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RESEARCH PAPER

Relations Between Almost *n*-Jordan Homomorphisms and Almost *n*-Homomorphisms

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Abstract

Let A and B be topological algebras equipped with separating sequences of submultiplicative seminorms. For $n \in \mathbb{N}$, $n \geq 2$, we investigate under what conditions an almost n-Jordan homomorphism $T : A \to B$ is an almost n-homomorphism. We also study the automatic continuity of almost n-Jordan homomorphisms and almost n-homomorphisms. Finally, we present some results concerning additive n-Jordan homomorphisms on rings.

Keywords n-Jordan homomorphism · n-homomorphism · Almost n-Jordan homomorphism · Almost n-homomorphism · Submultiplicative seminorm · Automatic continuity

Mathematics Subject Classification 47C05 · 46H05 · 46H40

1 Introduction

In this paper we always assume that $n \in \mathbb{N}$, $n \geq 2$. In this section we present the definitions, some preliminary results and the historical background of the subject.

An additive map (operator) T between rings A and B is called an n-homomorphism if

$$T(a_1 a_2 \cdots a_n) = Ta_1 Ta_2 \cdots Ta_n,$$

for every $a_1, a_2, \ldots, a_n \in A$. In particular, T is called an n-Jordan homomorphism if $T(a^n) = (Ta)^n$, for every $a \in A$. If $n = 2$ then T is simply called a homomorphism or a Jordan homomorphism, respectively. When A and B are algebras, we assume that T is linear.

The concept of n-Jordan homomorphisms between rings was introduced by Herstein (1956), and the notion of n-homomorphisms between algebras was introduced by Hejazian et al. (2005). One of the main problems in this area is that "under what conditions an n-Jordan homomorphism between algebras or rings is an n-homomorphism". Another main problem is that "under what conditions n-Jordan homomorphisms or n-homomorphisms between algebras are continuous, when these algebras are equipped with suitable topologies". Some authors have imposed many conditions on A and B to fulfil these properties. For more general results one may refer to Honary et al. (2021) and Honary (2022). When A and B are commutative algebras or rings, it has been shown that every n-Jordan homomorphism is an n-homomorphism. For example, see, Gselmann (2014), Honary et al. (2021), Cheshmavar et al. (2020) and Bodaghi and Inceboz (2018).

A sequence of seminorms (p_k) on an algebra A is called submultiplicative if $p_k(xy) \leq p_k(x)p_k(y)$ for all $k \in \mathbb{N}$ and every $x, y \in A$. A sequence (p_k) of seminorms on A is called separating if for each non-zero $a \in A$ there exists p_k such that $p_k(a) \neq 0$. Let $(A, (p_k))$ and $(B, (q_k))$ be algebras equipped with the separating sequences of submultiplicative seminorms (p_k) and (q_k). A linear map $T : (A, (p_k)) \to (B, (q_k))$ is called an almost n-homomorphism if there exists a non-negative ε (which may depend on n) such that

$$q_k(T(a_1 a_2 \cdots a_n) - Ta_1 Ta_2 \cdots Ta_n) \leq \varepsilon \prod_{i=1}^{n} p_k(a_i),$$

for every $a_1, a_2, \ldots, a_n \in A$ and all $k \in \mathbb{N}$. Moreover, T is called a weakly almost n-homomorphism if there exists a non-negative ε (which may depend on n) such that for each $k \in \mathbb{N}$ there exists $n(k) \in \mathbb{N}$ satisfying the following inequality

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$$q_k(T(a_1\cdots a_n) - Ta_1\cdots Ta_n) \le \varepsilon \prod_{i=1}^{n} p_{n(k)}(a_i),$$

for every $a_1, \ldots, a_n \in A$. In particular, T is called an almost n-Jordan homomorphism if there exists a non-negative ε (which may depend on n) such that

$$q_k(T(a^n) - (Ta)^n) \le \varepsilon(p_k(a))^n,$$

for each $k \in \mathbb{N}$ and every $a \in A$. Similarly, T is called a weakly almost n-Jordan homomorphism if there exists a non-negative ε (which may depend on n) such that for each $k \in \mathbb{N}$ there exists $n(k) \in \mathbb{N}$ satisfying the inequality

$$q_k(T(a^n) - (Ta)^n) \le \varepsilon(p_{n(k)}(a))^n,$$

for every $a \in A$.

In the case that A and B are commutative Fréchet algebras and $n = 2$, it was shown in Omidi et al. (2020, Theorem 2.8) that, if the map $T : A \to B$ is an almost Jordan homomorphism, then it is an almost homomorphism. By adopting a different method, we prove this property for any (weakly) almost n-Jordan homomorphism between commutative topological algebras in Corollaries 2.7 and 2.8. But for the non-commutative case, we have only partial answers for this problem, by imposing some conditions on the topological algebras A, B and the map T. Another problem in this area is to find conditions on the topological algebras A and B so that a (weakly) an almost n-Jordan homomorphism or a (weakly) an almost n-homomorphism $T : A \to B$ be continuous. K. Jarosz (1985, Proposition 5.5) was the first one who proved that an almost complex homomorphism on a Banach algebra is continiuous. Since then this result has been extended for more general cases by several authors. For more information on these kinds of results, one can refer, for example, to Šemrl (2002), Kulkarni and Sukumar (2010), Gselmann (2014), Honary et al. (2015, 2016, 2017). In this paper, among some other results, we present suitable conditions to fulfil these properties.

A subset V of a complex algebra A is called *multiplicatively convex*, or briefly, m-convex, if V is convex and idempotent, i.e., $VV \subseteq V$. A subset V of A is called *balanced* if $\lambda V \subseteq V$ for all scalars λ such that $|\lambda| \le 1$. A topological algebra is locally convex if there is a base of neighbourhoods of zero consisting of convex sets.

Since each base of convex neighbourhoods of zero consists a base of absolutely convex (convex and balanced) sets, we may always assume that a locally convex algebra has a base of neighbourhoods of zero consisting of absolutely convex sets.

A topological algebra is called a locally multiplicatively convex algebra, or briefly an lmc algebra, if there is a base of neighbourhoods of the origin consisting of sets which are absolutely convex and idempotent. An interesting class of topological algebras is the class of Fréchet algebras, defined as a complete metrizable lmc algebra. It is easy to see that a separating sequence of submultiplicative seminorms on an algebra A induces a topology on A so that A turns out to be a locally multiplicatively convex (lmc) topological algebra. This sequence, which induces a topology on the algebra A, is called the generating sequence of the topological algebra A.

Remark 1.1 If (p_k) is a generating sequence of an algebra A, then the sequence (P_k) with $P_k(x) = \max_{1\le i\le k} p_i(x)$, is an increasing sequence of submultiplicative seminorms on A, in the sense that $P_k(x) \le P_{k+1}(x)$ for all $x \in A$ and every $k \in \mathbb{N}$. Moreover, (P_k) also generates the same topology on A. Hence, whenever (p_k) is a generating sequence of an algebra A, we may always assume that (p_k) is an increasing sequence on A.

From now on by $(A, (p_k))$ we always assume that A is a topological algebra with the generating submultiplicative sequence of seminorms (p_k) on A, which is increasing.

As it is indicated in Goldmann (1990, 3.1.6, pp. 62–63), the topology of a Fréchet algebra A can be generated by a separating sequence of submultiplicative seminorms (p_k) on A. Moreover, by the remark above, we may assume that the sequence (p_k) is increasing. Note that a sequence (x_n) in a topological algebra $(A, (p_k))$ converges to $x \in A$ if and only if $p_k(x_n - x) \to 0$ for each $k \in \mathbb{N}$, as $n \to \infty$.

Banach algebras are important examples of Fréchet algebras. In fact, whenever the generating sequence (p_k) of the Fréchet algebra A, is defined by $p_k(x) = \|x\|$ for all $k \in N$, the algebra A turns out to be a Banach algebra.

We now recall the following property for the continuity of a linear map $T : (A, (p_k)) \to (B, (q_k))$, which is useful in the sequel. However, when A and B are Fréchet algebras, this remark has also been mentioned in Goldmann (1990, Remarks 3.2.2, pp. 72–73). One may also refer to Fragoulopoulou (2005, p. 8).

Remark 1.2 A linear map $T : (A, (p_k)) \to (B, (q_k))$ is continuous if and only if for each $k \in \mathbb{N}$ there exist $n(k) \in \mathbb{N}$ and a positive c_k such that $q_k(Ta) \le c_k\, p_{n(k)}(a)$ for all $a \in A$.

Based on the previous result, we bring the following definition:

Definition 1.1 A linear map $T : (A, (p_k)) \to (B, (q_k))$ is called quasi uniformly continuous if there exists $c > 0$ such that for each $k \in \mathbb{N}$ there exists $n(k) \in \mathbb{N}$ satisfying $q_k(Ta) \le c\, p_{n(k)}(a)$ for all $a \in A$.

Proposition 1.1 *If a linear map $T:(A,(p_k))\to(B,(q_k))$ is quasi uniformly continuous, then it is a weakly almost n-homomorphism for each $n\in\mathbb{N}$, $n\geq 2$.*

Proof By the hypothesis, there exists $c>0$ such that for each $k\in\mathbb{N}$, $q_k(Ta)\leq c\ p_{n(k)}(a)$ for some $n(k)$ and for every $a\in A$. Let $a_1,a_2,\ldots,a_n$ be arbitrary elements of A. Then for each $k\in\mathbb{N}$ we have

$$\begin{aligned}q_k(T(a_1a_2\cdots a_n)-Ta_1Ta_2\cdots Ta_n)&\leq q_k(T(a_1a_2\cdots a_n))\\&\quad+q_k(Ta_1)q_k(Ta_2)\cdots q_k(Ta_n)\\&\leq c\ p_{n(k)}(a_1a_2\cdots a_n)+c^np_{n(k)}(a_1)p_{n(k)}(a_2)\cdots p_{n(k)}(a_n)\\&\leq (c+c^n)\prod_{i=1}^n p_{n(k)}(a_i).\end{aligned}$$

Hence, $T:A\to B$ is a weakly almost n-homomorphism, with $\varepsilon=c+c^n$, which depends on n. □

It is clear from Remark 1.2 that if, in particular, $(B,\|\cdot\|)$ is a normed algebra, then every continuous linear map $T:(A,(p_k))\to(B,\|\cdot\|)$ is quasi uniformly continuous. Hence, we conclude the following result:

Corollary 1.2 *If $T:(A,(p_k))\to B$ is a continuous linear map, where B is a normed algebra, then T is a weakly almost n-homomorphism for each $n\geq 2$.*

Proposition 1.3 *Let $0\leq p<n$ and $T:(A,(p_k))\to(B,(q_k))$ be a linear map satisfying the following inequality for some $\varepsilon\geq 0$*

$$q_k(T(a_1a_2\cdots a_n)-Ta_1Ta_2\cdots Ta_n)\leq\varepsilon\left(\sum_{i=1}^n p_k(a_i)\right)^p,$$

for each $k\in\mathbb{N}$ and every $a_1,a_2,\ldots,a_n\in A$. Then T is an n-homomorphism.

One may compare this result with Corollary 2.2.

Proof For every $m\in\mathbb{N}$ we have

$$m^nq_k(T(a_1a_2\cdots a_n)-Ta_1Ta_2\cdots Ta_n)\leq\varepsilon\left(\sum_{i=1}^n p_k(ma_i)\right)^p$$

Hence

$$q_k(T(a_1a_2\cdots a_n)-Ta_1Ta_2\cdots Ta_n)\leq\frac{\varepsilon}{m^{n-p}}\left(\sum_{i=1}^n p_k(a_i)\right)^p$$

for all $m,k\in\mathbb{N}$. If $m\to\infty$ then $q_k(T(a_1a_2\cdots a_n)-Ta_1Ta_2\cdots Ta_n)=0$ for every $k\in\mathbb{N}$. Since (q_k) is separating, it follows that

$$T(a_1a_2\cdots a_n)=Ta_1Ta_2\cdots Ta_n.$$

Hence, T is an n-homomorphism. □

Definition 1.2 A ring A is of characteristic different from n ($char\,A\neq n$) if the equality $nx=0$ implies that $x=0$ for every $x\in A$. A ring A is of characteristic greater than n ($char\,A>n$) if the equality $n!x=0$ implies that $x=0$ for all $x\in A$.

The following result, which appeared in Honary et al. (2021, Theorem 2.2), is very useful in the sequel.

Theorem 1.4 *Let A and B be rings and $T:A\to B$ be an additive n-Jordan homomorphism. Then*

$$T\left(\sum_{\sigma\in S_n}a_{\sigma(1)}a_{\sigma(2)}\cdots a_{\sigma(n)}\right)=\sum_{\sigma\in S_n}Ta_{\sigma(1)}Ta_{\sigma(2)}\cdots Ta_{\sigma(n)},$$

where S_n is the set of all permutations (*bijections*) $\sigma:\{1,2,\ldots,n\}\to\{1,2,\ldots,n\}$.

In particular, if $char\,B>n$ then

$$\begin{aligned}&T\left(xy^{n-1}+yxy^{n-2}+\cdots+y^{n-1}x\right)\\&\quad=Tx(Ty)^{n-1}+TyTx(Ty)^{n-2}+\cdots+(Ty)^{n-1}Tx,\end{aligned}$$

for every $x,y\in A$

2 Basic Results on Almost *n*-Jordan Homomorphisms and Almost *n*-Homomorphisms

In this section we investigate under what conditions on the lmc-algebras A and B, an almost n-Jordan homomorphism $T:A\to B$ is an almost n-homomorphism. We also show that under certain conditions on the topological algebras A and B, an almost n-Jordan homomorphism $T:A\to B$ is either continuous or it is an n-homomorphism.

We first present a key result, which is very useful in the sequel.

Theorem 2.1 *If $T:(A,(p_m))\to(B,(q_m))$ is an almost n-Jordan homomorphism for some $\varepsilon\geq 0$, then we have*

$$\begin{aligned}&q_m\left(T\left(\sum_{\sigma\in S_n}a_{\sigma(1)}a_{\sigma(2)}\cdots a_{\sigma(n)}\right)\right.\\&\quad\left.-\sum_{\sigma\in S_n}Ta_{\sigma(1)}Ta_{\sigma(2)}\cdots Ta_{\sigma(n)}\right)\leq\varepsilon 2^n\left(\sum_{i=1}^n p_m(a_i)\right)^n,\end{aligned}$$

for all $m\in\mathbb{N}$ and every $a_1,a_2,\ldots,a_n\in A$.

Moreover, if $T:A\to B$ is a weakly almost n-Jordan homomorphism, then for every $m\in\mathbb{N}$ there exists $n(m)\in\mathbb{N}$ such that

$$q_m\left(T\left(\sum_{\sigma\in S_n} a_{\sigma(1)}a_{\sigma(2)}\cdots a_{\sigma(n)}\right)\right.$$
$$\left. - \sum_{\sigma\in S_n} Ta_{\sigma(1)}Ta_{\sigma(2)}\cdots Ta_{\sigma(n)}\right) \leq \varepsilon 2^n\left(\sum_{i=1}^n p_{n(m)}(a_i)\right)^n,$$

for every $a_1, a_2, \ldots, a_n \in A$.

Proof By Honary et al. (2021, Theorem 2.1) we have

$$\sum_{\sigma\in S_n} a_{\sigma(1)}a_{\sigma(2)}\cdots a_{\sigma(n)} = \sum_{k=0}^{n-1}(-1)^k\sum_{|I|=n-k}\left(\sum_{i\in I} a_i\right)^n, \quad (1)$$

for every $a_1, a_2, \ldots, a_n \in A$, where I is a subset of $\{1, 2, \ldots, n\}$ and $|I|$ denotes the cardinal of I. Similarly, for B we have

$$\sum_{\sigma\in S_n} Ta_{\sigma(1)}Ta_{\sigma(2)}\cdots Ta_{\sigma(n)} = \sum_{k=0}^{n-1}(-1)^k\sum_{|I|=n-k}\left(\sum_{i\in I} Ta_i\right)^n. \quad (2)$$

Since $T : A \to B$ is linear, from (1) it follows that

$$T\left(\sum_{\sigma\in S_n} a_{\sigma(1)}a_{\sigma(2)}\cdots a_{\sigma(n)}\right) = \sum_{k=0}^{n-1}(-1)^k\sum_{|I|=n-k} T\left(\left(\sum_{i\in I} a_i\right)^n\right). \quad (3)$$

Now by (2) and (3) we conclude the following equality:

$$T\left(\sum_{\sigma\in S_n} a_{\sigma(1)}a_{\sigma(2)}\cdots a_{\sigma(n)}\right) - \sum_{\sigma\in S_n} Ta_{\sigma(1)}Ta_{\sigma(2)}\cdots Ta_{\sigma(n)}$$
$$= \sum_{k=0}^{n-1}(-1)^k\sum_{|I|=n-k}\left[T\left(\left(\sum_{i\in I} a_i\right)^n\right) - \left(\sum_{i\in I} Ta_i\right)^n\right].$$

Therefore, the following inequalities hold for each $m \in \mathbb{N}$:

$$q_m\left(T\left(\sum_{\sigma\in S_n} a_{\sigma(1)}a_{\sigma(2)}\cdots a_{\sigma(n)}\right) - \sum_{\sigma\in S_n} Ta_{\sigma(1)}Ta_{\sigma(2)}\cdot Ta_{\sigma(n)}\right)$$
$$\leq \sum_{k=0}^{n-1}\sum_{|I|=n-k} q_m\left[T\left(\left(\sum_{i\in I} a_i\right)^n\right) - \left(\sum_{i\in I} Ta_i\right)^n\right]$$
$$\leq \sum_{k=0}^{n-1}\sum_{|I|=n-k}\varepsilon\left(p_m(\sum_{i\in I} a_i)\right)^n$$
$$\leq \sum_{k=0}^{n-1}\sum_{|I|=n-k}\varepsilon\left(\sum_{i\in I} p_m(a_i)\right)^n$$
$$\leq \sum_{k=0}^{n-1}\sum_{|I|=n-k}\varepsilon\left(\sum_{i=1}^n p_m(a_i)\right)^n$$
$$\leq \varepsilon 2^n\left(\sum_{i=1}^n p_m(a_i)\right)^n.$$

In the case that T is a weakly almost n-Jordan homomorphism, the proof is immediate. □

Corollary 2.2 *With the same hypothesis as in Theorem 2.1, if A and B are commutative, then for every $k \in \mathbb{N}$ and every $a_1, a_2, \ldots, a_n \in A$ we have*

$$q_k(T(a_1a_2\cdots a_n) - Ta_1Ta_2\cdots Ta_n) \leq \frac{\varepsilon 2^n}{n!}\left(\sum_{i=1}^n p_k(a_i)\right)^n.$$

Moreover, if T is a weakly almost n-Jordan homomorphism, then for every $k \in \mathbb{N}$ there exists $n(k) \in \mathbb{N}$ such that

$$q_k(T(a_1a_2\cdots a_n) - Ta_1Ta_2\cdots Ta_n) \leq \frac{\varepsilon 2^n}{n!}\left(\sum_{i=1}^n p_{n(k)}(a_i)\right)^n,$$

for every $a_1, a_2, \ldots, a_n \in A$

Proof It is clear that $\sum_{\sigma\in S_n} a_{\sigma(1)}a_{\sigma(2)}\cdots a_{\sigma(n)} = n!a_1 a_2\cdots a_n$, and similarly

$$\sum_{\sigma\in S_n} Ta_{\sigma(1)}Ta_{\sigma(2)}\cdots Ta_{\sigma(n)} = n!Ta_1Ta_2\cdots Ta_n.$$

Hence, the result follows by the Theorem. □

Note that the above inequalities are not sharp. For example, when A and B are commutative normed algebras and $T : A \to B$ is an almost 3-Jordan homomorphism for some $\varepsilon \geq 0$, we can show that

$$\|T(a_1a_2a_3) - Ta_1Ta_2Ta_3\| \leq \varepsilon(\|a_1\| + \|a_2\| + \|a_3\|)^3,$$

for every $a_1, a_2, a_3 \in A$. See Zivari-Kazempour (2020b, Theorem 10).

Corollary 2.3 *Let $(A, \|\cdot\|)$ be a normed algebra and $T : (A, \|\cdot\|) \to (B, (q_k))$ be an almost n-Jordan homomorphism for some $\varepsilon \geq 0$. Then*

$$q_k\left(T\left(\sum_{\sigma\in S_n} a_{\sigma(1)}a_{\sigma(2)}\cdots a_{\sigma(n)}\right)\right.$$
$$\left. - \sum_{\sigma\in S_n} Ta_{\sigma(1)}Ta_{\sigma(2)}\cdots Ta_{\sigma(n)}\right) \leq \varepsilon(2n)^n\prod_{i=1}^n\|a_i\|,$$

for each $k \in \mathbb{N}$ and every $a_1, a_2, \ldots, a_n \in A$. In particular, if A and B are commutative, then T is an almost n-homomorphism.

Proof Let $a_1, a_2, \ldots, a_n \in A$. Since the inequality clearly holds when $a_i = 0$ for some i, we may assume that $a_i \neq 0$ for all i. If we take $b_i = \frac{a_i}{\|a_i\|}$, then $\|b_i\| = 1$ and hence by Theorem 2.1 we have

$$q_k\left(T\left(\sum_{\sigma\in S_n} b_{\sigma(1)}b_{\sigma(2)}\cdots b_{\sigma(n)}\right) - \sum_{\sigma\in S_n} Tb_{\sigma(1)}Tb_{\sigma(2)}\cdots Tb_{\sigma(n)}\right) \le \varepsilon 2^n\left(\sum_{i=1}^{n}\|b_i\|\right)^n = \varepsilon(2n)^n.$$

Hence,

$$q_k\left(T\left(\sum_{\sigma\in S_n} a_{\sigma(1)}a_{\sigma(2)}\cdots a_{\sigma(n)}\right) - \sum_{\sigma\in S_n} Ta_{\sigma(1)}Ta_{\sigma(2)}\cdots Ta_{\sigma(n)}\right) \le \varepsilon(2n)^n\prod_{i=1}^{n}\|a_i\|.$$

In the case that A and B are commutative, it follows that

$$q_k(T(a_1a_2\cdots a_n) - Ta_1Ta_2\cdots Ta_n) \le \frac{\varepsilon(2n)^n}{n!}\prod_{i=1}^{n}\|a_i\|.$$

Consequently, T is an almost n-homomorphism. □

Note that Corollary 2.3 will be extended to more general cases later on. See Theorem 2.6, Corollaries 2.7 and 2.8.

Remark 2.1 If $T : (A,(p_k)) \to (B, \|\cdot\|)$ is a weakly almost n-Jordan homomorphism for some $\varepsilon \ge 0$, then by the definition, for each $k\in\mathbb{N}$ there exists $n(k)\in\mathbb{N}$ satisfying the inequality

$$\|T(a^n) - (Ta)^n)\| \le \varepsilon(p_{n(k)}(a))^n,$$

for every $a\in A$. If $r = min\{n(k) : k\in\mathbb{N}\}$, it follows that $\|T(a^n) - (Ta)^n)\| \le \varepsilon(p_r(a))^n$, for every $a\in A$. We use this fixed r in the following results.

A ring or an algebra A is called a domain if for every $a, b\in A$ the equality $ab = 0$ implies that either $a = 0$ or $b = 0$. If a domain is also commutative, it is called an integral domain.

Theorem 2.4 *Let $T : (A,(p_k)) \to (B, \|\cdot\|)$ be a weakly almost n-Jordan homomorphism for some $\varepsilon \ge 0$, and let r be the natural number introduced in the remark above. If A is commutative and B is an integral domain, then one of the following holds*:

(i) For every $b\in A$, if $p_r(b) = 0$ then $Tb = 0$,
(ii) T is an n-homomorphism.

Proof By Corollary 2.2 for every $a_1, a_2, \ldots, a_n \in A$ we have

$$\|T(a_1a_2\cdots a_n) - Ta_1Ta_2\cdots Ta_n\| \le \frac{\varepsilon 2^n}{n!}\left(\sum_{i=1}^{n} p_r(a_i)\right)^n. \quad (4)$$

We now follow a similar method as in the proof of Honary et al. (2017, Lemma 3.1) and apply (4) to prove the following inequalities:

For an arbitrary $z\in A$ we have

$$\begin{aligned}
&\left\|(Tz)^{n-1}(T(a_1a_2\cdots a_n) - Ta_1Ta_2\cdots Ta_n)\right\|\\
&= \left\|T(a_1a_2\cdots a_n)(Tz)^{n-1} - Ta_1Ta_2\cdots Ta_n(Tz)^{n-1}\right\|\\
&\le \left\|T(a_1a_2\cdots a_n)(Tz)^{n-1} - T(a_1a_2\cdots a_nz^{n-1})\right\|\\
&\quad + \left\|T(a_1a_2\cdots a_{n-1}(a_nz^{n-1})) - Ta_1Ta_2\cdots Ta_{n-1}T(a_nz^{n-1})\right\|\\
&\quad + \left\|Ta_1Ta_2\cdots Ta_{n-1}T(a_nz^{n-1}) - Ta_1Ta_2\cdots Ta_n(Tz)^{n-1}\right\|\\
&\le \frac{\varepsilon 2^n}{n!}\Big[(p_r(a_1a_2\cdots a_n) + (n-1)p_r(z))^n\\
&\quad + (p_r(a_1) + p_r(a_2) + \cdots + p_r(a_{n-1}) + p_r(a_nz^{n-1}))^n\\
&\quad + \|Ta_1Ta_2\cdots Ta_{n-1}\|(p_r(a_n) + (n-1)p_r(z))^n\Big]
\end{aligned}$$

Now assume that (i) does not hold. Then there exists $b\in A$ such that $p_r(b) = 0$ and $Tb \ne 0$. Hence, for an arbitrary $m\in\mathbb{N}$ we have $T(mb)\ne 0$ and $p_r(mb) = mp_r(b) = 0$. Since p_r is submultiplicative, it follows that $p_r(a_nb^{n-1}) \le p_r(a_n)p_r(b)^{n-1} = 0$. If we take $z = mb$, from the inequalities above, we conclude that

$$\begin{aligned}
&\left\|(T(mb))^{n-1}(T(a_1a_2\cdots a_n) - Ta_1Ta_2\cdots Ta_n)\right\|\\
&\le \frac{\varepsilon 2^n}{n!}\left[(p_r(a_1a_2\cdots a_n))^n.\right.\\
&\quad \left.+\left(\sum_{i=1}^{n-1} p_r(a_i)\right)^n + \|Ta_1Ta_2\cdots Ta_{n-1}\|(p_r(a_n))^n\right]
\end{aligned}$$

Therefore,

$$\left\|(Tb)^{n-1}(T(a_1a_2\cdots a_n) - Ta_1Ta_2\cdots Ta_n)\right\| \le \frac{\varepsilon 2^n\left[(p_r(a_1a_2\cdots a_n))^n + \left(\sum_{i=1}^{n-1} p_r(a_i)\right)^n + \|Ta_1Ta_2\cdots Ta_{n-1}\|(p_r(a_n))^n\right]}{n!m^{n-1}}.$$

If $m\to\infty$, it follows that

$$(Tb)^{n-1}(T(a_1a_2\cdots a_n) - Ta_1Ta_2\cdots Ta_n) = 0.$$

Since $Tb\ne 0$ and B is an integral domain, we conclude that $T(a_1a_2\cdots a_n) = Ta_1Ta_2\cdots Ta_n$. Therefore, T is an n-homomorphism. □

Remark 2.2 It is interesting to note that if we omit the commutativity property of A and B in Theorem 2.4, and take T as a weakly almost n-homomorphism, then the result is still valid. For the proof we can follow the same method as in the proof of Honary et al. (2017, Lemma 3.1), to show the following inequality:

$$\|(Tz)^{n-1}(T(a_1\cdots a_n) - Ta_1\cdots Ta_n\| \le \varepsilon p_r(a_1)p_r(z)^{n-1}(2p_r(a_2)\cdots p_r(a_n) + \|Ta_2\cdots Ta_n\|).$$

If in Theorem 2.4, (i) does not hold, then there exists $a\in A$ such that $p_r(a) = 0$ and $T(a)\ne 0$. If we take $z = a$ in the inequality above, we conclude that

$$\|(Ta)^{n-1}(T(a_1\cdots a_n)-Ta_1\cdots Ta_n)\|=0.$$

Since B is a domain it follows that T is an n-homomorphism. This result is actually an extension of Honary et al. (2017, Lemma 3.2).

When A is a Fréchet algebra (not necessarily commutative) and $T:A\to\mathbb{C}$ is a weakly almost multiplicative linear functional, the result above was proved in Honary et al. (2016, Theorem 2.7). In the special case that A is a Banach algebra and $T:A\to\mathbb{C}$ is an almost homomorphism, the continuity of T is an old result due to Jarosz (1985, Proposition 5.5). This result was also extended to more general cases in the recent years.

The following result is an extension of Omidi et al. (2020, Theorem 2.11).

Theorem 2.5 *Let $T:(A,(p_k))\to\mathbb{C}$ be a weakly almost n-Jordan homomorphism. If A is commutative then T is either continuous or it is an n-homomorphism.*

Proof By Remark 1.2, T is continuous if and only there exist $m\in\mathbb{N}$ and $M>0$ such that $|Ta|\le Mp_m(a)$ for each $a\in A$. Now assume that T is not continuous. We show that T is an n-homomorphism. By the property above for each $m\in\mathbb{N}$ there exists $c_m\in A$ satisfying the inequality $|Tc_m|>mp_r(c_m)$, where r is the natural number introduced in the Remark 2.1. We now consider two cases.

We first assume that there exists $m_0\in\mathbb{N}$ such that $p_r(c_{m_0})=0$. If we take $b=c_{m_0}$, then $|T(b)|=|Tc_{m_0}|>m_0p_r(c_{m_0})=m_0p_r(b)=0$, which implies that $Tb\neq0$. Consequently, case (i) in Theorem 2.4 does not occur and hence T is an n-homomorphism.

If $p_r(c_m)\neq0$ for every $m\in\mathbb{N}$, then $|Tb_m|>m$ for all $m\in\mathbb{N}$, where $b_m=\frac{c_m}{p_r(c_m)}$ and $p_r(b_m)=1$. By the proof of Theorem 2.4, for every $a_1,a_2,\ldots,a_n\in A$, we have

$$\begin{aligned}&|T(b_m)^{n-1}||T(a_1a_2\cdots a_n)-Ta_1Ta_2\cdots Ta_n|\\&\quad\le\frac{\varepsilon2^n}{n!}\Big[(p_r(a_1a_2\cdots a_n)+(n-1))^n\\&\qquad+(p_r(a_1)+p_r(a_2)+\cdots+p_r(a_{n-1})+p_r(a_n))^n\\&\qquad+|Ta_1Ta_2\cdots Ta_{n-1}||(p_r(a_n)+n-1)^n\Big]\\&\quad=\frac{\varepsilon2^n}{n!}L_n=M_n.\end{aligned}$$

Therefore,

$$|T(a_1a_2\cdots a_n)-Ta_1Ta_2\cdots Ta_n|\le\frac{M_n}{|T(b_m)|^{n-1}}\le\frac{M_n}{m^{n-1}}.$$

Since $\frac{M_n}{m^{n-1}}\underset{m\to\infty}{\longrightarrow}0$, it follows that

$$T(a_1a_2\cdots a_n)=Ta_1Ta_2\cdots Ta_n.$$

□

Remark 2.3 It is interesting to note that the theorem above is still valid if A is not necessarily commutative and $T:(A,(p_k))\to\mathbb{C}$ is a weakly almost n-homomorphism. For the proof one can use the inequality in Remark 2.2 and proceed with a similar argument as in the theorem above. This result also appeared in Honary et al. (2017, Theorem 3.3), but our method is rather different from Honary et al. (2017) and yields a shorter proof. Note that this result is also an extension of Honary et al. (2016, Theorem 2.7).

In the case that A is a Banach algebra and B is a semisimple commutative Fréchet algebra, every weakly almost n-homomorphism $T:A\to B$ is automatically continuous (Honary et al. 2017, Theorem 3.10). Moreover, for $n=2$ this result is still valid if A is a functionally continuous Fréchet algebra (Honary et al. 2016, Theorem 2.9).

However, it is not yet known whether Theorem 2.5 is still valid if $\mathbb{C}$ is replaced by a normed algebra B, which is an integral domain.

We now present the following *key result*, which is the extension of Corollary 2.3 and will be very useful in the sequel.

Theorem 2.6 *If $T:(A,(p_k))\to(B,(q_k))$ is an almost n-Jordan homomorphism for some $\varepsilon\ge0$, then the following inequality holds for each $k\in\mathbb{N}$ and for every $a_1,a_2,\ldots,a_n\in A$:*

$$q_k\left(T\left(\sum_{\sigma\in S_n}a_{\sigma(1)}a_{\sigma(2)}\cdots a_{\sigma(n)}\right)-\sum_{\sigma\in S_n}T(a_{\sigma(1)})T(a_{\sigma(2)})\cdots T(a_{\sigma(n)})\right)\le\varepsilon(2n)^n\prod_{i=1}^n p_k(a_i).$$

Proof Three cases may occur. Let $k\in\mathbb{N}$ be an arbitrary element.

Case 1. For every i $(1\le i\le n)$, $p_k(a_i)\neq0$. By Theorem 2.1, we have

$$q_k\left(T\left(\sum_{\sigma\in S_n}\frac{a_{\sigma(1)}}{p_k(a_{\sigma(1)})}\frac{a_{\sigma(2)}}{p_k(a_{\sigma(2)})}\cdots\frac{a_{\sigma(n)}}{p_k(a_{\sigma(n)})}\right)-\sum_{\sigma\in S_n}\frac{T(a_{\sigma(1)})}{p_k(a_{\sigma(1)})}\frac{T(a_{\sigma(2)})}{p_k(a_{\sigma(2)})}\cdots\frac{T(a_{\sigma(n)})}{p_k(a_{\sigma(n)})}\right)\le\varepsilon(2n)^n.$$

Hence,

$$q_k\left(T\left(\sum_{\sigma\in S_n} a_{\sigma(1)}a_{\sigma(2)}\cdots a_{\sigma(n)}\right) - \sum_{\sigma\in S_n} Ta_{\sigma(1)}Ta_{\sigma(2)}\cdots Ta_{\sigma(n)}\right) \le \varepsilon(2n)^n\prod_{i=1}^{n} p_k(a_i).$$

Case 2. We assume that there are ℓ $(1\le \ell < n)$ elements of the set $\{a_1,\ldots,a_n\}$ such that $p_k(a_i)\neq 0$ and for the other elements of $\{a_1,\ldots,a_n\}$, $p_k(a_i)=0$. Without loss of generality, we may assume that $p_k(a_i)\neq 0$ whenever $1\le i\le \ell$ and $p_k(a_i)=0$ whenever $\ell < i\le n$.

Let $m\in\mathbb{N}$ be arbitrary and take $b_i=\frac{ma_i}{p_k(ma_i)}$ for $1\le i\le \ell$ and $b_i=ma_i$ for $\ell<i\le n$. Hence $p_k(b_i)=1$ for $1\le i\le \ell$ and $p_k(b_i)=0$ for $\ell+1\le i\le n$.

By Theorem 2.1, we have

$$q_k\left(T\left(\sum_{\sigma\in S_n} b_{\sigma(1)}b_{\sigma(2)}\cdots b_{\sigma(n)}\right) - \sum_{\sigma\in S_n} Tb_{\sigma(1)}Tb_{\sigma(2)}\cdots Tb_{\sigma(n)}\right) \le \varepsilon 2^n\left(\sum_{i=1}^{\ell} p_k(b_i)+\sum_{i=\ell+1}^{n} p_k(b_i)\right)^n = \varepsilon 2^n\ell^n=\varepsilon(2\ell)^n$$

Consequently,

$$q_k\left(T\left(\sum_{\sigma\in S_n}\frac{m^n a_{\sigma(1)}a_{\sigma(2)}\cdots a_{\sigma(n)}}{m^\ell p_k(a_1)p_k(a_2)\cdots p_k(a_\ell)}\right) - \sum_{\sigma\in S_n}\frac{m^n T(a_{\sigma(1)})T(a_{\sigma(2)})\cdots T(a_{\sigma(n)})}{m^\ell p_k(a_1)p_k(a_2)\cdots p_k(a_\ell)}\right) \le \varepsilon(2\ell)^n.$$

Therefore,

$$q_k\left(T\left(\sum_{\sigma\in S_n} a_{\sigma(1)}a_{\sigma(2)}\cdots a_{\sigma(n)}\right) - \sum_{\sigma\in S_n} T(a_{\sigma(1)})T(a_{\sigma(2)})\cdots T(a_{\sigma(n)})\right) \le \frac{\varepsilon(2\ell)^n}{m^{n-\ell}}\prod_{i=1}^{\ell} p_k(a_i)$$

Since $n-\ell>0$, the right hand side of this inequality tends to zero as $m\to\infty$. Hence, the left hand side of the inequality is also zero. Since $\prod_{i=1}^{n} p_k(a_i)=0$ in this case, the required inequality holds in this case too.

Case 3. If $p_k(a_i)=0$ for every i $(1\le i\le n)$, then $\left(\sum_{i=1}^{n} p_k(a_i)\right)^n=0$. Hence by Theorem 2.1 the left hand side of the required inequality is also zero.

Therefore, the required inequality holds in all cases. □

The following interesting result is an extension of Omidi et al. (2020, Theorem 2.8)

Corollary 2.7 *If $(A,(p_k))$ and $(B,(q_k))$ are commutative, then every almost n-Jordan homomorphism $T:(A,(p_k))\to(B,(q_k))$ is an almost n-homomorphism.*

Proof By the commutativity property of A and B, the following inequality holds by the theorem:

$$q_k(T(a_1a_2\cdots a_n)-Ta_1Ta_2\cdots Ta_n)\le\frac{\varepsilon(2n)^n}{n!}\prod_{i=1}^{n}p_k(a_i),$$

for all $a_1,a_2,\ldots,a_n\in A$. Hence T is an almost n-homomorphism. □

The following result is also an easy consequence of the theorem above.

Corollary 2.8 *If $T:(A,(p_k))\to(B,(q_k))$ is a weakly almost n-Jordan homomorphism, then for every $k\in\mathbb{N}$ there exists $n(k)\in\mathbb{N}$ such that*

$$q_k\left(T\left(\sum_{\sigma\in S_n} a_{\sigma(1)}a_{\sigma(2)}\cdots a_{\sigma(n)}\right) - \sum_{\sigma\in S_n} T(a_{\sigma(1)})T(a_{\sigma(2)})\cdots T(a_{\sigma(n)})\right) \le \varepsilon(2n)^n\prod_{i=1}^{n} p_{n(k)}(a_i),$$

for all $a_1,a_2,\ldots,a_n\in A$. In particular, if A and B are commutative, then T is a weakly almost n-homomorphism.

Remark 2.4

1. i) Since the topology of Fréchet algebras can be generated by sequences of separating submultiplicative seminorms, Corollaries 2.7 and 2.8 are valid whenever A and B are commutative Fréchet algebras.
2. (ii) As a consequence of Corollaries 2.7 and 2.8, if $(A,(p_k))$ and $(B,(q_k))$ are commutative then a map $T:(A,(p_k))\to(B,(q_k))$ is a (weakly) almost n-Jordan homomorphism if and only if it is a (weakly) almost n-homomorphism. Hence, for the continuity of (weakly) almost n-Jordan homomorphisms between commutative topological algebras $(A,(p_k))$ and $(B,(q_k))$, it is enough to consider the continuity of (weakly) almost n-homomorphisms between $(A,(p_k))$ and $(B,(q_k))$.

By the remark above and Honary et al. (2017, Theorem 3.10), we conclude the following result, which is an extension of Zivari-Kazempour (2020b, Theorem 11).

Theorem 2.9 *Let A be a commutative Banach algebra and $(B,(q_k))$ be a semisimple commutative Fréchet algebra. If $T:A\to(B,(q_k))$ is a weakly almost n-Jordan homomorphism, then it is automatically continuous.*

Note that if we drop the commutativity of A in the theorem above, but we assume that T is a weakly almost n-homomorphism, the result is still valid, as it is shown in Honary et al. (2017, Theorem 3.10). However, these results are extensions of Zivari-Kazempour (2020b, Theorem 7) and Shayanpour et al. (2016, Theorem 2.4). Moreover, the following result is an extension of Honary et al. (2017, Theorem 3.10).

Theorem 2.10 *Let $(A, (p_k))$ be a functionally continuous Fréchet algebra and $(B, (q_k))$ be a semisimple commutative Fréchet algebra. If $T : (A, (p_k)) \to (B, (q_k))$ is a weakly almost n- homomorphism, then it is automatically continuous. Moreover, if A is commutative then every weakly almost n-Jordan homomorphism $T : (A, (p_k)) \to (B, (q_k))$ is automatically continuous.*

Proof Let $\varphi \in M_B$, where M_B is the class of all continuous multiplicative linear functionals on B. By Honary et al. (2017, Corollary 3.8) $\varphi \circ T : (A, (p_k)) \to \mathbb{C}$ is a weakly almost n-multiplicative linear functional on A, and since A is functionally continuous, it is continuous by Honary et al. (2017, Corollary 3.6). Now we follow the same method as Honary et al. (2017, Theorem 3.10) to conclude that T is automatically continuous. Note that the equality $radB = \bigcap\{\ker\varphi : \varphi \in M_B\}$ is also valid for commutative Fréchet algebras.

In the case that A and B are commutative and $T : (A, (p_k)) \to (B, (q_k))$ is a weakly almost n-Jordan homomorphism, T is a weakly almost n-homomorphism by Corollary 2.8 and hence, it is automatically continuous. □

3 Certain Properties of *n*-Jordan Homomorphisms on Rings

In this section we present two results concerning additive n-Jordan homomorphisms on rings. For further properties of n-Jordan homomorphisms and n-homomorphisms on rings and algebras, one may refer to Honary (2022) and Honary et al. (2021).

Definition 3.1 For the rings A and B, a mixed n-Jordan homomorphism is an additive map $T : A \to B$ such that $T(a^n b) = (Ta)^n Tb$, for all $a, b \in A$.

The notion of mixed n-Jordan homomorphism was first introduced in Neghabi et al. (2020).

The following result is an extension of Zivari-Kazempour (2020a, Theorem 3.3).

Theorem 3.1 *Let $T : A \to B$ be an additive n-Jordan homomorphism from a ring A into a ring B such that $char\, B > n$. If $T(A)$ is commutative and $T(ab) = T(ba)$ for every $a, b \in A$, then T is a mixed $(n-1)$-Jordan homomorphism.*

Proof By Theorem 1.4, we have

$$T\left(a^{n-1}b + a^{n-2}ba + \cdots + aba^{n-2} + ba^{n-1}\right) = nT(a)^{n-1}T(b),$$

for every $a, b \in A$.

On the other hand, from the hypotheses we conclude that for every $0 \le i, j \le n-1$, where $i + j = n-1$, we have

$$T\left(a^i b a^j\right) = T\left(a^{n-1}b\right) = T\left(ba^{n-1}\right),$$

and

$$(Ta)^i T(b)(Ta)^j = (Ta)^{n-1}Tb = Tb(Ta)^{n-1},$$

for all $a, b \in A$.

Since A and B may not have identities, as a convention we assume that, $a^0 b = ba^0 = b$ and $(Ta)^0 c = c(Ta)^0 = c$ for every $a, b \in A$ and every $c \in B$. Consequently,

$$nT(a^{n-1}b) = n(Ta)^{n-1}Tb,$$

for every $a, b \in A$. Since $char\, B > n$, B is of characteristic different from n and hence, $T(a^{n-1}b) = (Ta)^{n-1}Tb$, that is, T is a mixed $(n-1)$–Jordan homomorphism. □

The next result is an extension of Zivari-Kazempour (2020a, Theorem 3.4).

Theorem 3.2 *Suppose that A and B are rings, where $char\, B > n$ and $T : A \to B$ is an additive n-Jordan homomorphism such that $T(A)$ is commutative and $T(ab) = T(ba)$ for every $a, b \in A$. Then T is an n-homomorphism if $\ker T$ is an ideal in A.*

Proof By the hypothesis, for every $a_1, a_2, b \in A$ we have $a_1 b - ba_1 \in \ker T$ and since $\ker T$ is an ideal, $a_2 a_1 b - a_2 b a_1 \in \ker T$. On the other hand, $T(a_2 a_1 b) = T(a_1 b a_2)$. Hence $a_1 b a_2 - a_2 b a_1 \in \ker T$ for every $a_1, a_2, b \in A$. It follows that for all $a_1, a_2, \ldots, a_i, \ldots, a_j, \ldots, a_n \in A$, where $1 \le i, j \le n$. we have $T(a_i a_{i+1} \cdots a_{j-1} a_j) = T(a_j a_{i+1} \cdots a_{j-1} a_i)$, $(i + 1 < j)$.

Since $\ker T$ is an ideal we conclude that

$$\left(a_1 a_2 \cdots a_{i-1}\right)\left(a_i a_{i+1} \cdots a_{j-1} a_j - a_j a_{i+1} \cdots a_{i-1} a_i\right)\left(a_{j+1} \cdots a_n\right) \in \ker T.$$

Hence

$$T\left(a_1 a_2 \cdots a_{i-1} a_i a_{i+1} \cdots a_{j-1} a_j a_{j+1} \cdots a_n\right) = T\left(a_1 a_2 \cdots a_{i-1} a_j a_{i+1} \cdots a_{j-1} a_i a_{j+1} \cdots a_n\right).$$

In fact, we have shown that if we substitute a_i by a_j in the product $a_1 a_2 \cdots a_i \cdots a_j \cdots a_n$, then $T\left(\prod_{i=1}^n a_i\right)$ does not change. Therefore, if $\sigma \in S_n$ is an arbitrary permutation, then $T\left(a_{\sigma(1)}a_{\sigma(2)}\cdots a_{\sigma(n)}\right) = T(a_1a_2\cdots a_n)$ for every $a_1, a_2, \ldots, a_n \in A$. By Theorem 1.4, we have

$$T\left(\sum_{\sigma\in S_n} a_{\sigma(1)}a_{\sigma(2)}\cdots a_{\sigma(n)}\right) = \sum_{\sigma\in S_n} Ta_{\sigma(1)}Ta_{\sigma(2)}\cdots Ta_{\sigma(n)}.$$

By the argument above, $T\left(\sum_{\sigma\in S_n} a_{\sigma(1)} \quad a_{\sigma(2)}\cdots a_{\sigma(n)}\right) = n!T(a_1a_2\cdots a_n)$. Since $T(A)$ is commutative we have $\sum_{\sigma\in S_n} Ta_{\sigma(1)}Ta_{\sigma(2)}\cdots Ta_{\sigma(n)} = n!Ta_1Ta_2\cdots Ta_n$. Since $char\, B > n$, it follows that $T(a_1a_2\cdots a_n) = Ta_1Ta_2\cdots Ta_n$, that is, T is an n-homomorphism. □

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References

Bodaghi A, Inceboz H (2018) n-Jordan homomorphisms on commutative algebras. Acta Math Univ Comen 87(1):141–146

Cheshmavar J, Hosseini SK, Park C (2020) Some results on n-Jordan homomorphisms. Bull Korean Math Soc 57(1):31–35. https://doi.org/10.4134/BKMS.b180719

Fragoulopoulou M (2005) Topological algebras with involution. Elsevier, Amsterdam

Goldmann H (1990) Uniform Fréchet algebras. Elsevier Science Publishers, North Holland

Gselmann E (2014) On approximate n-Jordan homomorphisms. Ann Math Sil 28:47–58

Hejazian S, Mirzavaziri M, Moslehian MS (2005) n-Homomorphisms. Bull Iran Math Soc 31(1):13–23

Herstein IN (1956) Jordan homomorphisms. Trans Am Math Soc 81:331–341

Honary TG (2022) Certain properties of Jordan homomorphisms, n-Jordan homomorphisms and n-homomorphisms on rings and Banach algebras. Quaest Math 45(2):173–183. https://doi.org/10.2989/16073606.2020.1850540

Honary TG, Omidi M, Sanatpour AH (2015) Automatic continuity of almost multiplicative maps between Fréchet algebras. Bull Iran Math Soc 41(6):1497–1509

Honary TG, Omidi M, Sanatpour AH (2016) Automatic continuity of almost multiplicative linear functionals on Fréchet algebras. Bull Korean Math Soc 53(3):641–957

Honary TG, Omidi M, Sanatpour AH (2017) Almost n-multiplicative maps between Fréchet algebras. Int J Nonlinear Anal Appl 8(1):187–195

Honary TG, Hosseinzadeh H, Mohammadi S (2021) Relations between n-Jordan homomorphisms and n-homomorphisms. Bull Iran Math Soc 47(3):689–700. https://doi.org/10.1007/s41980-020-00407-4

Jarosz K (1985) Perturbations of Banach Algebras, vol 1120. Lecture Notes in Mathematics. Springer, Berlin

Kulkarni SH, Sukumar D (2010) Almost multiplicative functions on commutative Banach algebras. Stud Math 197(1):93–99

Neghabi M, Bodaghi A, Zivari-Kazempour A (2020) Characterization of mixed n-Jordan and pseudo n-Jordan homomorphisms. Filomat 34(6):1989–2002. https://doi.org/10.2298/FIL2006989N

Omidi MR, Oulfatian B, Zivari-Kazemour A (2020) Almost Jordan homomorphisms on Fréchet algebras. Quaest Math 43(10):1467–1476

Šemrl P (2002) Almost multiplicative functions and almost linear multiplicative functionals. Aequ Math 63:180–192

Shayanpour H, Ansari-Piri E, Heidarpour Z, Zohri A (2016) Approximately n-multiplicative functionals on Banach algebras. Mediterr J Math 13:1907–1920

Zivari-Kazempour A (2020a) Characterization of n-Jordan homomorphisms and automatic continuity of 3-Jordan homomorphisms on Banach algebras. Iran J Sci Technol Trans Sci 44:213–218. https://doi.org/10.1007/s40995-019-00814-1

Zivari-Kazempour A (2020b) Automatic continuity of almost 3-homomorphisms and almost 3-Jordan homomorphisms. Adv Oper Theory 5:1340–1349. https://doi.org/10.1007/s43036-020-00043-z