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RESEARCH PAPER

On q-Statistical Summability Method and Its Properties

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Abstract

In this paper, we discussed a regular summability method called q -statistical convergence. Two new sequence spaces m_*^q and s_*^q are also obtained. A condition for a q-statistically convergent sequences to be q-Cesàro summable is given. Necessary and sufficient conditions for real sequences and the sequences in m_*^q to be q-statistical convergent are obtained. Further, we prove that the set of all q-statistical convergent sequences is dense and of first Baire catagory in the Fréchet metric space and $s_*^q(L)$ is a set of second Borel class in the space m_*^q.

Keywords Sequence spaces · Banach spaces · Statistical convergence · Regular matrix

Mathematics Subject Classification 40B05 · 40H05 · 54A20

1 Introduction and Preliminaries

Till 19th century, mathematician had little temptation to use divergent series. According to Abel the divergent series is the interpretation of the devil and it is shameful to base on them any demonstration whatsoever. However, in 1890, Cesàro published a paper about the multiplication of series, which Hardy demonstrated first time that a theory of divergent series was formulated explicitly and it was kind of a start for proper summability theory. Cesàro's idea proved to be very fruitful, infact one of the first few applications was the beautiful Fejer theorem, which was developed by applying Cesàro's idea to Fourier series. The simplest form of Cesàro idea is given in Theorem 1.1 below.

Theorem 1.1 *The limit of sequence* (x_n) *can be defined to be* $\lim_n y_n$, *where* y_n *is the sequence of Cesàro means of the sequence* (x_n) *given by*

$$y_n = \frac{1}{n+1}\sum_{i=0}^{n} x_i. \tag{1.1}$$

Different summability methods have been introduced over the years. But the most efficient one would be those which are regular, i.e., the limit of a convergent sequence or sum of a series will not be changed if it exists. Toeplitz (1911) gives the following conditions for an infinite matrix to be regular.

Theorem 1.2 *A matrix* $A = (a_{mn})$ *is regular if and only if the following holds,*

(i) $\lim_{m\to\infty} a_{mn} = 0, \ n = 0, 1, 2, \ldots,$

(ii) $\lim_{m\to\infty} \left(\sum_{n=0}^{\infty} a_{mn}\right) = 1,$

(iii) $\sup_m \sum_{n=0}^{\infty} |a_{mn}| < \infty.$

Statistical convergence is also a type of regular summability method that gives statistical limit to some divergent and all convergent sequences. The idea of statistical convergence was introduced by Fast (1951) and

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Steinhaus (1951) in 1951 and later developed by Schoenberg (1959). See also Fridy (1985) and Connor (1988).

A sequence $x = (x_k)$ is said to be statistically convergent to a number L , if for a given $\epsilon > 0$,

$$\lim_n \frac{1}{n}|\{k \leq n : |x_k - L| \geq \epsilon\}| = 0,$$

where the vertical bar indicates the number of elements in a set.

Freedman and Sember (1981) showed that each non-negative regular matrix A can be associated by a density function

$$\delta_A(K) = \liminf_{n\to\infty}(A\chi_K)_n. \tag{1.2}$$

The q-calculus emerged as a very useful tool and a very fruitful connection between Mathematics and Physics. We recall here some basic definitions and notations about the q-calculus Kac and Cheung (2002). We take $\mathbb{C}$ as the set of complex numbers and $\mathbb{N}$ the set of positive integers.

Definition 1.1 Let $q \in \mathbb{C} \setminus \{0, 1\}$. Then, the q-number is defined by

$$[\theta]_q = \begin{cases} \dfrac{1-q^\theta}{1-q} & (\theta \in \mathbb{C} \setminus \{0\}) \\ 1 & (\theta = 0) \\ \sum_{s=0}^{r-1} q^s = 1 + q + q^2 + \cdots + q^s & (\theta \in \mathbb{N}). \end{cases} \tag{1.3}$$

Definition 1.2 For number $q \in \mathbb{C} \setminus \{0, 1\}$, the q -factorial is defined by

$$[\theta]_q! = \begin{cases} 1 & (\theta = 0) \\ \prod_{\theta=1}^{r} [\theta]_q & (\theta \in \mathbb{N}). \end{cases} \tag{1.4}$$

On replacing A by C_1 and lim inf by an ordinary limit in (1.2), we obtain the well-known natural density function and statistical convergence. Aktuğlu and Bekar Aktuğlu and Bekar (2011) gives the most suitable q-analog of Cesàro matrix and also showed that it is regular. Replacing A by this new q-Cesàro matrix they obtained q -density and defined q-statistical convergence.

Definition 1.3 Let $x = (x_k)$ be a number sequence. It is said to be Cesàro summable to L, if

$$\lim_n \frac{1}{n}\sum_{k=0}^{n} x_k = L.$$

Definition 1.4 The q-analog of Cesàro matrix is given by $C_1(q^k) = (c^1_{nk}(q^k))$, where

$$c^1_{nk}(q^k) = \begin{cases} \dfrac{q^k}{[n+1]_q}, & \textit{if } k \leq n, \\ 0, & \textit{otherwise}. \end{cases} \tag{1.5}$$

Definition 1.5 Let $K \subseteq \mathbb{N}$. For $q \geq 1$, the q-density of K is given by

$$\delta_q(K) = \delta_{C_1^q}(K) = \lim_{n\to\infty}\inf(C_1^q\chi_K)_n.$$

Definition 1.6 A number sequence $x = (x_k)$ is said to be q-statistically convergent to L, if for every $\epsilon > 0$, $\delta_q(K) = 0$, where $K = \{k : k \leq n : |x_k - L| \geq \epsilon\}$.

The set of all q-statistically convergent sequences is denoted by $S\ q$. For q-statistically convergence for double sequences, see Çinar and Et (2020). Recently, the notion of q-statistical convergence has been applied in Approximation Theory Al-Abied et al. (2021), Ayman Mursaleen and Serra-Capizzano (2022), Cai et al. (2022), Chen et al. (2022).

In this paper, we discuss q-statistical convergence using q-Cesàro matrix given by Aktuğlu and Bekar in Aktuğlu and Bekar (2011). We have extended some results of statistical convergence of Salat Šalát (1980) and Schoenberg Schoenberg (1959). We have given necessary and sufficient condition for a sequence to be q -statistical convergent. We have also shown that the set of all q -statistically convergent sequences is dense and of first Baire catagory in the Fréchet metric space. We define and study two new spaces m^q_* and s^q_* as q-analogs.

2 Main Result

Proposition 2.1 *If A and B are two subsets of $\mathbb{N}$ such that $A \subset B$, then $\delta_q(A) \leq \delta_q(B)$.*

Proof Since $A \subset B$,

$$\begin{aligned}\delta_q(A) &= \liminf_{n\to\infty}\sum_{k\in A}\frac{q^{k-1}}{[n]_q} \\ &\leq \liminf_{n\to\infty}\sum_{k\in B}\frac{q^{k-1}}{[n]_q} = \delta_q(B).\end{aligned}$$

Proposition 2.2 *If $A \subseteq \mathbb{N}$, then $\delta_q(A) + \delta_q(A^c) = 1$.*

Lemma 2.1 *If $x = (x_k)$ is q-statistically convergent to L and (x_k) is bounded, then x is q-Cesàro summable to L.*

Proof Let $x = (x_k)$ be q-statistically convergent to L. Then, for every $\epsilon > 0$

$$\delta_q(K_\epsilon) = \delta_q(\{k : |x_k - L| \geq \epsilon\}) = 0, \text{ or}$$

$$\delta_q(K_\epsilon) = \liminf_{n\to\infty} \sum_{k\in K_\epsilon} \frac{q^{k-1}}{[n]_q} = 0.$$

Also, since x is bounded, there exists $M > 0$ such that

$$|x_k| < M, \quad \forall k \in \mathbb{N}.$$

For a given $\epsilon > 0$, let $N_n = |\{1 \leq j \leq n : |q^{j-1}x_j| \geq \epsilon\}|$ for fix q. Without loss of generality we may assume $L = 0$. Now,

$$\begin{aligned}\left|\sum_{k=1}^{n} c_{nk}^1(q^k)x_k\right| &= \left|\sum_{k=1}^{n} \frac{q^{k-1}}{[n]_q} x_k\right| \\ &= \left|\frac{x_1 + qx_2 + q^2x_3 + \cdots + q^{n-1}x_n}{[n]_q}\right| \\ &\leq \frac{1}{[n]_q}\left\{|x_1| + |qx_2| + \left|q^2x_3\right| + \cdots + \left|q^{n-1}x_n\right|\right\} \\ &= \frac{M}{[n]_q} \sum_{k\leq n:|x_k|\geq\epsilon} q^{k-1} + \frac{\epsilon}{[n]_q} \sum_{k\leq n:|x_k|<\epsilon} q^{k-1}.\end{aligned} \tag{2.1}$$

when n is very big, the right hand side will be less than 2ϵ. Hence, x is q-Cesàro summable to L.

Lemma 2.2 *If (x_k) is q-statistically convergent to L and f(x) is continuous at $x = L$ for all $x \in \mathbb{R}$, then $f(x_k)$ is q-statistically convergent to f(L).*

Proof Assume that $x = (x_k)$ is q-statistically convergent to L. Let $\{K_\epsilon = k : |x_k - L| \geq \epsilon\}$. Then for every $\epsilon > 0$,

$$\liminf_{n\to\infty} \sum_{k\in K_\epsilon} \frac{q^{k-1}}{[n]_q} = 0.$$

Since $f(x)$ is continuous at $x = L$, for a given $\epsilon > 0$, $\exists\delta > 0$ such that

$$|f(x) - f(L)| < \epsilon \text{ whenever } |x - L| < \delta.$$

Thus,

$$|f(x) - f(L)| \geq \epsilon \text{ implies } |x - L| \geq \delta.$$

In particular,

$$|f(x_k) - f(L)| \geq \epsilon \text{ implies } |x_k - L| \geq \delta.$$

Therefore,

$$\{k : |f(x_k) - f(L)| \geq \epsilon\} \subset \{k : |x_k - L| \geq \delta\}.$$

Using Proposition 2.1, we get

$$\begin{aligned}&\delta_q(\{k : |f(x_k) - f(L)| \\ &\quad \geq \epsilon\}) \leq \delta_q(\{k : |x_k - L| \geq \delta\}) \\ &\quad = \lim_{n\to\infty} \inf \sum_{k\in K_\epsilon} \frac{q^{k-1}}{[n]_q} = 0.\end{aligned}$$

So, $f(x_k)$ is q-statistically convergent to $f(L)$.

Theorem 2.1 *The sequence $x = (x_k)$ is q-statistical convergence to L if and only if for each $r \in \mathbb{R}$*

$$\lim_{n\to\infty} \frac{1}{[n]_q}\left(e^{irx_1} + qe^{irx_2} + \ldots + q^{n-1}e^{irx_n}\right) = e^{irL}. \tag{2.2}$$

Proof of necessity. Indeed e^{irx} is a continuous function at a fixed value of r. To prove the necessity, let's take $f(x) = e^{irx}$ in Lemma 2.2. Then, we get that $f(x_k) = e^{irx_k}$ is q-statistically convergent to $f(L) = e^{irL}$.

Since (e^{irx_k}) is a bounded sequence and it is a q-statistically convergent to e^{irL}, using Lemma 2.1, (e^{irx_k}) is q-Cesàro summable to e^{irL} or

$$\lim_{n\to\infty} \frac{1}{[n]_q}\left(e^{irx_1} + qe^{irx_2} + \cdots + q^{n-1}e^{irx_n}\right) = e^{irL}.$$

Proof of sufficiency. To show that for $r \in \mathbb{R}$,

$$\lim_{n\to\infty} \frac{1}{[n]_q}\left(e^{irx_1} + qe^{irx_2} + \cdots + q^{n-1}e^{irx_n}\right) = e^{irL},$$

implies that q-stat $x_k = L$, it suffices to show that it holds for $L = 0$. From Eq. (2.2) we get

$$\lim_{n\to\infty} \frac{1}{[n]_q}\left(e^{ir(x_1-L)} + qe^{ir(x_2-L)} + \cdots + q^{n-1}e^{ir(x_n-L)}\right) = 1.$$

We may prove that q-stat$(x_k - L) = 0$. But q-$statL = L$. Adding these two we get the desired result.Now, we assume that

$$\lim_{n\to\infty} \frac{1}{[n]_q}\left(e^{irx_1} + qe^{irx_2} + \cdots + q^{n-1}e^{irx_n}\right) = 1, \tag{2.3}$$

and will show that

$$\liminf_{n} \frac{1}{[n]_q} \sum_{k:|x_k|\geq\epsilon} q^{k-1}x_k = 0, \tag{2.4}$$

or x is q-statistically convergent to 0. Let us take a continuous function $M(x)$ Schoenberg (1959), where

$$M(x)=\begin{cases}0, & \text{if } x<-1,\\ 1+x, & \text{if } -1\le x<0,\\ 1-x, & \text{if } 0\le x<1,\\ 0, & \text{if } x\ge 1.\end{cases} \tag{2.5}$$

With different approaches, for instance by Cauchy's calculus of residues, one can show that for $(-\infty<x<\infty)$, $M(x)$ allows the following integral representation,

$$M(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{\sin r/2}{r/2}\right)^2 e^{ixr}dr. \tag{2.6}$$

Let $r=\epsilon l$ in Eq. (2.6),

$$\begin{aligned}M(x)&=\frac{\epsilon}{2\pi}\int_{-\infty}^{\infty}\left(\frac{\sin \epsilon l/2}{\epsilon l/2}\right)^2 e^{ix\epsilon l}dl.\\ M(x/\epsilon)&=\frac{\epsilon}{2\pi}\int_{-\infty}^{\infty}\left(\frac{\sin \epsilon r/2}{\epsilon r/2}\right)^2 e^{ixr}dr.\end{aligned} \tag{2.7}$$

Since Eq. (2.7) is an absolutely convergent integral, i.e., in the sense of Lebesgue we can write

$$\frac{1}{[n]_q}\sum_{k=1}^{n}q^{k-1}M(x_k/\epsilon)=\frac{\epsilon}{2\pi}\int_{-\infty}^{\infty}\left(\frac{\sin\epsilon r/2}{\epsilon r/2}\right)^2\left\{\frac{1}{[n]_q}\sum_{k=1}^{n}q^{k-1}e^{ix_k r}\right\}dr.$$

If Eq. (2.2) holds, then for all real r and n

$$\left|\frac{1}{[n]_q}\sum_{k=1}^{n}q^{k-1}e^{ix_k r}\right|\le 1.$$

Using the Bounded Convergence Theorem we obtain

$$\lim_{n\to\infty}\frac{1}{[n]_q}\sum_{k=1}^{n}q^{k-1}M(x_k/\epsilon)=\frac{\epsilon}{2\pi}\int_{-\infty}^{\infty}\left(\frac{\sin\epsilon r/2}{\epsilon r/2}\right)^2dr=M(0)=1. \tag{2.8}$$

$$\begin{aligned}\sum_{k=1}^{n}q^{k-1}M(x_k/\epsilon)&=\{M(x_1/\epsilon)+qM(x_2/\epsilon)+\cdots+q^{n-1}M(x_n/\epsilon)\}\\ &=\sum_{k\le n:|x_k|\ge\epsilon}q^{k-1}M(x_k/\epsilon)+\sum_{k\le n:|x_k|<\epsilon}q^{k-1}M(x_k/\epsilon).\end{aligned} \tag{2.9}$$

For $|x_k|\ge\epsilon$, $M(x_k/\epsilon)=0$ and for $|x_k|<\epsilon$, $M(x_k/\epsilon)=1-\frac{1}{|x_k|/\epsilon}<1$. On applying this observation in Eq. (2.8) we get

$$\sum_{k=1}^{n}q^{k-1}M(x_k/\epsilon)\le\sum_{k\le n:|x_k|<\epsilon}q^{k-1}. \tag{2.10}$$

Then,

$$\lim_{n\to\infty}\frac{1}{[n]_q}\sum_{k=1}^{n}q^{k-1}M(x_k/\epsilon)\le\lim_{n\to\infty}\frac{1}{[n]_q}\sum_{k\le n:|x_k|<\epsilon}q^{k-1}. \tag{2.11}$$

From Eqs. (2.8) and (2.11),

$$\lim_{n\to\infty}\frac{1}{[n]_q}\sum_{k\le n:|x_k|<\epsilon}q^{k-1}\ge 1.$$

So,

$$\liminf_{n\to\infty}\frac{1}{[n]_q}\sum_{k\le n:|x_k|<\epsilon}q^{k-1}\ge 1. \tag{2.12}$$

Equation (2.12) gives the q-density of the set $\{k\le n:|x_k|<\epsilon\}$, so it cannot be greater than 1. Using Proposition 2.2 we established Eq. (2.4). This completes the proof of our theorem.

Now we will show some results between the set of q-statistically convergent sequences and Fréchet metric space of all real sequences. Let s denote the Fréchet metric space of all real sequences with the metric d_n, where

$$d_n=\sum_{k=0}^{\infty}\frac{1}{2^k}\frac{|x_k-y_k|}{1+|x_k-y_k|}, \tag{2.13}$$

for all $x=(x_k),y=(y_k)\in s$.

Theorem 2.2 *The set of all q-statistically convergent sequences is dense in s.*

Proof Let $S(C_1^q)$ be the set of all q-statistically convergent sequences. Let $x=(x_k)\in S(C_1^q)$ and $y=(y_k)$ be the sequence of real numbers differ from x only in a finite number of terms. Obviously $y\in S(C_1^q)$. As s is a complete metric space with respect to the translation-invariant metric given by Equation (2.13), $S(C_1^q)$ is dense in s. □

Lemma 2.3 (Salat) Šalát (1980) *Let g_k $(k=0,1,2,\ldots)$ be a complex valued continuous functions on $\mathbb{R}$ and c_1, c_2 be two distinct complex numbers such that for each sufficiently large k, we have c_1, $c_2\in g_k(\mathbb{R})$. Let (a_{nk}) be a triangular matrix with the following properties:*

(P_1) *For each fixed k, we have* $\lim_{n\to\infty}a_{nk}=0$;

(P_2) $\lim_{n\to\infty}\sum_{k=0}^{n}a_{nk}=1.$

Then the set s_1 of all such $x=(\xi_k)\in s$ for which there exists a finite limit $\lim_{n\to\infty}\sum_{k=0}^{n}a_{nk}g_k(\xi_k)$ is a set of the first Baire category in s.

Theorem 2.3 *The set of all q-statistically convergent sequences is a set of first Baire category in the space s.*

Proof Let s_1^q be the set of all $x = (x_k)$ in s such that the limit

$$\lim_{n\to\infty} \frac{1}{[n]_q}\left(e^{ix_1} + qe^{ix_2} + \cdots + q^{n-1}e^{ix_n}\right) \tag{2.14}$$

is finite. In Lemma 2.3, let's put $g_n(r) = e^{ir}$ $(n = 1, 2, \ldots)$ and

$$(a_{mn}) = C_1^q = \begin{cases} \frac{q^{k-1}}{[n]_q}, & \text{if } k < n, \\ 0, & \text{otherwise.} \end{cases} \tag{2.15}$$

It is clear that q-Cesàro matrix C_1^q satisfies properties P_1 and P_2. Also,

$$\lim_{n\to\infty} \sum_{k=0}^{n} \frac{q^{k-1}}{[n]_q} e^{ix_k} = \lim_{n\to\infty} \frac{1}{[n]_q}\left(e^{ix_1} + qe^{ix_2} + \ldots + q^{n-1}e^{ix_n}\right) \tag{2.16}$$

is finite. Hence s_1^q is a set of first Baire category in s. But Theorem 2.1 tells us that the set of all q-statistically convergent sequence is a subset of the space s, which concludes that the set of all q-statistically convergent sequences is a set of first Baire category in the space s, which completes the proof. □

Let us denote a subspace m_*^q of the space s by

$$m_*^q = \left\{ (x_k) \in s : \frac{1}{[n]_q} \sum_{k=0}^{n} q^{k-1}|x_k| \text{ is bounded} \right\}. \tag{2.17}$$

Also for $L \in \mathbb{R}$, let us denote the set $s_*^q(L)$ by

$$s_*^q(L) = \{(x_k) \in s : (x_k) \text{ is q-statistically convergent to L}\}. \tag{2.18}$$

In our next result, we will show that $s_*^q(L)$ is a set of second Borel class in the space m_*^q.

Lemma 2.4 *The sequence $x = (x_k) \in m_*^q$ is q-statistically convergent to L if and only if for each $r \in \mathbb{Q}$*

$$\lim_{n\to\infty} \frac{1}{[n]_q}\left(e^{irx_1} + qe^{irx_2} + \cdots + q^{n-1}e^{irx_n}\right) = e^{irL}. \tag{2.19}$$

Proof Let $r \in \mathbb{Q}$. Being a rational number, r is also a real number and hence the necessary part of theorem holds according to Theorem 2.1.

For the sufficient part, let Equation (2.19) hold for each $r \in \mathbb{Q}$. We shall prove that for each $r' \in \mathbb{R}$

$$\lim_{n\to\infty} \frac{1}{[n]_q}\left(e^{ir'x_1} + qe^{ir'x_2} + \ldots + q^{n-1}e^{ir'x_n}\right) = e^{ir'L}. \tag{2.20}$$

From this, $x = (x_k)$ will be q-statistically convergent to L according to Theorem 2.1. For $t \in \mathbb{R}$, let

$$A_n^q(r', r) = \frac{1}{[n]_q} \sum_{k=0}^{n} q^{k-1} e^{ir'x_k} - \frac{1}{[n]_q} \sum_{k=0}^{n} q^{k-1} e^{irx_k}. \tag{2.21}$$

After squaring and then taking square root of both sides of Equation (2.21) we get

$$|A_n^q(r', r)| = \frac{1}{[n]_q} \sum_{k=0}^{n} q^{k-1} \sqrt{\left((\cos r'x_k - \cos rx_k)^2 + (\sin r'x_k - \sin rx_k)^2\right)}$$
$$= \frac{1}{[n]_q} \sum_{k=0}^{n} q^{k-1} \sqrt{4 \sin^2 \frac{(r - r')x_k}{2}}.$$

On applying Mean Value Theorem we get

$$|A_n^q(r', r)| \le \frac{\sqrt{2}}{[n]_q} |r - r'| \sum_{k=0}^{n} q^{k-1}|x_k|.$$

Since (x_k) is a sequence in m_*^q, there exists a $K > 0$ such that

$$\frac{1}{[n]_q} \sum_{k=0}^{n} q^{k-1}|x_k| \le K. \tag{2.22}$$

Hence,

$$|A_n^q(r', r)| \le \sqrt{2}K|r - r'|. \tag{2.23}$$

$$\frac{1}{[n]_q} \sum_{k=0}^{n} q^{k-1} e^{ir'x_k} = \frac{1}{[n]_q} \sum_{k=0}^{n} q^{k-1} e^{irx_k} + A_n^q(r', r). \tag{2.24}$$

$$\left| \frac{1}{[n]_q} \sum_{k=0}^{n} q^{k-1} e^{ir'x_k} - e^{ir'L} \right|$$
$$\le \left| \frac{1}{[n]_q} \sum_{k=0}^{n} q^{k-1} e^{irx_k} - e^{irL} \right| + \left| e^{irL} - e^{ir'L} \right| + \left| A_n^q(r', r) \right|. \tag{2.25}$$

For a given $\epsilon > 0$, from Eq. (2.19) there exists an $n_0 \in \mathbb{N}$ such that

$$\left| \frac{1}{[n]_q} \sum_{k=0}^{n} q^{k-1} e^{irx_k} - e^{irL} \right| \le \frac{\epsilon}{2} \tag{2.26}$$

for each $n \ge n_0$. Also from Equation (2.232.242.25) and using the continuity of exponential functions we can choose a rational number r such that

$$|A_n^q(r', r)| \le \frac{\epsilon}{4} \tag{2.27}$$

and

$$\left| e^{irL} - e^{ir'L} \right| \le \frac{\epsilon}{4}. \tag{2.28}$$

Hence

$$\left| \frac{1}{[n]_q} \sum_{k=0}^{n} q^{k-1} e^{ir'x_k} - e^{ir'L} \right| \le \epsilon \tag{2.29}$$

for each $n \geq n_0$.

Since r' was an arbitrary real number, we directly get our desired result using Theorem 2.1. □

Theorem 2.4 *The set $s_*^q(L)$ is a set of second Borel class in the space m_*^q.*

Proof Let $r \in \mathbb{Q}$. Then from Lemma 2.4, we can write

$$s_*^q(L) = \bigcap_{r\in\mathbb{Q}} \bigcap_{i=1}^{\infty} \bigcup_{j=1}^{\infty} \bigcup_{n=j+1}^{\infty} M(n,i), \tag{2.30}$$

where

$$M(n,i) = \left\{ x = (x_k) \in m_*^q : \left| \frac{1}{[n]_q} \sum_{k=0}^{n} e^{irx_k} - e^{irL} \right| \leq \frac{1}{i} \right\}. \tag{2.31}$$

But for each n, i in $\mathbb{N}$, $M(n, i)$ is closed in m_*^q. Hence, we directly get our result from Equations (2.30) and (2.31). □

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