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RESEARCH PAPER

Multiresolution Analysis from a Riesz Family of Shifts of a Refinable Function in $L^2(G)$

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Abstract

Let G be a second countable locally compact abelian group, L be a uniform lattice in G and S_L be a fundamental domain for L in G. Let $L^p_\circ(G) = \{\varphi : G \longrightarrow \mathbb{C};\quad \|\sum_{k\in L} |\varphi(k^{-1}x)|\|_{L^p(S_L)} < \infty\}$ $(1 \leqslant p \leqslant \infty)$. In this paper we aim among other things, to introduce the Banach space $L^p_\circ(G)$ $(1 \leqslant p \leqslant \infty)$, with the norm $|\cdot|_p$, and for $p = 2$ and a refinable function $\varphi \in L^2_\circ(G)$ and the Riesz family generated by the shifts of φ by L in G, construct a multiresolution analysis in $L^2(G)$. Also some examples are provided to support our construction.

Keywords Reisz family · Multiresolution analysis · Refinable function · LCA group

Mathematics Subject Classification 47A55 · 39B52 · 34K20 · 39B82

1 Introduction

The idea of multiresolution analysis (MRA) was introduced by Meyer (1990) and Mallat (1989). It was a framework for construction of orthogonal wavelet bases and it was improved by many authors including those in Daubechies (1992) and Hernandez and Weiss (1996). From the physical point of view, MRA is a modern signal processing device in a mathematical manner that allows one to analyze the properties of signals at different resolution levels. In the recent years, the concept of MRA has become an important tool in pure and applied mathematics and many branches of engineering (Chibani and Houacine 1998; Dahlke 1994; Daubechies 1992; Jeng et al. 2009; Papadakis et al. 2003). Moreover, comprehensive studies have been conducted in which MRA has been investigated for the Euclidian group $\mathbb{R}^s$ $(s \in \mathbb{N})$, e.g., Bownik and Garrigos (2004), Daubechies (1992), Jia and Micchelli (1991), Mallat (1989), Meyer (1990) and Zhou (1996). For example, Jia and Micchelli Jia and Micchelli (1991) proved that the Riesz family generated by the integer shifts of a certain basis refinable function are sufficient to lead to an MRA of $L^p(\mathbb{R}^s)$ for $1 \leqslant p < \infty$ (for general scaling matrices and $p = 2$, see also Jia and Micchelli 1992; Madych 1992). Later, Zhou (1996) developed this theory for $p = \infty$. Furthermore, in Baggett (2000) and Bagget et al. (1999) Baggett, Medina, and Merrill generalized the concept of MRA in terms of wavelet dimension function properties. They investigated its relation to wavelets (for further details on MRA, see Arefijamaal and Ghaani Farashahi 2013; Arefijamaal and Kamyabi-Gol 2009; Ghaani Farashahi 2017a, b; Hernandez and Weiss 1996). Later Dahlke (1994) generalized the definition of MRA to locally compact abelian group (LCA) groups, and showed under certain conditions, the generalized B-splines generated an MRA (see also Galindo and Sanz 2001; Kamyabi and Raisi 2010).

This paper deals with the construction of a multiresolution approximation in the Hilbert space $L^2(G)$ (G is a LCA group) via the Riesz family generated by the shifts of a certain refinable function φ (see Mohammadian 2017). In contrast to Kamyabi and Raisi (2010), the “orthogonality” condition is replaced by the weaker condition “Riesz

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family" generated by the shifts of a refinable function with respect to a uniform lattice L in G.

The rest of the paper is organized as follows: In the second section, first, we review some facts on LCA group G and then define and investigate the specified Banach spaces $L^p_\circ(G)$, $1 \leqslant p \leqslant \infty$, which are notably interesting by themselves and are also needed in our study of MRA. Section 3 is devoted to introducing the definition and two propositions for construction MRA in the next section. Finally, Sect. 4 contains construction and study the multiresolution approximation in $L^2(G)$. This construction is based on the Riesz family generated by the shifts of a refinable function via a lattice in the second countable LCA group G. Some examples are provided to clarify our construction.

2 Preliminaries and Related Background

Let G be an LCA group with the identity 1_G and the dual group $\widehat{G}$. For a closed subgroup H of G, let $H^\perp$:=$\{\xi \in \widehat{G};\ \xi(H) = \{1\}\}$, denote the annihilator of H in $\widehat{G}$. A discrete subgroup L of G is called a uniform lattice if it is co-compact (i.e., $\frac{G}{L}$ is compact). From (Folland 1995, Theorem 4.39) it follows that the subgroup $L^\perp$ is also a uniform lattice in $\widehat{G}$. Note that it is not decisive that all groups should have uniform lattices, and the examples which illustrate the concept are p-adic groups. Now a fundamental domain for a uniform lattice L in G is a measurable set S_L in G, such that every $x \in G$ can be uniquely written as $x = ks$, for $k \in L$ and $s \in S_L$. For a uniform lattice L, it is known that, there exists a relatively compact fundamental domain S_L which has a positive measure. Moreover, $L^2(G/L) \cong L^2(S_L)$, when G is a second countable LCA (Kamyabi Gol and Raeisi Tousi 2008; Kaniuth and Kutyniok 2008). For a uniform lattice L, L-invariant subspaces are very useful (e.g., see Kamyabi Gol and Raeisi Tousi 2008; Ron and Shen 1995). We recall that a closed subspace $V \subseteq L^2(G)$ is called L-invariant if $f \in V$ implies $T_k f \in V$, where T_k is the translation operator on $L^2(G)$ defined by $T_k f(x) = f(k^{-1}x)$ for all $x \in G$, $k \in L$. It is well known that any LCA group G possesses Haar measures and it is unique up to positive constants. Now if α is a topological automorphism on G, then the Radon measure λ_α defined by $\lambda_\alpha(E) = \lambda(\alpha(E))$ (E Borel set, λ Haar measure on G) is also a Haar measure of G. So by the uniqueness of Haar measure, there exists a positive constant δ_α (depending on α) such that $\lambda_\alpha(E) = \delta_\alpha \lambda((E))$. Now consider the dilation operator $D : L^2(G) \longrightarrow L^2(G)$ by $Df(x) = \delta_\alpha^{\frac{1}{2}} f(\alpha(x))$, (the fact δ_α is a proper positive constant depending on α makes the operator D an isometric isomorphism), and for $j \in \mathbb{Z}$, $D^j f(x)$ is defined as $\delta_\alpha^{\frac{j}{2}} f(\alpha^j(x))$.

Now, we introduce the notion of multiresolution approximation in $L^2(G)$, following Mallat (1989). A sequence $\{V_j\}_{j\in\mathbb{Z}}$ of closed subspaces of $L^2(G)$ forms a multiresolution approximation of $L^2(G)$ if it satisfies the following conditions:

(i) $V_j \subseteq V_{j+1}$, $\forall j \in \mathbb{Z}$.
(ii) $f \in V_j \Longrightarrow D^j T_k D^{j^{-1}} f \in V_j$, for all $j \in \mathbb{Z}, k \in L$.
(iii) $f \in V_j \Longleftrightarrow \delta_\alpha^{-\frac{1}{2}} Df \in V_{j+1}$.
(iv) There is an isomorphism from $l^2(L)$ onto V_0 which commutes with shift operators.
(v) $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$.
(vi) $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(G)$.

From (iv), one may find that there exists a unique function $g \in L^2(G)$ such that for any $j \in \mathbb{Z}$, $\{\delta_\alpha^{\frac{j}{2}} g(k^{-1}\alpha^j(\cdot))\}_{k\in L}$ is a wavelet orthonormal basis of V_j (see also Mallat 1989).

Note that $l(L)$ is the linear space of all funtions on L, and $l^p(L)$ $(1 \leqslant p \leqslant \infty)$, is the linear space of all functions on L, as $f = (f(k))_{k\in L}$ such that for $1 \leqslant p < \infty$, $\|f\|^p = \sum_{k\in L} |f(k)|^p < \infty$, and for $p = \infty$, f is bounded. Also a function φ on G is compactly supported if the closure of the set of all points $x \in G$ at which $\varphi(x) \neq 0$ is compact.

For $a \in l(L)$, the symbol of a is defined by $\tilde{a}(\xi) := \sum_{k\in L} a(k)\overline{\xi(k)}$ for $\xi \in \widehat{G}$. It is worthwhile to note that the symbol of a, $\tilde{a}$, is a continuous function on $S_{L^\perp}$ if $a \in l^1(L)$ and that $l^1(L)$ is a unital commutative involutive Banach algebra (δ is unit element).

We recall that for a locally compact group G, a topological automorphism $\alpha : G \to G$ is said to be contractive if $\lim_{n\to\infty} \alpha^n(x) = 1_G$ for all $x \in G$.

Also, a related concept with contractivity is expansiveness. Following Siebert (1986), a topological automorphism α of G is said to be expansive if there exists a neighborhood U of 1_G such that $\bigcap_{n\geqslant 1} \alpha^{-n}(U) = \{1_G\}$. If α is contractive then α^{-1} is expansive in view of Lemma 1.

Lemma 1 (*Siebert 1986*) *For a locally compact group G, let $\alpha \in Aut(G)$ be contractive and let U be a closed neighborhood of 1_G in G. For any $n \in \mathbb{Z}$, we put $U_n = \bigcap_{k\leqslant n, k\in\mathbb{Z}} \alpha^k(U)$. Then we have*

(a) *$U_{n+1} \subseteq U_n$ and $\alpha(U_n) = U_{n+1}$ for all $n \in \mathbb{Z}$.*
(b) *$\bigcup_{n\in\mathbb{Z}} U_n = G$.*
(c) *every U_n has non-void interior.*
(d) *for every compact subset C of G there exist some $n_0 \in \mathbb{N}$ such that $\alpha^n(C) \subseteq U$ for all $n \geqslant n_0$.*

Note that the definition of contractivity of α agrees with part (d) of Lemma 1. For a topological automorphism α on G, we denote by $\hat{\alpha}$ the topological automorphism on $\widehat{G}$ defined by $\hat{\alpha}(\xi)(x) = \xi(\alpha(x))$, (see Bagget et al. 1999). It can be shown that α is contractive if and only if $\hat{\alpha}$ is contractive.

Now, we introduce the Banach spaces $L^p_\circ(G)$, $1 \leqslant p \leqslant \infty$. For uniform lattice L in G and function φ on G, let

$$\varphi^\circ(x) := \sum_{k\in L} |\varphi(k^{-1}x)|,$$

then φ° is a L-periodic function. Write

$$|\varphi|_p := \|\varphi^\circ\|_{L^p(S_L)} = \Big(\int_{S_L} \big|\sum_{k\in L}\varphi(k^{-1}x)\big|^p \mathrm{d}x\Big)^{\frac{1}{p}}.$$

It is easy to see that $|.|_p$ is a norm. Put

$$L^p_\circ(G) = \{\varphi : G \longrightarrow \mathbb{C}; \quad |\varphi|_p < \infty\} \quad (1 \leqslant p \leqslant \infty).$$

One can check that $L^p_\circ(G)$ equipped with the norm $|\cdot|_p$, is also a Banach space (see Jia and Micchelli 1991 for the case $L^p_\circ(\mathbb{R}^s)$, s is a positive integer), and obviously $\|\varphi\|_p \leqslant |\varphi|_p$, for all $1 \leqslant p \leqslant \infty$. Indeed, for $1 \leqslant p < \infty$,

$$\begin{aligned}|\varphi|_p^p &= \int_{S_L} (\sum_{k\in L}|\varphi(k^{-1}x)|)^p \mathrm{d}x\\ &\geqslant \int_{S_L}\sum_{k\in L}|\varphi(k^{-1}x)|^p \mathrm{d}x\\ &= \int_G |\varphi(k^{-1}x)|^p = \|\varphi\|_p^p.\end{aligned}$$

For $p = \infty$,

$$\begin{aligned}|\varphi|_\infty &= \|\varphi^\circ\|_{L^\infty(S_L)}\\ &= \mathrm{esssup}\{\sum_{k\in L}|\varphi(k^{-1}x)|,\ x\in S_L\}\\ &= \mathrm{esssup}\{\sum_{k\in L}|\varphi(k^{-1}x)|,\ x\in G\}\\ &\quad (since\ \varphi^\circ is\ L-periodic)\\ &\geqslant \mathrm{esssup}\{|\varphi(x)|,\ x\in G\} = \|\varphi\|_\infty.\end{aligned}$$

Note that $L^1_\circ(G) = L^1(G)$. Because for $\varphi \in L^1(G)$,

$$\begin{aligned}|\varphi|_1 = \|\varphi\|_{L^1(S_L)} &= \int_{S_L}\big|\sum_{k\in L}\varphi(k^{-1}x)\big|\mathrm{d}x\\ &\leqslant \int_G |\varphi(x)|\mathrm{d}x = \|\varphi\|_1 < \infty.\end{aligned}$$

So $\varphi \in L^1_\circ(G)$. Also, if $\varphi \in L^p(G)$ is compactly supported, then $\varphi \in L^p_\circ(G)$, for all $1 \leqslant p \leqslant \infty$. Indeed by (Mohammadian et al. 2016, Lemma 3.2),

$$\begin{aligned}|\varphi|_p = \|\varphi^\circ\|_{L^p(S_L)} &= \big\|\sum_{k\in L}|\varphi(k^{-1}\cdot)|\big\|_{L^p(S_L)}\\ &\leqslant \sum_{k\in L}\|\varphi(k^{-1}\cdot)\|_{L^p(S_L)}\\ &= \sum_{k\in L}\|\varphi\|_{L^p(S_L)} \leqslant \infty.\end{aligned}$$

Now, for $\varphi \in L^p_\circ(G)$, $1 \leqslant p \leqslant \infty$, and $a \in l^\infty(L)$, the semidiscrete convolution $\varphi *' a$ is defined by $\sum_{k\in L}\varphi(k^{-1}\cdot)a(k)$. We also denote by $\varphi *'$ the mapping $a \to \varphi *' a$ $(a \in l^\infty(L))$.

Now the following theorem shows that $\varphi *'$ maps $l^q(L)$ to $L^p(G)$ where $1 \leqslant q \leqslant p \leqslant \infty$, and $l^1(L)$ to $L^p_\circ(G)$.

Theorem 1 *With notations as above for $\varphi \in L^p_\circ(G)$ we have,*

(i) $|\varphi *' a|_p \leqslant |\varphi|_p \|a\|_1$
(ii) $\|\varphi *' a\|_p \leqslant |\varphi|_p \|a\|_q$, $(q \leqslant p)$

Proof Part (i) is obtained easily for every $1 \leqslant p \leqslant \infty$, since,

$$\begin{aligned}(\varphi *' a)^\circ &= \sum_{k\in L}|(\varphi *' a)(k^{-1}.)|\\ &\leqslant \sum_{k\in L}\sum_{l\in L}\big|\varphi\big((kl)^{-1}\cdot\big)\big||a(l)|\\ &= \sum_{l\in L}|a(l)|\sum_{k\in L}\big|\varphi\big((kl)^{-1}\cdot\big)\big|\\ &= \varphi^\circ\|a\|_1.\end{aligned}$$

Now

$$\begin{aligned}|\varphi *' a|_p &= \|(\varphi * a)^\circ\|_{L^p(S_L)}\\ &\leqslant \|\ \|a\|_1\varphi^\circ\|_{L^p(S_L)}\\ &\leqslant \|a\|_1\|\varphi^\circ\|_{L^p(S_L)}\\ &= \|a\|_1|\varphi|_p.\end{aligned}$$

Part (ii), for $p = \infty$,

$$\begin{aligned}\|\varphi *' a\|_\infty &= \mathrm{esssup}\{\big|\sum_{k\in L}\varphi(k^{-1}x)a(k)\big|,\ x\in G\}\\ &\leqslant \mathrm{esssup}\{\|a\|_\infty\sum|\varphi(k^{-1}x)|,\ x\in G\}\\ &= \|a\|_\infty \mathrm{esssup}\{\sum|\varphi(k^{-1}x)|,\ x\in S_L\}\\ &\quad (\text{since} \sum_{k\in L}|\varphi(k^{-1}x)|\ \text{is}\ L-\text{periodic})\\ &\leqslant \|a\|_\infty\|\varphi^\circ\|_{L^\infty(S_L)} = \|a\|_\infty|\varphi|_\infty \leqslant \|\varphi\|_\infty\|a\|_1.\end{aligned}$$

For $1 \leqslant p < \infty$, let $I = \|\varphi *' a\|_p$, and so

$$\begin{aligned} I^p &= \int_G |(\varphi *' a)(x)|^p \mathrm{d}x \\ &= \sum_{l\in L} \int_{S_L} |(\varphi *' a)(xl)|^p \mathrm{d}x \\ &= \int_{S_L} \sum_{l\in L} |(\varphi *' a)(xl)|^p \mathrm{d}x. \end{aligned}$$

For $x \in G$, as a fixed point, let c be the sequence $(\varphi(xl))_{l\in L}$, that $c \in l^1(L)$. Then

$$(\varphi *' a)(xl) = \sum_{k\in L} \varphi(k^{-1}xl)a(k) = a * c(l),$$

where $a * c$ denotes the discrete convolution of a and c. By Young's inequality (Folland 1984, Proposition 8.9), it follows that

$$\sum_{l\in L} |(\varphi *' a)(xl)|^p = \|a * c\|_p^p \leqslant \|a\|_p^p (\varphi^\circ(x))^p.$$

Consequently, we have

$$\begin{aligned} I^p &\leqslant \|a\|_p^p \int_{S_L} (\varphi^\circ(x))^p \mathrm{d}x \\ &= \|a\|_p^p |\varphi|_p^p \\ &\leqslant \|a\|_q^p |\varphi|_p^p. \end{aligned}$$

This completes the proof. □

We conclude this section with some definitions.

Recall that the shifts of φ, via the lattice L in G is called a Riesz family of $L^p(G)$, if there exist constants $A_p, B_p > 0$ such that

$$A_p\|a\|_p \leqslant \|\varphi *' a\|_p \leqslant B_p\|a\|_p \quad (1 \leqslant p \leqslant \infty),$$

for all $a \in l^p(L)$.

The right-hand side of the above inequality is valid by Theorem 1, so it is enough to say that the set of shifts of φ, via the lattice L in G, forms a Riesz family of $L^p(G)$ if there exists a constant $A_p > 0$ such that,

$$A_p\|a\|_p \leqslant \|\varphi *' a\|_p \quad (1 \leqslant p \leqslant \infty),$$

for all $a \in l^p(L)$.

Let $S_p(\varphi)$ be the image of $l^p(L)$ of the mapping $\varphi *'$. In this case, the set of shifts of φ via the lattice L in G is a Riesz basis of $S_p(G)$. See also (Christensen 2016, Theorem 3.6.6).

3 Propositions

Throughout this section and afterward, G denotes a second countable locally compact abelian group with a fixed Haar measure λ, and α is a topological automorphism on G such that $\lambda(\alpha(E)) = \delta_\alpha\lambda(E)$ for all Borel subsets E of G (δ_α is a positive constant depending on α). Furthermore, we assume that α^{-1} is contractive and L is a uniform lattice in G.

Now for a refinable function $\varphi \in L^2_\circ(G)$, we consider $V_0 = S_2(\varphi)$ and $V_j = D^j V_0$, where D is the dilation operator defined on $L^2(G)$ by $Df(x) = \delta_\alpha^{\frac{1}{2}} f(\alpha(x))$.

Note that a function $\varphi \in L^2_\circ(G)$ is said to be refinable, if it satisfies the following refinement equation:

$$\begin{aligned} \varphi &= \sum_{k\in L} b(k) D T_k \varphi(\cdot) \\ &= \sum_{k\in L} \delta_\alpha^{\frac{1}{2}} b(k)\varphi(k^{-1}\alpha(\cdot)), \end{aligned} \tag{1}$$

for some $b \in l^1(L)$, that is called the mask of the refinement equation.

The following proposition shows that for a refinable function $\varphi \in L^1(G)$, $\hat{\varphi}(\eta) = 0$ for all $\eta \in L^\perp \backslash \{1_{\widehat{G}}\}$, where $\hat{\varphi}$ is the Fourier transform of φ. Although establishing this condition is closely related to contractivity of a topological automorphism $\hat{\alpha}^{-1}$. According to this proposition, $\sum_{k\in L} \varphi(k^{-1}\cdot)$ is a constant. If in addition, the shifts of φ via the lattice L in G forms a Riesz family of $L^2(G)$, then this constant must be nonzero. This fact will be shown in proposition 2.

Proposition 1 *If $\varphi \in L^1(G)$ is refinable and $\alpha : G \to G$ is a topological automorphism such that $\hat{\alpha}^{-1}$ is contractive and $\hat{\alpha}(L^\perp) \subseteq L^\perp$, then $\hat{\varphi}(\eta) = 0$ for all $\eta \in L^\perp \backslash \{1_{\widehat{G}}\}$. Moreover,*

$$\sum_{k\in L} \varphi(k^{-1}\cdot) = \hat{\varphi}(1_{\widehat{G}}).$$

Proof By taking the Fourier transforms of the refinement equation (1), we have

$$\begin{aligned} \hat{\varphi}(\xi) &= \sum_{k\in L} b(k)\delta_\alpha^{-1}\overline{\xi(\alpha^{-1}(k))}\hat{\varphi}(\hat{\alpha}^{-1}(\xi)) \quad (\xi \in \widehat{G}) \\ &= \delta_\alpha^{-1}\tilde{b}(\hat{\alpha}^{-1}(\xi))\hat{\varphi}(\hat{\alpha}^{-1}(\xi)), \end{aligned} \tag{2}$$

where $\tilde{b}(\xi) = \sum_{k\in L} b(k)\overline{\xi(k)}$, is the symbol of b. Then by induction, we get

$$\hat{\varphi}(\xi) = \prod_{j=1}^{k} (\delta_\alpha^{-1}\tilde{b}(\hat{\alpha}^{-j}(\xi)))\hat{\varphi}(\hat{\alpha}^{-k}(\xi)) \quad (\xi \in \widehat{G}). \tag{3}$$

Consider two cases $|\tilde{b}(1_{\widehat{G}})| < \delta_\alpha$, and $|\tilde{b}(1_{\widehat{G}})| \geqslant \delta_\alpha$. If $|\tilde{b}(1_{\widehat{G}})| < \delta_\alpha$, then by choosing $\xi = 1_{\widehat{G}}$ in (2), we obtain $\hat{\varphi}(1_{\widehat{G}}) = 0$. Moreover, contractivity of $\hat{\alpha}^{-1}$ and the continuity of $\tilde{b}$ imply that $\tilde{b}(\hat{\alpha}^{-j}(\xi)) \to \tilde{b}(1_{\widehat{G}})$ for any fixed

$\xi \in \widehat{G}$, and sufficiently large j. Therefore, for any $\xi \in \widehat{G}$ and sufficiently large j, $|\delta_\alpha^{-1}\tilde{b}(\hat{\alpha}^{-j}(\xi))| < 1$. By letting $k \to \infty$ we obtain

$$\prod_{j=1}^{k} \delta_\alpha^{-1}\tilde{b}(\hat{\alpha}^{-j}(\xi)) \to 0.$$

Thus $\varphi = 0$.

Now suppose $|\tilde{b}(1_{\widehat{G}})| \geqslant \delta_\alpha$. By replacing ξ by $\hat{\alpha}^k(\eta)$ $(\eta \in L^\perp \backslash \{1_{\widehat{G}}\})$, in Eq. (3), we have

$$\begin{aligned}\hat{\varphi}(\hat{\alpha}^k(\eta)) &= \left(\delta_\alpha^{-1}\tilde{b}\left(\hat{\alpha}^{-j}\left(\hat{\alpha}^k(\eta)\right)\right)\right)^k \hat{\varphi}(\eta)\\ &= \left(\delta_\alpha^{-1}\tilde{b}(1_{\widehat{G}})\right)^k \hat{\varphi}(\eta),\end{aligned}$$

then

$$|\hat{\varphi}(\hat{\alpha}^k(\eta))| \geqslant |\hat{\varphi}(\eta)|. \tag{4}$$

Also $\varphi \in L^1(G)$ implies $\varphi' := \sum_{k\in L} \varphi(k^{-1}\cdot) \in L^1(\frac{G}{L})$, so $\widehat{\varphi'} = \hat{\varphi}\,|_{L^\perp} \in c_0(\widehat{(\frac{G}{L})}) \cong c_0(L^\perp)$. Since $\hat{\alpha}$ is an expansive automorphism, then $\hat{\varphi}(\hat{\alpha}^k(\eta)) \in c_0(L^\perp)$ when $k \to \infty$ in (4). So $\hat{\varphi}(\eta) = 0$ for all $\eta \in L^\perp \backslash \{1_{\widehat{G}}\}$. For $\varphi'(\cdot) = \sum_{k\in L} \varphi(k^{-1}\cdot)$, we have $\widehat{\varphi'} \in L^1(L^\perp)$.

Now Poisson summation (Folland 1995, theorem 4.42), implies that,

$$\sum_{k\in L} \varphi(k^{-1}\cdot) = \sum_{\eta\in L^\perp} \hat{\varphi}'(\eta)\eta(\cdot) = \sum_{\eta\in L^\perp} \hat{\varphi}(\eta)\eta(\cdot) = \hat{\varphi}(1_{\widehat{G}}),$$

and the proof is complete. □

For example, it is easy to see that for $\varphi = \chi_{[0,1]}$, which is refinable and belongs to $L^1(\mathbb{R})$, we have $\sum_{k\in\mathbb{Z}} \varphi(\cdot - k) = 1 = \hat{\varphi}(0)$.

Along with Proposition 1, the next proposition plays a key role in reaching the result $\hat{\varphi}(1_{\widehat{G}}) \neq 0$. Thus after normalization we may assume that $\sum_{k\in L} \varphi(k^{-1}\cdot) = \hat{\varphi}(1_{\widehat{G}}) = 1$.

Proposition 2 *Let $\varphi \in L^2_\circ(G)$, and the shifts of φ via the lattice L in G forms a Riesz family of $L^2(G)$. Then*

$$\sup_{\eta\in L^\perp} |\hat{\varphi}(\xi\eta)| > 0,$$

for all $\xi \in \widehat{G}$.

Proof Suppose that for some $\xi \in \widehat{G}$, $\hat{\varphi}(\xi\eta) = 0$ for all $\eta \in L^\perp$. In the sequel, we show that the shifts of φ via the lattice L in G does not form a Riesz family of $L^2(G)$.

Consider $x \longrightarrow \overline{\xi(x)}\varphi(x)$ $(x \in G)$. Without loss of generality, we may assume that $\hat{\varphi}(\eta) = 0$ for all $\eta \in L^\perp$. Therefore, by Poisson summation formula, we have

$$\sum_{k\in L} \varphi(k^{-1}x) = \sum_{\eta\in L^\perp} \hat{\varphi}(\eta)\eta(x) = 0 \qquad (x \in G). \tag{5}$$

Let U be a symmetric compact neighborhood of the identity 1_G.

Set $U_m = UU\ldots U$ (m factors). Then every U_m is compact that contains finitely many $k \in L$, by (Mohammadian et al. 2016, Lemma 3.2). For each $n \in \mathbb{N}$, let a_n be the sequence on L defined by,

$$a_n(k) = \begin{cases} 1 & k \in U_{n^2} \\ 0 & \text{o.w.} \end{cases}.$$

To prove that the shifts of φ via the lattice L in G do not form a Riesz family, it is enough to show that

$$\frac{\|\varphi *' a_n\|_2}{\|a_n\|_2} \to 0 \qquad \textit{whenever} \quad n \to \infty$$

To this end, consider the functions $\varphi_{_N}$ and $\psi_{_N}$ on G defined as follows,

$$\varphi_{_N}(x) := \begin{cases} \varphi(x) & x \in U_{_N} \\ 0 & \text{o.w.} \end{cases}.$$

Without loss of generality we may choose $N \in \mathbb{N}$ such that $S_L \subseteq U_{_N}$,

$$\psi_{_N}(x) := \begin{cases} \sum_{k\in L} (\varphi - \varphi_{_N})(k^{-1}x) & x \in S_L \\ 0 & \text{o.w.} \end{cases}.$$

Set $\psi := \varphi_{_N} + \psi_{_N}$ which is compactly supported in $U_{_N}$, that is,

$$\psi(x) = 0 \quad \textit{for} \quad x \in U^c_{_N}. \tag{6}$$

Construction of $\psi_{_N}$ guarantees that

$$|\psi_{_N}|_2 \leqslant |\varphi - \varphi_{_N}|_2.$$

Hence we have

$$\begin{aligned}|\varphi - \psi|_2 &\leqslant |\varphi - \varphi_{_N}|_2 + |\psi_{_N}|_2\\ &\leqslant 2|\varphi - \varphi_{_N}|_2.\end{aligned} \tag{7}$$

The above relation and Theorem 1 gives the following estimate,

$$\begin{aligned}\|(\varphi - \psi) *' a_{_N}\|_2 &\leqslant |\varphi - \psi|_2 \|a_{_N}\|_2\\ &\leqslant 2|\varphi - \varphi_{_N}|_2 \|a_{_N}\|_2.\end{aligned}$$

Therefore,

$$\frac{\|\varphi *' a_{_N}\|_2}{\|a_{_N}\|_2} \leqslant \frac{\|\psi *' a_{_N}\|_2}{\|a_{_N}\|_2} + 2|\varphi - \varphi_{_N}|_2. \tag{8}$$

On the other hand, $|\varphi - \varphi_{_N}|_2 \leqslant 2|\varphi|_2$. As $\varphi \in L^1(G)$, Dominated convergence theorem implies that $|\varphi - \varphi_{_N}|_2 \to$

0 as $N \to \infty$. It remains to estimate $\frac{\|\psi *' a_N\|_2}{\|a_N\|_2}$. From (5) and construction of ψ we have,

$$\sum_{k\in L} \psi(k^{-1}x) = \sum_{k\in L} \varphi(k^{-1}x) = 0 \qquad (x \in G), \tag{9}$$

because

$$\sum_{k\in L} \psi_N(k^{-1}x) = \sum_{k\in L} (\varphi - \varphi_N)(k^{-1}x) \qquad (x \in G).$$

By (6) and (9) we obtain $\psi *' a_N = \sum_{k\in L} a_N(k)\psi(k^{-1}x) = 0$, for all $x \in U_{N^2}U_N^c \cup U_N^c U_{N^2}^c$. Indeed,

$$\begin{aligned}\sum_{k\in U_{N^2}\cap L} a_N(k)\psi(k^{-1}x) + \sum_{k\in U_{N^2}^c\cap L} a_N(k)\psi(k^{-1}x) &= \sum_{k\in U_{N^2}\cap L} \psi(k^{-1}x)\\ &= -\sum_{k\in U_{N^2}^c\cap L} \psi(k^{-1}x).\end{aligned}$$

Therefore, $\psi *' a_N$ is supported in $E = (U_{N^2}U_N^c)^c \cap (U_N^c U_{N^2}^c)^c$. It follows that

$$\begin{aligned}\|\psi *' a_N\|_2^2 &\leqslant \int_E (\sum_{k\in L} |a_N(k)||\psi(k^{-1}x)|)^2 \mathrm{d}x\\ &\leqslant \int_E \left(\psi^\circ(x)\right)^2 \mathrm{d}x\\ &= \sum_{k\in E\cap L} \int_{S_L} \left(\psi^\circ(x)\right)^2 \mathrm{d}x.\end{aligned}$$

We have $\psi(k^{-1}x) \neq 0$ if $k^{-1}x \in U_N$ (U_N is symmetric), so $k \in U_N x$. If $x \in E$, $k \in U_N E \subseteq U_N U_N = U_{2N}$, then

$$\|\psi *' a_N\|_2^2 \leqslant |\psi|_2^2 card\{k,\ k \in U_{2N} \cap L\},$$

and $\|a_N\|_2^2 = \sum_{k\in U_{N^2}\cap L} 1 \geqslant card\{k,\ k \in U_{N^2} \cap L\}$.

Therefore, we obtain the following estimate

$$\frac{\|\psi *' a_N\|_2}{\|a_N\|_2} \leqslant \frac{|\psi|_2 card\{k \in L, k \in U_{2N} \cap L\}}{card\{k \in U_{N^2} \cap L\}}. \tag{10}$$

By (7),

$$\begin{aligned}|\psi|_2 &\leqslant |\varphi|_2 + |\varphi - \psi|_2\\ &\leqslant |\varphi|_2 + 2|\varphi - \varphi_N|.\end{aligned} \tag{11}$$

Consequently, from (10) and (11),

$$\frac{\|\psi *' a_n\|_2}{\|a_n\|_2} \to 0 \quad whenever \quad n \to \infty,$$

which completes the proof. □

4 Multiresolution Analysis

In this section, we construct an MRA of $L^2(G)$ by a Riesz family of shifts of φ via the lattice L in G, for a refinable function $\varphi \in L^2_\circ(G)$. As mentioned priory, we consider $V_0 = S_2(\varphi)$ and $V_j = D^j V_0$, where D is the dilation operator defined on $L^2(G)$ by $Df(x) = \delta_\alpha^{\frac{1}{2}} f(\alpha(x))$. We recall α is a topological automorphism on G and for Haar measure λ on G, δ_α is a positive constant depending on α, such that $\lambda(\alpha(E)) = \delta_\alpha \lambda(E)$ for all Borel subsets E of G.

Theorem 2 *With the notation as above let $\varphi \in L^2(G)$, $V_0 = S_2(G)$ and $V_j = D^j V_0$. If φ is refinable and shifts of φ via the lattice L in G forms a Riesz family, then $(V_j)_{j\in\mathbb{Z}}$ forms a multiresolution approximation of $L^2(G)$.*

Proof By the definition of V_0, and that φ is refinable, in the definition of multiresolution approximation, (i) is obtained. (ii), (iii) are also followed by the definition of V_j. (iv) is clear by the definition of $V_0 = S_2(\varphi)$ and the function $\varphi *'$ is an isomorphism.

For the property (v), let $f \in \bigcap_{j\in\mathbb{Z}} V_j$, we have $D^{-j}f \in V_0$. Hence there is an $a \in l^2(L)$ such that $D^{-j}f = \varphi *' a$. Now by applying the hypothesis that the shifts of φ via the lattice L in G is a Riesz family, there exists a constant $B > 0$ such that

$$\begin{aligned}\|a\|_2 &\leqslant B^{-1}(\int_G |D^{-j}f(x)|^2 \mathrm{d}x)^{\frac{1}{2}}.\\ &= B^{-1}\left(\int_G |\delta_\alpha^{\frac{-j}{2}} f(\alpha^{-j}(x))|^2 \mathrm{d}x\right)^{\frac{1}{2}}\\ &= B^{-1}\|f\|_2.\end{aligned} \tag{12}$$

On the other hand,

$$\begin{aligned}|D^{-j}f(x)| &= |\delta_\alpha^{\frac{-j}{2}} f(\alpha^{-j}(x))|\\ &= |(\varphi *' a)(x)|\\ &\leqslant |\sum_{k\in L} \varphi(k^{-1}x)a(k)|\\ &\leqslant \|a\|_\infty \sum_{k\in L} |\varphi(k^{-1}x)|\\ &\leqslant \|a\|_2 \varphi^\circ(x),\end{aligned}$$

for all $x \in G$. Therefore,

$$|f(x)|^2 \leqslant \delta_\alpha^j \|a\|_2^2 (\varphi^\circ(\alpha^j(x)))^2.$$

Now suppose that V is a compact neighborhood of 1_G. We get

$$\int_V |f(x)|^2 \mathrm{d}x \leqslant \delta_\alpha^j \|a\|_2^2 \int_V |\varphi^\circ(\alpha^j(x))|^2 \mathrm{d}x$$
$$= \|a\|_2^2 \int_{\alpha^j V} |\varphi^\circ(x)|^2 \mathrm{d}x.$$

This together with relation (12) implies that,

$$\int_V |f(x)|^2 \mathrm{d}x \leqslant B^{-2} \|f\|_2^2 \int_{\alpha^j V} |\varphi^\circ(x)|^2 \mathrm{d}x.$$

Now for any $\varepsilon > 0$, let U be a neighborhood of 1_G, such that $\lambda(U) < \varepsilon$. Note that φ° is L-periodic and belongs to $L^2(G)$. So the contractivity of α^{-1} implies that for sufficiently small j, $\alpha^{-j}(V) \subseteq U$. Then

$$\int_V |f(x)|^2 \mathrm{d}x \leqslant B^{-2} \|f\|_2^2 \int_U |\varphi^\circ(x)|^2 \mathrm{d}x.$$

Therfore $f = 0$.

For the property (vi), let us consider the refinable function $\varphi \in L^2_\circ(G) \subseteq L^1(G)$ such that the set of shifts of φ forms a Riesz family. Propositions 1 and 2 guarantee $\hat{\varphi}(1_{\widehat{G}}) \neq 0$. After normalization, we may assume $\hat{\varphi}(1_{\widehat{G}}) = 1$. Let U be a compact neighborhood of 1_G, and $U_N = U \ldots U$, (N *factors*). For $j \in \mathbb{Z}$, we define the operator $\mathcal{T}_j$ as follows,

$$\mathcal{T}_j f(\cdot) = D^j[\varphi *' D^{-j}(f|_L)](\cdot) = \sum_{k \in L} f(\alpha^{-j}(k)) \varphi(k^{-1}\alpha^j(\cdot)),$$

in which $f \in C_c(G)$ is supported in U_N, $N \in \mathbb{N}$. Then by using the fact that the set of shifts of φ via the lattice L in G, forms a Riesz family, we have

$$\Big\| \sum_{k \in L} f(\alpha^{-j}(k)) \varphi(k^{-1}\alpha^j(\cdot)) \Big\|_2 \leqslant \delta_\alpha^{-\frac{j}{2}} B \|D^{-j}(f|_L)\|_{l^2(L)}$$
$$\leqslant \delta_\alpha^{-\frac{j}{2}} B \,\mathrm{card}(Q_N)^{\frac{1}{2}} \|f\|_\infty, \tag{13}$$

where $Q_N := \{k \in L;\ \alpha^{-j}(k) \in U_N\}$. Note that the sets $\alpha^{-j}(kS_L)$, $k \in L$, are pairwise disjoint. Moreover, contractivity of α^{-1} implies that, $\alpha^{-j}\overline{(S_L)} \subseteq U$ for sufficiently large j. Thus

$$\bigcup_{k \in Q_N} \alpha^{-j}(kS_L) \subseteq U_N.\alpha^{-j}(S_L) \subseteq U_N.U = U_{N+1}. \tag{14}$$

For each $k \in Q_N$, $\lambda(\alpha^{-j}(kS_L)) = \delta_\alpha^{-j}\lambda(S_L) = \delta_\alpha^{-j}$. This fact together with (14) reveals that, $\mathrm{card}(Q_N).\delta_\alpha^{-j} \leqslant \lambda(U_{N+1})$. This inequality with (13), show that for sufficiently large j,

$$\|\mathcal{T}_j f\|_2 \leqslant \lambda(U_{N+1})^{\frac{1}{2}} B \|f\|_\infty.$$

Hence $\mathcal{T}_j f \in L^2(G)$. The proof is completed by if it is shown that $\mathcal{T}_j f$ converges to $\hat{\varphi}(1_{\widehat{G}})f$ weakly in $L^2(G)$ as $j \to \infty$, that is

$$\lim_{j \to \infty} \int_G \mathcal{T}_j f(x)\bar{g}(x)\mathrm{d}x = \hat{\varphi}(1_{\widehat{G}}) \int_G f(x)\bar{g}(x)\mathrm{d}x,$$

for any $g \in \mathcal{S}_\circ(G)$, the Segal algebra ($\mathcal{S}_\circ(G)$ is dense in $L^2(G)$, see Feichtinger 1979, 1977, 1981).

According to Plancherel formula,

$$\int_G \mathcal{T}_j f(x)\bar{g}(x)\mathrm{d}x = \int_{\widehat{G}} \widehat{\mathcal{T}_j f}(\xi)\hat{\bar{g}}(\xi)\mathrm{d}\xi$$
$$= \int_{\widehat{G}} \delta_\alpha^{-j} \sum_{k \in L} f(\alpha^{-j}(k)) \overline{\xi(\alpha^{-j}(k))} \hat{\varphi}(\hat{\alpha}^{-j}(\xi))\hat{\bar{g}}(\xi)\mathrm{d}\xi. \tag{15}$$

As $\varphi \in L^1(G)$, so $\|\hat{\varphi}\|_\infty \leqslant \|\varphi\|_1$. Hence the integrand (15) is bounded by

$$\lambda(U_{N+1})\|\varphi\|_1 \|f\|_\infty |\hat{\bar{g}}(\xi)|.$$

Since $f \in C_c(G)$ and $\hat{g} \in L^1(\widehat{G})$, by Dominated convergence theorem, the integrand (15) converges pointwise to $\hat{f}(\xi)\hat{\varphi}(1_{\widehat{G}})\hat{\bar{g}}(\xi)$ as $j \to \infty$. Then by applying Proposition 1 and Proposition 2, we may assume $\hat{\varphi}(1_{\widehat{G}}) = 1$, and the desired result is obtained. □

Example 1 Consider Haar wavelet $\varphi = \chi_{[0,1]}$. It satisfies the refinement equation (1) and the set of shifts of φ, via the lattice $\mathbb{Z}$ in $\mathbb{R}$ forms a Riesz family of $L^2(\mathbb{R})$, therefore by Theorem 2, shifts of φ construct an MRA. Moreover, it can be checked that by (Kamyabi and Raisi 2010, Corollary 3.5, Proposition 4.5), $\overline{\bigcup_{j \in \mathbb{Z}} V_j} = L^2(\mathbb{R})$ and $\bigcap_{j \in \mathbb{Z}} V_j = \{0\}$.

Example 2 Consider,

$$G = \{x = (x_n)_{n \in \mathbb{Z}},\ x_n \in Z_2 = \{0,1\},\ \exists N \in \mathbb{Z}\ \ s.t.\ \forall n > N,\ x_n = 0\},$$

with the operation given by

$$(x^1 + x^2)_n = x_n^1 + x_n^2 \quad mod\ 2,$$

then G is an LCA group. We identify G with $[0, \infty)$ as a measure space by $x \to |x|$ where $|x| = \sum_{j \in \mathbb{Z}} x_j 2^j$. This induces the Haar measure of $[0, \infty)$ on G. We will be interested in the following subgroups,

$$L = \{x \in G, x_j = 0\ for\ j < 0\},$$
$$\frac{G}{L} = \{x \in G, x_j = 0\ for\ j \geqslant 0\}.$$

The subgroup $\frac{G}{L}$ is known as the Cantor group. We have that L is countable, closed, discrete and that $\frac{G}{L}$ is compact (see Lang 1996 for more details). Consider the Hilbert space $H = L^2(G, \lambda_G)$. The dilation $D: H \to H$ and translation $T: H \to H$ are defined respectively by $(Df(x))_j = f(x_{j-1})$ and $T_k f(x) = f(x - k)$ for $f \in H, x \in G, k \in L$. The dual group of G, $\widehat{G}$, is isomorphic to G, and that characters are given by

$$<x, \xi> = \prod_{j\in\mathbb{Z}} (-1)^{\xi_{-1-j}x_j},$$

for $x \in G, \xi \in \widehat{G}$.

Let the scaling function be $\varphi(x) = \chi_{\frac{G}{L}}(x)$, the characteristic function of $\frac{G}{L}$. We have $(D^{-1}\varphi)(x) = \varphi(x) + \varphi(x+1)$, so $\chi_{\frac{G}{L}}$ is satisfied the refinable equation and shifts of φ via the lattice L in G, are an orthonormal basis of H. suppose $V_0 = S_2(\varphi)$ and $V_j = D^j V_0$, then by Theorem 2, $\{V_j\}_{j\in\mathbb{Z}}$, construct a multiresolution approximation of H.

Example 3 Let

$$\varphi(x) = \begin{cases} 1, & \text{if } 0 \leqslant x < 1 \\ \frac{1}{2}, & \text{if } 1 \leqslant x < 2 \, . \\ 0, & \text{o.w.} \end{cases}$$

Then it can be checked that the set of integer translates of φ, $\{\varphi(\cdot - \alpha)\}_{\alpha\in\mathbb{Z}}$, forms a Riesz family, and φ satisfies the refinement equation $\varphi = \sum_{j\in\mathbb{Z}} b(j)\varphi(2\cdot - j)$, where the mask b is given by $b(z) = (z+1)(z^2+2)/(z+2)$. Thus the integer shifts of φ construct an MRA.

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