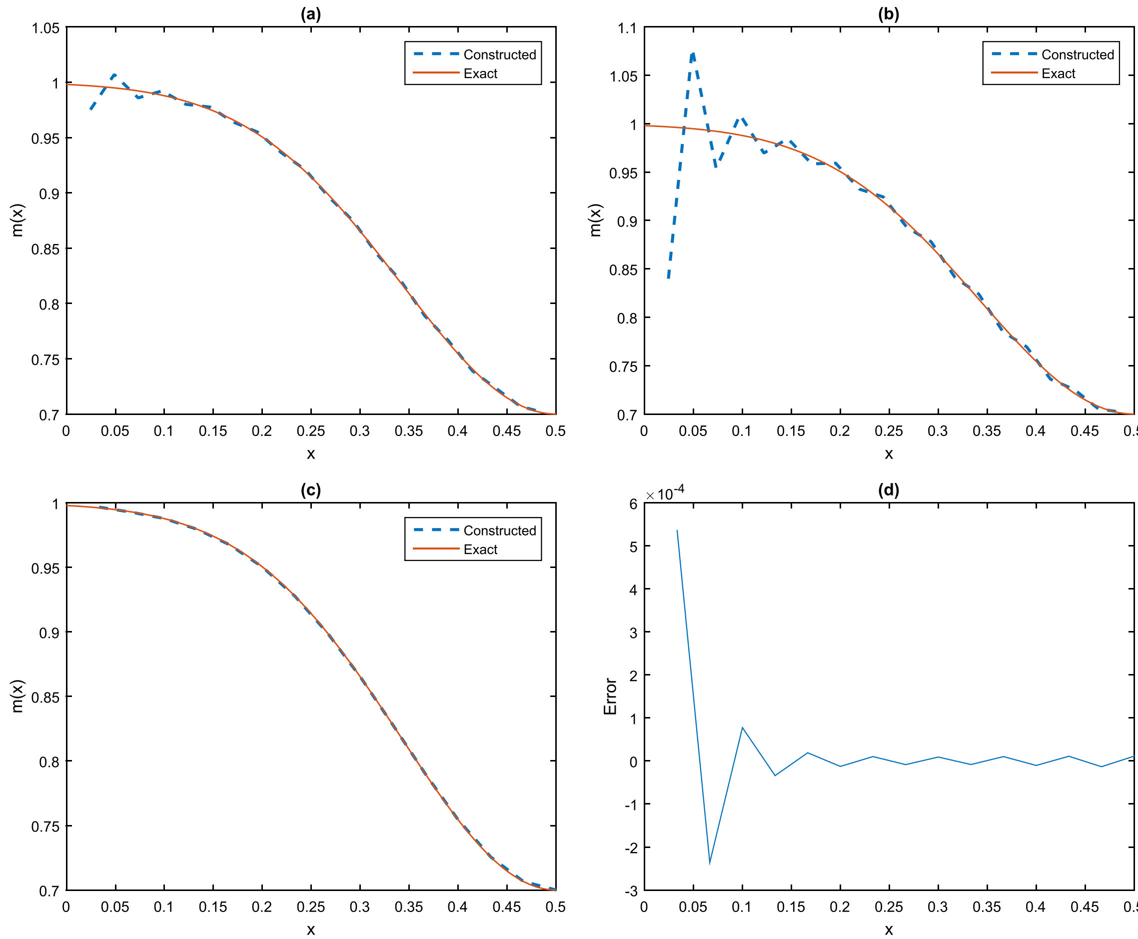

Fig. 2 Results for $m_2(x)$ with $n = 20$. **a** With correction term $\epsilon_{2,i}$, **b** without correction term, **c** with correction term $\beta\epsilon_{2,i}$, **d** the error $(\tilde{m}_i - m_i)$ with correction term $\beta\epsilon_{2,i}$

Table 8 Results of the Example 2 for $m_3(x)$ using $P_i(\mathbf{m}) = 0$.

x_i	n			
	5	10	15	20
0.2	$1.3e-3$	$4.7e-4$	$1.5e-4$	$3.4e-4$
0.4	$7.1e-3$	$4.5e-5$	$1.6e-5$	$3.8e-5$
0.6	$4.0e-3$	$1.2e-5$	$1.4e-5$	$2.7e-5$
0.8	$3.2e-3$	$1.0e-5$	$2.5e-5$	$3.6e-5$
1.0	$2.8e-3$	$9.4e-6$	$3.3e-5$	$4.1e-5$
β	0.5576	0.4405	0.4337	0.4312
$\Vert\tilde{\mathbf{m}} - \mathbf{m}\Vert_2$	$9.3e-3$	$4.1e-3$	$1.4e-3$	$2.8e-3$
$\Vert\tilde{\mathbf{m}} - \mathbf{m}\Vert_{2,n}$	$3.9e-2$	$1.0e-2$	$2.1e-2$	$3.8e-2$

Table 9 Results of the Example 2 for $m_4(x)$ using $P_i(\mathbf{m}) = 0$

x_i	n			
	5	10	15	20
0.2	$2.4e-2$	$2.6e-3$	$1.4e-3$	$2.2e-4$
0.4	$2.7e-2$	$5.1e-4$	$2.4e-4$	$1.5e-5$
0.6	$3.2e-2$	$1.1e-3$	$1.9e-5$	$4.2e-6$
0.8	$3.4e-2$	$1.5e-3$	$5.3e-5$	$1.0e-5$
1.0	$3.5e-2$	$1.7e-3$	$9.1e-5$	$5.3e-6$
β	1.0007	1.1361	1.2019	1.1760
$\Vert\tilde{\mathbf{m}} - \mathbf{m}\Vert_2$	$6.9e-2$	$6.2e-3$	$4.7e-3$	$2.6e-3$
$\Vert\tilde{\mathbf{m}} - \mathbf{m}\Vert_{2,n}$	$7.2e-2$	$1.4e-2$	$2.0e-2$	$2.8e-2$

Proof Let $\Vert.\Vert$ denote a norm on $\mathbb{R}^{n+1}$. Suppose that $S_r(\mathbf{m}_0) := \{\mathbf{m} \mid \Vert\mathbf{m} - \mathbf{m}_0\Vert < r\}$. First we prove that $\mathbf{P}(\mathbf{m})$ is an analytic function with respect to $\mathbf{m}$ and a differentiable function with respect to $m(x)$. The matrix $-B^{-1}A$ is positive definite (Chawla and Katti 1980), thus positivity of $\mathbf{m}_k$ implies that the eigenvalues $\Lambda_i^1(\mathbf{m})$ of matrix pair $(-B^{-1}A, M)$ are positive and simple (see (Gladwell 2004), chapter 3). Thus $\mathbf{P}(\mathbf{m})$ is an analytic function of $\mathbf{m}$ (Sun 1990). Also by Remark 4, the eigenvalues $\lambda_i(m(x))$ are differentiable function of $m(x)$. Since G is nonsingular and

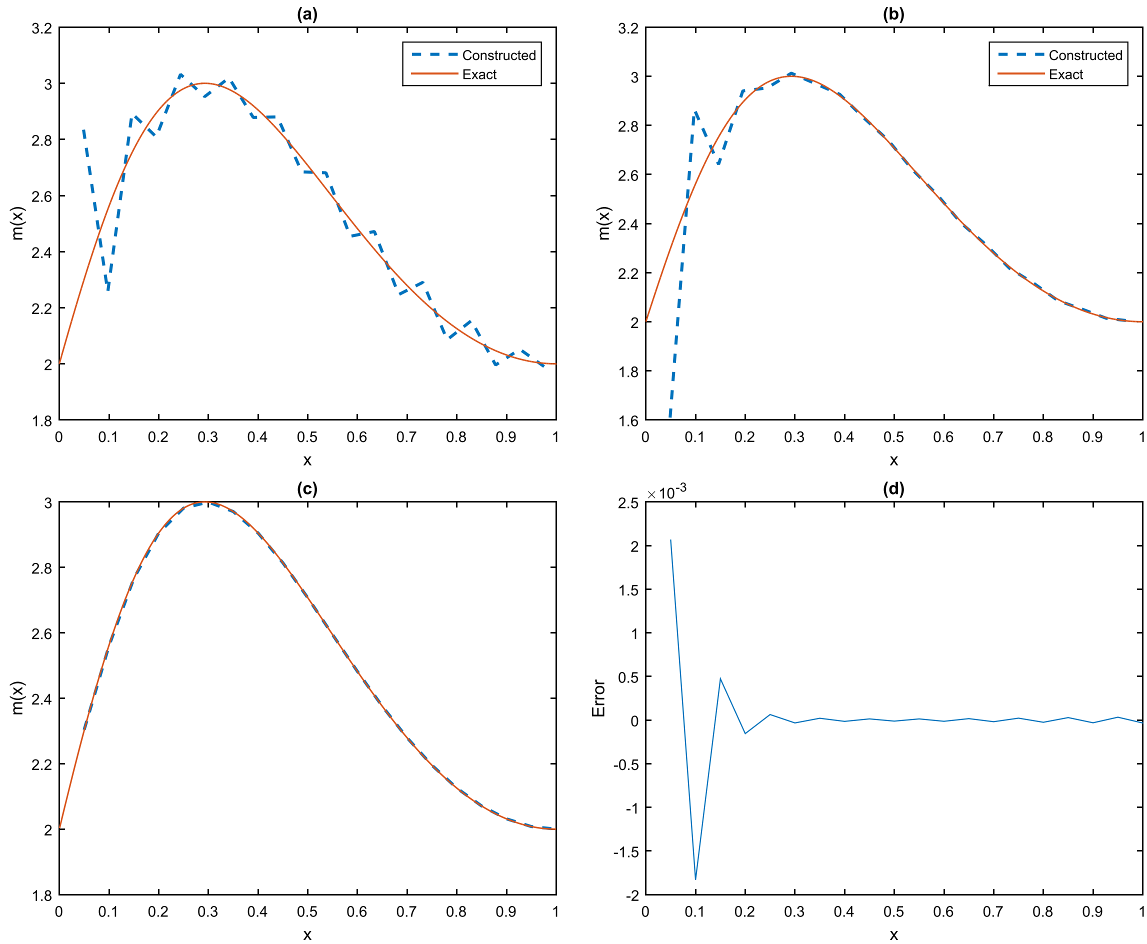

Fig. 3 Results for $m_3(x)$ with $n = 20$. **a** With correction term $\epsilon_{2,i}$, **b** without correction term, **c** with correction term $\beta\epsilon_{2,i}$, **d** the error $(\tilde{m}_i - m_i)$ with correction term $\beta\epsilon_{2,i}$

$\mathbf{P}$ is analytic with respect to $\mathbf{m}$, there exists a constant $K > 0$ such that $\|G^{-1}(\mathbf{P}'(\mathbf{m}) - \mathbf{P}'(\mathbf{m}_0))\| \leq K\|\mathbf{m} - \mathbf{m}_0\|$. Suppose that $\eta = \|G^{-1}\mathbf{P}(\mathbf{m}_0)\|$, $\rho = K\eta$ and $r_- = \frac{1-\sqrt{1-2\rho}}{K}$. If $\|m(x) - 1\|_p = 0$, then for $i = 1, 2, \ldots, n+1$ we have

$$P_i(\mathbf{m}_0) = \Lambda_i^\circ(\mathbf{m}_0) - \lambda_i(m(x)) + \beta\epsilon_2(i,h) = 0.$$

Thus for all $p \geq 1$, there exists $c_p(n)$, such that for all $m \in L_p[0,1]$, if $\|m(x) - 1\|_p \leq c_p(n)$, then $0 < \rho < \frac{1}{2}$ and $r_- < r$. By theory of modified Newton's method (Stoer and Bulirsch 2002), we conclude that all $\mathbf{m}_k$ lie in $S_{r_-}(\mathbf{m}_0)$ and the sequence $\{\mathbf{m}_k\}$ converges to a solution of $\mathbf{P}(\mathbf{m}) = 0$. □

Corollary 1 *By Remark 2, we can solve the* ***Inverse problem*** 2 *using the method of* ***Inverse problem*** 1 *on the interval* [0, 2].

Using the following algorithm we can solve the inverse problem to construct $m(x)$.

Algorithm 1 Compute $m(x)$

Input the eigendata $\{\lambda_i\}_{i=1}^{n+1}$ and initial guess $\mathbf{m}_0$,
Compute the Jacobian matrix G from Lemma 6,
Compute $\mathbf{m}_k$ by recursive sequence (37) or (42).

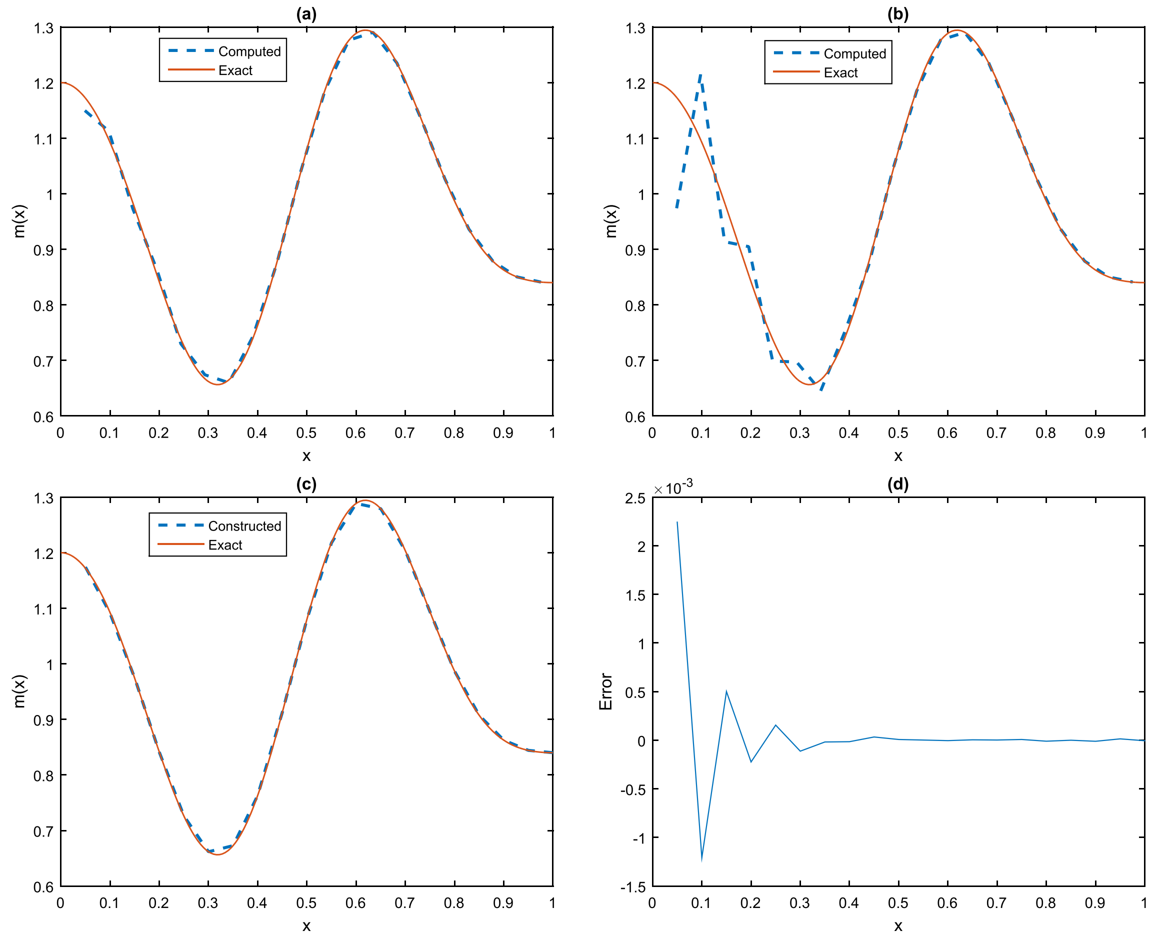

Fig. 4 Results for $m_4(x)$ with $n = 20$. **a** With correction term $\epsilon_{2,i}$, **b** without correction term, **c** with correction term $\beta\epsilon_{2,i}$, **d** the error $(\tilde{m}_i - m_i)$ with correction term $\beta\epsilon_{2,i}$

4 Numerical Results

In this section, we propose some numerical examples to show the efficiency of the presented algorithm for construction of mass function by solving the nonlinear system $P_i(\mathbf{m}):=\Lambda_i^1(\mathbf{m}) - \lambda_i(m(x)) + \beta\epsilon_{2,i} = 0$. The computed solution of the nonlinear system is denoted by $\tilde{\mathbf{m}}$. We denote the L_2 error of constructed $m(x)$ by $\|\tilde{\mathbf{m}} - \mathbf{m}\|_2$ when the parameter β is obtained by solving nonlinear system $P_i(\mathbf{m}) = 0$ and by $\|\tilde{\mathbf{m}} - \mathbf{m}\|_{2,n}$ when the parameter β is approximated by (36). Moreover, in the numerical examples we try to construct $m(x)$ by solving uncorrected system $T_i(\mathbf{m}):=\Lambda_i^1(\mathbf{m}) - \lambda_i(m(x)) = 0$ and system $H_i(\mathbf{m}):=\Lambda_i^1(\mathbf{m}) - \lambda_i(m(x)) + \epsilon_{2,i} = 0$ with correction term $\epsilon_{2,i}$. The results of these systems are compared in the numerical examples. All computations were performed with MATLAB R2015a on an Intel(R) Core(TM) i5 desktop computer.

Example 1 We consider the mass functions $m_1(x) = 3 + (x - 0.5)^2$ and $m_2(x) = 1 - 0.3e^{-20(x-0.5)^2}$ on the interval [0, 1]. These functions are symmetric with respect to $x = 0.5$, thus we can construct them using one spectrum $\{\lambda_i\}_i^{n+1}$. Tables 6 and 7 show the L_2 errors and absolute errors in some points. Also parameter β is obtained in these tables. The results of $m_2(x)$ for $n = 15$ and $n = 20$ are obtained using regularized quasi-Newton's method (42). Figures 1 and 2 show the numerical results corresponding to uncorrected scheme $(T_i(\mathbf{m}) = 0)$ and corrected scheme with correction terms ϵ $(H_i(\mathbf{m}) = 0)$ and $\beta\epsilon$ $(P_i(\mathbf{m}) = 0)$. Numerical results show the good efficiency of the correction term $\beta\epsilon_{2,i}$. The mass function $m_2(x)$ is constructed in Jiang and Xu (2019). There is an agreement between the results of our method and Jiang and Xu (2019).

Example 2 We consider the non symmetric mass functions $m_3(x) = 2 + \sin(\pi(x-1)^2)$ and

$$m_4(x) = 0.98 - 0.04\cos(\pi x) - 0.03\cos(2\pi x) + 0.26\cos(3\pi x) + 0.07\cos(4\pi x) - 0.04\cos(5\pi x),$$

on the interval [0, 1]. We need two spectra $\{\lambda_i\}_1^n$ and $\{\mu_i\}_1^{n+1}$ for constructing these functions. The L_2 errors and absolute errors in some selected points are listed is Tables 8 and 9. For uncorrected and corrected schemes the numerical results are compared and depicted in Figs. 3 and 4. The results of $m_4(x)$ for $n = 15, 20$ are obtained using regularized quasi-Newton's method (42). The mass function $m_4(x)$ is constructed in Jiang and Xu (2021). There is an agreement between the results of our method and Jiang and Xu (2021).

5 Conclusion

In this paper, we used Numerov's method along with a new correction term to approximate the eigenvalues and construct mass function of the string equation. It was shown that, unlike to canonical Sturm-Liouville and vibrating Rod equations, for the string equation the correction term $\epsilon_{2,i}$ does not work and we must consider the correction term as $\beta\epsilon_{2,i}$. Therefore, we introduced the parameter β that plays an important role in approximating the eigenvalues and constructing the mass function $m(x)$. Numerical results show that the correction technique is able to reduce the asymptotic error constant of Numerov's method, significantly. This leads to efficient results for direct and inverse problems of the string equation.

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