Iran J Sci Technol Trans Sci (2020) 44:213–218
https://doi.org/10.1007/s40995-019-00814-1

RESEARCH PAPER

Characterization of *n*-Jordan Homomorphisms and Automatic Continuity of 3-Jordan Homomorphisms on Banach Algebras

A. Zivari-Kazempour[1]

Received: 1 August 2019 / Accepted: 23 December 2019 / Published online: 20 January 2020

Abstract

We show that every Banach algebra $\mathcal{A}$ is 3-Jordan functionally continuous. More especially, every 3-Jordan homomorphism from a Banach algebra $\mathcal{A}$ into a commutative semisimple Banach algebra $\mathcal{B}$ is automatically continuous. We also prove the same result for n-Jordan homomorphism ($n \geqslant 4$) with the additional hypothesis that the Banach algebra $\mathcal{A}$ is unital. A characterization of n-Jordan homomorphism for the special case $n \in \{2, 3, 4\}$ is also given.

Keywords n-Homomorphism · n-Jordan homomorphism · Automatic continuity · n-Jordan functionally continuous · Semisimple

Mathematics Subject Classification Primary 47B48 · Secondary 46L05 · 46H25

1 Introduction and Preliminaries

Let $\mathcal{A}$ and $\mathcal{B}$ be complex Banach algebras and $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ be a linear map. Then, φ is called an *n-homomorphism* if for all $a_1, a_2, \ldots, a_n \in \mathcal{A}$,

$$\varphi(a_1 a_2 \ldots a_n) = \varphi(a_1)\varphi(a_2)\ldots\varphi(a_n).$$

The concept of an n-homomorphism was studied for complex algebras in Bračič and Moslehian (2007) and Hejazian et al. (2005). A linear map φ between Banach algebras $\mathcal{A}$ and $\mathcal{B}$ is called an *n-Jordan homomorphism* if $\varphi(a^n) = \varphi(a)^n$, for all $a \in \mathcal{A}$. This notion was introduced by Herstein (1956).

A 2-homomorphism (2-Jordan homomorphism) is called simply a homomorphism (Jordan homomorphism). It is clear that every n-homomorphism is an n-Jordan homomorphism, but in general the converse is false. The connection of Jordan homomorphisms and homomorphisms was firstly studied in Jacobson and Rickart (1950). There are plenty of known examples of n-Jordan homomorphism which are not n-homomorphism. For $n = 2$, it is proved in Jacobson and Rickart (1950) that some Jordan homomorphism on the polynomial rings cannot be homomorphism.

It is shown in Gordji (2009) that every n-Jordan homomorphism between two commutative Banach algebras is an n-homomorphism for $n \in \{3, 4\}$. Note that for $n = 2$, the proof is clear. Lee (2013) and Gselmann (2014) generalized this result and proved it for all $n \in \mathbb{N}$. Later, this problem was solved in Bodaghi and İnceboz (2018) based on the property of the Vandermonde matrix, which is different from the methods that are used in Gselmann (2014) and Lee (2013).

Obviously, each homomorphism is an n-homomorphism for every $n \geqslant 2$, but the converse does not hold in general. For instance, if $h : \mathcal{A} \longrightarrow \mathcal{B}$ is a homomorphism, then $g := -h$ is a 3-homomorphism which is not a homomorphism Bračič and Moslehian (2007).

Zelazko (1968) presented the following result [see also Miura et al. (2005)].

Theorem 1.1 *Suppose that $\mathcal{A}$ is a Banach algebra, which need not be commutative, and suppose that $\mathcal{B}$ is a semisimple commutative Banach algebra. Then, each Jordan homomorphism $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ is a homomorphism.*

This result has been proved by the author in Zivari-Kazempour (2016) for 3-Jordan homomorphism with the additional hypothesis that the Banach algebra $\mathcal{A}$ is unital. In other words, he presented the next theorem.

✉ A. Zivari-Kazempour
zivari@abru.ac.ir; zivari6526@gmail.com

[1] Department of Mathematics, Ayatollah Borujerdi University, Borujerd, Iran

Theorem 1.2 *Suppose that* $\mathcal{A}$ *is a unital Banach algebra, which need not be commutative, and suppose that* $\mathcal{B}$ *is a semisimple commutative Banach algebra. Then, each 3-Jordan homomorphism* $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ *is a 3-homomorphism.*

After that, An (2018) extended the above theorem for all $n \in \mathbb{N}$ and showed that for unital ring $\mathcal{A}$ and ring $\mathcal{B}$ with $\mathrm{char}(\mathcal{B}) > n$, every n-Jordan homomorphism from $\mathcal{A}$ into $\mathcal{B}$ is an n-homomorphism (n-anti-homomorphism) provided that every Jordan homomorphism from $\mathcal{A}$ into $\mathcal{B}$ is a homomorphism (anti-homomorphism).

For non-unital Banach algebra $\mathcal{A}$, Bodaghi and İnceboz in Bodaghi and İnceboz (2019) proved Theorem 1.2 for $n \in \{3, 4\}$ by considering an extra condition that $\varphi([a^2, b]) = 0$, for all $a, b \in \mathcal{A}$, where $[a, b] = ab - ba$ is the Lie product of a and b.

The following theorem is a well-known result, due to Šilov, concerning the automatic continuity of homomorphisms between Banach algebras Bonsall and Duncan (1973).

Theorem 1.3 (Dales (2000), Theorem 2.3.3) *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be Banach algebras such that* $\mathcal{B}$ *is commutative and semisimple. Then, every homomorphism* $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ *is automatically continuous.*

In 1967, a classical result of B. E. Johnson shows that if $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ is a surjective homomorphism between a Banach algebra $\mathcal{A}$ and a semisimple Banach algebra $\mathcal{B}$, then φ is automatically continuous. Then, the Johnson's result was extended to n-homomorphism in Honari and Shayanpour (2010), with the extra condition that the Banach algebra $\mathcal{B}$ is factorizable, and then it was extended to non-factorizable Banach algebras in Gordji et al. (2015).

In Hejazian et al. (2005), the authors asked whether every $*$-preserving n-homomorphism between C^*-algebras is continuous. For $n = 3$, Bračič and Moslehian (2007) responded to the above question and presented the next result.

Theorem 1.4 *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be* C^**-algebras and* $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ *be an involution-preserving 3-homomorphism. Then* φ *is norm decreasing.*

By Theorem 1.4, every involution-preserving 3-homomorphism between C^*-algebras is continuous. For any arbitrary natural number n, this question solved by Park and Trout (2009), in the affirmative by proving that every involutive n-homomorphism between C^*-algebras is norm contractive.

Homomorphisms and their automatic continuity of Banach algebras have been widely studied by many authors. One may refer to the monographs of Dales (2000), Jarosz (1985) and Palmer (1994). Some significant results concerning Jordan homomorphisms and their automatic continuity on Banach algebras obtained by the author in Zivari-Kazempour (2016, 2018a, b).

In Sect. 2, we investigate the automatic continuity of n-Jordan homomorphism and we show that every 3-Jordan homomorphism from a Banach algebra $\mathcal{A}$ into a commutative semisimple Banach algebra $\mathcal{B}$ is continuous. We also prove the same result for n-Jordan homomorphism ($n \geqslant 4$) with the additional hypothesis that the Banach algebra $\mathcal{A}$ is unital. As a direct consequence, we obtain the automatic continuity of n-homomorphisms.

In Sect. 3, for $n \in \{2, 3, 4\}$, under certain conditions we show that each n-Jordan homomorphism between algebras $\mathcal{A}$ and $\mathcal{B}$ is an n-homomorphism.

2 Automatic Continuity of *n*-Jordan Homomorphisms

A Banach algebra $\mathcal{A}$ is called *n -functionally continuous*, if every n-multiplicative linear functional on $\mathcal{A}$ is continuous, and it is called *n -Jordan functionally continuous*, if every n-Jordan linear functional on $\mathcal{A}$ is continuous.

A 2-functionally continuous (2-Jordan functionally continuous) algebra is just functionally continuous (Jordan functionally continuous), in the usual sense.

Theorem 2.1 (Shayanpour et al. (2015), Corollary 2.2) *A topological algebra* $\mathcal{A}$ *is n-functionally continuous if and only if it is functionally continuous.*

By Theorems 1.3 and 2.1, every Banach algebra $\mathcal{A}$ is n-functionally continuous.

Theorem 2.2 (Zivari-Kazempour (2018b), Proposition 2.1) *Let* $\mathcal{A}$ *be a Banach algebra. Then, every Jordan functional* $\varphi : \mathcal{A} \longrightarrow \mathbb{C}$ *is norm decreasing and hence it is continuous.*

By the above theorem, every Banach algebra $\mathcal{A}$ is Jordan functionally continuous. Furthermore, if φ is a Jordan homomorphism from a Banach algebra $\mathcal{A}$ into a commutative semisimple Banach algebra $\mathcal{B}$, then φ is continuous. The following question can be raised for $n \geqslant 3$.

Is every Banach algebra $\mathcal{A}$ *n-Jordan functionally continuous?*

Recently, the author has proved that if $\mathcal{A}$ and $\mathcal{B}$ are unital Banach algebras, where $\mathcal{B}$ is commutative and semisimple, then every unital n-Jordan homomorphism $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ is automatically continuous [see Corollary 2.9 of Zivari-Kazempour (2018b)]. Thus, the answer of this question is affirmative with the extra condition that the mapping $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ is unital.

In the next result, which is the main one in the paper, we present the positive answer to this question for $n = 3$.

Theorem 2.3 *Every Banach algebra $\mathcal{A}$ is 3-Jordan functionally continuous.*

Proof Let $\varphi : \mathcal{A} \longrightarrow \mathbb{C}$ be a 3-Jordan functional. Then $\varphi(x^3) = \varphi(x)^3$, for all $x \in \mathcal{A}$. Replacing x by $x + y$, we get

$$\begin{aligned}&\varphi(xy^2 + y^2x + x^2y + yx^2 + xyx + yxy)\\ &\quad = 3\varphi(x)\varphi(y)^2 + 3\varphi(x)^2\varphi(y),\end{aligned} \tag{1}$$

and switching x by $-x$, in (1), gives

$$\begin{aligned}&\varphi(-xy^2 - y^2x + x^2y + yx^2 + xyx - yxy)\\ &\quad = -3\varphi(x)\varphi(y)^2 + 3\varphi(x)^2\varphi(y).\end{aligned} \tag{2}$$

By (1) and (2), we have

$$\varphi(x^2y + xyx + yx^2) = 3\varphi(x)^2\varphi(y), \tag{3}$$

for all $x, y \in \mathcal{A}$.

Suppose that there exist $a \in \mathcal{A}$ such that $\|a\| < 1$ and $\varphi(a) = 1$. Take $b = \sum_{n=1}^{\infty} a^n$. Then, $a^2b = aba = ba^2 = b - a - a^2$ and so by (3) we get

$$\begin{aligned}\varphi(b - a - a^2) = \varphi(a^2b) &= \frac{1}{3}\varphi\left(a^2b + aba + ba^2\right)\\ &= \frac{1}{3}\left(3\varphi(a)^2\varphi(b)\right) = \varphi(b).\end{aligned}$$

Thus, $\varphi(a^2) = -\varphi(a) = -1$. Let X be a Banach subalgebra of $\mathcal{A}$, generated by the above element a of norm $\|a\| < 1$. For all $x \in X$, define $\psi : X \longrightarrow \mathbb{C}$ by $\psi(x) := \varphi(x)$. Then, ψ is a 3-Jordan functional, that is $\psi(x^3) = \psi(x)^3$, for all $x \in X$. Since X is commutative, we obtain

$$\psi(xyz) = \psi(x)\psi(y)\psi(z), \tag{4}$$

for all $x, y, z \in X$. Replacing z by a^2 in (4) gives

$$\psi(xya^2) = -\psi(x)\psi(y), \tag{5}$$

for all $x, y \in X$. Since $\psi(a) = 1$, by (4) and (5), we have

$$\begin{aligned}\psi(x)\psi(ay) &= \psi(x)\psi(ay)\psi(a)\\ &= \psi(xaya) = \psi(xya^2) = -\psi(x)\psi(y),\end{aligned}$$

and since $\psi \neq 0$ we get $\psi(ay) = -\psi(y)$, for all $y \in X$. As such

$$\psi(x)\psi(y) = \psi(a)\psi(x)\psi(y) = \psi(axy) = -\psi(xy),$$

for all $x, y \in X$. Consequently, $-\psi$ is a multiplicative linear functional, and hence, it is continuous with $\|\psi\| \leqslant 1$. This is a contradiction with $\|a\| < 1$ and $\psi(a) = 1$. Thus, for all $a \in \mathcal{A}$ with $\|a\| \leqslant 1$, we have $|\varphi(a)| \leqslant 1$. Therefore, φ is norm decreasing, and hence, it is continuous. This finishes the proof. □

Corollary 2.4 *Let φ be a 3-Jordan homomorphism from a Banach algebra $\mathcal{A}$ into a commutative semisimple Banach algebra $\mathcal{B}$. Then, φ is automatically continuous.*

Proof Let $h \in \mathfrak{M}(\mathcal{B})$, where $\mathfrak{M}(\mathcal{B})$ is the maximal ideal space of $\mathcal{B}$. Then, $h \circ \varphi : \mathcal{A} \longrightarrow \mathbb{C}$ is a 3-Jordan homomorphism and so it is automatically continuous by Theorem 2.3. Now suppose that $a_n \subseteq \mathcal{A}$, $a_n \longrightarrow 0$ and $\varphi(a_n) \longrightarrow b$. Hence,

$$h(b) = \lim_n h(\varphi(a_n)) = \lim_n h \circ \varphi(a_n) = 0.$$

Therefore, $h(b) = 0$, for each $h \in \mathfrak{M}(\mathcal{B})$. Since $\mathcal{B}$ is semisimple, we have $b = 0$. Thus, φ is continuous by the closed graph theorem. □

Since every C^*-algebra is semisimple, we deduce the next result.

Corollary 2.5 *Every 3-Jordan homomorphism from a Banach algebra $\mathcal{A}$ into a commutative C^*-algebra $\mathcal{B}$ is automatically continuous.*

From Corollary 2.4, we have the following result.

Corollary 2.6 *Let φ be a 3-homomorphism from a Banach algebra $\mathcal{A}$ into a commutative semisimple Banach algebra $\mathcal{B}$. Then, φ is automatically continuous.*

Now we generalize Theorem 2.3, for $n \geqslant 4$, with the additional hypothesis that the Banach algebra $\mathcal{A}$ is unital.

Theorem 2.7 *Every unital Banach algebra $\mathcal{A}$ is n-Jordan functionally continuous.*

Proof Suppose that $\varphi : \mathcal{A} \longrightarrow \mathbb{C}$ is an n-Jordan homomorphism. Then, by Corollary 2.5 of An (2018), φ is an n-homomorphism. Now it follows from Theorems 1.3 and 2.1 that φ is automatically continuous. Thus, $\mathcal{A}$ is n-Jordan functionally continuous. □

The next result generalizes Corollary 2.5 of An (2018), without surjectivity condition.

Corollary 2.8 *Let φ be an n-Jordan homomorphism from a unital Banach algebra $\mathcal{A}$ into a commutative semisimple Banach algebra $\mathcal{B}$. Then, φ is automatically continuous.*

Since every n-Jordan homomorphism from commutative Banach algebra $\mathcal{A}$ into $\mathbb{C}$ is an n-homomorphism Bodaghi and İnceboz (2018), we have the following.

Theorem 2.9 *Let $\mathcal{A}$ be a commutative Banach algebra. Then, $\mathcal{A}$ is n-Jordan functionally continuous.*

Corollary 2.10 *Let $\mathcal{A}$ and $\mathcal{B}$ be two commutative Banach algebra, and let $\mathcal{B}$ be semisimple. Then, every n-Jordan homomorphism $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ is automatically continuous.*

The following result has been proved for the case $n = 2$ and $n = 3$ in Zivari-Kazempour (2018b), Theorem 2.7, and it was claimed that the result can be established for $n \geqslant 4$, by a similar discussion. We give a short proof for the general case $n \in \mathbb{N}$ as follows:

Theorem 2.11 *Every unital $(n+1)$-Jordan homomorphism $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ is an n-Jordan homomorphism.*

Proof We firstly have

$$\varphi((a+me)^{n+1}) = (\varphi(a+me))^{n+1} \tag{6}$$

for all $a, b \in \mathcal{A}$, where m is an integer with $1 \leqslant m \leqslant n$ and e is the unit of $\mathcal{A}$. It follows from the equality (6) and assumption that

$$\sum_{i=1}^{n} m^{n+1-i}\binom{n+1}{i}[\varphi(a^i) - \varphi(a)^i] = 0, \qquad (1 \leqslant m \leqslant n), \tag{7}$$

for all $a \in \mathcal{A}$ where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. We can rewrite the equalities in (7) as follows:

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 2^n & 2^{n-1} & \cdots & 2 \\ 3^n & 3^{n-1} & \cdots & 3 \\ \cdots & \cdots & \cdots & \cdots \\ n^n & n^{n-1} & \cdots & n \end{bmatrix} \begin{bmatrix} \Gamma_1(a) \\ \Gamma_2(a) \\ \Gamma_3(a) \\ \cdots \\ \Gamma_n(a) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \cdots \\ 0 \end{bmatrix}, \tag{8}$$

for all $a \in \mathcal{A}$, where

$$\Gamma_i(a) = \binom{n+1}{i}[\varphi(a^i) - \varphi(a)^i],$$

for all $1 \leqslant i \leqslant n$. It is shown in [Bodaghi and İnceboz (2018) , Lemma 2.1] that the above square matrix is invertible. This implies that $\Gamma_i(a) = 0$ for all $1 \leqslant i \leqslant n$ and all $a \in \mathcal{A}$. In particular, $\Gamma_n(a) = 0$. This means that φ is an n-Jordan homomorphism. □

In view of Theorems 2.3 and 2.7, we have the following question.

Question 2.12 *Is every Banach algebra $\mathcal{A}$ n-Jordan functionally continuous without any additional hypothesis?*

3 Characterization of *n*-Jordan Homomorphisms

Let $\mathcal{A}$ and $\mathcal{B}$ be complex algebras and $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ be a linear map. Then, φ is called a *mixed n-Jordan homomorphism* if for all $a, b \in \mathcal{A}$,

$$\varphi(ab^n) = \varphi(a)\varphi(b)^n.$$

This notion was introduced in Neghabi et al. (2020). For $n = 2$, we speak about mixed Jordan homomorphism. Clearly, every n-homomorphism is an mixed $(n-1)$-Jordan homomorphism for $n \geqslant 3$, and every mixed n-Jordan homomorphism is $(n+1)$-Jordan homomorphism, but the converse is not true in general. The following example illustrates this fact.

Example 3.1 Let

$$\mathcal{A} = \left\{ \begin{bmatrix} u & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix} : \quad u, a, b, c \in \mathbb{C} \right\},$$

and define $\varphi : \mathcal{A} \longrightarrow \mathcal{A}$ by

$$\varphi\left(\begin{bmatrix} u & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} -u & 0 & 0 \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix}.$$

Then, for all $n \geqslant 3$ and for any $U \in \mathcal{A}$, we have

$$\varphi(U^n) = \varphi(U)^n.$$

Thus, φ is n-Jordan homomorphism for all $n \geqslant 3$, but φ is not mixed $(2n-3)$-Jordan homomorphism.

Let $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ be a map between Banach algebras $\mathcal{A}$ and $\mathcal{B}$. Then, we say that $\mathcal{B}$ is φ-commutative if for all $a, b \in \mathcal{A}$, $[\varphi(a), \varphi(b)] = 0$. Note that every commutative Banach algebra is id-commutative, where id is the identity map.

Example 3.2 Consider the Banach algebras $\mathcal{A}$ as in Example 3.1 and let

$$\mathcal{B} = \left\{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} : \quad a, b \in \mathbb{C} \right\},$$

with the usual product. Define the linear map $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ by

$$\varphi\left(\begin{bmatrix} u & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} u & 0 \\ 0 & 0 \end{bmatrix}.$$

Then, $\mathcal{B}$ is non-commutative Banach algebra, but it is φ-commutative.

Theorem 3.3 *Let $n \in \{3, 4, 5\}$ be fixed, and let φ be an n-Jordan homomorphism from algebra $\mathcal{A}$ into φ-commutative algebra $\mathcal{B}$ such that $\varphi([a, b]) = 0$ for any $a, b \in \mathcal{A}$. Then, φ is $(n-1)$-mixed Jordan homomorphism.*

Proof The case $n = 3$ is Theorem 2.7 of Neghabi et al. (2020). Suppose that $n = 4$ and let

$$\varphi(x^4) = \varphi(x)^4, \qquad (x \in \mathcal{A}). \tag{9}$$

Replacing x by $a + b$ in (9), we obtain

$$p + q = 6\varphi(a)^2\varphi(b)^2 + 4\varphi(a)\varphi(b)^3 + 4\varphi(a)^3\varphi(b), \tag{10}$$

where

$$p = \varphi(a^2b^2 + abab + ab^2a + ba^2b + baba + b^2a^2),$$

and

$$q = \varphi(a^3b + a^2ba + aba^2 + ab^3 + ba^3 + bab^2 + b^2ab + b^3a).$$

Switching b by $-b$ in (10), we get

$$p - q = 6\varphi(a)^2\varphi(b)^2 - 4\varphi(a)\varphi(b)^3 - 4\varphi(a)^3\varphi(b), \tag{11}$$

for all $a, b \in \mathcal{A}$. The equalities (10) and (11) show that

$$p = 6\varphi(a)^2\varphi(b)^2, \qquad (a, b \in \mathcal{A}). \tag{12}$$

Since $\varphi([a, b]) = 0$ for all $a, b \in \mathcal{A}$, we have

$$\varphi([a, ab^2]) = \varphi([ab, ba]) = \varphi([b, ba^2]) = \varphi([a, bab]) = 0. \tag{13}$$

It follows from (12) and (13) that

$$2\varphi(a^2b^2) + \varphi(abab) = 3\varphi(a)^2\varphi(b)^2, \quad (a, b \in \mathcal{A}). \tag{14}$$

Replacing a by $a + b$ in (14), we get

$$\varphi(3ab^3 + 2bab^2 + b^2ab) = 6\varphi(a)\varphi(b)^2, \tag{15}$$

for all $a, b \in \mathcal{A}$. By assumption, $\varphi([b, ab^2]) = \varphi([b^2, ab]) = 0$, and hence, (15) implies that

$$\varphi(ab^3) = \varphi(a)\varphi(b)^3,$$

for all $a, b \in \mathcal{A}$. Therefore, φ is 3-mixed Jordan homomorphism. By the same method, we can prove the result for $n = 5$. □

Theorem 3.4 *Let $n \in \{2, 3, 4\}$ be fixed, and let φ be an n-Jordan homomorphism from algebra $\mathcal{A}$ into φ-commutative algebra $\mathcal{B}$ such that $\varphi([a, b]) = 0$ for any $a, b \in \mathcal{A}$. If ker φ is an ideal of $\mathcal{A}$, then φ is an n-homomorphism.*

Proof The case $n = 2$ is trivial. Suppose that $n = 3$ and let $\varphi(x^3) = \varphi(x)^3$, for all $x \in \mathcal{A}$. Replacing x by $a + b$, we get

$$\varphi(ab^2 + b^2a + a^2b + ba^2 + aba + bab) = 3\varphi(a)\varphi(b)^2 + 3\varphi(a)^2\varphi(b), \tag{16}$$

and switching b by $-b$, in (16), gives

$$\varphi(ab^2 + b^2a - a^2b - ba^2 - aba + bab) = 3\varphi(a)\varphi(b)^2 - 3\varphi(a)^2\varphi(b). \tag{17}$$

By (16) and (17), we have

$$\varphi(ab^2 + b^2a + bab) = 3\varphi(a)\varphi(b)^2, \tag{18}$$

for all $a, b \in \mathcal{A}$. Replacing b by $b - c$ in (18), we deduce

$$\varphi(abc + acb + bac + bca + cab + cba) = 6\varphi(a)\varphi(b)\varphi(c). \tag{19}$$

Since $[a, b] \in \ker\varphi$ for all $a, b \in \mathcal{A}$, we have

$$\varphi([ab, c]) = 0, \quad \varphi([a, bc]) = 0. \tag{20}$$

It follows from (19) and (20) that

$$3\varphi(abc) + \varphi(acb + bac + cba) = 6\varphi(a)\varphi(b)\varphi(c), \tag{21}$$

for all $a, b, c \in \mathcal{A}$. Similarly, we have

$$\varphi([ac, b]) = 0, \quad \varphi([ba, c]) = 0. \tag{22}$$

By (21) and (22), we get

$$\varphi(abc) + \varphi(acb) = 2\varphi(a)\varphi(b)\varphi(c). \tag{23}$$

By assumption, $\ker\varphi$ is an ideal of $\mathcal{A}$, and hence, $\varphi(a[b, c]) = 0$. Thus, $\varphi(abc) = \varphi(acb)$ and so (23) implies that

$$\varphi(abc) = \varphi(a)\varphi(b)\varphi(c), \quad (a, b, c \in \mathcal{A}).$$

Hence, φ is 3-homomorphism.

Now assume that $n = 4$ and let $\varphi(x^4) = \varphi(x)^4$, for all $x \in \mathcal{A}$. By similar argument which has been used in the proof of Theorem 3.3, we get the relation (14). That is,

$$2\varphi(x^2y^2) + \varphi(xyxy) = 3\varphi(x)^2\varphi(y)^2, \tag{24}$$

for all $x, y \in \mathcal{A}$. Replacing x by $a + b$ in (24) gives

$$2\varphi(aby^2 + bay^2) + \varphi(ayby + byay) = 6\varphi(a)\varphi(b)\varphi(y)^2, \quad (a, b \in \mathcal{A}). \tag{25}$$

Replacing y by $x + y$ in (25), to get

$$2\varphi(abxy + abyx + baxy + bayx) + \varphi(axby + aybx + bxay + byax) = 12\varphi(a)\varphi(b)\varphi(x)\varphi(y). \tag{26}$$

Since $\ker\varphi$ is an ideal of $\mathcal{A}$ and $[a, b] \in \ker\varphi$ for all $a, b \in \mathcal{A}$, we get

$$\varphi([a, b]xy) = 0, \quad \varphi([a, b]yx) = 0, \quad \varphi(ab[x, y]) = 0, \tag{27}$$

for all $a, b, x, y \in \mathcal{A}$. It follows from (26) and (27) that

$$8\varphi(abxy) + \varphi(axby + aybx + bxay + byax) = 12\varphi(a)\varphi(b)\varphi(x)\varphi(y). \tag{28}$$

Similarly, we have

$$\varphi(a[bx, y]) = 0. \tag{29}$$

From (27) and the equation $\varphi(a[by, x]) = 0$ we obtain

$$\varphi(a[b,x]y) = 0. \tag{30}$$

It follows from (28), (29) and (30) that

$$10\varphi(abxy) + \varphi(bxay + byax) = 12\varphi(a)\varphi(b)\varphi(x)\varphi(y), \tag{31}$$

for all $a, b, x, y \in \mathcal{A}$. By using (27) and the equations

$$\varphi(b[ax, y]) = \varphi(b[ay, x]) = 0,$$

we deduce from (31) that $\varphi(abxy) = \varphi(a)\varphi(b)\varphi(x)\varphi(y)$ for all $a, b, x, y \in \mathcal{A}$. Thus, φ is an 4-homomorphism. This completes the proof. □

In general, the kernel of an n-Jordan homomorphism may not be an ideal. For example, let $\mathcal{A}$ be the algebra of all 3×3 matrices having 0 on and below the diagonal. In this algebra, product of any 3 elements is equal to 0, so any linear map from $\mathcal{A}$ into itself is a 3-Jordan homomorphism, but its kernel does not need to be an ideal. For Jordan homomorphism $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$, it is shown in Palmer (1994) that kerφ is an ideal, if $\mathcal{B}$ is semiprime and φ is surjective.

The following two results are reported by the author in Zivari-Kazempour (2018a, b), respectively.

Theorem 3.5 *Let φ be an unital n-Jordan homomorphism from Banach algebra $\mathcal{A}$ into a semiprime Banach algebra $\mathcal{B}$. Then, ker φ is an ideal of $\mathcal{A}$ if either φ is surjective, or $\mathcal{B}$ is commutative.*

Theorem 3.6 *Let φ be a 3-Jordan homomorphism from unital Banach algebra $\mathcal{A}$ into a Banach algebra $\mathcal{B}$ such that $\varphi([a,b]) = 0$ for all $a, b \in \mathcal{A}$. Then, ker φ is an ideal of $\mathcal{A}$.*

Note that Theorem 3.6 can be proved for the case $n = 4$. Therefore, by Theorems 3.6 and 3.4 we have the next result.

Corollary 3.7 *Let $n \in \{2, 3, 4\}$ be fixed, and let φ be an n-Jordan homomorphism from unital algebra $\mathcal{A}$ into φ-commutative algebra $\mathcal{B}$ such that $\varphi([a,b]) = 0$ for any $a, b \in \mathcal{A}$. Then, φ is an n-homomorphism.*

Remark 3.8 It seems that Theorem 3.4 holds for any arbitrary natural number $n \geqslant 5$, but the proof method of Theorem 3.4 is long and tedious. Therefore, that way is not suitable for such numbers.

Acknowledgements The author gratefully acknowledges the helpful comments of the anonymous referees. This research was partially supported by a Grant from Ayatollah Borujerdi University with No. 15664–170143.

References

An G (2018) Characterization of n-Jordan homomorphism. Linear Multilinear Algebra 68(4):671–680

Bodaghi A, İnceboz H (2018) n-Jordan homomorphisms on commutative algebras. Acta Math Univ Comenianae 87(1):141–146

Bodaghi A, İnceboz H (2019) Extension of Zelazko's theorem to n-Jordan homomorphisms. Adv Pure Appl Math 10(2):165–170

Bonsall FF, Duncan J (1973) Complete normed algebra. Springer, New York

Bračič J, Moslehian MS (2007) On automatic continuity of 3-homomorphisms on Banach algebras. Bull Malays Math Sci Soc 30(2):195–200

Dales HG (2000) Banach algebras and automatic continuity. London mathematical society, monograph 24. Clarendon Press, Oxford

Gordji ME (2009) n-Jordan homomorphisms. Bull Aust Math Soc 80(1):159–164

Gordji ME, Jabbari A, Karapinar E (2015) Automatic continuity of surjective n-homomorphisms on Banach algebras. Bull Iran Math Soc 41(5):1207–1211

Gselmann E (2014) On approximate n-Jordan homomorphisms. Annales Math Silesianae 28:47–58

Hejazian S, Mirzavaziri M, Moslehian MS (2005) n-homomorphisms. Bull Iran Math Soc 31(1):13–23

Herstein IN (1956) Jordan homomorphisms. Trans Am Math Soc 81(1):331–341

Honari TG, Shayanpour H (2010) Automatic continuity of n-homomorphisms between Banach algebras. Q Math 33(2):189–196

Jacobson N, Rickart CE (1950) Jordan homomorphisms of rings. Trans Am Math Soc 69(3):479–502

Jarosz K (1985) Perturbation of Banach algebras. Lecture notes in mathematics. Springer, Berlin

Lee YH (2013) Stability of n-Jordan homomorphisms from a normed algebra to a Banach algebra. Abstr Appl Anal 2013, Article ID 691025, 1–5

Miura T, Takahasi SE, Hirasawa G (2005) Hyers–Ulam–Rassias stability of Jordan homomorphisms on Banach algebras. J. Inequal. Appl. 2005(4):435–441

Neghabi M, Bodaghi A, Zivari-Kazempour A (2020) Characterization of mixed n-Jordan homomorphisms and pseudo n-Jordan homomorphisms, Preprint

Palmer T (1994) Banach algebras and the general theory of $*$-algebras, vol I. University Press, Cambridge

Park E, Trout J (2009) On the nonexistence of nontrivial involutive n-homomorphisms of C^*-algebras. Trans Am Math Soc 361(4):1949–1961

Shayanpour H, Honary TG, Hashemi MS (2015) Certain properties of n-characters and n-homomorphisms on topological algebras. Bull Malays Math Sci Soc 38:985–999

Zelazko W (1968) A characterization of multiplicative linear functionals in complex Banach algebras. Studia Math 30:83–85

Zivari-Kazempour A (2016) A characterization of 3-Jordan homomorphism on Banach algebras. Bull Aust Math Soc 93(2):301–306

Zivari-Kazempour A (2018) A characterization of Jordan and 5-Jordan homomorphisms between Banach algebras. Asian Eur J Math 11(2):1–10

Zivari-Kazempour A (2018) Automatic continuity of n-Jordan homomorphisms on Banach algebras. Commun Korean Math Soc 33(1):165–170