



# The Influence of Viscosity on the Growth Rate of Rayleigh–Taylor Instability

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## Abstract

Nowadays, studying of hydrodynamic instabilities such as Rayleigh–Taylor instability, Richtmyer–Meshkov instability and Kelvin–Helmholtz instability is of great importance to scientists. These instabilities have plenty of applications in fluid mechanics, meteorology, astronomy and inertial confinement fusion (ICF). They play an important role in many physical phenomena. These instabilities are considered as negative factors in fusion reactions and play a significant role in decreasing the rate of ICF reactions. Among hydrodynamic instabilities, Rayleigh–Taylor instability is of particular importance. Different factors affect the growth rate of this instability. One of them is the viscosity effect of fluid molecules that reduces the growth rate of this instability and increases the rate of ICF reactions. In this research, which is a new idea and has not been presented or published anywhere up to now, the mass and the linear momentum conservation equations, known as Euler equations, are solved analytically by considering the viscosity effect of fluid molecules. In this way, by using boundary conditions and solving nonlinear equations, the growth rate of Rayleigh–Taylor instability as a function of the perturbation wave number is obtained at the interface of two fluids with different densities. This quantity is an important criterion for measuring the growth of fluid disturbances.

**Keywords** Hydrodynamic instability · Rayleigh–Taylor instability · Viscosity effect · Growth rate · Atwood number

## 1 Introduction

Rayleigh–Taylor instability is the most important type of hydrodynamic instabilities studied by Lord Rayleigh (1883) and G. I. Taylor (Asthana et al. 2012, 1950). When a heavy fluid is on the light one in a gravitational field or when a lower density fluid accelerates the dense one, the Rayleigh–Taylor instability occurs at the interface of these fluids (Atzeni and Meyer-ter-Vehn 2004). This instability is initially as small perturbations at the interface of fluids and then grows exponentially with time as bubbles from lighter fluid to heavier one and also as spikes from heavier fluid to lighter one (Banerjee et al. 2013). Finally, there will be two disturbed fluids, and the growth of perturbations enters the nonlinear phase (Abarzhi et al. 2003).

In fluid mechanics, from the stand point of thermodynamics and statistical mechanics, this instability happens, while a density gradient is in the opposite direction to the pressure gradient (Eliezer 2002). In a special case, the condition for the occurrence of RTI in terms of entropy is expressed as  $\vec{\nabla}s \cdot \vec{\nabla}p > 0$  in which  $s$  is specific entropy. But when  $\vec{\nabla}s \cdot \vec{\nabla}p < 0$ , the perturbations at the interface of fluids oscillate sinusoidally with finite amplitudes, which is called steady state. On the other hand, in an incompressible fluid, the specific entropy can be expressed as  $s = s_0 + c_V \ln(p/\rho^\gamma)$ . Therefore, according to the above-mentioned equation, RTI at the interface of two fluids occurs when two fluids with different densities experience a pressure gradient in the opposite direction of the density gradient as  $\vec{\nabla}\rho \cdot \vec{\nabla}p < 0$  (Drake 2018).

viscosity is the most important property in the study of fluid flows (Silveira and Orlandi 2017). It is a property of fluid by which the fluid resists to cutting, so that the viscosity of gases increases with temperature and the viscosity of liquids decreases with it. Air and water have little viscosity. The resistance of a fluid to cutting depends on its

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coherence and its momentum transfer (Kim and Kim 2016). The molecules of liquids are closer to each other than gases; as a result, their cohesive forces are larger than gases. Therefore, it seems that coherence is the main reason of liquid viscosity and as its coherence decreases with temperature, the viscosity also reduces with it (Gerashchenko and Livescu 2016). It should also be noted that viscosity at normal pressures is independent of pressure and depends only on temperature. But at high pressures, the viscosity of gases and especially liquids changes irregularly with pressure (Banerjee et al. 2011). The consequences obtained from experimental works and simulations show that the influence of viscosity on RTI may severely quench the growth rate of this instability (Park et al. 2010), so that the calculations performed in this paper confirm these results.

## 2 Euler Equations and the Effects of Fluid Viscosity

The hydrodynamic equations describing fluid motion in the study of hydrodynamic instabilities are mass and linear momentum conservation equations which are called Euler equations and they will be as follows (Drake 2018):

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{u}) = 0 \tag{1}$$

$$\rho \frac{\partial \vec{u}}{\partial t} + \rho (\vec{u} \cdot \vec{\nabla}) \vec{u} = -\vec{\nabla} P + \vec{\nabla} \cdot \underline{\sigma}_v + \rho \vec{g} \tag{2}$$

In these equations,  $\rho$  is the fluid density,  $\mathbf{u}$  is the flow velocity,  $P$  is pressure, and  $\rho \vec{g}$  is the external gravitational force applied to fluid. In this paper, the Cartesian coordinate system is shown with unit vectors  $\hat{x}$ ,  $\hat{y}$  and  $\hat{z}$ . Also, the fluid system is defined as having the interface on the  $xy$  plane, and the gravitational field (acceleration) is normal to the interface of fluids. So we have:

$$\vec{g} = -g \hat{z} \tag{3}$$

In the analysis of hydrodynamic instabilities such as Rayleigh–Taylor instability, we neglect the force of surface tension. Because of the high temperature in high energy density systems, there are no molecular interactions that generate this force. We also ignore radiation pressure, electromagnetic fields, and the other forces, but hold the viscous force to look for viscosity effects. So, according to the above statements, it can be said that the viscosity tensor shown by  $\underline{\sigma}_v$  has the following elements:

$$\sigma_{vij} = \rho v \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3} \delta_{ij} \frac{\partial u_k}{\partial x_k} \right) + \zeta \delta_{ij} \frac{\partial u_k}{\partial x_k} \tag{4}$$

in which  $\delta_{ij}$  is the Kronecker delta function,  $v$  is the kinematic viscosity, and  $\zeta$  is the second coefficient of

viscosity which is not important here, because the sentences containing it will be deleted in the divergence operation. The sentence containing the viscosity tensor is substantially simplified, because due to the condition of fluid incompressibility, we will have  $\partial u_i / \partial x_i = 0$ . We also assume that the only non-zero derivative of  $v$  is  $dv/dz (= dv/dx_3)$ . Eventually with all of the assumptions, we have made, the  $k$  component of  $\vec{\nabla} \cdot \underline{\sigma}_v$  will be as follows:

$$\left( \vec{\nabla} \cdot \underline{\sigma}_v \right)_k = \rho v \nabla^2 u_k + \frac{\partial(\rho v)}{\partial x_3} \left( \frac{\partial u_k}{\partial x_3} + \frac{\partial u_3}{\partial x_k} \right) \tag{5}$$

Our aim is to linearize the fluid, so that the unperturbed pressure and density are considered as  $P = P(z)$  and  $\rho = \rho(z)$ . Also, the first-order perturbations of pressure and density are assumed as  $\delta P(x, y, z)$  and  $\delta \rho(x, y, z)$ , respectively. It should be noted that we are working in a framework where the interface is stationary, so the zero order of flow velocity (unperturbed velocity) equals to zero and the first order of the flow velocity as a function of  $x, y, z$ , is  $\vec{u}(x, y, z)$ . We also consider the perturbations to be incompressible. Of course, it should be noted that the fluid itself may not need to be incompressible, which this assumption follows as  $\vec{\nabla} \cdot \vec{u} = 0$ . Finally, with the assumptions we made, we can now obtain the first-order continuity equation as follows:

$$\frac{\partial(\delta \rho)}{\partial t} + \vec{u} \cdot \vec{\nabla} \rho = 0 \tag{6}$$

Also, since the expression  $\rho (\vec{u} \cdot \vec{\nabla}) \vec{u}$  in the left side of Eq. (2) is removed due to the incompatibility assumption, the first-order momentum equation will be:

$$\rho \frac{\partial \vec{u}}{\partial t} = -\vec{\nabla}(\delta P) + \vec{\nabla} \cdot \underline{\sigma}_v - g \delta \rho \hat{z} \tag{7}$$

We define the  $x, y$  and  $z$  components of the flow velocity as  $u, v$  and  $\omega$ . Consequently, if we include the expression obtained from Eq. (5) in our calculations, Eq. (7) can be written as three equations in terms of its components as follows:

$$\rho \frac{\partial u}{\partial t} = -\frac{\partial}{\partial x}(\delta P) + \rho v \nabla^2 u + \frac{\partial(\rho v)}{\partial z} \left( \frac{\partial u}{\partial z} + \frac{\partial \omega}{\partial x} \right) \tag{8}$$

$$\rho \frac{\partial v}{\partial t} = -\frac{\partial}{\partial y}(\delta P) + \rho v \nabla^2 v + \frac{\partial(\rho v)}{\partial z} \left( \frac{\partial v}{\partial z} + \frac{\partial \omega}{\partial y} \right) \tag{9}$$

$$\rho \frac{\partial \omega}{\partial t} = -\frac{\partial}{\partial z}(\delta P) + \rho v \nabla^2 \omega + 2 \frac{\partial(\rho v)}{\partial z} \left( \frac{\partial \omega}{\partial z} \right) - g \delta \rho \tag{10}$$

Also, the compressibility condition and the  $z$  component of Eq. (6) can be written as follows:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \omega}{\partial z} = 0 \quad (11)$$

$$\frac{\partial}{\partial t}(\delta\rho) = -\omega \frac{\partial \rho}{\partial z} \quad (12)$$

The last five equations are the ones that describe the linear phase of the Rayleigh–Taylor instability. We are looking for the waves that show the surface modulation modes and so amplitudes with indefinite changes with  $\mathbf{z}$ , but are proportional to  $\exp(ik_x x + ik_y y + nt)$  with respect to  $\mathbf{x}$ ,  $\mathbf{y}$  and  $\mathbf{t}$ , so that  $\mathbf{k}_x$  and  $\mathbf{k}_y$  are the  $\mathbf{x}$  and  $\mathbf{y}$  components of wave vector and  $n$  is the exponential growth rate of instability. Therefore, with substitution of  $\exp(ik_x x + ik_y y + nt)$  statement in the amplitude of all modulations including velocity, density and pressure and by simplifying them, we will get the following equations:

$$\rho u n = -ik_x(\delta P) + \rho v \left( \frac{\partial^2}{\partial z^2} - k^2 \right) u + \frac{\partial(\rho v)}{\partial z} \left( \frac{\partial u}{\partial z} + ik_x \omega \right) \quad (13)$$

$$\rho v n = -ik_y(\delta P) + \rho v \left( \frac{\partial^2}{\partial z^2} - k^2 \right) v + \frac{\partial(\rho v)}{\partial z} \left( \frac{\partial v}{\partial z} + ik_y \omega \right) \quad (14)$$

$$\rho \omega n = -\frac{\partial}{\partial z}(\delta P) + \rho v \left( \frac{\partial^2}{\partial z^2} - k^2 \right) \omega + 2 \frac{\partial(\rho v)}{\partial z} \left( \frac{\partial \omega}{\partial z} \right) - g \delta P \quad (15)$$

$$ik_x u + ik_y v = -\frac{\partial \omega}{\partial z} \quad (16)$$

$$n \delta \rho = -\omega \frac{\partial \rho}{\partial z} \quad (17)$$

In the above equations, we will have  $k^2 = k_x^2 + k_y^2$ . By multiplying Eq. (13) in  $-ik_x$  and also by multiplying Eq. (14) in  $-ik_y$  and then by summing the two obtained equations and by using Eq. (16) to simplify them, we will attain:

$$\rho n \frac{\partial \omega}{\partial z} = -k^2 \delta P + \rho v \left( \frac{\partial^2}{\partial z^2} - k^2 \right) \frac{\partial \omega}{\partial z} + \frac{\partial(\rho v)}{\partial z} \left( \frac{\partial^2}{\partial z^2} + k^2 \right) \omega \quad (18)$$

Also, by using Eq. (17) for removing  $\delta P$  from Eq. (15), we will obtain:

$$\rho \omega n = -\frac{\partial}{\partial z}(\delta P) + \rho v \left( \frac{\partial^2}{\partial z^2} - k^2 \right) \omega + 2 \frac{\partial(\rho v)}{\partial z} \left( \frac{\partial \omega}{\partial z} \right) + \omega \frac{g \partial \rho}{n \partial z} \quad (19)$$

Since all quantities in the above two equations except  $\delta P$  are non-perturbed quantities, therefore, by removing it, we can obtain an equation in terms of the parameters specified for  $\omega$  as follows:

$$\begin{aligned} & \frac{\partial}{\partial z} \left[ -\rho n \frac{\partial \omega}{\partial z} + \rho v \left( \frac{\partial^2}{\partial z^2} - k^2 \right) \frac{\partial \omega}{\partial z} + \frac{\partial(\rho v)}{\partial z} \left( \frac{\partial^2}{\partial z^2} + k^2 \right) \omega \right] \\ & = -k^2 \left[ \rho \omega n + \rho v \left( \frac{\partial^2}{\partial z^2} - k^2 \right) \omega + 2 \frac{\partial(\rho v)}{\partial z} \left( \frac{\partial \omega}{\partial z} \right) + \omega \frac{g \partial \rho}{n \partial z} \right] \end{aligned} \quad (20)$$

This equation with boundary conditions provides the conditions we need to calculate the growth rate which can be applied in both sides of the interface. Boundary conditions play a central role in calculations. A clear boundary condition is that the fluids must remain in contact with each other, and this requires  $\omega$  to be continuous across the interface. So we can integrate from fundamental equations along the boundary to find boundary conditions.

### 3 Useful Integrals for Obtaining Boundary Conditions

Suppose the interface of fluids is in the  $\mathbf{z} = \mathbf{0}$  plane. Consider that the area with  $z < 0$  is denoted by subscript 1 and the area with  $z > 0$  is denoted by subscript 2. We assume that  $q(z)$  is an arbitrary function that is continuous and derivable everywhere and  $f(z)$  and  $h(z)$  are also arbitrary functions that are continuous and derivable everywhere except at the interface. So we can write  $f(z)$  as follows  $f(z) = f_1(z) H(-z) + f_2(z) H(z)$ , where  $H(z)$  is a step function that equals to zero for  $z < 0$  and equals to 1 for  $z > 0$  and  $\mathbf{f}_1$  and  $\mathbf{f}_2$  are continuous and derivable functions. After integrating over a small area across the interface, when the width of the desired area goes to zero, we take the limit of the integral over a small area across the interface. With these assumptions, the following integrals are used to obtain the boundary conditions:

$$\lim_{\varepsilon \rightarrow 0} \int_{-\varepsilon}^{\varepsilon} f(z) dz = \lim_{\varepsilon \rightarrow 0} \left[ \varepsilon f_2 \left( \frac{\varepsilon}{2} \right) - \varepsilon f_1 \left( \frac{\varepsilon}{2} \right) \right] = 0 \quad (21)$$

$$\lim_{\varepsilon \rightarrow 0} \int_{-\varepsilon}^{\varepsilon} \left( \frac{\partial f(z)}{\partial z} \right) dz = f_2(0) - f_1(0) \quad (22)$$

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_{-\varepsilon}^{\varepsilon} q(z) \frac{\partial f(z)}{\partial z} dz &= q_0 (f_2 - f_1)_{z=0} + \lim_{\varepsilon \rightarrow 0} \int_{-\varepsilon}^{\varepsilon} \varepsilon \frac{\partial q}{\partial z} \frac{\partial f}{\partial z} dz \\ &= q_s (f_2 - f_1)_{z=0} \end{aligned} \quad (23)$$

$$\lim_{\varepsilon \rightarrow 0} \int_{-\varepsilon}^{\varepsilon} f(z) \frac{\partial q(z)}{\partial z} dz = q_0(f_2 - f_1)_{z=0} - \lim_{\varepsilon \rightarrow 0} \int_{-\varepsilon}^{\varepsilon} q(z) \frac{\partial f(z)}{\partial z} dz = 0 \quad (24)$$

In the above equations, the subscript  $z = 0$  shows that quantities must be calculated as they approach the interface.

#### 4 The Boundary Conditions Used in Linear Analysis of Rayleigh–Taylor Instability

By applying Eqs. (21)–(24) in Eqs. (18) and (19), we can find that  $\omega$  and its derivatives are continuous across the interface, and we will obtain the following equations:

$$\left[ \rho_2 v_2 \left( \frac{\partial^2}{\partial z^2} + k^2 \right) \omega_2 = \rho_1 v_1 \left( \frac{\partial^2}{\partial z^2} + k^2 \right) \omega_1 \right]_{z=0} \quad (25)$$

$$(\delta P_2 - \delta P_1)_{z=0} = 2 \left[ (\rho_2 v_2 - \rho_1 v_1) \frac{\partial \omega_1}{\partial z} \right]_{z=0} + \omega_0 \frac{g}{n} (\rho_2 - \rho_1)_{z=0} \quad (26)$$

In Eq. (26),  $\omega_0$  is the value of  $\omega$  at the interface. This equation is particularly necessary to analyze the Rayleigh–Taylor instability, but we must eliminate the pressure difference to obtain an equation that contains only one indefinite function. To do this case, we can attain another condition involving  $\delta P$  for the pressure by subtracting Eq. (18) from itself across the interface:

$$\begin{aligned} k^2(\delta P_2 - \delta P_1) &= \rho_1 \left[ n - v_1 \left( \frac{\partial^2}{\partial z^2} - k^2 \right) \right] \\ &\frac{\partial \omega_1}{\partial z} - \rho_2 \left[ n - v_2 \left( \frac{\partial^2}{\partial z^2} - k^2 \right) \right] \frac{\partial \omega_2}{\partial z} \\ &+ \frac{\partial(\rho_2 v_2)}{\partial z} \left( \frac{\partial^2}{\partial z^2} + k^2 \right) \omega_2 - \frac{\partial(\rho_1 v_1)}{\partial z} \left( \frac{\partial^2}{\partial z^2} + k^2 \right) \omega_1 \end{aligned} \quad (27)$$

By combining the above equation with Eq. (26) and eliminating the pressure, another boundary condition is obtained as follows:

$$\begin{aligned} &2k^2 \left[ (\rho_2 v_2 - \rho_1 v_1) \frac{\partial \omega_1}{\partial z} \right]_{z=0} + k^2 \omega_0 \frac{g}{n} (\rho_2 - \rho_1)_{z=0} \\ &= \rho_1 \left[ n - v_1 \left( \frac{\partial^2}{\partial z^2} - k^2 \right) \right] \frac{\partial \omega_1}{\partial z} - \rho_2 \left[ n - v_2 \left( \frac{\partial^2}{\partial z^2} - k^2 \right) \right] \frac{\partial \omega_2}{\partial z} \end{aligned} \quad (28)$$

For problems involving viscosity, another boundary condition is that the first derivative of  $\omega$  means  $\partial \omega / \partial z$ , just as the other derivatives of  $u$ , must remain continuous across the interface. The microscopic interactions of the

particles that create the viscosity insure this condition. So the boundary conditions we have to work with include Eqs. (25) and (28), continuity of  $\omega$  across the interface, continuity of  $\partial \omega / \partial z$  along the interface for viscous flows and any further restrictions inflicted by the geometry of the problem.

#### 5 Rayleigh–Taylor Instability in Two Uniform Fluids without Viscosity

The simplest instance we can assume in the linear analysis of Rayleigh–Taylor instability is a two-fluid identical system with a boundary at  $z = 0$  which has no viscosity. In this case for uniform fluids, the phrase  $\partial \rho / \partial z$  equals to zero and Eq. (20) is as follows:

$$\frac{\partial^2 \omega}{\partial z^2} = k^2 \omega \quad (29)$$

Since at great distances, the fluid must be unperturbed, that is, when  $z \rightarrow \pm \infty$ , then  $\omega \rightarrow 0$ , therefore, the solutions to Eq. (29) are as follows:

$$\begin{cases} \omega_1 = \omega_0 \exp(kz) & \text{for } z < 0 \\ \omega_2 = \omega_0 \exp(-kz) & \text{for } z > 0 \end{cases} \quad (30)$$

So  $\omega_0$  is the same at both solutions because  $\omega$  must be continuous at the interface. But the obtained differential equation just gives us the velocity profile, but it does not achieve the growth rate. So, in order to find the growth rate, we must use Eq. (28) as follows:

$$\omega_0 \frac{g}{n} (\rho_2 - \rho_1) = \frac{n}{k^2} (\rho_2 + \rho_1) k \omega_0 \quad (31)$$

$$n = \sqrt{\frac{\rho_2 - \rho_1}{\rho_2 + \rho_1}} k g = \sqrt{A_t k g} \quad (32)$$

In Eq. (32),  $A_t = (\rho_2 - \rho_1) / (\rho_2 + \rho_1)$  is Atwood number which generally changes from -1 to 1 and measures the amount of density jump at the interface. When  $A_t$  is negative it means that the denser fluid is at the bottom and the lighter one is at the top, so in this case, the RTI growth rate is a pure imaginary value in which the perturbations are only oscillating but not growing. If we consider finite viscosity, such fluctuations decline and this is called steady state. But when  $A_t$  is positive it means that the denser fluid is at the top and the lighter one is at the bottom, so in this case, the growth rate of Rayleigh–Taylor instability is a pure real value in which the perturbations grow exponentially with time. Equation (32) yields the simplest conclusion for the growth rate of Rayleigh–Taylor instability, and therefore, it is often referred to as the classical growth rate of Rayleigh–Taylor instability.

### 6 Effects of Fluid Viscosity on the Growth Rate of Rayleigh–Taylor Instability

If we suppose that the viscosity of fluids at both sides of interface is non-zero and the densities and viscosity are uniform at both areas; therefore, Eq. (20) can be written as follows:

$$\left[ n + v \left( k^2 - \frac{\partial^2}{\partial z^2} \right) \right] \frac{\partial^2 \omega}{\partial z^2} = \left[ n\omega + v \left( k^2 - \frac{\partial^2}{\partial z^2} \right) \omega \right] k^2 \tag{33}$$

The above equation can be expanded and written as follows:

$$\frac{1}{k^2} \frac{\partial^4 \omega}{\partial z^4} - \left( 1 + \frac{n}{k^2 v} \right) \frac{\partial^2 \omega}{\partial z^2} = -k^2 \left( 1 + \frac{n}{k^2 v} \right) \omega + \frac{\partial^2 \omega}{\partial z^2} \tag{34}$$

By defining the parameter:

$$s = k \left( 1 + \frac{n}{k^2 v} \right)^{1/2} \tag{35}$$

And for simplicity, we consider that the viscosity of the two fluids is the same, so we can define a general answer for Eq. (34) as follows:

$$\omega(z) = A e^{sz} + B e^{-sz} + C e^{kz} + D e^{-kz} \tag{36}$$

Since when  $z \rightarrow \pm \infty$ ,  $\omega$  tends to zero, it is obvious that **A** and **C** for  $z > 0$  and also **B** and **D** for  $z < 0$  equal to zero. Also, we have four boundary conditions which include continuity of  $\omega(z)$ , continuity of  $\partial\omega/\partial z$  and Eqs. (25) and (28) which by placing this general answer in them, four equations are obtained as follows:

$$A - B + C - D = 0 \tag{37}$$

$$A s + B s + C k + D k = 0 \tag{38}$$

$$n(\rho_2 B - \rho_1 A) + 2\rho_2 v k^2(B + D) - 2\rho_1 v k^2(A + C) = 0 \tag{39}$$

$$\frac{2A_t g k^2}{n} (A + C) - k n [C(1 - A_t) + D(1 + A_t)] + 4A_t k^2 (C k + A s) v = 0 \tag{40}$$

These four equations can be written as the multiplication of a matrix and a vector as follows:

$$\begin{vmatrix} 1 & -1 & 1 & -1 \\ s & s & k & k \\ -\rho_1(n + 2vk^2) & -\rho_2(n + 2vk^2) & -2\rho_1 vk^2 & 2\rho_2 vk^2 \\ \frac{2A_t g k^2}{n} + 4A_t k^2 sv & 0 & \frac{2A_t g k^2}{n} - nk(1 - A_t) + 4A_t k^3 v & 0 \end{vmatrix} = 0 \tag{41}$$

As a result, the following equation is obtained for the amount of the above-mentioned determinant:

$$n^2 (A_t^2 k - s) + 4n A_t^2 k^2 v (k - s) + 4A_t^2 k^3 v^2 (k - s)^2 - A_t g k (k - s) = 0 \tag{42}$$

In the above-mentioned equation, by substituting  $s$  instead of  $n$ , as  $n = v(s^2 - k^2)$ , and using Eq. (35), a fifth-order polynomial of  $s$  is obtained as follows:

$$-s^5 v^2 + s^4 k v^2 A_t^2 + 2s^3 k^2 v^2 (1 - 2A_t^2) + 6s^2 k^3 v^2 A_t^2 + s(A_t g k - k^4 v^2 (1 + 4A_t^2)) - A_t k^2 g + A_t^2 k^5 v^2 = 0 \tag{43}$$

According to the above equation, it can be realized when the viscosity is zero, the growth rate of instability becomes its classical value,  $\sqrt{A_t g k}$ , while Eq. (43) can be solved explicitly with a mathematical computational program, writing it in dimensionless form would be more useful. If we compare the sentences in terms of the coefficient  $s$ , it is clear that  $g$  is proportional to  $k^3 v^2$ . So we can use the normalized wave number as  $\tilde{k} = k / (g/v^2)^{1/3}$ . Also, by comparing the third sentence with the sixth one on the left side of the equation, we conclude that  $g$  is proportional to  $s^3 v^2$  and we can use the normalized value  $\tilde{s} = s / (g/v^2)$ . By deriving Eq. (35), the corresponding normalized growth rate will be obtained as  $\tilde{n} = n / (g^2/v)^{1/3}$ . By the performed normalization, the dispersion relation is expressed as:

$$0 = -\tilde{s}^5 + \tilde{s}^4 \tilde{k} A_t^2 + 2\tilde{s}^3 \tilde{k}^2 (1 - 2A_t^2) + 6\tilde{s}^2 \tilde{k}^3 A_t^2 + \tilde{s} \tilde{k} (A_t - \tilde{k}^3 (1 + 4A_t^2)) - A_t \tilde{k}^2 + A_t^2 \tilde{k}^5 = 0 \tag{44}$$

Also, Eq. (35) can be written in a normalized form as follows:

$$\tilde{s} = \tilde{k} \left( 1 + \frac{\tilde{n}}{\tilde{k}^2} \right)^{1/2} \tag{45}$$

With substituting Eq. (45) in Eq. (44), an equation in terms of  $\tilde{n}$  will be obtained as follows:

$$\begin{aligned}
 & - (\tilde{n} + \tilde{k}^2)^{5/2} + \tilde{k} A_t^2 (\tilde{n} + \tilde{k}^2)^2 + 2 \tilde{k}^2 (\tilde{n} + \tilde{k}^2)^{3/2} (1 - 2A_t^2) \\
 & + 6 \tilde{k}^3 A_t^2 (\tilde{n} + \tilde{k}^2) + (\tilde{n} + \tilde{k}^2)^{1/2} \tilde{k} (A_t - \tilde{k}^3 (1 + 4A_t^2)) \\
 & - A_t \tilde{k}^2 + A_t^2 \tilde{k}^5 = 0
 \end{aligned}
 \tag{46}$$

In the above equation, if the Atwood number is specified, then with a computational program, the normalized growth rate can be plotted in terms of the normalized wave number as a specified curve, curves (1), as shown in Figs. 1, 2 and 3. Also, for high viscosity, Eq. (35) can be expanded as  $s \approx k(1 + n/(2k^2v))$ , so by doing this expansion and putting it in Eq. (42), we will obtain the following equation:

$$n^2 + (2k^2v)n - A_t g k = 0
 \tag{47}$$

The above equation can be solved in terms of  $n$  and then written in dimensionless form as follows (Drake, 2018):

$$n = \sqrt{A_t k g + k^4 v^2} - k^2 v
 \tag{48}$$

$$\tilde{n} = \sqrt{A_t \tilde{k} + \tilde{k}^4} - \tilde{k}^2
 \tag{49}$$

The obtained growth rate is shown as curves in Figs. 1, 2, 3 and 4 for different values of the Atwood number.

Now, we derive Eq. (49) with respect to  $k$  and then set the result equal to zero. So the value of  $\tilde{k} = A_t^{1/3} / 2$  or  $k = (A_t g / v^2)^{1/3} / 2$  is obtained for the maximum amount of growth wave number. As a result, the maximum value of growth rate corresponding to this wave number in the presence of viscosity is obtained as follows:

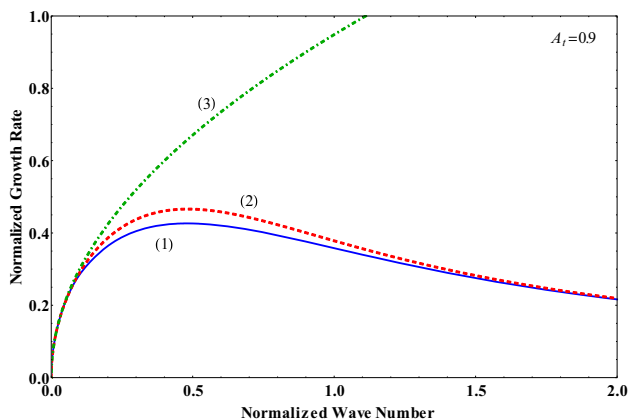


Fig. 1 Rayleigh–Taylor instability with viscosity: The curve (1) shows the actual growth rate, the curve (2) shows the high viscosity approximation, and the curve (3) shows the zero-viscosity result or classical growth rate for  $A_t = 0.9$ .

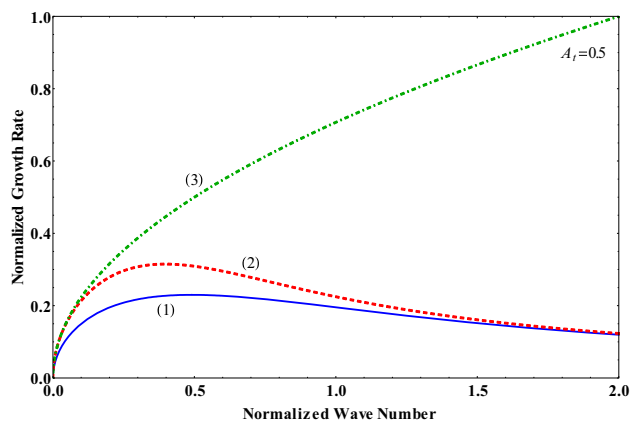


Fig. 2 Rayleigh–Taylor instability with viscosity: The curve (1) shows the actual growth rate, the curve (2) shows the high viscosity approximation, and the curve (3) shows the zero-viscosity result or classical growth rate for  $A_t = 0.5$ .

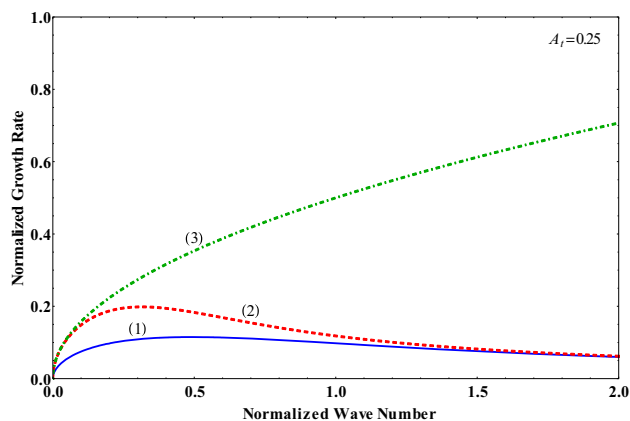


Fig. 3 Rayleigh–Taylor instability with viscosity: The curve (1) shows the actual growth rate, the curve (2) shows the high viscosity approximation, and the curve (3) shows the zero-viscosity result or classical growth rate for  $A_t = 0.25$ .

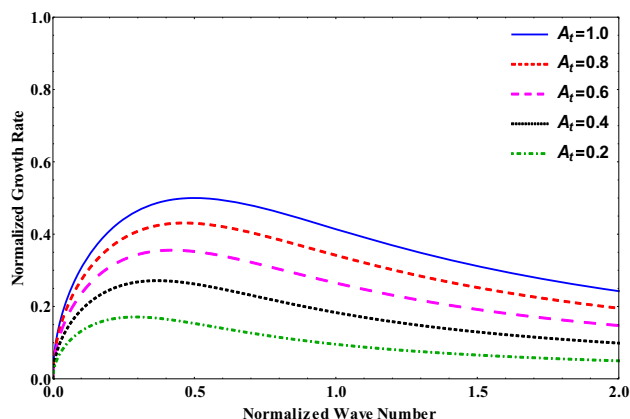


Fig. 4 Rayleigh–Taylor instability with viscosity for different values of Atwood number



$$n_{MAX} = \frac{3}{4} \sqrt{\frac{(A_t g)^{4/3}}{v^{2/3}}} - \frac{(A_t g)^{2/3}}{4 v^{1/3}} \quad (50)$$

or

$$n_{MAX} = \frac{(A_t g)^{2/3}}{2 v^{1/3}} \quad (51)$$

Figures 1, 2, 3 and 4 show that with increasing the Atwood number, the growth rate of Rayleigh–Taylor instability increases. Also, for large values of the dimensionless wave number, the growth rate of instability decreases and the diagrams (1) and (2) tend to move toward each other. In high energy density experiments, with an initial approximation,  $v$  has a value of  $0.01 \text{ cm}^2/\text{s}$  and  $g$  has a value of  $10^{15} \text{ cm}/\text{s}^2$ . Therefore, the maximum growth wave number in the presence of viscosity for  $A_t = 1$  is obtained from order of  $10^6 \text{ cm}^{-1}$ , so that the corresponding wavelength is from order of  $0.1 \mu\text{m}$ , and also, the maximum growth rate of instability in this case is almost from order of  $2 \times 10^{10} \text{ s}^{-1}$ . As we can be seen in Figs. 1, 2, 3 and 4, viscosity is more effective at larger wave numbers and therefore at smaller wavelengths than a fraction of microns, and substantially, it reduces the growth rate, but at the wavelengths larger than that, it has no influence. Experiments show that the most dangerous wavelengths that destroy the fuel are those that are about three times the thickness of cold fuel ( $\lambda \approx 3\Delta R$ ) (Atzeni and Meyer-ter-Vehn, 2004). Now assuming the shell is  $\Delta R \approx 5 - 20 \mu\text{m}$  thick, we can conclude that the most dangerous wavelengths that destroy a fuel pellet are about 15 to 60 microns. So, the viscosity is ineffective at the wavelengths that destroy the fuel pellet more (Eliezer, 2002).

## 7 Conclusion

In this article, for the first time, using linear hydrodynamic equations such as the mass and the linear momentum conservation equations, called Euler equations, considering the effects of viscosity of fluid molecules and applying boundary conditions, we have been able to determine the growth rate of Rayleigh–Taylor instability analytically, which is the most important factor in measuring the growth of fluid perturbations. Then, by using the physical relations and thermodynamic equations, we have been able to obtain the dimensionless growth rate of Rayleigh–Taylor instability analytically as a function of the dimensionless wave number. In this way, we conclude that the viscosity or the adhesion between the molecules of a fluid is considered to be a positive factor reducing the growth rate of

hydrodynamic instabilities such as Rayleigh–Taylor instability which is the most important of them.

**Authors' contributions** All authors contributed to the study conception and design. Material preparation, data collection and analysis were performed by Prof. Dr. AG and Mr. AM. The first draft of the manuscript was written by Mr. AM and Prof. Dr. AG commented on previous versions of the manuscript. All authors read and approved the final manuscript.

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**Availability of data and material** The authors confirm that the data supporting the findings of this study are available within the article and its supplementary materials.

## Declarations

**Conflict of interest** The authors declare that there is no conflict of interest.

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