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RESEARCH PAPER

Modified α-Bernstein–Durrmeyer-Type Operators

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Abstract

In this paper, we construct a Durrmeyer variant of the modified α-Bernstein-type operators introduced by Kajla and Acar (Ann Funct Anal 10(4):570–582, 2019), for $\alpha \in [0, 1]$. We investigate the degree of approximation via the approach of Peetre's K-functional and the Lipschitz-type maximal function. The quantitative Voronovskaja- and Grüss Voronovskaja-type theorems are discussed. Further, we determine the rate of convergence by the above operators for the functions with derivatives of bounded variation.

Keywords Peetre's K-functional · Ditzian–Totik modulus of smoothness · Voronovskaja-type theorem · Grüss Voronovskaja-type theorem · Functions of bounded variation

Mathematics Subject Classification 41A10 · 41A25

1 Introduction

Let $\psi : J \to \mathbb{R}$ be bounded on $J = [0, 1]$. In 1912, to prove Weierstrass approximation theorem (Weierstrass 1885), Bernstein (1912) defined a sequence of linear positive operators, known as Bernstein operators, as follows:

$$B_n(\psi; \varkappa) = \sum_{m=0}^{n} P_{n,m}(\varkappa)\psi\left(\frac{m}{n}\right), \quad \varkappa \in J \tag{1.1}$$

where

$$P_{n,m}(\varkappa) = \binom{n}{m}\varkappa^m(1-\varkappa)^{n-m}, \quad 0 \le m \le n.$$

It is easily verified that the Bernstein basis functions $P_{n,m}(\varkappa)$ satisfy the recurrence relation

$$P_{n,m}(\varkappa) = (1-\varkappa)P_{n-1,m}(\varkappa) + \varkappa P_{n-1,m-1}(\varkappa), \quad \forall\, 0 < m < n.$$

There are many applications of Bernstein operators in fields such as mathematics, physics, computer science and engineering. Due to their useful structure, many researchers discovered their various approximation properties and made significant contributions, the interested reader may refer to Păltănea (2004), Gonska (2007), Gonska and Raşa (2009), Gavrea and Ivan (2012), Tachev (2012), etc. Chen et al. (2017) defined a generalization of Bernstein operators (1.1) based on a real parameter 'α' satisfying $0 \le \alpha \le 1$, as

$$L_{n,\alpha}(\psi; \varkappa) = \sum_{m=0}^{n} P_{n,m,\alpha}(\varkappa)\psi\left(\frac{m}{n}\right), \quad \varkappa \in J \tag{1.2}$$

where the α-Bernstein basis function $P_{n,m,\alpha}(\varkappa)$, for $n \ge 2$, is given by

$$\begin{aligned} P_{n,m,\alpha}(\varkappa) = &\binom{n-2}{m}(1-\alpha)\varkappa^m(1-\varkappa)^{n-m-1} \\ &+ \binom{n-2}{m-2}(1-\alpha)\varkappa^{m-1}(1-\varkappa)^{n-m} \\ &+ \binom{n}{m}\alpha\varkappa^m(1-\varkappa)^{n-m} \end{aligned} \tag{1.3}$$

with $\binom{n-2}{-2} = \binom{n-2}{-1} = 0$. It is easily seen that $P_{n,m,\alpha}(\varkappa)$, verifies the recurrence relation

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$$P_{n,m,\alpha}(\varkappa) = (1-\varkappa)P_{n-1,m,\alpha}(\varkappa) + \varkappa P_{n-1,m-1,\alpha}(\varkappa), \quad \forall\ 0<m<n, \text{ and } n\geq 3.$$

Chen et al. (2017) studied the uniform convergence theorem and the Voronovskaja-type asymptotic formula, etc. For the special case $\alpha = 1$, the operators (1.2) reduce to (1.1). Acar and Kajla (2018) introduced a bivariate extension of (1.2) and studied some approximation properties of these operators and the associated GBS (Generalized Boolean Sum) operators. Kajla and Acar (2018) defined the Durrmeyer-type modification of (1.2) and studied the local and global approximation properties and Voronovskaja-type asymptotic theorem. For more details of the research work in this direction, we refer to cf. (Lupas and Lupas 1987; Nowak 2009; Aral et al. 2013; Acar et al. 2016; Gupta and Tachev 2017; Mohiuddine et al. 2017; Acu et al. 2018; Mohiuddine et al. 2020; Kajla et al. 2020; Rao et al. 2021; Mishra and Gandhi 2019; Mohiuddine 2020; Mohiuddine and Özger 2020; Özger et al. 2020, etc.). Khosravian-Arab et al. (2018) introduced a new family of Bernstein operators as follows:

$$B_n^{M,1}(\psi;\varkappa) = \sum_{m=0}^{n} P_{n,m}^{M,1}(\varkappa)\psi\left(\frac{m}{n}\right), \quad \varkappa \in J \tag{1.4}$$

where

$$P_{n,m}^{M,1}(\varkappa) = a(\varkappa,n)P_{n-1,m}^{M,1}(\varkappa) + a(1-\varkappa,n)P_{n-1,m-1}^{M,1}(\varkappa), \quad 1\leq m\leq n-1$$

$$P_{n,0}^{M,1}(\varkappa) = a(\varkappa,n)(1-\varkappa)^{n-1},$$

$$P_{n,n}^{M,1}(\varkappa) = a(1-\varkappa,n)\varkappa^{n-1}.$$

and

$$a(\varkappa,n) = a_0(n) + \varkappa\, a_1(n), \quad n = 0,1,2,3,\ldots \tag{1.5}$$

$a_0(n)$ and $a_1(n)$ being two unknown sequences, which may be defined in an appropriate manner. If $a_0(n) = 1$, and $a_1(n) = -1$, then (1.4) reduces to (1.1). Gupta et al. (2019) defined a Kantorovich version of the operators (1.4) and studied some better approximation properties. For $\psi \in C(J)$, the space of continuous functions on J with the uniform norm denoted by $\|.\|$, Kajla and Acar (2019) introduced a modification of the generalized Bernstein operators (1.4) by means of a parameter α, $0\leq\alpha\leq 1$ as follows:

$$B_{n,\alpha}^{M,1}(\psi;\varkappa) = \sum_{m=0}^{n} P_{n,m,\alpha}^{M,1}(\varkappa)\psi\left(\frac{m}{n}\right), \quad \forall\, \varkappa \in J \tag{1.6}$$

where

$$P_{n,m,\alpha}^{M,1}(\varkappa) = a(\varkappa,n)P_{n-1,m,\alpha}(\varkappa) + a(1-\varkappa,n)P_{n-1,m-1,\alpha}(\varkappa), \tag{1.7}$$

and $P_{n,m,\alpha}(\varkappa)$ is the same as defined in (1.3) and examined the uniform convergence and the asymptotic approximation. It is clear that for $\alpha = 1$, the operators (1.6) include the operators (1.4).

Inspired by the above research work, for $\psi \in C(J)$, we define a Durrmeyer-type modification of the operators (1.6) as follows:

$$K_{n,\alpha}^{M,1}(\psi;\varkappa) = \frac{n(n^2-1)}{(n-2)(n+1)+2\alpha} \sum_{m=0}^{n} P_{n,m,\alpha}^{M,1}(\varkappa)\int_0^1 P_{n,m,\alpha}(t)\psi(t)dt, \tag{1.8}$$

where $P_{n,m,\alpha}(\varkappa)$ is the α-Bernstein basis function as defined in (1.3). It is evident that for $a_0(n) = 1$, $a_1(n) = -1$, and $\alpha = 1$, the operator (1.8) includes the Bernstein–Durrmeyer operator introduced by Durrmeyer (1967) and subsequently studied by many other researchers (Derriennic 1981; Berens et al. 1992; Zhou 1992; Karsli 2019, etc.). Throughout this paper, we assume that the sequences $a_0(n)$ and $a_1(n)$ satisfy the relation $2a_0(n) + a_1(n) = 1$.

The purpose of this paper is to investigate the approximation degree of the operators (1.8) with the aid of the Peetre's K-functional and the Lipschitz-type maximal function. We also discuss the quantitative Voronovskaya- and Grüss–Voronovskaya-type theorem. The rate of convergence for functions with the derivative of bounded variation is also derived.

2 Preliminaries

Throughout this paper, we assume $\mathbb{N}_0 := \mathbb{N}\cup\{0\}$. Let $e_j(t) = t^j$, $j \in \mathbb{N}_0$. By a simple computation, we get

$$\int_0^1 P_{n,m}(t)t^j dt = \int_0^1 \binom{n}{m} t^{m+j}(1-t)^{(n-m)}dt = \frac{n!(m+j)!}{m!(n+j+1)!}. \tag{2.1}$$

Lemma 1 *(Kajla and Acar 2019) For $\sum_{m=0}^{n} m^j P_{n,m,\alpha}^{M,1}(\varkappa)$, $j \in \mathbb{N}_0$, we have the following identities:*

$$(i)\quad \sum_{m=0}^{n} P^{M,1}_{n,m,\alpha}(\varkappa) = 2a_0(n) + a_1(n) = 1$$

$$(ii)\quad \sum_{m=0}^{n} mP^{M,1}_{n,m,\alpha}(\varkappa) = n\varkappa + (1-2\varkappa)(a_0(n) + a_1(n))$$

$$(iii)\quad \sum_{m=0}^{n} m^2P^{M,1}_{n,m,\alpha}(\varkappa) = n^2\varkappa^2 + n\varkappa\Big(2a_0(n)(2-3\varkappa) + a_1(n)(3-5\varkappa)\Big) + (a_0(n) + a_1(n)) - 2\varkappa(1-\varkappa)\Big(2\alpha a_0(n) + (1+\alpha)a_1(n)\Big)$$

$$(iv)\sum_{m=0}^{n} m^3P^{M,1}_{n,m,\alpha}(\varkappa) = n^3\varkappa^3 + n^2\{3\varkappa^2a_0(n)(3-4\varkappa) + 3\varkappa^2a_1(n)(2-3\varkappa)\} + n\{(8\varkappa - 3\varkappa^2(5+4\alpha) + 2\varkappa^3(5+6\alpha))a_0(n) + (7\varkappa - 6\varkappa^2(3+\alpha) + 2\varkappa^3(7+3\alpha))a_1(n)\} + \{(1 + 2\varkappa(5-9\varkappa) + 18\varkappa^2(3\alpha-2) + 6\varkappa^3(4-6\alpha))a_0(n) + (1 + 4\varkappa(1-3\alpha) + 6\varkappa^2(1-2\alpha)(2\varkappa-3))a_1(n)\}$$

$$(v)\sum_{m=0}^{n} m^4P^{M,1}_{n,m,\alpha}(\varkappa) = n^4\varkappa^4 + 4n^3\varkappa^3(4-5\varkappa)a_0(n) + 2n^3\varkappa^3(5-7\varkappa)a_1(n) + 2n^2\varkappa^2[16 - 12\varkappa(\alpha+3) + \varkappa^2(23+12\alpha)]a_0(n) + n^2\varkappa^2[25 - 2\varkappa(33+6\alpha) + \varkappa^2(47+12\alpha)]a_1(n) + 4n\varkappa[4 - 24\alpha\varkappa + 22\varkappa^2(3\alpha-1) + \varkappa^3(17-42\alpha)]a_0(n) + n\varkappa[15 - \varkappa(29+60\alpha) + 4\varkappa^2(42\alpha-1) + 2\varkappa^3(7-54\alpha)]a_1(n) + [1 + 16\varkappa(3-4\alpha) + 32\varkappa^2(11\alpha-9) + 48\varkappa^3(5-6\alpha(2+\varkappa))]a_0(n) + [1 + 2\varkappa(17-25\alpha) + 2\varkappa^2(133\alpha-10) + 24\varkappa^3(7-9\alpha)(2+\varkappa)]a_1(n).$$

As a consequence of Lemma 1, we obtain:

Lemma 2 *The operators* $K^{M,1}_{n,\alpha}(.;\varkappa)$, *verify the following equalities*:

$$(i)\quad K^{M,1}_{n,\alpha}(1;\varkappa) = 2a_0(n) + a_1(n) = 1;$$

$$(ii)\quad K^{M,1}_{n,\alpha}(\mathrm{t};\varkappa) = \frac{n(n^2-1)}{(n-2)(n+1)+2\alpha}\left[\left(\frac{(1-\alpha)(n-3)}{n(n^2-1)}\right)\Big(n\varkappa + (1-2\varkappa)(a_0(n)+a_1(n))\Big) + \left(\frac{1-\alpha}{n(n+1)} + \frac{\alpha}{(n+1)(n+2)}\right)\right] = \alpha_n(\varkappa);$$

$$(iii)\quad K^{M,1}_{n,\alpha}(\mathrm{t};\varkappa) = \frac{n(n^2-1)}{(n-2)(n+1)+2\alpha}\Bigg[\left\{\frac{(1-\alpha)(n-4)}{(n-1)n(n+1)(n+2)} + \frac{\alpha}{(n+1)(n+2)(n+3)}\right\}\Big\{n^2\varkappa^2 + n\varkappa\Big(2a_0(n)(2-3\varkappa) + a_1(n)(3-5\varkappa)\Big) + (a_0(n)+a_1(n)) - 2\varkappa(1-\varkappa)\Big(2\alpha a_0(n) + (1+\alpha)a_1(n)\Big)\Big\} + \left\{\frac{3(n-2)(1-\alpha)}{(n-1)n(n+1)(n+2)} + \frac{3\alpha}{(n+1)(n+2)(n+3)}\right\}\{n\varkappa + (1-2\varkappa)(a_0(n)+a_1(n))\} + \left\{\frac{2(1-\alpha)}{n(n+1)(n+2)} + \frac{2\alpha}{(n+1)(n+2)(n+3)}\right\}\Bigg].$$

Let $\mu^{M,1}_{n,\alpha,r}(\varkappa) = K^{M,1}_{n,\alpha}((\mathrm{t}-\varkappa)^r;\varkappa)$, $r \in \mathbb{N}_0$. Further, let us assume that $\lim_{n\to\infty}a_i(n) = p_i$, for $i = 0, 1$. Then by a simple computation, using Lemma 2, we reach the following crucial result:

Lemma 3 *For the operators* $K^{M,1}_{n,\alpha}(.;\varkappa)$, *we have*

$$(i)\ \lim_{n\to\infty} n\mu^{M,1}_{n,\alpha,1}(\varkappa) = (1-2\varkappa)(1 + p_0 + p_1)$$

$$(ii)\ \lim_{n\to\infty} n\mu^{M,1}_{n,\alpha,2}(\varkappa) = (2p_0 + p_1 + 1)\varkappa(1-\varkappa)$$

$$(iii)\ \lim_{n\to\infty} n^2\mu^{M,1}_{n,\alpha,4}(\varkappa) = 12\varkappa^2(1-\varkappa)^2.$$

Remark From Lemma 3, for all sufficiently large n and $\varkappa \in J$, we have $\mu^{M,1}_{n,\alpha,2}(\varkappa) \le \frac{C}{n}\phi^2(\varkappa)$ and $\mu^{M,1}_{n,\alpha,4}(\varkappa) \le \frac{C}{n^2}\phi^4(\varkappa)$, where $\phi^2(\varkappa) = \varkappa(1-\varkappa)$ and C is a positive constant. Our following result shows that the operators $K^{M,1}_{n,\alpha}$ are bounded operators.

Lemma 4 *For $\psi \in C(J)$, and for each $\varkappa \in J$, we have*

$$\left|K^{M,1}_{n,\alpha}(\psi;\varkappa)\right| \le \|\psi\|.$$

Proof Applying the definition of the operators (1.8) and using Lemma 2, we have

$$\begin{aligned}\left|K^{M,1}_{n,\alpha}(\psi;\varkappa)\right| &\le \frac{n(n^2-1)}{(n-2)(n+1)+2\alpha}\sum_{m=0}^{n} P^{M,1}_{n,m,\alpha}(\varkappa)\\ &\quad \int_0^1 P_{n,m,\alpha}(\mathrm{t})|\psi(\mathrm{t})|\mathrm{dt}\\ &\le \|\psi\| K^{M,1}_{n,\alpha}(1;\varkappa) = \|\psi\|,\end{aligned}$$

which completes the proof. □

3 Main Results

First, we show that the operator $K^{M,1}_{n,\alpha}(\psi)$ is an approximation process for function $\psi \in C(J)$.

3.1 Basic Uniform Convergence Theorem

Theorem 1 *If $\psi \in C(J)$ and $\alpha \in J$, then*

$$\lim_{n\to\infty} K^{M,1}_{n,\alpha}(\psi;\varkappa) = \psi(\varkappa),$$

uniformly in $\varkappa \in J$.

Proof Applying Lemma 2, it follows that $\lim_{n\to\infty} K^{M,1}_{n,\alpha}(\mathrm{t}^j;\varkappa) = \varkappa^j$, $j = 0,1,2$ uniformly in $\varkappa \in J$. Hence, by the well-known Bohman–Korovkin theorem (Korovkin 1960), we obtain the desired result. □

3.2 Local Approximation

For $\psi \in C(J)$ and any $\delta > 0$, the Peetre's K-functional $K(\psi;\delta)$ is defined by

$$K(\psi;\delta) = \inf_{g\in C^2(J)}\{\|\psi - g\| + \delta\|g''\|,\ \delta > 0\}, \tag{3.1}$$

where $C^2(J) = \{g : g'' \in C(J)\}$. By DeVore and Lorentz (1993), ∃ a constant $M > 0$ such that

$$K(\psi;\delta) \le M\omega_2(\psi;\sqrt{\delta}), \quad \delta > 0, \tag{3.2}$$

where $\omega_2(\psi;\sqrt{\delta})$ is the second-order modulus of continuity of $\psi \in C(J)$, defined as

$$\begin{aligned}\omega_2(\psi;\sqrt{\delta}) = \sup_{0<h\le\sqrt{\delta}}\ \sup_{\varkappa,\varkappa\pm h\in J} &|\psi(\varkappa+h) - 2\psi(\varkappa)\\ &+ \psi(\varkappa-h)|.\end{aligned}$$

Also, for $\psi \in C(J)$, the first-order modulus of continuity is given by

$$\omega(\psi;\delta) = \sup_{0<|h|\le\delta}\ \sup_{\varkappa,\varkappa+h\in J}|\psi(\varkappa+h) - \psi(\varkappa)|.$$

Theorem 2 *For $\psi \in C(J)$, ∃ a constant $M > 0$ such that*

$$\begin{aligned}|K^{M,1}_{n,\alpha}(\psi;\varkappa) - \psi(\varkappa)| &\le M\omega_2\left(\psi;\sqrt{\xi^{M,1}_{n,\alpha}(\varkappa)}\right)\\ &\quad + \omega(\psi;\mu^{M,1}_{n,\alpha,1}(\varkappa)),\end{aligned}$$

where $\xi^{M,1}_{n,\alpha}(\varkappa) = \mu^{M,1}_{n,\alpha,2}(\varkappa) + (\mu^{M,1}_{n,\alpha,1}(\varkappa))^2$.

Proof Let us define an auxiliary operator as follows:

$$\bar{K}^{M,1}_{n,\alpha}(\psi;\varkappa) = K^{M,1}_{n,\alpha}(\psi;\varkappa) + \psi(\varkappa) - \psi(\alpha_n(\varkappa)), \tag{3.3}$$

where $\alpha_n(\varkappa)$ is defined as in Lemma 2. From (3.3), it is clear that $\bar{K}^{M,1}_{n,\alpha}(\psi;\varkappa)$ is a linear operator and applying Lemma 2

$$\bar{K}^{M,1}_{n,\alpha}(1;\varkappa) = 1,\quad \bar{K}^{M,1}_{n,\alpha}((\mathrm{t}-\varkappa);\varkappa) = 0. \tag{3.4}$$

Let $g \in C^2(J)$ and $\varkappa \in J$ be arbitrary. Then by Taylor's formula,

$$g(\mathrm{t}) = g(\varkappa) + (\mathrm{t}-\varkappa)g'(\varkappa) + \int_\varkappa^{\mathrm{t}}(\mathrm{t}-u)g''(u)\mathrm{d}u.$$

Now, applying the linear operator $\bar{K}^{M,1}_{n,\alpha}(.;\varkappa)$ on both sides of the above equation and using Eq. (3.4), we get

$$\begin{aligned}\bar{K}^{M,1}_{n,\alpha}(g;\varkappa) - g(\varkappa) &= \bar{K}^{M,1}_{n,\alpha}\left(\int_\varkappa^{\mathrm{t}}(\mathrm{t}-u)g''(u)\mathrm{d}u\right)\\ &= K^{M,1}_{n,\alpha}\left(\int_\varkappa^{\mathrm{t}}(\mathrm{t}-u)g''(u)\mathrm{d}u\right)\\ &\quad - \int_\varkappa^{\alpha_n(\varkappa)}(\alpha_n(\varkappa)-u)g''(u)\mathrm{d}u.\end{aligned}$$

Hence,

$$\left|\bar{K}_{n,\alpha}^{M,1}(g;\varkappa)-g(\varkappa)\right|\leq K_{n,\alpha}^{M,1}\left(\left|\int_{\varkappa}^{t}|t-u|\,|g''(u)|du\right|;\varkappa\right)+\left|\int_{\varkappa}^{\alpha_n(\varkappa)}|\alpha_n(\varkappa)-u|\,|g''(u)|du\right|$$
$$\leq\left\{K_{n,\alpha}^{M,1}((t-\varkappa)^2;\varkappa)+(\alpha_n(\varkappa)-\varkappa)^2\right\}\|g''\|$$
$$\leq\left\{K_{n,\alpha}^{M,1}((t-\varkappa)^2;\varkappa)+\left(K_{n,\alpha}^{M,1}(t;\varkappa)-\varkappa\right)^2\right\}\|g''\|$$
$$\leq\left\{\mu_{n,\alpha,2}^{M,1}(\varkappa)+(\mu_{n,\alpha,1}^{M,1}(\varkappa))^2\right\}\|g''\|$$
$$=\xi_{n,\alpha}^{M,1}(\varkappa)\|g''\|. \tag{3.5}$$

Now from Eq. (3.3) and using Lemma 2, we have

$$\left|\bar{K}_{n,\alpha}^{M,1}(\psi;\varkappa)\right|\leq 3\|\psi\|. \tag{3.6}$$

For $\varkappa\in J$, $\psi\in C(J)$ and any $g\in C^2(J)$, from Eqs. (3.5) and (3.6), we obtain

$$\left|K_{n,\alpha}^{M,1}(\psi;\varkappa)-\psi(\varkappa)\right|$$
$$\leq\left|\bar{K}_{n,\alpha}^{M,1}(\psi;\varkappa)-\psi(\varkappa)+\psi(\alpha_n(\varkappa))-\psi(\varkappa)\right|$$
$$\leq\left|\bar{K}_{n,\alpha}^{M,1}((\psi-g);\varkappa)\right|+\left|\bar{K}_{n,\alpha}^{M,1}(g;\varkappa)-g(\varkappa)\right|+|g(\varkappa)-\psi(\varkappa)|+|\psi(\alpha_n(\varkappa))-\psi(\varkappa)|$$
$$\leq 4\|\psi-g\|+\xi_{n,\alpha}^{M,1}(\varkappa)\|g''\|+\omega(\psi;|\alpha_n(\varkappa)-\varkappa|)$$
$$\leq 4\|\psi-g\|+\xi_{n,\alpha}^{M,1}(\varkappa)\|g''\|+\omega(\psi;|\mu_{n,\alpha,1}^{M,1}(\varkappa)|).$$

Taking infimum on the right-hand side over all $g\in C^2(J)$ and using the definition of Peetre's K-functional given by (3.1), we obtain

$$\left|K_{n,\alpha}^{M,1}(\psi;\varkappa)-\psi(\varkappa)\right|\leq 4K(\psi;\frac{\xi_{n,\alpha}^{M,1}(\varkappa)}{4})+\omega(\psi;|\mu_{n,\alpha,1}^{M,1}(\varkappa)|).$$

Hence considering the relation (3.2), we get

$$\left|K_{n,\alpha}^{M,1}(\psi;\varkappa)-\psi(\varkappa)\right|\leq M\omega_2(\psi;\frac{\sqrt{\xi_{n,\alpha}^{M,1}(\varkappa)}}{2})+\omega(\psi;|\mu_{n,\alpha,1}^{M,1}(\varkappa)|),$$

which completes the proof. □

Lipschitz-type space: In view of Szász (1950), for $\varkappa\in J$

$$Lip_M^{(\rho)}:=\left\{\psi\in C(J):|\psi(t)-\psi(\varkappa)|\leq M_\psi\frac{|t-\varkappa|^\rho}{(t+\varkappa)^{\frac{\rho}{2}}};t\in J,\varkappa\in(0,1]\right\},$$

where $\rho\in(0,2]$, *and* $M_\psi>0$ is a constant dependent only on ψ.

Theorem 3 *Let* $\psi\in Lip_M^{(\rho)}$. *Then for all* $\varkappa\in(0,1]$, *we have*

$$\left|K_{n,\alpha}^{M,1}(\psi;\varkappa)-\psi(\varkappa)\right|\leq M_\psi\left(\frac{\mu_{n,\alpha,2}^{M,1}(\varkappa)}{\varkappa}\right)^{\frac{\rho}{2}}.$$

Proof First of all, we show the result for the case $\rho=2$. By our hypothesis, we have

$$\left|K_{n,\alpha}^{M,1}(\psi;\varkappa)-\psi(\varkappa)\right|\leq\frac{n(n^2-1)}{(n-2)(n+1)+2\alpha}\sum_{m=0}^{n}P_{n,m,\alpha}^{M,1}(t)\int_0^1P_{n,m,\alpha}(t)|\psi(t)-\psi(\varkappa)|dt$$
$$\leq M_\psi\frac{n(n^2-1)}{(n-2)(n+1)+2\alpha}\sum_{m=0}^{n}P_{n,m,\alpha}^{M,1}(t)\int_0^1P_{n,m,\alpha}(t)\frac{(t-\varkappa)^2}{(t+\varkappa)}dt.$$

Now using the inequality $\frac{1}{t+\varkappa}\leq\frac{1}{\varkappa}$, $\forall\ t\in J$ *and* $\varkappa\in(0,1]$, we get

$$\left|K_{n,\alpha}^{M,1}(\psi;\varkappa)-\psi(\varkappa)\right|\leq\frac{M_\psi}{\varkappa}\frac{n(n^2-1)}{(n-2)(n+1)+2\alpha}\sum_{m=0}^{n}P_{n,m,\alpha}^{M,1}(t)\int_0^1P_{n,m,\alpha}(t)(t-\varkappa)^2dt$$
$$=M_\psi\left(\frac{\mu_{n,\alpha,2}^{M,1}(\varkappa)}{\varkappa}\right).$$

This proves the result for $\rho=2$. Now, we show the above theorem for $0<\rho<2$. By using Hölder inequality with $q_1=\frac{2}{\rho}$ and $q_2=\frac{2}{2-\rho}$, we get

$$\left|K_{n,\alpha}^{M,1}(\psi;\varkappa)-\psi(\varkappa)\right| \le \frac{n(n^2-1)}{(n-2)(n+1)+2\alpha}\sum_{m=0}^{n}P_{n,m,\alpha}^{M,1}(\mathrm{t}) \int_0^1 P_{n,m,\alpha}(\mathrm{t})|\psi(\mathrm{t})-\psi(\varkappa)|\mathrm{dt}$$
$$\le \left\{\sum_{m=0}^{n}P_{n,m,\alpha}^{M,1}(\varkappa)\left(\frac{n(n^2-1)}{(n-2)(n+1)+2\alpha}\int_0^1 P_{n,m,\alpha}(\mathrm{t})|\psi(\mathrm{t})-\psi(\varkappa)|\mathrm{dt}\right)^{\frac{2}{\rho}}\right\}^{\frac{\rho}{2}}$$
$$\le \left\{\left(\frac{n(n^2-1)}{(n-2)(n+1)+2\alpha}\right)\sum_{m=0}^{n}P_{n,m,\alpha}^{M,1}(\varkappa)\int_0^1 P_{n,m,\alpha}(\mathrm{t})|\psi(\mathrm{t})-\psi(\varkappa)|^{\frac{2}{\rho}}\mathrm{dt}\right\}$$
$$\le M_\psi\left\{\frac{n(n^2-1)}{(n-2)(n+1)+2\alpha}\sum_{m=0}^{n}P_{n,m,\alpha}^{M,1}(\varkappa)\int_0^1 P_{n,m,\alpha}(\mathrm{t})\frac{(\mathrm{t}-\varkappa)^2}{\mathrm{t}+\varkappa}\mathrm{dt}\right\}^{\frac{\rho}{2}}$$
$$\le \frac{M_\psi}{\varkappa^{\frac{\rho}{2}}}\left\{\frac{n(n^2-1)}{(n-2)(n+1)+2\alpha}\sum_{m=0}^{n}P_{n,m,\alpha}^{M,1}(\varkappa)\int_0^1 P_{n,m,\alpha}(\mathrm{t})(\mathrm{t}-\varkappa)^2\mathrm{dt}\right\}^{\frac{\rho}{2}}$$
$$\le M_\psi\left(\frac{\mu_{n,\alpha,2}^{M,1}(\varkappa)}{\varkappa}\right)^{\frac{\rho}{2}}.$$

This completes the proof. □

Next, we study a local direct estimate for the operators defined in (1.8). First of all, we define the Lipschitz-type maximal function of order ρ, given by Lenze (1988) as

$$\bar{\omega}_\rho(\psi;\varkappa)=\sup_{t\neq\varkappa,t\in J}\frac{|\psi(\mathrm{t})-\psi(\varkappa)|}{|\mathrm{t}-\varkappa|^\rho}\ \forall\,\varkappa\in J \text{ and } \rho\in(0,1]. \tag{3.7}$$

For similar studies, one can refer to Kajla and Agrawal (2015), Kajla et al. (2017), Kajla (2017) and Neer et al. (2017).

Theorem 4 *Let $\psi\in C(J)$ and $0<\rho\le 1$. Then, $\forall\,\varkappa\in J$, we have*

$$\left|K_{n,\alpha}^{M,1}(\psi;\varkappa)-\psi(\varkappa)\right|\le\bar{\omega}_\rho(\psi;\varkappa)\left(\mu_{n,\alpha,2}^{M,1}(\varkappa)\right)^{\frac{\rho}{2}}.$$

Proof In view of (3.7),

$$|\psi(\mathrm{t})-\psi(\varkappa)|\le\bar{\omega}_\rho(\psi;\varkappa)|\mathrm{t}-\varkappa|^\rho.$$

Applying the operator $K_{n,\alpha}^{M,1}(.;\varkappa)$ on the above inequality, then using Lemma 3 and the Hölder inequality with $q_1=\frac{2}{\rho}$, $q_2=\frac{2}{2-\rho}$, we have

$$|K_{n,\alpha}^{M,1}\psi(\mathrm{t})-\psi(\varkappa)|\le\bar{\omega}_\rho(\psi;\varkappa)K_{n,\alpha}^{M,1}(|\mathrm{t}-\varkappa|^\rho;\varkappa)$$
$$\le\bar{\omega}_\rho(\psi;\varkappa)\left(K_{n,\alpha}^{M,1}(\mathrm{t}-\varkappa)^2;\varkappa\right)^{\frac{\rho}{2}}\left(K_{n,\alpha}^{M,1}(1^2;\varkappa)\right)^{\frac{2-\rho}{2}}$$
$$=\bar{\omega}_\rho(\psi;\varkappa)\left(\mu_{n,\alpha,2}^{M,1}(\varkappa)\right)^{\frac{\rho}{2}}.$$

□

3.3 Asymptotic approximation by $K_{n,\alpha}^{M,1}$

We present a Voronovskaja-type asymptotic theorem for the operators $K_{n,\alpha}^{M,1}(\psi;\varkappa)$, defined in (1.8).

Theorem 5 *Let $\psi\in C(J)$ and $\alpha\in J$. If $\psi''(\varkappa)$ exists at a given point $\varkappa\in J$ then we have*

$$\lim_{n\to\infty}n\left(K_{n,\alpha}^{M,1}(\psi;\varkappa)-\psi(\varkappa)\right)=(1-2\varkappa)(1+p_0+p_1)\psi'(\varkappa)+(2p_0+p_1+1)\varkappa(1-\varkappa)\frac{\psi''(\varkappa)}{2}. \tag{3.8}$$

Furthermore, if $\psi''\in C(J)$ then Eq. (3.8) holds uniformly in $\varkappa\in J$.

Proof By using Taylor's formula, we have

$$\psi(\mathrm{t})=\psi(\varkappa)+(\mathrm{t}-\varkappa)\psi'(\varkappa)+\frac{(\mathrm{t}-\varkappa)^2}{2!}\psi''(\varkappa)+\phi(\mathrm{t},\varkappa)(\mathrm{t}-\varkappa)^2, \tag{3.9}$$

where $\phi(\mathrm{t},\varkappa)\in C(J)$ and $\phi(\mathrm{t},\varkappa)\to 0$, as $\mathrm{t}\to\varkappa$. Now operating by $K_{n,\alpha}^{M,1}(.;\varkappa)$ on both sides of (3.9) and using Lemma 3, we get

$$K_{n,\alpha}^{M,1}(\psi;\varkappa)-\psi(\varkappa)=\psi'(\varkappa)K_{n,\alpha}^{M,1}((\mathrm{t}-\varkappa);\varkappa)+\frac{\psi''(\varkappa)}{2!}K_{n,\alpha}^{M,1}((\mathrm{t}-\varkappa)^2;\varkappa)+K_{n,\alpha}^{M,1}(\phi(\mathrm{t},\varkappa)(\mathrm{t}-\varkappa)^2;\varkappa).$$

Hence,

$$\lim_{n\to\infty} n\left(K_{n,\alpha}^{M,1}(\psi;\varkappa)-\psi(\varkappa)\right)=(1-2\varkappa)(1+p_0+p_1)\psi'(\varkappa)+(2p_0+p_1+1)\varkappa(1-\varkappa)\frac{\psi''(\varkappa)}{2}+\lim_{n\to\infty} nK_{n,\alpha}^{M,1}(\phi(\mathrm{t},\varkappa)(\mathrm{t}-\varkappa)^2;\varkappa). \tag{3.10}$$

Since $\phi(\mathrm{t},\varkappa)\to 0$ as $\mathrm{t}\to\varkappa$, for a given $\epsilon>0$, $\exists$ a $\delta>0$ such that $|\phi(\mathrm{t},\varkappa)|<\epsilon$ whenever $|\mathrm{t}-\varkappa|<\delta$, and for $|\mathrm{t}-\varkappa|\geq\delta$, we have $|\phi(\mathrm{t},\varkappa)|\leq M\frac{(\mathrm{t}-\varkappa)^2}{\delta^2}$, where $M>0$. Let $\chi_\delta(\mathrm{t})$ be the characteristic function of the interval $(\varkappa-\delta,\varkappa+\delta)$. Now using Lemma (3), we obtain

$$\begin{aligned}&\left|K_{n,\alpha}^{M,1}(\phi(\mathrm{t},\varkappa)(\mathrm{t}-\varkappa)^2;\varkappa)\right|\\&\leq K_{n,\alpha}^{M,1}\left(|\phi(\mathrm{t},\varkappa)|(\mathrm{t}-\varkappa)^2\chi_\delta(\mathrm{t});\varkappa\right)\\&\quad+K_{n,\alpha}^{M,1}\left(|\phi(\mathrm{t},\varkappa)|(\mathrm{t}-\varkappa)^2(1-\chi_\delta(\mathrm{t}));\varkappa\right)\\&\leq \epsilon K_{n,\alpha}^{M,1}\left((\mathrm{t}-\varkappa)^2;\varkappa\right)+\frac{M}{\delta^2}K_{n,\alpha}^{M,1}\left((\mathrm{t}-\varkappa)^4;\varkappa\right)\\&=\epsilon\, O\left(\frac{1}{n}\right)+O\left(\frac{1}{n^2}\right),\quad \text{as } n\to\infty.\end{aligned}$$

Hence, due to the arbitrariness of $\epsilon>0$, we have $\lim_{n\to\infty} nK_{n,\alpha}^{M,1}(\phi(\mathrm{t},\varkappa)(\mathrm{t}-\varkappa)^2;\varkappa)=0$. Thus by (3.10), we get

$$\lim_{n\to\infty} n\left(K_{n,\alpha}^{M,1}(\psi;\varkappa)-\psi(\varkappa)\right)=(1-2\varkappa)(1+p_0+p_1)\psi'(\varkappa)+(2p_0+p_1+1)\varkappa(1-\varkappa)\frac{\psi''(\varkappa)}{2}.$$

This completes the proof of the first assertion of the theorem.

The second assertion follows due to the uniform continuity of ψ'' on J enabling δ to become independent of $\varkappa$, and all the other estimates hold uniformly in $\varkappa\in J$. □

3.4 Quantitative Voronovskaja-Type Theorem

Next, we calculate the order of approximation for the operators (1.8) by means of the Ditzian–Totik modulus of smoothness. Let $\phi(\varkappa)=\sqrt{\varkappa(1-\varkappa)}$. For $\psi\in C(J)$, the Ditzian–Totik modulus of smoothness of first order (Ditzian and Totik 2012) is defined as

$$\omega_\phi(\psi;\delta)=\sup_{0\leq h\leq\delta}\left\{\left|\psi\left(\varkappa+\frac{h\phi(\varkappa)}{2}\right)-\psi\left(\varkappa-\frac{h\phi(\varkappa)}{2}\right)\right|;\ \varkappa\pm\frac{h\phi(\varkappa)}{2}\in J\right\},$$

and the corresponding Peetre's K-functional is given by

$$K_\phi(\psi;\delta)=\inf_{g\in W_\phi(J)}\{\|\psi-g\|+\delta\|\phi g'\|\},\ (\delta>0),$$

where $W_\phi(J)=\{g: g\in AC_{loc}(J),\ \|\phi g'\|<\infty\}$ and $AC_{loc}(J)$ is the space of locally absolutely continuous functions on every closed and bounded interval $[a,b]\subseteq J$. It is well-known from Ditzian and Totik (2012) that $\exists$ a constant $M>0$, such that

$$M^{-1}\omega_\phi(\psi;\delta)\leq K_\phi(\psi;\delta)\leq M\omega_\phi(\psi;\delta). \tag{3.11}$$

Theorem 6 *For any $\psi\in C^2(J)$ and sufficiently large n, there holds the following two inequalities*:

$$(i)\ \left|n\left\{K_{n,\alpha}^{M,1}(\psi;\varkappa)-\psi(\varkappa)-\mu_{n,\alpha,1}^{M,1}(\varkappa)\psi'(\varkappa)-\frac{1}{2}\psi''(\varkappa)\mu_{n,\alpha,2}^{M,1}(\varkappa)\right\}\right|\leq M'\omega_\phi(\psi'';\phi(\varkappa)n^{-1/2}),$$

$$(ii)\ \left|n\left\{K_{n,\alpha}^{M,1}(\psi;\varkappa)-\psi(\varkappa)-\mu_{n,\alpha,1}^{M,1}(\varkappa)\psi'(\varkappa)-\frac{1}{2}\psi''(\varkappa)\mu_{n,\alpha,2}^{M,1}(\varkappa)\right\}\right|$$

where $M'$is some positive constant.

Proof For $\psi\in C^2(J)$ and $\mathrm{t},\varkappa\in J$, using Taylor's formula, we have

$$\psi(\mathrm{t})-\psi(\varkappa)=(\mathrm{t}-\varkappa)\psi'(\varkappa)+\int_\varkappa^{\mathrm{t}}(\mathrm{t}-u)\psi''(u)\mathrm{d}u.$$

Hence,

$$\psi(\mathrm{t})-\psi(\varkappa)-(\mathrm{t}-\varkappa)\psi'(\varkappa)-\frac{(\mathrm{t}-\varkappa)^2}{2}\psi''(\varkappa)=\int_\varkappa^{\mathrm{t}}(\mathrm{t}-u)(\psi''(u)-\psi''(\varkappa))\mathrm{d}u.$$

Now applying the linear positive operator $K_{n,\alpha}^{M,1}(.;\varkappa)$ on both sides of the above equation, we have

$$\left|K_{n,\alpha}^{M,1}(\psi;\varkappa)-\psi(\varkappa)-\mu_{n,\alpha,1}^{M,1}(\varkappa)\psi'(\varkappa)-\mu_{n,\alpha,2}^{M,1}(\varkappa)\frac{\psi''(\varkappa)}{2}\right| \leq K_{n,\alpha}^{M,1}\left(\left|\int_{\varkappa}^{t}|t-u||\psi''(u)-\psi''(\varkappa)|du\right|;\varkappa\right). \tag{3.12}$$

For any $g\in C^2(J)$, Finta (2011, p. 337) estimated the right-hand quantity of Eq. (3.12) as follows:

$$\left|\int_{\varkappa}^{t}|t-u||\psi''(u)-\psi''(\varkappa)|du\right|\leq 2\|\psi''-g\|(t-\varkappa)^2 + 2\|\phi g'\|\phi^{-1}(\varkappa)|t-\varkappa|^3. \tag{3.13}$$

Now, combining (3.12) and (3.13)

$$\left|K_{n,\alpha}^{M,1}(\psi;\varkappa)-\psi(\varkappa)-\mu_{n,\alpha,1}^{M,1}(\varkappa)\psi'(\varkappa)-\frac{\psi''(\varkappa)}{2}\mu_{n,\alpha,2}^{M,1}(\varkappa)\right| \leq 2\|\psi''-g\|K_{n,\alpha}^{M,1}((t-\varkappa)^2;\varkappa) + 2\|\phi g'\|\phi^{-1}(\varkappa)K_{n,\alpha}^{M,1}(|t-\varkappa|^3;\varkappa).$$

Now, using Cauchy–Schwarz's inequality and Remark 2, we obtain

$$\begin{aligned}&\left|K_{n,\alpha}^{M,1}(\psi;\varkappa)-\psi(\varkappa) -\mu_{n,\alpha,1}^{M,1}(\varkappa)\psi'(\varkappa)-\frac{\psi''(\varkappa)}{2}\mu_{n,\alpha,2}^{M,1}(\varkappa)\right|\\ &\leq 2\|\psi''-g\|\mu_{n,\alpha,2}^{M,1}(\varkappa) + 2\|\phi g'\|\phi^{-1}(\varkappa)\{\mu_{n,\alpha,2}^{M,1}(\varkappa)\}^{\frac{1}{2}}\{\mu_{n,\alpha,4}^{M,1}(\varkappa)\}^{\frac{1}{2}}\\ &\leq 2\|\psi''-g\|\mu_{n,\alpha,2}^{M,1}(\varkappa) + 2\|\phi g'\|\phi^{-1}(\varkappa)\{\mu_{n,\alpha,2}^{M,1}(\varkappa)\}^{\frac{1}{2}}\{\mu_{n,\alpha,4}^{M,1}(\varkappa)\}^{\frac{1}{2}}\\ &\leq 2\|\psi''-g\|\frac{C}{n}\phi^2(\varkappa)+\frac{2C}{n^{\frac{3}{2}}}\|\phi g'\|\phi^2(\varkappa)\\ &=\frac{2C}{n}\left(\phi^2(\varkappa)\|\psi''-g\|+\phi^2(\varkappa)\frac{\|\phi g'\|}{\sqrt{n}}\right).\end{aligned}$$

Since $\phi^2(\varkappa)\leq\phi(\varkappa)\leq 1,\ \forall\ \varkappa\in J$, we get

$$\left|K_{n,\alpha}^{M,1}(\psi;\varkappa)-\psi(\varkappa)-\mu_{n,1}\psi'(\varkappa)-\frac{\psi''(\varkappa)}{2}\mu_{n,2}(\varkappa)\right| \leq\frac{2C}{n}\left(\|\psi''-g\|+\phi(\varkappa)\frac{\|\phi g'\|}{\sqrt{n}}\right).$$

and

$$\left|K_{n,\alpha}^{M,1}(\psi;\varkappa)-\psi(\varkappa)-\mu_{n,\alpha,1}^{M,1}\psi'(\varkappa)-\frac{\psi''(\varkappa)}{2}\mu_{n,\alpha,2}^{M,1}(\varkappa)\right| \leq\phi^2(\varkappa)\frac{2C}{n}\left(\|\psi''-g\|+\frac{\|\phi g'\|}{\sqrt{n}}\right).$$

Taking the infimum on the right-hand side of the above relations over all $g\in W_\phi(J)$, we obtain

$$\left|n\left\{K_{n,\alpha}^{M,1}(\psi;\varkappa)-\psi(\varkappa)-\mu_{n,1}(\varkappa)\psi'(\varkappa)-\mu_{n,2}(\varkappa)\frac{\psi''(\varkappa)}{2}\right\}\right| \leq\begin{cases}M'K_\phi(\psi'';\phi(\varkappa)n^{-1/2}),\\ M'\phi^2(\varkappa)K_\phi(\psi'';n^{-1/2}),\end{cases}$$

where $M'=2C$. Now, applying relation (3.11), we reach the required result. □

3.5 Grüss–Voronovskaya-Type Theorem for the Operators $K_{n,\alpha}^{M,1}$

Theorem 7 *For $\psi, g\in C^2(J)$, there holds the following equality*:

$$\lim_{n\to\infty}n\left[K_{n,\alpha}^{M,1}(\psi g;\varkappa)-K_{n,\alpha}^{M,1}(\psi;\varkappa)K_{n,\alpha}^{M,1}(g;\varkappa)\right] =\psi'(\varkappa)g'(\varkappa)(2p_0+p_1+1)\varkappa(1-\varkappa),$$

uniformly $\varkappa\in J$.

Proof We have the double derivative of the product of two functions ψ and g as

$$(\psi g)''(\varkappa)=\psi''(\varkappa)g(\varkappa)+2\psi'(\varkappa)g'(\varkappa)+g''(\varkappa)\psi(\varkappa).$$

By making an appropriate arrangement, we get

$$\begin{aligned}&n\{K_{n,\alpha}^{M,1}((\psi g);\varkappa)-K_{n,\alpha}^{M,1}(\psi;\varkappa)K_{n,\alpha}^{M,1}(g;\varkappa)\}\\ &=n\Bigg\{K_{n,\alpha}^{M,1}((\psi g);\varkappa)-\psi(\varkappa)g(\varkappa)-(\psi g)'\mu_{n,\alpha,1}^{M,1}(\varkappa) -\frac{(\psi g(\varkappa))''}{2!}\mu_{n,\alpha,2}^{M,1}(\varkappa)\\ &-g(\varkappa)\left[K_{n,\alpha}^{M,1}(\psi;\varkappa)-\psi(\varkappa)-\frac{\psi'(\varkappa)}{1!}\mu_{n,\alpha,1}^{M,1}(\varkappa) -\frac{\psi''(\varkappa)}{2!}\mu_{n,\alpha,2}^{M,1}(\varkappa)\right]\\ &-K_{n,\alpha}^{M,1}(\psi;\varkappa)\left[K_{n,\alpha}^{M,1}(g;\varkappa)-g(\varkappa)-\frac{g'(\varkappa)}{1!}\mu_{n,\alpha,1}^{M,1}(\varkappa) -\frac{g''(\varkappa)}{2!}\mu_{n,\alpha,2}^{M,1}(\varkappa)\right]\\ &+2\frac{\psi'(\varkappa)g'(\varkappa)}{2!}\mu_{n,\alpha,2}^{M,1}(\varkappa)+\frac{g''(\varkappa)}{2!}\mu_{n,\alpha,2}^{M,1}(\varkappa) \left(\psi(\varkappa)-K_{n,\alpha}^{M,1}(\psi;\varkappa)\right)\\ &+\frac{g'(\varkappa)}{1!}\mu_{n,\alpha,1}^{M,1}(\varkappa)\left[\psi(\varkappa)-K_{n,\alpha}^{M,1}(\psi;\varkappa)\right]\Bigg\}.\end{aligned}$$

Using Theorem 1, for any $\psi\in C(J)$, $K_{n,\alpha}^{M,1}(\psi;\varkappa)\to\psi(\varkappa)$, as $n\to\infty$, uniformly in $\varkappa\in J$, and for $\psi''(\varkappa)\in C(J)$, from the proof of Theorem 5, it is clear that

$$\lim_{n\to\infty} n\left(K_{n,\alpha}^{M,1}(\psi;\varkappa) - \psi(\varkappa) - \frac{\psi'(\varkappa)}{1!}\mu_{n,\alpha,1}^{M,1}(\varkappa) - \frac{\psi''(\varkappa)}{2!}\mu_{n,\alpha,2}^{M,1}(\varkappa)\right) = 0,$$

uniformly in $\varkappa \in J$. Hence, applying Lemma 3, we obtain the desired result. □

3.6 Approximation of functions with derivatives of bounded variation (*DBV*)

Cheng (1983) obtained the rate of convergence of Bernstein polynomials for functions of bounded variation (BV). Bojanic and Cheng (1989) obtained the rate of convergence of Bernstein polynomials for functions with DBV. Bojanic and Khan (1991) estimated the rate of convergence of some operators for functions with DBV. Subsequently, many mathematicians studied in this direction (Bojanic and Cheng 1989, 1992; Zeng and Chen 2000; Gupta et al. 2003; Ibikli and Karsli 2005; Gupta et al. 2005). We shall obtain the rate of convergence of the operators $K_{n,\alpha}^{M,1}(\psi;\varkappa)$ defined by (1.8) for functions $\psi(\mathrm{t};\varkappa)$ having *DBV*. We show that the operators $K_{n,\alpha}^{M,1}(\psi;\varkappa)$ converge to the function $\psi(\varkappa)$, where $\psi'(\varkappa+)$ and $\psi'(\varkappa-)$ exist. Let $DBV(J)$ be the class of all absolutely continuous functions *f* on *J*, having a derivative ψ' equivalent with a function of *BV* on every finite subinterval of *J*. Note that the function $\psi \in DBV(J)$ can be represented as

$$\psi(\varkappa) = \int_0^{\varkappa} g(\mathrm{t})\mathrm{dt} + \psi(0),$$

where *g* is a function of *BV* on every finite subinterval of *J*. We observe that the operator (1.8) may be rewritten as

$$K_{n,\alpha}^{M,1}(\psi;\varkappa) = \int_0^1 N_{n,\alpha}^{M,1}(\mathrm{t},\varkappa)\psi(\mathrm{t})\mathrm{dt}, \tag{3.14}$$

where $N_{n,\alpha}^{M,1}(\mathrm{t},\varkappa)$ is the kernel defined as

$$N_{n,\alpha}^{M,1}(\mathrm{t},\varkappa) = \frac{n(n^2-1)}{(n-2)(n+1)+2\alpha}\sum_{m=0}^{n} P_{n,m,\alpha}^{M,1}(\varkappa)P_{n,m,\alpha}(\mathrm{t}). \tag{3.15}$$

Lemma 5 *For* $\forall\, \varkappa \in (0,1)$ *and sufficiently large* n, *we have*

$$(i)\quad \lambda_{n,\alpha}^{M,1}(\mathrm{t},\varkappa) = \int_0^{\mathrm{t}} N_{n,\alpha}^{M,1}(v,\varkappa)\mathrm{d}v \le \frac{\mu_{n,\alpha,2}^{M,1}(\varkappa)}{(\varkappa-\mathrm{t})^2},\quad 0\le \mathrm{t}<\varkappa,$$

$$(ii)\quad (1-\lambda_{n,\alpha}^{M,1}(\mathrm{t},\varkappa)) = \int_{\mathrm{t}}^1 N_{n,\alpha}^{M,1}(v,\varkappa)\mathrm{d}v \le \frac{\mu_{n,\alpha,2}^{M,1}(\varkappa)}{(\mathrm{t}-\varkappa)^2},\quad \varkappa\le \mathrm{t}<1.$$

Proof (i) By definition, we have

$$\lambda_{n,\alpha}^{M,1}(\mathrm{t},\varkappa) \le \int_0^{\mathrm{t}} \left(\frac{\varkappa-v}{\varkappa-\mathrm{t}}\right)^2 N_{n,\alpha}^{M,1}(v,\varkappa)\mathrm{d}v \le \frac{1}{(\varkappa-\mathrm{t})^2}\int_0^1 (v-\varkappa)^2 N_{n,\alpha}^{M,1}(v,\varkappa)\mathrm{d}v. = \frac{1}{(\varkappa-\mathrm{t})^2}\mu_{n,\alpha,2}^{M,1}(\varkappa).$$

Similarly, we can prove the other inequality (ii). □

Let

$$\psi'_{\varkappa}(\mathrm{t}) = \begin{cases} \psi'(\mathrm{t}) - \psi(\varkappa-), 0\le \mathrm{t}<\varkappa \\ 0, \quad \text{for } \mathrm{t}=\varkappa \\ \psi'(\mathrm{t}) - \psi'(\varkappa+), \quad \varkappa<\mathrm{t}<1 \end{cases} \tag{3.16}$$

Theorem 8 *Let* $\psi \in DBV(J)$, $\varkappa \in (0,1)$ *and* n *be sufficiently large. Then, we have*

$$|K_{n,\alpha}^{M,1}(\psi;\varkappa) - \psi(\varkappa)| \le \left|\frac{\psi'(\varkappa+)+\psi'(\varkappa-)}{2}\right| |\mu_{n,\alpha,1}^{M,1}(\varkappa)| + \frac{\mu_{n,\alpha,2}^{M,1}(\varkappa)}{\varkappa}\sum_{m=1}^{[\sqrt{n}]}\left(\bigvee_{\varkappa-\frac{\varkappa}{m}}^{\varkappa+\frac{\varkappa}{m}}\psi'_{\varkappa}\right) + \frac{\varkappa}{\sqrt{n}}\left(\bigvee_{\varkappa-\frac{\varkappa}{\sqrt{n}}}^{\varkappa+\frac{\varkappa}{\sqrt{n}}}\psi'_{\varkappa}\right) + \frac{\mu_{n,\alpha,2}^{M,1}(\varkappa)}{\varkappa^2}|\psi(2\varkappa)-\psi(\varkappa)-\varkappa\psi'(\varkappa+)| + \left(\frac{\|\psi\|+|\psi(\varkappa)|}{\varkappa^2}\right)\mu_{n,\alpha,2}^{M,1}(\varkappa) + |\psi'(\varkappa+)|\mu_{n,\alpha,2}^{M,1}(\varkappa) + \left|\frac{(\psi'(\varkappa+)-\psi'(\varkappa-))}{2}\right|\mu_{n,\alpha,2}^{M,1}(\varkappa),$$

Proof By the hypothesis (3.16), we have

$$\psi'(\mathrm{t}) = \frac{\psi'(\varkappa+)+\psi'(\varkappa-)}{2} + \psi'_{\varkappa}(\mathrm{t}) + \frac{\psi'(\varkappa+)-\psi'(\varkappa-)}{2}\mathrm{sgn}(\mathrm{t}-\varkappa) + \delta_{\varkappa}(\mathrm{t})\left(\psi'(\mathrm{t}) - \frac{\psi'(\varkappa+)+\psi'(\varkappa-)}{2}\right), \tag{3.17}$$

where

$$\delta_\varkappa(t) = \begin{cases} 1, & \text{for } t = \varkappa \\ 0, & \text{for } t \neq \varkappa. \end{cases}$$

Now using Lemma 2, Eqs. (3.14) and (3.17), we get

$$\begin{aligned} &K_{n,\alpha}^{M,1}(\psi;\varkappa) - \psi(\varkappa) \int_0^1 (\psi(t) - \psi(\varkappa)) N_{n,\alpha}^{M,1}(t,\varkappa)dt \\ &= \int_0^1 \left(\int_\varkappa^t \psi'(v)dv \right) N_{n,\alpha}^{M,1}(t,\varkappa)dt \\ &= \int_0^1 \Bigg[\int_\varkappa^t \Bigg\{ \frac{\psi'(\varkappa+) + \psi'(\varkappa-)}{2} + \psi'_\varkappa(v) \\ &+ \frac{\psi'(\varkappa+) - \psi'(\varkappa-)}{2} \operatorname{sgn}(v - \varkappa) \\ &\delta_\varkappa(v) \left(\psi'(v) - \frac{1}{2}(\psi'(\varkappa+) + \psi'(\varkappa-)) \right) \Bigg\} dv \Bigg] N_{n,\alpha}^{M,1}(t,\varkappa)dt. \end{aligned}$$

Since $\int_0^1 \left(\int_\varkappa^t \left(\psi'(v) - \frac{1}{2}(\psi'(\varkappa+) + \psi'(\varkappa-)) \right) \delta_\varkappa(v)dv \right) N_{n,\alpha}^{M,1}(t,\varkappa)dt = 0$, we have

$$\begin{aligned} &K_{n,\alpha}^{M,1}(\psi;\varkappa) - \psi(\varkappa) \left(\frac{\psi'(\varkappa+) + \psi'(\varkappa-)}{2} \right) \\ &\int_0^1 (t - \varkappa) N_{n,\alpha}^{M,1}(t,\varkappa)dt \\ &+ \int_0^1 \left(\int_\varkappa^t \psi'_\varkappa(v)dv \right) N_{n,\alpha}^{M,1}(t,\varkappa)dt \\ &\left(\frac{\psi'(\varkappa+) - \psi'(\varkappa-)}{2} \right) \int_0^1 |t - \varkappa| N_{n,\alpha}^{M,1}(t,\varkappa)dt. \end{aligned} \tag{3.18}$$

Now, we break the second term on the right-hand side of (3.18) as follows:

$$\begin{aligned} &\int_0^1 \left(\int_\varkappa^t \psi'_\varkappa(v)dv \right) N_{n,\alpha}^{M,1}(t,\varkappa)dt \\ &= -\int_0^\varkappa \left(\int_t^\varkappa \psi'_\varkappa(v)dv \right) N_{n,\alpha}^{M,1}(t,\varkappa)dt \\ &\quad + \int_\varkappa^1 \left(\int_\varkappa^t \psi'_\varkappa(v)dv \right) N_{n,\alpha}^{M,1}(t,\varkappa)dt \\ &= -I_1 + I_2. \end{aligned}$$

Then from (3.18), we get

$$\begin{aligned} &|K_{n,\alpha}^{M,1}(\psi;\varkappa) - \psi(\varkappa)| \left| \frac{\psi'(\varkappa+) + \psi'(\varkappa-)}{2} \right. \\ &\left| |K_{n,\alpha}^{M,1}((t - \varkappa);\varkappa)| + |I_1| + |I_2| \right. \\ &+ \left| \left(\frac{\psi'(\varkappa+) - \psi'(\varkappa-)}{2} \right) \right| K_{n,\alpha}^{M,1}(|t - \varkappa|;\varkappa). \end{aligned}$$

Now applying Cauchy–Schwarz inequality, we have

$$\begin{aligned} &|K_{n,\alpha}^{M,1}(\psi;\varkappa) - \psi(\varkappa)| \left| \frac{\psi'(\varkappa+) + \psi'(\varkappa-)}{2} \right| |\mu_{n,\alpha,1}^{M,1}(\varkappa)| \\ &+ |I_1| + |I_2| \\ &+ \left| \left(\frac{\psi'(\varkappa+) - \psi'(\varkappa-)}{2} \right) \right| \sqrt{\mu_{n,\alpha,2}^{M,1}(\varkappa)}. \end{aligned} \tag{3.19}$$

Using Lemma 5 and integration by parts,

$$\begin{aligned} I_1 &= \int_0^\varkappa \left(\int_t^\varkappa \psi'_\varkappa(v)dv \right) N_{n,\alpha}^{M,1}(t,\varkappa)dt \\ &= \int_0^\varkappa \left(\int_t^\varkappa \psi'_\varkappa(v)dv \right) \frac{\partial}{\partial t} \lambda_{n,\alpha}^{M,1}(t,\varkappa)dt \\ &= \int_0^\varkappa \psi'_\varkappa(t) \lambda_{n,\alpha}^{M,1}(t,\varkappa)dt. \end{aligned}$$

Consequently,

$$\begin{aligned} |I_1| &\le \int_0^\varkappa |\psi'_\varkappa(t)| \lambda_{n,\alpha}^{M,1}(t,\varkappa)dt \\ &= K_1(\varkappa) + K_2(\varkappa), \ (say). \end{aligned}$$

Since $\psi'_\varkappa(\varkappa) = 0$ by the hypothesis 3.16, it follows that

$$K_1(\varkappa) = \int_0^{(\varkappa - \frac{\varkappa}{\sqrt{n}})} |\psi'_\varkappa(t) - \psi'_\varkappa(\varkappa)| \lambda_{n,\alpha}^{M,1}(t,\varkappa)dt.$$

Now, using Lemma 5, we get

$$K_1(\varkappa) \le \mu_{n,\alpha,2}^{M,1}(\varkappa) \int_0^{(\varkappa - \frac{\varkappa}{\sqrt{n}})} |\psi'_\varkappa(t) - \psi'_\varkappa(\varkappa)| \frac{1}{(\varkappa - t)^2} dt.$$

By the definition of total variation of ψ and putting $t = (\varkappa - \frac{\varkappa}{v})$,

$$\begin{aligned} K_1(\varkappa) &\le \mu_{n,\alpha,2}^{M,1}(\varkappa) \int_0^{(\varkappa - \frac{\varkappa}{\sqrt{n}})} \left(\bigvee_t^\varkappa \psi'_\varkappa \right) \frac{1}{(\varkappa - t)^2} dt \\ &= \mu_{n,\alpha,2}^{M,1}(\varkappa) \int_1^{\sqrt{n}} \left(\bigvee_{\varkappa - \frac{\varkappa}{v}}^\varkappa \psi'_\varkappa \right) \frac{dv}{\varkappa}. \end{aligned}$$

Hence,

$$\begin{aligned} K_1(\varkappa) &\le \frac{\mu_{n,\alpha,2}^{M,1}(\varkappa)}{\varkappa} \sum_{m=1}^{[\sqrt{n}]} \int_m^{m+1} \left(\bigvee_{\varkappa - \frac{\varkappa}{m}}^\varkappa \psi'_\varkappa \right) dv. \\ &\le \frac{\mu_{n,\alpha,2}^{M,1}(\varkappa)}{\varkappa} \sum_{m=1}^{[\sqrt{n}]} \left(\bigvee_{\varkappa - \frac{\varkappa}{m}}^\varkappa \psi'_\varkappa \right). \end{aligned}$$

Since by Lemma 5, $\lambda_{n,\alpha}^{M,1}(t,\varkappa) \le 1$ and $\psi'_\varkappa(\varkappa) = 0$, we obtain

$$K_2(\varkappa) = \int_{(\varkappa-\frac{\varkappa}{\sqrt{n}})}^{\varkappa} |\psi'_\varkappa(\mathrm{t})|\lambda_{n,\alpha}^{M,1}(\mathrm{t},\varkappa)\mathrm{dt}$$
$$\leq \int_{(\varkappa-\frac{\varkappa}{\sqrt{n}})}^{\varkappa} |\psi'_\varkappa(\mathrm{t}) - \psi'_\varkappa(\varkappa)|\mathrm{dt}.$$

Now through the definition of total variation of ψ

$$K_2(\varkappa) \leq \int_{(\varkappa-\frac{\varkappa}{\sqrt{n}})}^{\varkappa} \left(\bigvee_{\mathrm{t}}^{\varkappa} \psi'_\varkappa(\mathrm{t})\right)\mathrm{dt}$$
$$\leq \left(\bigvee_{(\varkappa-\frac{\varkappa}{\sqrt{n}})}^{\varkappa} \psi'_\varkappa\right) \int_{(\varkappa-\frac{\varkappa}{\sqrt{n}})}^{\varkappa} \mathrm{dt}$$
$$= \frac{\varkappa}{\sqrt{n}} \left(\bigvee_{(\varkappa-\frac{\varkappa}{\sqrt{n}})}^{\varkappa} \psi'_\varkappa\right).$$

Hence combining the estimates of $K_1(\varkappa)$ and $K_2(\varkappa)$, we get

$$|I_1| \leq \frac{\mu_{n,\alpha,2}^{M,1}(\varkappa)}{\varkappa} \sum_{m=1}^{[\sqrt{n}]} \left(\bigvee_{\varkappa-\frac{\varkappa}{m}}^{\varkappa} \psi'_\varkappa\right) + \frac{\varkappa}{\sqrt{n}} \left(\bigvee_{\varkappa-\frac{\varkappa}{\sqrt{n}}}^{\varkappa} \psi'_\varkappa\right). \tag{3.20}$$

Using Lemma 5, we can write

$$|I_2| = \left| \int_{\varkappa}^{1} \left(\int_{\varkappa}^{\mathrm{t}} \psi'_\varkappa(v)\mathrm{d}v\right) N_{n,\alpha}^{M,1}(\mathrm{t},\varkappa)\mathrm{dt}\right|$$
$$\leq \left| \int_{\varkappa}^{2\varkappa} \left(\int_{\varkappa}^{\mathrm{t}} \psi'_\varkappa(v)\mathrm{d}v\right) \frac{\partial}{\partial t}\left(1 - \lambda_{n,\alpha}^{M,1}(\mathrm{t},\varkappa)\right)\mathrm{dt}\right|$$
$$\left| \int_{2\varkappa}^{1} \left(\int_{\varkappa}^{\mathrm{t}} \psi'_\varkappa(v)\mathrm{d}v\right) N_{n,\alpha}^{M,1}(\mathrm{t},\varkappa)\mathrm{dt}\right|.$$

Now applying integration by parts and hypothesis (3.16)

$$|I_2| \left| \left(\int_{\varkappa}^{2\varkappa} \psi'_\varkappa(v)\mathrm{d}v\right)(1 - \lambda_{n,\alpha}^{M,1}(2\varkappa,\varkappa)) \right.$$
$$\left. - \int_{\varkappa}^{2\varkappa} \psi'_\varkappa(\mathrm{t}).(1 - \lambda_{n,\alpha}^{M,1}(\mathrm{t},\varkappa))\mathrm{dt}\right|$$
$$+ \left| \int_{2\varkappa}^{1} \left(\int_{\varkappa}^{\mathrm{t}} (\psi'(v) - \psi'(\varkappa+))\mathrm{d}v\right) N_{n,\alpha}^{M,1}(\mathrm{t},\varkappa)\mathrm{dt}\right|.$$
$$\left| \int_{\varkappa}^{2\varkappa} \psi'_\varkappa(v)\mathrm{d}v\right|.\frac{\mu_{n,\alpha,2}^{M,1}}{\varkappa^2} + \int_{\varkappa}^{2\varkappa} |\psi'_\varkappa(\mathrm{t})|.(1 - \lambda_{n,\alpha}^{M,1}(\mathrm{t},\varkappa))\mathrm{dt}$$
$$\left| \int_{2\varkappa}^{1} (\psi(\mathrm{t}) - \psi(\varkappa))N_{n,\alpha}^{M,1}(\mathrm{t},\varkappa)\mathrm{dt}\right|$$
$$+ |\psi'(\varkappa+)| \left| \int_{2\varkappa}^{1} (\mathrm{t} - \varkappa)N_{n,\alpha}^{M,1}(\mathrm{t},\varkappa)\mathrm{dt}\right|$$

$E_1 + E_2 + E_3 + E_4$, (say).

Now, using hypothesis (3.16)

$$E_1 = \frac{\mu_{n,\alpha,2}^{M,1}(\varkappa)}{\varkappa^2} \left| \int_{\varkappa}^{2\varkappa} \left(\psi'(v) - \psi'(\varkappa+)\right)\mathrm{d}v\right|$$

and

$$E_2 = \int_{\varkappa}^{2\varkappa} |\psi'_\varkappa(\mathrm{t})|.(1 - \lambda_{n,\alpha}^{M,1}(\mathrm{t},\varkappa))\mathrm{dt}$$
$$= \int_{\varkappa}^{\varkappa+\frac{\varkappa}{\sqrt{n}}} |\psi'_\varkappa(\mathrm{t})|.(1 - \lambda_{n,\alpha}^{M,1}(\mathrm{t},\varkappa))\mathrm{dt}$$
$$+ \int_{\varkappa+\frac{\varkappa}{\sqrt{n}}}^{2\varkappa} |\psi'_\varkappa(\mathrm{t})|.(1 - \lambda_{n,\alpha}^{M,1}(\mathrm{t},\varkappa))\mathrm{dt} = J_1 + J_2.$$

Now in view of Lemma 5, $1 - \lambda_{n,\alpha}^{M,1}(\mathrm{t},\varkappa) \leq 1$ and from hypothesis (3.16), $\psi'_\varkappa(\varkappa) = 0$, therefore

$$J_1 \leq \int_{\varkappa}^{\varkappa+\frac{\varkappa}{\sqrt{n}}} |\psi'_\varkappa(\mathrm{t}) - \psi'_\varkappa(\varkappa)|\mathrm{dt}$$
$$\leq \int_{\varkappa}^{\varkappa+\frac{\varkappa}{\sqrt{n}}} \left(\bigvee_{\varkappa}^{t} \psi'_\varkappa\right) \leq \left(\bigvee_{\varkappa}^{\varkappa+\frac{\varkappa}{\sqrt{n}}} \psi'_\varkappa\right) \int_{\varkappa}^{\varkappa+\frac{\varkappa}{\sqrt{n}}} \mathrm{dt}$$
$$= \frac{\varkappa}{\sqrt{n}} \left(\bigvee_{\varkappa}^{\varkappa+\frac{\varkappa}{\sqrt{n}}} \psi'_\varkappa\right).$$

Now again using Lemma 5 and definition (3.16), we get

$$J_2 = \int_{\varkappa+\frac{\varkappa}{\sqrt{n}}}^{2\varkappa} |\psi'_\varkappa(\mathrm{t})|.(1 - \lambda_{n,\alpha}^{M,1}(\mathrm{t},\varkappa))\mathrm{dt}$$
$$\leq \mu_{n,\alpha,2}^{M,1}(\varkappa) \int_{\varkappa+\frac{\varkappa}{\sqrt{n}}}^{2\varkappa} |\psi'_\varkappa(\mathrm{t}) - \psi'_\varkappa(\varkappa)| \frac{\mathrm{dt}}{(\mathrm{t}-\varkappa)^2}.$$

From the definition of total variation of ψ and put $\mathrm{t} = \varkappa + \frac{\varkappa}{v}$,

$$J_2 \leq \mu_{n,\alpha,2}^{M,1}(\varkappa) \int_{\varkappa+\frac{\varkappa}{\sqrt{n}}}^{2\varkappa} \left(\bigvee_{\varkappa}^{\mathrm{t}} \psi'_\varkappa\right) \frac{\mathrm{dt}}{(\mathrm{t}-\varkappa)^2}$$
$$= \mu_{n,\alpha,2}^{M,1}(\varkappa) \int_{1}^{\sqrt{n}} \left(\bigvee_{\varkappa}^{\varkappa+\frac{\varkappa}{v}} \psi'_\varkappa\right) \frac{\mathrm{d}v}{\varkappa} \leq \frac{\mu_{n,\alpha,2}^{M,1}(\varkappa)}{\varkappa} \sum_{m=1}^{[\sqrt{n}]} \left(\bigvee_{\varkappa}^{\varkappa+\frac{\varkappa}{m}} \psi'_\varkappa\right).$$

Hence,

$$E_2 \leq \frac{\varkappa}{\sqrt{n}} \left(\bigvee_{\varkappa}^{\varkappa+\frac{\varkappa}{\sqrt{n}}} \psi'_\varkappa\right) + \frac{\mu_{n,\alpha,2}^{M,1}(\varkappa)}{\varkappa} \sum_{m=1}^{[\sqrt{n}]} \left(\bigvee_{\varkappa}^{\varkappa+\frac{\varkappa}{m}} \psi'_\varkappa\right).$$

Now, we calculate E_4. Using Cauchy–Schwarz inequality,

$$E_4 = |\psi'(\varkappa+)| \left| \int_{2\varkappa}^{1} (\mathrm{t} - \varkappa)N_{n,\alpha}^{M,1}(\mathrm{t},\varkappa)\mathrm{dt}\right|$$
$$\leq |\psi'(\varkappa+)| \int_{2\varkappa}^{1} |\mathrm{t} - \varkappa|N_{n,\alpha}^{M,1}(\mathrm{t},\varkappa)\mathrm{dt}$$
$$\leq |\psi'(\varkappa+)| \int_{0}^{1} |\mathrm{t} - \varkappa|N_{n,\alpha}^{M,1}(\mathrm{t},\varkappa)\mathrm{dt}$$
$$= |\psi'(\varkappa+)| \sqrt{K_{n,\alpha}^{M,1}((\mathrm{t}-\varkappa)^2;\varkappa)} = |\psi'(\varkappa+)| \sqrt{\mu_{n,\alpha,2}^{M,1}(\varkappa)}.$$

In order to estimate E_3, we note that $t \geq 2\varkappa$, hence $(\mathrm{t} - \varkappa) \geq \varkappa$, therefore

$$\begin{aligned}E_3 &\leq \int_{2\varkappa}^{1} |\psi(t)| N_{n,\alpha}^{M,1}(t,\varkappa)dt + \int_{2\varkappa}^{1} |\psi(\varkappa)| N_{n,\alpha}^{M,1}(t,\varkappa)dt \\ &\leq \|\psi\| \int_{2\varkappa}^{1} N_{n,\alpha}^{M,1}(t,\varkappa)dt + |\psi(\varkappa)| \int_{2\varkappa}^{1} N_{n,\alpha}^{M,1}(t,\varkappa)dt \\ &\leq (\|\psi\| + |\psi(\varkappa)|) \int_{2\varkappa}^{1} N_{n,\alpha}^{M,1}(t,\varkappa)dt \\ &\leq (\|\psi\| + |\psi(\varkappa)|) \int_{2\varkappa}^{1} \frac{(t-\varkappa)^2}{\varkappa^2} N_{n,\alpha}^{M,1}(t,\varkappa)dt \\ &\leq \left(\frac{\|\psi\| + |\psi(\varkappa)|}{\varkappa^2}\right) \mu_{n,\alpha,2}^{M,1}(\varkappa).\end{aligned}$$

Hence collecting the estimates of $E_1 - E_4$, we have

$$\begin{aligned}|I_2| &\leq \frac{\mu_{n,\alpha,2}^{M,1}(\varkappa)}{\varkappa^2} |\psi(2\varkappa) - \psi(\varkappa) - \varkappa\psi'(\varkappa+)| \\ &\quad + \frac{\varkappa}{\sqrt{n}} \left(\bigvee_{\varkappa}^{\varkappa+\frac{\varkappa}{\sqrt{n}}} \psi'_\varkappa \right) + \frac{\mu_{n,\alpha,2}^{M,1}(\varkappa)}{\varkappa} \sum_{m=1}^{[\sqrt{n}]} \left(\bigvee_{\varkappa}^{\varkappa+\frac{\varkappa}{m}} \psi'_\varkappa \right) \\ &\quad + \left(\frac{\|\psi\| + |\psi(\varkappa)|}{\varkappa^2}\right) \mu_{n,\alpha,2}^{M,1}(\varkappa) + |\psi'(\varkappa+)| \mu_{n,\alpha,2}^{M,1}(\varkappa).\end{aligned} \tag{3.21}$$

Now combining Eqs. (3.19)–(3.21), we reach the desired result. □

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