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RESEARCH PAPER

Signed Domination Number of Some Graphs

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Abstract

Let $G=(V,E)$ be a simple graph. A function $f:V\longrightarrow\{-1,1\}$ is a signed dominating function if for every vertex $v\in V$, the closed neighborhood of v contains more vertices with function value 1 than with -1. The weight of a function f is $\omega(f)=\sum_{v\in V}f(v)$. The signed domination number of G, $\gamma_s(G)$, is the minimum weight of a signed dominating function on G. A signed dominating function of weight $\gamma_s(G)$ is called $\gamma_s(G)$-function. A $\gamma_s(G)$-function can also be represented by a set of ordered pairs $S_f=\{(v,f(v)):v\in V\}$. A subset T of S_f is called a forcing subset of S_f if S_f is the unique extension of T to a $\gamma_s(G)$-function. The forcing signed domination number of S_f $f(S_f,\gamma_s)=\min\{|T|: T \text{ is a forcing subset of } S_f\}$ and the forcing signed domination number of G, $f(G,\gamma_s)$, is defined by $f(G,\gamma_s)=\min\{f(S_f,\gamma_s): S_f \text{ is a } \gamma_s(G)\text{-function}\}$. In this paper, we deal with the signed domination number of several classes of graph. Also the forcing signed domination number of some graphs are determined.

Keywords Signed domination number · Forcing signed domination number · Cartesian product · Direct product · Join graph

1 Introduction

Let G be a simple graph with vertex set V and edge set E. The graph G is a complete t-partite graph if there is a partition $V=V_1\cup\ldots\cup V_t$ of the vertex set, such that two vertices v_1 and v_2 are adjacent if and only if v_1 and v_2 are in the different parts of the partition. If $|V_k|=n_k$ $(1\le k\le t)$, then G is denoted by $K_{n_1,\ldots,n_t}$. Let G_1 and G_2 be two graphs with vertex sets V_1 and V_2 and edge sets E_1 and E_2, respectively. The *Cartesian product* $G=G_1\Box G_2$ has the vertex set $V_1\times V_2$, and two vertices (u_1,u_2) and (v_1,v_2) of G are adjacent if and only if either $u_1=v_1$ and $u_2v_2\in E_2$ or $u_2=v_2$ and $u_1v_1\in E_1$. The *direct product* of G and H, denoted by $G\times H$ whose vertex set is $V(G)\times V(H)$, and for which vertices (g,h) and (g',h') are adjacent precisely if $gg'\in E(G)$ and $hh'\in E(H)$. We say that $G=(V,E)$ is a join graph if G is the complete union of two graphs $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$. In other words, $V=V_1\cup V_2$ and $E=E_1\cup E_2\cup\{uv|u\in V_1,v\in V_2\}$. If G is the join graph of G_1 and G_2, then we write $G=G_1+G_2$.

For a vertex $v\in V$, the closed neighborhood $N[v]$ of v is the set consisting of v and all of its neighbors. For a function $g:V\longrightarrow\{-1,1\}$ and a vertex $v\in V$ we define $g(N[v])=\sum_{u\in N[v]}g(u)$. A *signed dominating function* of G is a function $g:V\longrightarrow\{-1,1\}$ such that $g(N[v])>0$ for all $v\in V$. The *weight* of a function g is $\omega(g)=\sum_{v\in V}g(v)$. The *signed domination number*, $\gamma_s(G)$, is the minimum weight of all signed dominating functions on G. A signed dominating function of weight $\gamma_s(G)$ is called a $\gamma_s(G)$-function. This concept was defined in Dunbar et al. (1995) and has been studied by several authors (see for instance Dunbar et al. 1995; Favaron 1996; Füredi and Mubayi 1999; Haas and Wexler 2004; Volkman and Zelinka 2005; Zelinka 2006). We denote $g(N[v])$ by $g[v]$. Also for signed dominating function g, the set $\{v\in V: g(v)=-1\}$ denoted by V_g^-.

A signed dominating function g of G can also be represented by $S_g=\{(v,g(v))\,|\,v\in V\}$. Let g be a $\gamma_s(G)-$function. A subset T of S_g is called a *forcing subset* of S_g, if S_g is the unique extension of T to a $\gamma_s(G)-$function. The *forcing signed domination number* of S_g,

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$f(S_g, \gamma_s)$, is defined by $f(S_g, \gamma_s) = \min\{|T| \mid T$ is a forcing subset of $S_g\}$. The *forcing signed domination number* of G is defined by $f(G, \gamma_s) = \min\{f(S_g, \gamma_s) \mid S_g$ is a $\gamma_s(G)-$function $\}$. The forcing signed domination number was introduced by Sheikholeslami in Sheikholeslami (2007) in which $f(G, \gamma_s)$ determined for several classes of graphs.

In this paper, we compute the signed domination number of several classes of graph, including $P_n + K_1$, $P_2 \Box K_{1,n}$, $P_2 \Box K_n$, $K_n \times K_m$ and $K_{n,\ldots,n}$. Also, the forcing signed domination numbers of some graphs are determined.

2 Main Results

We begin with the following lemma to obtain the forcing signed domination number of complete graph K_n.

Lemma 1 Füredi and Mubayi (1999) *Let G be a complete graph of order n. Then*

$$\gamma_s(G) = \begin{cases} 1 & \text{if } n \text{ is odd;} \\ 2 & \text{if } n \text{ is even.} \end{cases}$$

Theorem 1 *The forcing signed domination number of complete graph K_n is $\lceil \frac{n}{2} \rceil - 1$.*

Proof Let $V(K_n) = \{v_1, v_2, \ldots, v_n\}$ and f be a γ_s-function of K_n. We have $|V_f^-| = \lceil \frac{n}{2} \rceil - 1$. Without loss of generality, suppose that $f(v_i) = -1$ for $i = 1, 2, \ldots, \lceil \frac{n}{2} \rceil - 1$ which implies that $T = V_f^-$ is a forcing set of S. Hence, $f(G, \gamma_S) \leq \lceil \frac{n}{2} \rceil - 1$. On the contrary, assume that T' is a forcing subset and $|T'| < \lceil \frac{n}{2} \rceil - 1$. Since all vertices are adjacent, there are at least $\lceil \frac{n}{2} \rceil + 1$ distinct extensions of T' to a γ_S-function. This is a contradiction. □

In the following two theorems, we consider the join graph of P_n and K_1 and we determine the signed domination number as well as the forcing signed domination number of $P_n + K_1$. Let $V(P_n + K_1) = v_0, v_1, \ldots, v_n$ where $deg(v_0) = n$. If $n = 1, 2$, then $P_n + K_1$ is isomorphic to P_2, K_3 and so $\gamma_s(P_n + K_1) = 2, 1$, respectively. Also it is not hard to see that $\gamma_s(P_3 + K_1) = 2$. For $n \geq 4$, we deduce the following theorem.

Theorem 2 *For $n \geq 4$, $\gamma_S(P_n + K_1) = n + 1 - 2\lceil \frac{n}{3} \rceil$.*

Proof Define $g : V \longrightarrow \{-1, 1\}$ where $g(v_i) = -1$ if and only if $i = 3k + 1$ and $0 \leq k \leq \lfloor \frac{n-1}{3} \rfloor$. For any $1 \leq i \leq n$, $\left| N[v_i] \cap V_g^- \right| = 1$ and $deg(v_i) \geq 2$, so g is a signed dominating function and $\omega(g) = n + 1 - 2\lceil \frac{n}{3} \rceil$. Hence, $\gamma_S(P_n + K_1) \leq n + 1 - 2\lceil \frac{n}{3} \rceil$. On the other side, let h be a γ_s-function of $P_n + K_1$ where $h(v_0) = -1$. Since $2 \leq deg(v_i) \leq 3$ for $1 \leq i \leq n$, so $h(v_i) = 1$. Hence, $\omega(h) = n - 1$ and as a result of that $\omega(h) > n + 1 - 2\lceil \frac{n}{3} \rceil$. This is a contradiction. Thus, $h(v_0) = 1$. If $h(v_i) = h(v_j) = -1$, then $|i - j| \geq 3$. As a consequence $|V_h^-| \leq \lceil \frac{n}{3} \rceil$. Therefore, $\omega(h) \geq n + 1 - 2\lceil \frac{n}{3} \rceil$. This completes the proof. □

Lemma 2 Sheikholeslami (2007) *For a graph G, $f(G, \gamma_s) = 0$ if and only if G has a unique γ_s-function. Moreover, $f(G, \gamma_s) = 1$ if and only if G does not have a unique γ_s-function but some pair $(v, \pm 1)$ belongs to exactly one γ_s-function.*

Theorem 3 *For $n \geq 1$,*

$$f(P_n + K_1, \gamma_S) = \begin{cases} 0 & n \equiv 1 \pmod{3}; \\ 1 & \text{otherwise.} \end{cases}$$

Proof Let k be a positive integer. There are three cases:

Case 1: Let $n = 3k + 1$ and consider the γ_s-function g which is defined in proof of Theorem 2. Let h be another γ_s-function of $P_n + K_1$. We show that $h = g$. On the contrary, suppose that $g \neq h$. So for some $0 \leq i \leq k - 1$, $V_h^- \cap \{v_{3i}, v_{3i+2}\} \neq \emptyset$. Hence, $h(v_{3i\pm1}) = h(v_{3i\pm2}) = 1$ or $h(v_{3i}) = h(v_{3i\pm1}) = 1$. So $|V_h^-| \leq k$. Thus, $\omega(h) > \gamma_s(P_n + K_1)$. This is a contradiction. Thus, $g = h$ and by Lemma 2, $f(P_n + K_1, \gamma_s) = 0$.

Case 2: Let $n = 3k$ and let h be a γ_s-function of $P_n + K_1$ such that $V_h^- = \{v_3, v_6, \ldots, v_n\}$. Then, in each induced subgraph of $\{v_1, v_2, \ldots, v_n\}$ which is isomorphic to P_3, there is exactly one vertex with label -1. Let $T = \{(v_3, -1)\}$. Then, T is a forcing subset of h and $f(P_n + K_1, \gamma_s) \leq 1$. Since there are more than one γ_s-function for $P_n + K_1$, $f(P_n + K_1, \gamma_s) \geq 1$ and so $f(P_n + K_1, \gamma_s) = 1$.

Case 3: Let $n = 3k + 2$ and let g, h be two γ_s-functions of $P_n + K_1$ such that $V_g^- = \{v_2, v_5, \ldots, v_n\}$ and $V_h^- = \{v_1, v_4, \ldots, v_{n-1}\}$. By Lemma 2, $f(P_n + K_1, \gamma_s) \geq 1$. Let $T = \{(v_2, -1)\}$ be a subset of g. Since $|V_g^-| = \lceil \frac{n}{3} \rceil$, so T is a forcing subset. Thus, $f(P_n + K_1, \gamma_s) \leq 1$. This completes the proof.

□

Now we consider the Cartesian product of P_2 and $K_{1,n}$. Let $V = \{u_0, \ldots, u_n, v_0, v_1, \ldots, v_n\}$ be the vertex set of $P_2 \Box K_{1,n}$ where the induced subgraph on $\{u_0, u_1, \ldots, u_n\}$ and $\{v_0, v_1, \ldots, v_n\}$ are $K_{1,n}$. In Theorems 4 and 7, we determine signed domination and forcing signed domination numbers of $P_2 \Box K_{1,n}$.

Theorem 4 *For* $n \geq 1$, $\gamma_S(P_2 \Box K_{1,n}) = 2$.

Proof Let $g : V \longrightarrow \{-1, 1\}$ such that $g(v) = -1$ if and only if $v \in \{v_1, \ldots v_{\lceil \frac{n}{2} \rceil}, u_{\lceil \frac{n}{2} \rceil + 1}, \ldots, u_n\}$. Since $1 \leq g[u_i], g[v_i] \leq 3$ for $0 \leq i \leq n$, g is a signed dominating function. Hence, $\gamma_s(P_2 \Box K_{1,n}) \leq \omega(g) = 2$. Let h be a γ_s-function and $\omega(h) < 2$. Then, $|V_h^-| > n$. So $h[u_0] = 0$ or $h[v_0] = 0$. This is a contradiction. Therefore, $\gamma_s(P_2 \Box K_{1,n}) = 2$. □

Theorem 5 Zelinka (2006) *Let* $K_{a,b}$ *be a complete bipartite graph with* $b \leq a$. *Then*

$$\gamma_s(K_{a,b}) = \begin{cases} a+1 & \text{if } b = 1; \\ b & \text{if } 2 \leq b \leq 3 \text{ and } a \text{ is even;} \\ b+1 & \text{if } 2 \leq b \leq 3 \text{ and } a \text{ is odd;} \\ 4 & \text{if } b \geq 4 \text{ and } a, b \text{ are both even;} \\ 6 & \text{if } b \geq 4 \text{ and } a, b \text{ are both odd;} \\ 5 & \text{if } b \geq 4 \text{ and } a, b \text{ have different parity.} \end{cases}$$

Theorems 4 and 5 imply the following Corollary.

Corollary 1 *For every positive integer* n, $\gamma_s(P_2 \Box K_{1,n}) < \gamma_s(P_2) \times \gamma_s(K_{1,n})$.

Theorem 6 Sheikholeslami (2007) *For* $n \in \{3, 4, 5\}$, $f(C_n, \gamma_s) = 1$ *and for* $n \geq 6$,

$$f(C_n, \gamma_s) = \begin{cases} 1 & \text{if } n \equiv 0 \pmod{3}; \\ 2 & \text{if } n \equiv 1, 2 \pmod{3}. \end{cases}$$

Theorem 7 *For* $n \geq 1$, $f(P_2 \Box K_{1,n}, \gamma_s) = \lceil \frac{n}{2} \rceil$.

Proof Let f be a γ_s-function of $P_2 \Box K_{1,n}$. Then, $|V_f^-| = n$. If $n = 1$, then $P_2 \Box K_{1,n} \simeq C_4$ and by Theorem 6, $f(P_2 \Box K_{1,n}, \gamma_s) = 1$. Let $n \geq 2$. Since $|V_f^-| = n$ and $deg(v_i) = deg(u_i) = 2$ for $1 \leq i \leq n$, so $u_0, v_0 \notin V_f^-$. On the other hand, $|N(x) \cap V_f^-| \leq \lceil \frac{n}{2} \rceil$ where $x \in \{u_0, v_0\}$. It is clear that $f(v_i) = -f(u_i)$ for $1 \leq i \leq n$. Hence, $(v_i, f(v_i))$ forces $(u_i, f(u_i))$. Let $T \subseteq S_f$ where $|T| < \lceil \frac{n}{2} \rceil$. Then, there are at least two distinct extensions of T. Thus, T is not a forcing subset of f and so the forcing signed domination number of $P_2 \times K_{1,n}$ is $\lceil \frac{n}{2} \rceil$. □

Theorem 8 Sheikholeslami (2007) *Let* G *be a graph with* $\Delta \leq 3$, g *be a signed dominating function of* G *and* $u, v \in V(G)$. *If* $g(u) = g(v) = -1$, *then* $d(u, v) \geq 3$.

Let $V = V_1 \cup V_2$ be the vertices set of $P_2 \Box K_n$ such that $V_1 = \{v_1, \ldots, v_n\}$ and $V_2 = \{u_1, \ldots, u_n\}$ where induced subgraph on V_i is K_n for $1 \leq i \leq 2$. Also u_j, v_j are adjacent for $1 \leq j \leq n$. Now, we obtain the signed domination number of $P_2 \Box K_n$.

Theorem 9 *For any positive integer* n,

$$\gamma_s(P_2 \Box K_n) = \begin{cases} 2 & n = 2; \\ 4 & 2 < n \text{ is even or } n = 3; \\ 6 & 3 < n \text{ is odd.} \end{cases}$$

Proof If $n = 2$, then $P_2 \Box K_n \simeq C_4$ and so $\gamma_s(P_2 \Box K_n) = 2$. If $n = 3$, then $P_2 \Box K_n$ is a 3-regular graph of diameter two. Let f be a γ_s-function of $P_2 \Box K_n$. By Theorem 8, $|V_f^-| = 1$. Thus, $\gamma_s(P_2 \Box K_n) = 4$.

Consider two following cases when $n > 3$.

Case 1: Let n be even. Define $f : V(P_2 \Box K_n) \longrightarrow \{-1, 1\}$, where $f(v) = -1$ if and only if $v \in \{v_1, \ldots, v_{\frac{n}{2}-1}\} \cup \{u_{\frac{n}{2}}, \ldots, u_{n-2}\}$. For each $v \in V$, $1 \leq f[v] \leq 3$ and also $\omega(f) = 4$. Thus, $\gamma_s(P_2 \Box K_n) \leq 4$. Let g be a γ_s-function of $P_2 \Box K_n$. If $n = 4$, then $|V_i \cap V_g^-| \leq 2$ for $1 \leq i \leq 2$. Also if $|V_1 \cap V_g^-| = 2$, then $V_2 \cap V_g^- = \emptyset$ and so $\omega(g) = 4$. Hence, $\gamma_s(P_2 \Box K_4) = 4$. Let $n > 4$. Since the Cartesian product of P_2 and K_n is n-regular graph, so there are at most $\frac{n}{2}$ vertices of label -1 in the closed neighborhood of each vertex. Without loss of generality, let $|V_1 \cap V_g^-| = \frac{n}{2}$. Then, $V_2 \cap V_g^- = \emptyset$. Hence, $\omega(g) = n > 4$. This is contradiction by $\gamma_s(P_2 \Box K_n) \leq 4$. Thus, for $1 \leq i \leq 2$, $|V_i \cap V_g^-| \leq \frac{n}{2} - 1$ and so $\omega(g) = \gamma_s(P_2 \Box K_n) \geq 4$.

Case 2: Let n be odd. Define $f : V(P_2 \Box K_n) \longrightarrow \{-1, 1\}$, where $f(v) = -1$ if and only if $v \in \{v_1, \ldots, v_{\lfloor \frac{n}{2} \rfloor - 1}\} \cup \{u_{\lfloor \frac{n}{2} \rfloor}, \ldots, u_{n-3}\}$. For each $v \in V$, $2 \leq f[v] \leq 4$ and also $\omega(f) = 6$. Thus, $\gamma_s(P_2 \Box K_n) \leq 6$. Let h be a γ_s-function of $P_2 \Box K_n$. If $n = 5$, then $|V_i \cap V_h^-| \leq 2$ for $1 \leq i \leq 2$. Also if $|V_1 \cap V_h^-| = 2$, then $V_2 \cap V_h^- = \emptyset$ and so $\gamma_s(P_2 \Box K_4) = 6$. Let $n > 5$. Let $|V_1 \cap V_h^-| = \lfloor \frac{n}{2} \rfloor$. If for some $1 \leq j \leq n$, $h(u_j) = -1$, then $h[v_j] = 0$ which is a contradiction. Hence, $V_2 \cap V_h^- = \emptyset$ and so $\omega(h) = n + 1 > 6$. This is not true. Thus, for $1 \leq i \leq 2$, $|V_i \cap V_h^-| \leq \lfloor \frac{n}{2} \rfloor - 1$ and so $\gamma_s(P_2 \Box K_n) \geq 6$. This completes the proof.

□

By Theorems 1 and 9, we have following Corollary.

Corollary 2 *For* $n > 2$, $\gamma_S(P_2 \Box K_n) \geq \gamma_s(P_2) \times \gamma_s(K_n)$.

Theorem 10 *Let* G *be a complete* n*-partite graph with* n^2 *vertices where* $n \geq 3$. *Then*

$$\gamma_s(G) = \begin{cases} 3 & \text{if n is odd;} \\ 4 & \text{if n is even.} \end{cases}$$

Proof Let G be a complete n-partite graph. Let $V = A_1 \cup \ldots \cup A_n$ where $A_i = \{v_{ij} : 1 \le j \le n\}$ and induced subgraph on A_i has no edge for $1 \le i \le n$. Let n be odd. We define $f : V \to \{-1, 1\}$ such that $f(v) = -1$ if and only if $v \in \{v_{ij} : 1 \le i \le \lfloor \frac{n}{2} \rfloor - 1$ and $1 \le j \le \lceil \frac{n}{2} \rceil\} \cup \{v_{ij} : \lfloor \frac{n}{2} \rfloor \le i \le n$ and $1 \le j \le \lfloor \frac{n}{2} \rfloor\}$. It is not hard to see that for any $v \in V$, $|N[v] \cap V_f^-| = |V_f^-| - \lfloor \frac{n}{2} \rfloor + 1$ and $f[v] \ge 1$ and so f is a signed dominating function of weight $\omega(f) = 3$. As a consequence, $\gamma_S(G) \le 3$. On the contrary, suppose that there is a γ_s-function g such that $\omega(g) < 3$. Hence, $|V_g^-| > \frac{1}{2}(n^2 - 3)$. Let $|V_g^-| = \frac{1}{2}(n^2 - 1)$. Two following steps help us to reach the result.

- If $A_i \cap V_g^- = \emptyset$ for some $1 \le i \le n$, then $g[v] = 2 - n$ for any $v \in A_i$ and since $n \ge 3$, $g[v] < 0$. This is not true. Hence, $A_i \cap V_g^- \ne \emptyset$ for every $1 \le i \le n$.
- If $|A_i \cap V_g^-| \ge \lceil \frac{n}{2} \rceil$ for every $1 \le i \le n$, then $|V_g^-| \ge \frac{1}{2}(n^2 + n)$. This is impossible. So there is $1 \le j \le n$ such that $|A_j \cap V_g^-| \le \lfloor \frac{n}{2} \rfloor$. Let $u \in A_j \cap V_g^-$. Then,

$$g[u] = n^2 - n + 1 - 2\left(\frac{1}{2}(n^2 - 1) - |A_j \cap V_g^-| + 1\right) \le 2\lfloor \frac{n}{2} \rfloor - n \le -1.$$

This is a contradiction. Therefore, $\gamma_S(G) = 3$ when n is odd.

Now suppose that n is even and we define $f : V \longrightarrow \{-1, 1\}$ such that $f(v) = -1$ if and only if $v \in \{v_{ij} : 1 \le i \le n - 2$ and $1 \le j \le \frac{n}{2}\} \cup \{v_{ij} : n - 1 \le i \le n$ and $1 \le j \le \frac{n}{2} - 1\}$. It is easily seen that $1 \le f[v] \le 5$ for each $v \in V$. Hence, f is a signed dominating function and $\omega(f) = 4$.

Let g be a γ_s-function where $|V_g^-| = |V_f^-| + 1$. We show that for any $1 \le i \le n$, $A_i \cap V_g^- \ne \emptyset$. On the contrary, let $A_k \cap V_g^- = \emptyset$ for some $1 \le k \le n$ and $u \in A_k$. Then, $g[u] = 3 - n \le 0$. This is a contradiction. Thus, for any $1 \le i \le n$, $A_i \cap V_g^- \ne \emptyset$. Let $w \in A_i \cap V_g^-$. Then, $g[w] = n^2 - n + 1 - 2\Big||V_g^-| - |A_i \cap V_g^-| + 1\Big| \ge 1$. So $|A_i \cap V_g^-| \ge \frac{n}{2}$ for any $1 \le i \le n$ and so $\frac{n^2}{2} - 1 = |V_g^-| \ge n(\frac{n}{2})$ which is impossible. Therefore, $\gamma_s(G) = 4$ where n is even. This completes the proof. □

Theorem 11 Dunbar et al. (1995), Henning and Slater (1996) *Let G be a k−regular graph of order n. If k is odd, then* $\gamma_s(G) \ge \frac{2n}{k+1}$ *and if k is even, then* $\gamma_s(G) \ge \frac{n}{k+1}$.

Let $V = \{v_{ij} : 1 \le i \le n\,,\ 1 \le j \le m\}$ be the vertex set of $K_n \times K_m$ for any positive integers $1 < n < m$. Then, v_{ij} and v_{ks} are adjacent if and only if $i \ne k$ and $j \ne s$. Also for any $1 \le i \le n$ and $1 \le j \le m$, the induced subgraphs on $A_i = \{v_{ij} : 1 \le j \le m\}$ and $B_j = \{v_{ij} : 1 \le i \le n\}$ are empty. With these notations in mind, we will prove the following result:

Theorem 12 *For any positive integers* $1 < n < m$,

$$\gamma_s(K_n \times K_m) = \begin{cases} 2 & n = 2, m \text{ is odd;} \\ 4 & n = 2, m \text{ is even;} \\ 3 & n = 3, m \text{ is odd;} \\ 5 & n \ne 3, mn \text{ is odd;} \\ 6 & 3 < n < m \text{ have different parity;} \\ 8 & 2 \ne n, m \text{ are both even.} \end{cases}$$

Proof Consider following cases:

Case 1. If $n = 2$, then $K_n \times K_m$ is a bipartite graph and $(m-1)$-regular. Define $f: V \longrightarrow \{-1, 1\}$ such that $f(v_{ij}) = -1$ if and only if $1 \le i \le 2$ and $1 \le j \le \lceil \frac{m}{2} \rceil - 1$. Since for any i and j, $\left| N[v_{ij}] \cap V_f^- \right| \le \lceil \frac{m}{2} \rceil - 1$, so $f[v_{ij}] \ge 1$. Thus, f is a signed dominating function and $\omega(f) = 2, 4$ and so $\gamma_s(K_n \times K_m) \le 2, 4$ depending on m is odd or even, respectively. On the other hand, by Theorem 11, if m is odd, then $\gamma_s(K_n \times K_m) \ge 2$, otherwise $\gamma_s(K_n \times K_m) \ge 4$.

Case 2. If $n = 3$ and m is odd. Define $f: V \longrightarrow \{-1, 1\}$ such that $f(v_{ij}) = -1$ if and only if $1 \le i \le 3$ and $1 \le j \le \lfloor \frac{m}{2} \rfloor$. Hence, $|N[v_{ij}] \cap V_f^-| \le 2\lfloor \frac{m}{2} \rfloor$ for any i and j. So f is a signed dominating function of weight $\omega(f) = 3$.

If g is a γ_s-function and $\gamma_s(K_3 \times K_m) < \omega(f)$, then $|V_g^-| > |V_f^-|$. Let $|V_g^-| = |V_f^-| + 1 = \frac{3m-1}{2}$. Let $1 \le i \le 3$ and $A_i \cap V_g^- = \emptyset$. Then, $g[u] = 3 - m$ for each $u \in A_i$. So for any $1 \le i \le 3$, $A_i \cap V_g^- \ne \emptyset$. The same argument shows $B_j \cap V_g^- \ne \emptyset$ for any $1 \le j \le m$. If there is $1 \le \ell \le m$ such that $|B_\ell \cap V_g^-| = 2$, then for $v_{i\ell} \in V_g^-$ we have

$$g[v_{i\ell}] = 2m - 1 - 2\Big(|V_g^-| - |A_i \cap V_g^-| - |B_\ell \cap V_g^-| + 2\Big) \ge 1$$

As a consequence $|A_i \cap V_g^-| \ge \frac{m+1}{2}$. Since $|V_g^-| = \sum_{i=1}^{3} |A_i \cap V_g^-|$, so $|V_g^-| \ge \frac{3m+3}{2}$. This is not true. Thus, for any $1 \le j \le m$, $|B_j \cap V_g^-| = 1$ and so $|V_g^-| = m$. Again this is not true. Therefore, $\gamma_s(K_3 \times K_m) = 3$.

Case 3. Let $n \ne 3$ and mn is odd. Define $f: V \longrightarrow \{-1, 1\}$ such that $f(v) = -1$ if and only if $v \in \{v_{ij} : i \equiv 1 + k (\text{mod n}), j \equiv 1 + k (\text{mod m}), 1 \le k \le \frac{mn-5}{2}\}$. Thus, for any i, j, $\lfloor \frac{m}{2} \rfloor \le |A_i \cap V_f^-| \le \lceil \frac{m}{2} \rceil$ and $\lfloor \frac{n}{2} \rfloor \le |B_j \cap V_f^-| \le \lceil \frac{n}{2} \rceil$. Hence, for $v \in V$, $|N[v] \cap V_f^-| \le \frac{mn-1}{2} - \lfloor \frac{m}{2} \rfloor - \lfloor \frac{n}{2} \rfloor$ and so $f[v] \ge 1$. So f is a signed dominating function of weight $\omega(f) = 5$. Hence, $\gamma_s(K_n \times K_m) \le 5$.

Let g be a signed dominating function where $\omega(g) < 5$. Then, $|V_g^-| > \frac{mn-5}{2}$. Let $|V_g^-| = \frac{mn-3}{2}$. Suppose that $A_i \cap V_g^- = \emptyset$. If there is B_j for some $1 \le j \le m$ such that $|B_j \cap V_g^-| \le \lfloor \frac{n}{2} \rfloor$, then

$$g[v_{ij}] = (m-1)(n-1) + 1 - 2\Big(|V_g^-| - |B_j \cap V_g^-|\Big) \le 4 - n < 0.$$

Hence, for any $1 \le j \le m$, $|B_j \cap V_g^-| \ge \lceil \frac{n}{2} \rceil$ and then $|V_g^-| \ge m\lceil \frac{n}{2} \rceil$. This makes a contradiction. Therefore, $A_i \cap V_g^- \ne \emptyset$ for any $1 \le i \le n$. By similar argument, we can see $B_j \cap V_g^- \ne \emptyset$ for $1 \le j \le m$. If there are i and j such that $|A_i \cap V_g^-| \le \lfloor \frac{m}{2} \rfloor$, $|B_j \cap V_g^-| \le \lfloor \frac{n}{2} \rfloor$ and also $v_{ij} \in V_g^-$, then

$$g[v_{ij}] = (m-1)(n-1) + 1 - 2\Big(|V_g^-| - |A_i \cap V_g^-| - |B_j \cap V_g^-| + 2\Big).$$

Thus $g[v_{ij}] \le -1$. This is a contradiction. Otherwise, i.e., for any i, j, where $|A_i \cap V_g^-| \le \lfloor \frac{m}{2} \rfloor$, $|B_j \cap V_g^-| \le \lfloor \frac{n}{2} \rfloor$, $g(v_{ij}) = +1$, then $|V_g^-| \ge \lfloor \frac{m}{2} \rfloor \lceil \frac{n}{2} \rceil + \lfloor \frac{n}{2} \rfloor \lceil \frac{m}{2} \rceil$ which is not true. Therefore, for any $1 \le i \le n$, $|A_i \cap V_g^-| \ge \lceil \frac{m}{2} \rceil$ or $|B_j \cap V_g^-| \ge \lceil \frac{n}{2} \rceil$ for any $1 \le j \le m$. But both cases are impossible. Hence, $\gamma_s(K_n \times K_m) = 5$.

Case 4. Let $n > 3$ and m, n have different parity. We define function $f : V \longrightarrow \{-1, 1\}$ such that $f(v) = -1$ if and only if

$$v \in \{v_{ij} : i \equiv 1 + k(\text{mod n}), j \equiv 1 + k(\text{mod m}), 1 \leq k \leq \frac{mn-6}{2}\}.$$

Likewise Case 3, for any i, j we have $\lceil \frac{m}{2} \rceil - 1 \leq |A_i \cap V_f^-| \leq \lceil \frac{m}{2} \rceil$ and $\lceil \frac{n}{2} \rceil - 1 \leq |B_j \cap V_f^-| \leq \lceil \frac{n}{2} \rceil$. Thus, $|N[v] \cap V_f^-| \leq \frac{mn}{2} - \lfloor \frac{m}{2} \rfloor - \lfloor \frac{n}{2} \rfloor$ for any $v \in V$. So f is a signed dominating function of weight $\omega(f) = 6$. Hence, $\gamma_s(K_n \times K_m) \leq 6$. As above we can see $\gamma_s(K_n \times K_m) = 6$.

Case 5. Let $2 < n < m$ are both even. Define $f : V \longrightarrow \{-1, 1\}$ such that $f(v) = -1$ if and only if

$$v \in \left\{v_{ij} : i \equiv 1 + k(\text{mod n}), j \equiv 1 + k (\text{mod m}), 1 \leq k \leq \frac{mn-8}{2}\right\}.$$

With the same argument in Case 3, it is easily seen that $\gamma_s(K_n \times K_m) = 8$.

□

By Theorem 12 and Lemma 1, we deduce the following Corollary.

Corollary 3 *For* $2 \leq n < m$, $\gamma_s(K_n \times K_m) \geq \gamma_s(K_n) \times \gamma_s(K_m)$.

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