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RESEARCH PAPER

Necessary Stationary Conditions for Multiobjective Optimization Problems with Nondifferentiable Convex Vanishing Constraints

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Abstract

In this paper, we consider a multiobjective optimization problem with vanishing constraints, in which its objective functions are continuously differentiable and its constraints are convex, not necessarily differentiable. We introduce two new Abadie-type constraint qualifications and present some necessary condition for properly efficient solutions of the problem, using convex subdifferential.

Keywords Stationary conditions · Multiobjective optimization · Abadie constraint qualification · Vanishing constraints.

Mathematics Subject Classification 90C34 · 90C40 · 49J52

1 Introduction

Given continuously differentiable functions $f_j : \mathbb{R}^n \to \mathbb{R}$ as $j \in J := \{1, \ldots, p\}$, and convex functions $g_i, h_i : \mathbb{R}^n \to \mathbb{R}$ as $i \in I := \{1, \ldots, m\}$, we define the "multiobjective mathematical programming with vanishing constraints" (MMPVC in brief) as

$$\text{(MMPVC):} \quad \min f(x) := \big(f_1(x), \ldots, f_p(x)\big)$$
$$\text{s.t. } x \in S := \{x \in \mathbb{R}^n \mid h_i(x) \geq 0,$$
$$g_i(x)h_i(x) \leq 0, \quad i \in I\}.$$

The above assumptions about objective and constraint functions are standing throughout the whole paper.

If $p = 1$, then MMPVC coincides with the "mathematical programming with vanishing constraints" (MPVC) which is introduced in Achtziger and Kanzow (2008), Hoheisel and Kanzow (2007). The MPVCs received attention from different fields. Some of their applications in topological optimization and geometry have been introduced in Achtziger and Kanzow (2008), Shikhman (2012). Karush–Kuhn–Tucker (KKT)-type optimality conditions for MPVCs, named stationary conditions, are presented in some studies (see Achtziger and Kanzow 2008; Hoheisel and Kanzow 2008, 2009 and Kazemi and Kanzi 2018 for smooth and nonsmooth cases, respectively).

If $h_i(x)g_i(x) \leq 0$ is replaced by $h_i(x)g_i(x) = 0$ for each $i \in I$, MMPVC reduces to "multiobjective programming problem with equilibrium constraints," (MMPEC). Stationary conditions for smooth and nonsmooth MMPECs are established under various constraint qualifications (CQ); see, e.g., (Ansari Ardali et al. 2016; Bigi et al. 2016; Movahedian 2017; Movahedian and Nobakhtian 2010) for $p = 1$, and (Luu 2016) for $p > 1$.

It is easy to see that MMPVC is a generalization of MMPEC and MPVC. To the best of our knowledge, it is not any work available dealing with stationary conditions for MMPVCs. The aim of this paper is to extend some stationary conditions for optimality of MMPVCs. In addition to classic multiobjective optimization, we can consider different kinds of optimality (efficiency) for MMPVC, including weakly efficient solution, efficient solution, strictly efficient solution, isolated efficient solution, and properly efficient solution. In this paper, we focus on properly efficient solutions for MMPVCs.

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The structure of subsequent sections of this paper is as follows: In Sect. 2, we define required definitions and preliminary results which are requested in sequel. Section 3 is devoted to the main results of paper, containing some Abadie-type CQs and some kinds of necessary stationary conditions for the problem.

2 Preliminaries

This section contains some preliminary results in convex analysis from (Rockafellar 1970; Rockafellar and Wets 1998).

First, we recall that the nonnegative real numbers $[0, +\infty)$, the nonpositive real number $(-\infty, 0]$, the standard inner product of vectors $x, y \in \mathbb{R}^n$, and the zero vector of $\mathbb{R}^n$ are, respectively, denoted by $\mathbb{R}_+$, $\mathbb{R}_-$, $\langle x, y\rangle$, and 0_n.

Considering $\Omega \subseteq \mathbb{R}^n$, the negative polar cone of Ω is defined as

$$\Omega^0 := \{x \in \mathbb{R}^n \mid \langle x, u\rangle \le 0, \quad \forall u \in \Omega\}.$$

The closure, the convex cone (containing origin), and the closed convex cone of Ω are, respectively, denoted by $cl(\Omega)$, $cone(\Omega)$, and $clcone(\Omega)$. Also, the orthogonal set, contingent cone of Ω at $\bar{x} \in cl(\Omega)$, and the Fréchet normal cone of Ω at $\bar{x}$ are, respectively, defined as

$$\Omega^\perp := \{x \in \mathbb{R}^n \mid \langle x, u\rangle = 0, \quad \forall u \in \Omega\},$$

$$\Gamma_\Omega(\bar{x}) := \Big\{y \in \mathbb{R}^n \mid \exists t_\ell \downarrow 0,\ \exists y_\ell \to y \ \text{ such that } \ \bar{x} + t_\ell y_\ell \in \Omega \ \ \forall \ell \in \mathbb{N}\Big\},$$

and $\widehat{N}_\Omega(\bar{x}) := (\Gamma_\Omega(\bar{x}))^0$.

Theorem 1 (bipolar theorem Rockafellar and Wets 1998) *Suppose that Ω is a subset on $\mathbb{R}^n$. Then,*

$$(\Omega^0)^0 = clcone(\Omega).$$

Theorem 2 (Rockafellar and Wets 1998) *Let $\psi : \mathbb{R}^n \to \mathbb{R}$be a continuously differentiable function at $\bar{x} \in \Omega \subseteq \mathbb{R}^n$. If the minimum of ψ on Ω is attained at $\bar{x}$, then*

$$-\nabla\psi(x_0) \in \widehat{N}_\Omega(\bar{x}).$$

Theorem 3 (Rockafellar 1970) *Suppose that the linear function $\psi : \mathbb{R}^n \longrightarrow \mathbb{R}$ is defined as $\psi(x) = \langle z, x\rangle$ for a given $z \in \mathbb{R}^n$. If $A \subseteq \mathbb{R}$ is a given convex set and we define $\psi^{-1}\big(cl(A)\big) := \{x \in \mathbb{R}^n \mid \psi(x) \in cl(A)\}$, then*

$$\big(\psi^{-1}\big(cl(A)\big)\big)^0 = A^0 z.$$

Theorem 4 (Rockafellar and Wets 1998) *Suppose that L is an arbitrary index set and that $\Omega_\ell \subseteq \mathbb{R}^n$ is closed convex cone for each $\ell \in L$. Then*

$$\Big(\bigcup_{\ell\in L}\Omega_\ell\Big)^0 = \bigcap_{\ell\in L}\Omega_\ell^0, \qquad \Big(\bigcap_{\ell\in L}\Omega_\ell\Big)^0 = clcone\Big(\bigcup_{\ell\in L}\Omega_\ell^0\Big).$$

Theorem 5 (Rockafellar 1970) *Let $\Omega_1, \ldots, \Omega_k$ be closed convex cones in $\mathbb{R}^n$. One may conclude that*

$$cone\Big(\bigcup_{\ell=1}^k \Omega_\ell\Big) = \sum_{\ell=1}^k \Omega_\ell.$$

Let $\varphi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a convex function, and $x_0 \in dom\varphi := \{x \in \mathbb{R}^n \mid \varphi(x) < +\infty\}$. The subdifferential of φ at x_0 is defined as

$$\partial\varphi(x_0) := \{\xi \in \mathbb{R}^n \mid \varphi(x) - \varphi(x_0) \ge \langle \xi, x - x_0\rangle, \quad \forall x \in \mathbb{R}^n\}.$$

It should be noted that the subdifferential set $\partial\varphi(x_0)$ is always nonempty, compact, and convex in $\mathbb{R}^n$.

3 Main Results

At starting point of this section, we recall from (Geoffrion 1968; Gopfert et al. 2003) that a feasible point $\bar{x} \in S$ is called a properly efficient solution to MMPVC when there exist some positive scalars $\eta_1, \ldots, \eta_p > 0$ such that

$$\sum_{j=1}^p \eta_j f_j(\bar{x}) \le \sum_{j=1}^p \eta_j f_j(x), \quad \forall x \in S.$$

Considering a feasible point $\hat{x} \in S$ (this point will be fixed throughout this paper), we define the following index sets:

$$I_{00} := \{i \in I \mid h_i(\hat{x}) = 0,\ g_i(\hat{x}) = 0\},$$
$$I_{0+} := \{i \in I \mid h_i(\hat{x}) = 0,\ g_i(\hat{x}) > 0\},$$
$$I_{0-} := \{i \in I \mid h_i(\hat{x}) = 0,\ g_i(\hat{x}) < 0\},$$
$$I_{+0} := \{i \in I \mid h_i(\hat{x}) > 0,\ g_i(\hat{x}) = 0\},$$
$$I_{+-} := \{i \in I \mid h_i(\hat{x}) > 0,\ g_i(\hat{x}) < 0\}.$$

Following (Kazemi and Kanzi 2018), we consider two linearized cones $\mathcal{L}^0$ and $\mathcal{L}^\sharp := \mathcal{L}^0 \cap \Lambda$ for MMPVC, where

$$\mathcal{L} := \Big(\bigcup_{I_{0+}}\partial h_i(\hat{x})\Big) \cup \Big(-\bigcup_{I_{0+}}\partial h_i(\hat{x})\Big) \cup \Big(-\bigcup_{I_{0-}\cup I_{00}}\partial h_i(\hat{x})\Big) \cup \Big(\bigcup_{I_{+0}}\partial g_i(\hat{x})\Big),$$

$$\Lambda := \Big\{ v \in \mathbb{R}^n \mid \langle v, \xi_i \rangle \langle v, \zeta_i \rangle \leq 0, \\ \forall \xi_i \in \bigcup_{I_{00}} \partial g_i(\hat{x}),\ \forall \zeta_i \in \bigcup_{I_{00}} \partial h_i(\hat{x}) \Big\}.$$

It is worth mentioning that unlike to $\mathcal{L}^0$, the linearized cone $\mathcal{L}^\sharp$ is not convex.

Motivated by Achtziger and Kanzow (2008), Hoheisel and Kanzow (2009), Kazemi and Kanzi (2018), we define two Abadie-type constraint qualifications for MMPVC.

Definition 1 We say that MMPVC satisfies the ACQ (resp. $ACQ_\sharp$), if $\Gamma_S(\hat{x}) \subseteq \mathcal{L}^0$ $\left(\text{resp. } \Gamma_S(\hat{x}) \subseteq \mathcal{L}^\sharp\right)$.

Trivially, the following implication holds by $\mathcal{L}^\sharp \subseteq \mathcal{L}^0$,

$ACQ \implies ACQ_\sharp$.

Remark 1 We observe that $ACQ_\sharp$ is named MPVC-ACQ in some studies; see Hoheisel and Kanzow (2009) and Kazemi and Kanzi (2018) for MPVCs with smooth and nonsmooth data, respectively. Since the concepts of ACQ and $ACQ_\sharp$ are described and discussed in detailed manner in Hoheisel and Kanzow (2009), Kazemi and Kanzi (2018), we do not repeat that descriptions in the present article. Also, there are provided MPVC-tailored constraint qualifications which are sufficient conditions for ACQ and $ACQ_\sharp$ in (Achtziger and Kanzow 2008. Theorems 2 and 3), (Hoheisel and Kanzow 2008, Sect. 4), and (Kazemi and Kanzi 2018, Theorem 3.1).

The following simple theorem is a normal extension of (Kazemi and Kanzi 2018, Theorem 4.1).

Theorem 6 *Let $\hat{x}$ be a properly efficient solution to MMPVC, and ACQ holds at $\hat{x}$.*

(i) *There exist some positive scalars λ_j^f, $j \in J$, such that*

$$-\sum_{j=1}^{p} \lambda_j^f \nabla f_j(\hat{x}) \in clcone\big(\mathcal{L}\big).$$

(ii) *If, in addition, $cone\big(\mathcal{L}\big)$ is a closed cone, we can find some coefficients λ_i^h and λ_i^g as $i \in I$, such that:*

$$0_n \in \sum_{j=1}^{p} \lambda_j^f \nabla f_j(\hat{x}) + \sum_{i=1}^{m} \Big(\lambda_i^g \partial g_i(\hat{x}) - \lambda_i^h \partial h_i(\hat{x})\Big), \tag{1}$$

$$\lambda_i^g \geq 0,\ i \in I_{+0}; \\ \lambda_i^g = 0,\ i \in I_{0+} \cup I_{0-} \cup I_{00} \cup I_{+-}; \tag{2}$$

$$\lambda_i^h \text{ is free},\ i \in I_{0+};\ \lambda_i^h \geq 0,\ i \in I_{0-} \cup I_{00}; \\ \lambda_i^h = 0,\ i \in I_{+-} \cup I_{+0}. \tag{3}$$

Proof (i) The definition of properly efficiency leads us to fine some positive scalars $\lambda_j^f > 0$, for $j \in J$, such that $\hat{x}$ is a minimizer of $\sum_{j=1}^{p} \lambda_j^f f_j(x)$ on S. Thus, owing to Theorem 2, we get

$$-\sum_{j=1}^{p} \lambda_j^f \nabla f_j(\hat{x}) \in \widehat{N}_S(\hat{x}). \tag{4}$$

On the other hand, by ACQ and bipolar Theorem 1 we conclude that

$$N_S(\hat{x}) = (\Gamma_S(\hat{x}))^0 \subseteq \big(\mathcal{L}^0\big)^0 = clcone\big(\mathcal{L}\big).$$

This inclusion and (4) imply the result.

(ii) The structure of convex cones implies that

$$cone(\mathcal{L}) = \bigcup_{\alpha_i, \beta_i, \gamma_i \in \mathbb{R}_+} \Big\{ \sum_{i \in I_{0+}} \alpha_i \partial h_i(\hat{x}) + \sum_{i \in I_{0+}} \beta_i \Big(-\partial h_i(\hat{x})\Big) \\ + \sum_{i \in I_{0-} \cup I_{00}} \alpha_i \Big(-\partial h_i(\hat{x})\Big) + \sum_{i \in I_{+0}} \gamma_i \partial g_i(\hat{x}) \mid \alpha_i, \beta_i, \gamma_i \geq 0 \Big\} = \\ \bigcup_{\lambda_i^g, \lambda_i^h} \left\{ \sum_{i=1}^{m} \big(\lambda_i^g \partial g_i(\hat{x}) - \lambda_i^h \partial h_i(\hat{x})\big) \;\middle|\; \begin{array}{ll} \lambda_i^g \geq 0, & i \in I_{+0} \\ \lambda_i^g = 0, & i \in I \setminus I_{+0} \\ \lambda_i^h \text{ is free}, & i \in I_{0+} \\ \lambda_i^h \geq 0, & i \in I_{0-} \cup I_{00} \\ \lambda_i^h = 0, & i \in I_{+-} \cup I_{+0} \end{array} \right\}, \tag{5}$$

where,

$$\lambda_i^g := \gamma_i,\quad i \in I_{+0}, \quad \text{and} \quad \lambda_i^h := \begin{cases} \beta_i - \alpha_i, & i \in I_{0+} \\ \alpha_i, & i \in I_{0-} \cup I_{00} \end{cases}.$$

The closedness condition, virtue of (5), and part (i) conclude the result. □

It is worth mentioning that when $p = 1$, conditions (1)–(3) which are named "strongly stationary condition" (resp. KKT condition) (Hoheisel and Kanzow 2007; Kazemi and Kanzi 2018) (resp. Achtziger and Kanzow 2008; Hoheisel and Kanzow 2008), present an important optimality condition for MPVCs which are an appropriate alternative for classic KKT condition. Another important point of Theorem 6 is that all the coefficients λ_j^f are nonzero, which guaranties the effect of each objective functions in the relation of (1); to see the theoretical significance of this topic, we can refer to (Kanzi 2015, 2018). The present paper is the first that studies this kind of stationary condition for MMPVCs.

Since $ACQ_\sharp$ is weaker than ACQ, we cannot expect the strongly stationary condition to hold at properly efficient solution $\hat{x}$ where $ACQ_\sharp$ is satisfied. In the rest of this section, we get another optimality condition under $ACQ_\sharp$, which is weaker than strongly stationary condition. Since the $ACQ_\sharp$ is easier to happen than ACQ, this new stationary condition will be more user-friendly in applications. We observe that owing to nonconvexity of $\mathcal{L}_\sharp$, we cannot follow the simple strategy of Theorem 6 for giving the new stationary condition, and for achieve it, we need some preliminaries.

For each $w \in \mathbb{R}^n$, $i \in I$, and $I_* \subseteq I$, let

$$B_i(w) := \{\langle w, \xi_i\rangle \mid \xi_i \in \partial g_i(\hat{x})\} \times \{\langle w, \zeta_i\rangle \mid \zeta_i \in \partial h_i(\hat{x})\} \subseteq \mathbb{R}^2,$$
$$B_{I_*}(w) := \bigcup_{i\in I_*} B_i(w).$$

The following technical lemma plays a key role in the reminder of this article.

Lemma 1 *Suppose that the constraints of MMPVC with index $i \in I_{00}$ are first written, then the constraints with index $i \in I_{0+}$, then $i \in I_{0-}$, then $i \in I_{+0}$, and finally $i \in I_{+-}$. Assume also that $Y \subseteq \mathbb{R}^{2m}$ is defined as*

$$Y := \prod_{I_{00}}(\mathbb{R}_- \times \mathbb{R}_+) \times \prod_{I_{0+}}(\mathbb{R} \times \{0\}) \times \prod_{I_{0-}}(\mathbb{R} \times \mathbb{R}_+) \times \prod_{I_{+0}}(\mathbb{R}_- \times \mathbb{R}) \times \prod_{I_{+-}}(\mathbb{R} \times \mathbb{R}). \tag{6}$$

Then, one has

$$\left\{w \in \mathbb{R}^n \mid \prod_{i=1}^m B_i(w) \subseteq Y\right\}^0 = cl\left[\bigcup_{\lambda_i^g,\lambda_i^h}\left\{\sum_{i=1}^m\left(\lambda_i^g\partial g_i(\hat{x}) - \lambda_i^h\partial h_i(\hat{x})\right) \;\middle|\; \begin{array}{ll} \lambda_i^g \geq 0, & i \in I_{00}\cup I_{+0}\\ \lambda_i^g = 0, & i \in I_{0+}\cup I_{0-}\cup I_{+-}\\ \lambda_i^h \text{ is free}, & i \in I_{0+}\\ \lambda_i^h \geq 0, & i \in I_{0-}\cup I_{00}\\ \lambda_i^h = 0, & i \in I_{+-}\cup I_{+0}\end{array}\right\}\right].$$

Proof Due to Theorem 4, the following equalities are fulfilled:

$$\begin{aligned}
&\left\{w \in \mathbb{R}^n \mid \prod_{i=1}^m B_i(w) \subseteq Y\right\}^0 = \\
&\left\{w \in \mathbb{R}^n \mid B_{I_{00}}(w) \subseteq \mathbb{R}_- \times \mathbb{R}_+,\ B_{I_{0+}}(w) \subseteq \mathbb{R} \times \{0\},\ B_{I_{0-}}(w) \subseteq \mathbb{R} \times \mathbb{R}_+,\right.\\
&\qquad \left. B_{I_{+0}}(w) \subseteq \mathbb{R}_- \times \mathbb{R},\ B_{I_{+-}}(w) \subseteq \mathbb{R} \times \mathbb{R}\right\}^0 = \\
&\left[\left(\bigcap_{i\in I_{00}}\left\{w \in \mathbb{R}^n \mid B_i(w) \subseteq \mathbb{R}_- \times \mathbb{R}_+\right\}\right)\right.\\
&\cap\left(\bigcap_{i\in I_{0+}}\left\{w \in \mathbb{R}^n \mid B_i(w) \subseteq \mathbb{R} \times \{0\}\right\}\right)\cap\\
&\left(\bigcap_{i\in I_{0-}}\left\{w \in \mathbb{R}^n \mid B_i(w) \subseteq \mathbb{R} \times \mathbb{R}_+\right\}\right)\\
&\cap\left(\bigcap_{i\in I_{+0}}\left\{w \in \mathbb{R}^n \mid B_i(w) \subseteq \mathbb{R}_- \times \mathbb{R}\right\}\right)\cap\\
&\qquad\left.\left(\bigcap_{i\in I_{+-}}\left\{w \in \mathbb{R}^n \mid B_i(w) \subseteq \mathbb{R} \times \mathbb{R}\right\}\right)\right]^0 = \\
&clcone\left[\left(\bigcup_{i\in I_{00}}\left\{w \in \mathbb{R}^n \mid B_i(w) \subseteq \mathbb{R}_- \times \mathbb{R}_+\right\}^0\right)\cup\right.\\
&\left(\bigcup_{i\in I_{0+}}\left\{w \in \mathbb{R}^n \mid B_i(w) \subseteq \mathbb{R} \times \{0\}\right\}^0\right)\\
&\cup\left(\bigcup_{i\in I_{0-}}\left\{w \in \mathbb{R}^n \mid B_i(w) \subseteq \mathbb{R} \times \mathbb{R}_+\right\}^0\right)\cup\\
&\left(\bigcup_{i\in I_{+0}}\left\{w \in \mathbb{R}^n \mid B_i(w) \subseteq \mathbb{R}_- \times \mathbb{R}\right\}^0\right)\\
&\left.\cup\left(\bigcup_{i\in I_{+-}}\left\{w \in \mathbb{R}^n \mid B_i(w) \subseteq \mathbb{R} \times \mathbb{R}\right\}^0\right)\right].
\end{aligned}\tag{7}$$

The definition of $B_i(w)$ and Theorem 4 imply that, for each $i \in I_{00}$, one has

$$\begin{aligned}
&\left\{w \in \mathbb{R}^n \mid B_i(w) \subseteq \mathbb{R}_- \times \mathbb{R}_+\right\}^0\\
&= \left[\left\{w \in \mathbb{R}^n \mid \{\langle w, \xi_i\rangle \mid \xi_i \in \partial g_i(\hat{x})\} \subseteq \mathbb{R}_-\right\}\cap\right.\\
&\qquad\left.\left\{w \in \mathbb{R}^n \mid \{\langle w, \zeta_i\rangle \mid \zeta_i \in \partial h_i(\hat{x})\} \subseteq \mathbb{R}_+\right\}\right]^0\\
&= clcone\left[\left\{w \in \mathbb{R}^n \mid \{\langle w, \xi_i\rangle \mid \xi_i \in \partial g_i(\hat{x})\} \subseteq \mathbb{R}_-\right\}^0\cup\right.\\
&\qquad\left.\left\{w \in \mathbb{R}^n \mid \{\langle w, \zeta_i\rangle \mid \zeta_i \in \partial h_i(\hat{x})\} \subseteq \mathbb{R}_+\right\}^0\right]
\end{aligned}$$

$$= clcone\Big[\Big(\bigcap_{\xi_i\in\partial g_i(\hat{x})}\{w\in\mathbb{R}^n \mid \langle w,\xi_i\rangle\in\mathbb{R}_-\}\Big)^0\cup \Big(\bigcap_{\zeta_i\in\partial h_i(\hat{x})}\{w\in\mathbb{R}^n \mid \langle w,\zeta_i\rangle\in\mathbb{R}_+\}\Big)^0\Big]$$
$$= clcone\Big[clcone\Big(\bigcup_{\xi_i\in\partial g_i(\hat{x})}\{w\in\mathbb{R}^n \mid \langle w,\xi_i\rangle\in\mathbb{R}_+\}^0\Big)\cup clcone\Big(\bigcup_{\zeta_i\in\partial h_i(\hat{x})}\{w\in\mathbb{R}^n \mid \langle w,\zeta_i\rangle\in\mathbb{R}_-\}^0\Big)\Big]$$
$$= clcone\Big[\Big(\bigcup_{\xi_i\in\partial g_i(\hat{x})}\{w\in\mathbb{R}^n \mid \langle w,\xi_i\rangle\in\mathbb{R}_-\}^0\Big)\cup \Big(\bigcup_{\zeta_i\in\partial h_i(\hat{x})}\{w\in\mathbb{R}^n \mid \langle w,\zeta_i\rangle\in\mathbb{R}_+\}^0\Big)\Big]. \tag{8}$$

For each $\xi_i\in\partial g_i(\hat{x})$ and $\zeta_i\in\partial h_i(\hat{x})$, we consider the functions $\widehat{g}_{\xi_i},\widehat{h}_{\zeta_i}:\mathbb{R}^n\to\mathbb{R}$ as $\widehat{g}_{\xi_i}(w):=\langle w,\xi_i\rangle$ and $\widehat{h}_{\zeta_i}(w):=\langle w,\zeta_i\rangle$. Thus, equality (8) can be rewritten as

$$\Big\{w\in\mathbb{R}^n \mid B_i(w)\subseteq\mathbb{R}_-\times\mathbb{R}_+\Big\}^0 = clcone\Big[\Big(\bigcup_{\xi_i\in\partial g_i(\hat{x})}\widehat{g}_{\xi_i}^{-1}(\mathbb{R}_-)\Big)\cup\Big(\bigcup_{\zeta_i\in\partial h_i(\hat{x})}\widehat{h}_{\zeta_i}^{-1}(\mathbb{R}_+)\Big)\Big].$$

The last equality and Theorem 3 yield

$$\Big\{w\in\mathbb{R}^n \mid B_i(w)\subseteq\mathbb{R}_-\times\mathbb{R}_+\Big\}^0 = clcone\Big[\Big(\bigcup_{\xi_i\in\partial g_i(\hat{x})}\xi_i(\mathbb{R}_-)^0\Big)\cup\Big(\bigcup_{\zeta_i\in\partial h_i(\hat{x})}\zeta_i(\mathbb{R}_+)^0\Big)\Big] = clcone\Big[\Big(\bigcup_{\xi_i\in\partial g_i(\hat{x})}\xi_i\mathbb{R}_+\Big)\cup\Big(\bigcup_{\zeta_i\in\partial h_i(\hat{x})}\zeta_i\mathbb{R}_-\Big)\Big] = clcone\Big[\mathbb{R}_+\partial g_i(\hat{x})\cup\mathbb{R}_-\partial h_i(\hat{x})\Big] = cl\Big[\mathbb{R}_+\partial g_i(\hat{x})+\mathbb{R}_-\partial h_i(\hat{x})\Big], \tag{9}$$

where the last equality holds by Theorem 5.

From (9) and similar processes for $i\in I_{0+}$, $i\in I_{0-}$, $i\in I_{+0}$, $i\in I_{+-}$, we deduce that

$$\begin{cases}\bigcup_{i\in I_{00}}\Big\{w\in\mathbb{R}^n \mid B_i(w)\subseteq\mathbb{R}_-\times\mathbb{R}_+\Big\}^0=\bigcup_{i\in I_{00}}cl\Big[\mathbb{R}_+\partial g_i(\hat{x})+\mathbb{R}_-\partial h_i(\hat{x})\Big],\\ \bigcup_{i\in I_{0+}}\Big\{w\in\mathbb{R}^n \mid B_i(w)\subseteq\mathbb{R}\times\{0\}\Big\}^0=\bigcup_{i\in I_{0+}}cl\Big[\{0\}\partial g_i(\hat{x})+\mathbb{R}\partial h_i(\hat{x})\Big],\\ \bigcup_{i\in I_{0-}}\Big\{w\in\mathbb{R}^n \mid B_i(w)\subseteq\mathbb{R}\times\mathbb{R}_+\Big\}^0=\bigcup_{i\in I_{0-}}cl\Big[\{0\}\partial g_i(\hat{x})+\mathbb{R}_-\partial h_i(\hat{x})\Big],\\ \bigcup_{i\in I_{+0}}\Big\{w\in\mathbb{R}^n \mid B_i(w)\subseteq\mathbb{R}_-\times\mathbb{R}\Big\}^0=\bigcup_{i\in I_{+0}}cl\Big[\mathbb{R}_+\partial g_i(\hat{x})+\{0\}\partial h_i(\hat{x})\Big],\\ \bigcup_{i\in I_{+-}}\Big\{w\in\mathbb{R}^n \mid B_i(w)\subseteq\mathbb{R}\times\mathbb{R}\Big\}^0=\bigcup_{i\in I_{+-}}cl\Big[\{0\}\partial g_i(\hat{x})+\{0\}\partial h_i(\hat{x})\Big].\end{cases} \tag{10}$$

Now, (7), (10), and Theorem 5 conclude that

$$\Big\{w\in\mathbb{R}^n \mid \prod_{i=1}^m B_i(w)\subseteq Y\Big\}^0 = clcone\Big[\Big(\bigcup_{i\in I_{00}}cl\Big[\mathbb{R}_+\partial g_i(\hat{x})+\mathbb{R}_-\partial h_i(\hat{x})\Big]\Big)\cup\Big(\bigcup_{i\in I_{0+}}cl\Big[\{0\}\partial g_i(\hat{x})+\mathbb{R}\partial h_i(\hat{x})\Big]\Big)\cup\Big(\bigcup_{i\in I_{0-}}cl\Big[\{0\}\partial g_i(\hat{x})+\mathbb{R}_-\partial h_i(\hat{x})\Big]\Big)\cup\Big(\bigcup_{i\in I_{+0}}cl\Big[\mathbb{R}_+\partial g_i(\hat{x})+\{0\}\partial h_i(\hat{x})\Big]\Big)\cup\Big(\bigcup_{i\in I_{+-}}cl\Big[\{0\}\partial g_i(\hat{x})+\{0\}\partial h_i(\hat{x})\Big]\Big)\Big]$$

$$= cl\Bigg[\sum_{i\in I_{00}}\Big(\mathbb{R}_+\partial g_i(\hat{x}) + \mathbb{R}_-\partial h_i(\hat{x})\Big) + \sum_{i\in I_{0+}}\Big(\{0\}\partial g_i(\hat{x}) + \mathbb{R}\partial h_i(\hat{x})\Big) + \sum_{i\in I_{0-}}\Big(\{0\}\partial g_i(\hat{x}) + \mathbb{R}_-\partial h_i(\hat{x})\Big) + \sum_{i\in I_{+0}}\Big(\mathbb{R}_+\partial g_i(\hat{x}) + \{0\}\partial h_i(\hat{x})\Big) + \sum_{i\in I_{+-}}\Big(\{0\}\partial g_i(\hat{x}) + \{0\}\partial h_i(\hat{x})\Big)\Bigg]$$

$$= cl\Bigg[\bigcup_{\mu_i^g,\mu_i^h}\Bigg\{\sum_{i=1}^m(\mu_i^g\partial g_i(\hat{x}) + \mu_i^h\partial h_i(\hat{x}))\ \Bigg|\ \begin{array}{ll}\mu_i^g\in\mathbb{R}_+, & i\in I_{00}\cup I_{+0}\\ \mu_i^g\in\{0\}, & i\in I_{0+}\cup I_{0-}\cup I_{+-}\\ \mu_i^h\in\mathbb{R}, & i\in I_{0+}\\ \mu_i^h\in\mathbb{R}_-, & i\in I_{0-}\cup I_{00}\\ \mu_i^h\in\{0\}, & i\in I_{+-}\cup I_{+0}\end{array}\Bigg\}\Bigg] =$$

$$cl\Bigg[\bigcup_{\lambda_i^g,\lambda_i^h}\Bigg\{\sum_{i=1}^m(\lambda_i^g\partial g_i(\hat{x}) - \lambda_i^h\partial h_i(\hat{x}))]quad\ \Bigg|\ \begin{array}{ll}\lambda_i^g\geq 0, & i\in I_{00}\cup I_{+0}\\ \lambda_i^g=0, & i\in I_{0+}\cup I_{0-}\cup I_{+-}\\ \lambda_i^h \text{ is free}, & i\in I_{0+}\\ \lambda_i^h\geq 0, & i\in I_{0-}\cup I_{00}\\ \lambda_i^h=0, & i\in I_{+-}\cup I_{+0}\end{array}\Bigg\}\Bigg],$$

where $\lambda_i^g := \mu_i^g$ and $\lambda_i^h := -\mu_i^h$, for all $i\in I$. The proof is complete. □

Since the negative polar of each subset of $\mathbb{R}^n$ is always a convex cone, Lemma 1 guaranties that $\sqsupset$, which is defined below, is a (not necessarily closed) convex cone in $\mathbb{R}^n$,

$$\sqsupset := \bigcup_{\lambda_i^g,\lambda_i^h}\Bigg\{\sum_{i=1}^m(\lambda_i^g\partial g_i(\hat{x}) - \lambda_i^h\partial h_i(\hat{x}))\ \Bigg|\ \begin{array}{ll}\lambda_i^g\geq 0, & i\in I_{00}\cup I_{+0}\\ \lambda_i^g=0, & i\in I_{0+}\cup I_{0-}\cup I_{+-}\\ \lambda_i^h \text{ is free}, & i\in I_{0+}\\ \lambda_i^h\geq 0, & i\in I_{0-}\cup I_{00}\\ \lambda_i^h=0, & i\in I_{+-}\cup I_{+0}\end{array}\Bigg\}.$$

Theorem 7 *Let $\hat{x}$ be a properly efficient solution to MMPVC, and $ACQ_\sharp$ holds at $\hat{x}$.*

(i) *There exist some positive scalars λ_j^f, $j\in J$, such that*

$$-\sum_{j=1}^p\lambda_j^f\nabla f_j(\hat{x})\in cl\Bigg[\bigcup_{\lambda_i^g,\lambda_i^h}\Bigg\{\sum_{i=1}^m(\lambda_i^g\partial g_i(\hat{x}) - \lambda_i^h\partial h_i(\hat{x}))\ \Bigg|\ \begin{array}{ll}\lambda_i^g\geq 0, & i\in I_{00}\cup I_{+0}\\ \lambda_i^g=0, & i\in I_{0+}\cup I_{0-}\cup I_{+-}\\ \lambda_i^h \text{ is free}, & i\in I_{0+}\\ \lambda_i^h\geq 0, & i\in I_{0-}\cup I_{00}\\ \lambda_i^h=0, & i\in I_{+-}\cup I_{+0}\end{array}\Bigg\}\Bigg].$$

(ii) *If, in addition, $\sqsupset$ is a closed cone, we can find some coefficients λ_i^h and λ_i^g as $i\in I$, such that:*

$$0_n\in\sum_{j=1}^p\lambda_j^f\nabla f_j(\hat{x}) + \sum_{i=1}^m\Big(\lambda_i^g\partial g_i(\hat{x}) - \lambda_i^h\partial h_i(\hat{x})\Big), \tag{11}$$

$$\lambda_i^g\geq 0,\ i\in I_{00}\cup I_{+0};\quad \lambda_i^g=0,\ i\in I_{0+}\cup I_{0-}\cup I_{+-}; \tag{12}$$

$$\lambda_i^h \text{ is free},\ i\in I_{0+};\ \lambda_i^h\geq 0,\ i\in I_{0-}\cup I_{00};\quad \lambda_i^h=0,\ i\in I_{+-}\cup I_{+0}. \tag{13}$$

Proof (i) Owing to (4), we can find some positive scalars $\lambda_j^f>0$ as $j\in J$ such that

$$-\sum_{j=1}^p\lambda_j^f\nabla f_j(\hat{x})\in\widehat{N}_S(\hat{x}).$$

From the above inclusion and Lemma 1, it is enough to prove that

$$\widehat{N}_S(\hat{x})\subseteq\Bigg\{w\in\mathbb{R}^n\ |\ \prod_{i=1}^m B_i(w)\subseteq Y\Bigg\}^0,$$

in which Y is defined as (6) and the ordering of constraints is same as considered in Lemma 1. The last inclusion is true when

$$\Bigg\{w\in\mathbb{R}^n\ |\ \prod_{i=1}^m B_i(w)\subseteq Y\Bigg\}\subseteq\varGamma_S(\hat{x}). \tag{14}$$

We define $\varphi(x):\mathbb{R}^n\longrightarrow\mathbb{R}^{2m}$ and $\pi\subseteq\mathbb{R}^{2m}$ as

$$\varphi(x) := \big(g_1(x), h_1(x),\ldots,g_m(x),h_m(x)\big),$$
$$\pi := \{(a_1,b_1,\ldots,a_m,b_m)\in\mathbb{R}^{2m}\ |\ b_i\geq 0,\ a_ib_i\leq 0,\quad\forall i\in I\}.$$

According to (Hoheisel and Kanzow 2008, Lemma 3.2), we conclude that

$$(u_1, v_1, \ldots, u_m, v_m) \in \Gamma_\pi(\varphi(\hat{x})) \iff$$
$$(u_i, v_i) \in \begin{cases} \mathbb{R} \times \{0\}, & i \in I_{0+}, \\ \mathbb{R} \times \mathbb{R}_+, & i \in I_{0-}, \\ \mathbb{R}_- \times \mathbb{R}, & i \in I_{+0}, \\ \mathbb{R} \times \mathbb{R}, & i \in I_{+-}, \\ \{(r,s) \in \mathbb{R} \times \mathbb{R} \mid s \geq 0,\ rs \leq 0\}, & i \in I_{00}. \end{cases} \tag{15}$$

Thus, $Y \subseteq \Gamma_\pi(\varphi(\hat{x}))$, and as a result

$$\left\{ w \in \mathbb{R}^n \mid \prod_{i=1}^m B_i(w) \subseteq Y \right\} \subseteq \left\{ w \in \mathbb{R}^n \mid \prod_{i=1}^m B_i(w) \subseteq \Gamma_\pi(\varphi(\hat{x})) \right\}.$$

Therefore, for proving (14), it is enough to show that

$$\left\{ w \in \mathbb{R}^n \mid \prod_{i=1}^m B_i(w) \subseteq \Gamma_\pi(\varphi(\hat{x})) \right\} \subseteq \Gamma_S(\hat{x}). \tag{16}$$

To prove the above, suppose that $v \in \left\{ w \in \mathbb{R}^n \mid \prod_{i=1}^m B_i(w) \subseteq \Gamma_\pi(\varphi(\hat{x})) \right\}$ is arbitrarily chosen. Then, by (15) we have

$$B_{I_{0+}}(v) \subseteq \mathbb{R} \times \{0\}, \quad B_{I_{0-}}(v) \subseteq \mathbb{R} \times \mathbb{R}_+, \quad B_{I_{+0}}(v) \subseteq \mathbb{R}_- \times \mathbb{R},$$
$$B_{I_{+-}}(v) \subseteq \mathbb{R} \times \mathbb{R}, \quad B_{I_{00}}(v) \subseteq \{(r,s) \in \mathbb{R} \times \mathbb{R} \mid s \geq 0,\ rs \leq 0\}.$$

Thus, we get

$$\begin{cases} \langle v, \zeta_i \rangle = 0, & \forall \zeta_i \in \partial h_i(\hat{x}),\ \forall i \in I_{0+}, \\ \langle v, \zeta_i \rangle \geq 0, & \forall \zeta_i \in \partial h_i(\hat{x}),\ \forall i \in I_{0-}, \\ \langle v, \xi_i \rangle \leq 0, & \forall \xi_i \in \partial g_i(\hat{x}),\ \forall i \in I_{+0}, \\ \begin{cases} \langle v, \zeta_i \rangle \geq 0, \\ \langle v, \zeta_i \rangle \langle v, \xi_i \rangle \leq 0, \end{cases} & \forall \zeta_i \in \partial h_i(\hat{x}),\ \forall \xi_i \in \partial g_i(\hat{x}),\ \forall i \in I_{00}, \end{cases}$$

$$\left\{ \begin{array}{l} \Longrightarrow\ v \in \left(\bigcup_{i \in I_{0+}} \partial h_i(\hat{x}) \right)^{\perp}, \\ \Longrightarrow\ v \in \left(\bigcup_{i \in I_{0-}} \left(-\partial h_i(\hat{x}) \right) \right)^0, \\ \Longrightarrow\ v \in \left(\bigcup_{i \in I_{+0}} \partial g_i(\hat{x}) \right)^0, \\ \Longrightarrow\ \begin{cases} v \in \left(\bigcup_{i \in I_{00}} \left(-\partial h_i(\hat{x}) \right) \right)^0, \\ v \in \Lambda, \end{cases} \end{array} \right\} \Longrightarrow\ v \in \mathcal{L}^0 \cap \Lambda = \mathcal{L}^\sharp.$$

We thus proved that $\left\{ w \in \mathbb{R}^n \mid \prod_{i=1}^m B_i(w) \subseteq \Gamma_\pi(\varphi(\hat{x})) \right\} \subseteq \mathcal{L}^\sharp$. This inclusion and $ACQ_\sharp$ assumption at $\hat{x}$ justify (16), and the proof of (*i*) is complete. (*ii*) follows from (*i*) and closedness assumption of $\beth$. □

It is worth mentioning that when $p = 1$, conditions (11)–(13) are referred by "VC stationary condition" in Hoheisel and Kanzow (2008), Kazemi and Kanzi (2018). Clearly, the VC stationary condition is weaker than strongly stationary condition.

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