Iran J Sci Technol Trans Sci (2019) 43:1873–1879
https://doi.org/10.1007/s40995-018-0647-0

RESEARCH PAPER

# Certain Subclasses of Bi-Univalent Functions Associated with the Horadam Polynomials

H. M. Srivastava[1,2] · Şahsene Altınkaya[3] · Sibel Yalçın[3]

Received: 15 May 2018 / Accepted: 9 October 2018 / Published online: 2 November 2018


**Abstract**
In Geometric Function Theory, there have been many interesting and fruitful usages of a wide variety of special functions and special polynomials. Here, in this article, we propose to make use of the Horadam polynomials which are known to include, as their particular cases, such potentially useful polynomials as (for example) the Fibonacci polynomials, the Lucas polynomials, the Pell polynomials, the Pell–Lucas polynomials, and the Chebyshev polynomials of the second kind. We aim first at introducing a new class of bi-univalent functions defined by means of the Horadam polynomials. For functions belonging to this new bi-univalent function class, we then derive coefficient inequalities and consider the celebrated Fekete–Szegö problem. We also provide relevant connections of our results with those considered in earlier investigations.

**Keywords** Analytic functions · Univalent functions · Bi-univalent functions · Horadam polynomials · Recurrence relations · Generating function · Taylor–Maclaurin coefficients · Fekete–Szegö problem · Principle of subordination · Chebyshev polynomials · Gauss hypergeometric function

**Mathematics Subject Classification** Primary 11B39 · 30C45 · 33C45; Secondary 30C50 · 33C05

## 1 Introduction and Preliminaries

Recently, Hörçum and Gökçen Koçer (2009) considered the Horadam polynomials $h_n(x)$, which are given by the following recurrence relation (see also Horadam and Mahon 1985):

$$h_n(x) = pxh_{n-1}(x) + qh_{n-2}(x) \quad (n \in \mathbb{N} \setminus \{1,2\};\ \mathbb{N} := \{1,2,3,\cdots\}), \tag{1.1}$$

with

$$h_1(x) = a \qquad \text{and} \qquad \mathrm{h}_2(\mathrm{x}) = \mathrm{bx} \tag{1.2}$$

for some real constants $a$, $b$, $p$ and $q$.

We first present some particular cases of the polynomials $h_n(x)$ (see, for details, Horadam and Mahon 1985 and Hörçum and Gökçen Koçer 2009):

1. For $a = b = p = q = 1$, we get the Fibonacci polynomials $F_n(x)$;
2. For $a = 2$ and $b = p = q = 1$, we get the Lucas polynomials $L_n(x)$;
3. If $a = q = 1$ and $b = p = 2$, then we get the Pell polynomials $P_n(x)$;
4. If $a = b = p = 2$ and $q = 1$, then we get the Pell–Lucas polynomials $Q_n(x)$;
5. If $a = 1$, $b = p = 2$ and $q = -1$, then we get the Chebyshev polynomials $U_n(x)$ of the second kind.

Such polynomials as (for example) the Fibonacci polynomials, the Lucas polynomials, the Chebyshev polynomials,

✉ H. M. Srivastava
harimsri@math.uvic.ca

Şahsene Altınkaya
sahsenealtinkaya@gmail.com

Sibel Yalçın
syalcin@uludag.edu.tr

1 Department of Mathematics and Statistics, University of Victoria, Victoria, British Columbia V8W 3R4, Canada

2 Department of Medical Research, China Medical University Hospital, China Medical University, Taichung 40402, Taiwan, ROC

3 Department of Mathematics, Faculty of Arts and Science, Bursa Uludag University, TR-16059 Bursa, Turkey

the Pell polynomials, the Pell–Lucas polynomials, the Lucas–Lehmer polynomials, and the families of orthogonal polynomials and other special polynomials as well as their generalizations are potentially important in a variety of disciplines in the mathematical, physical, statistical, and engineering sciences. These polynomials have been studied in several papers from a theoretical point of view (see, e.g., Horadam and Mahon 1985; Lupaş 1999; Filipponi and Horadam 1990, 1993; Vellucci and Bersani 2016 and Wang and Zhang 2012).

**Theorem 1** (see Hörçum and Gökçen Koçer 2009) *Let $\Pi(x,z)$ be the generating function of the Horadam polynomials $h_n(x)$. Then,*

$$\Pi(x,z) := \sum_{n=1}^{\infty} h_n(x) z^{n-1} = \frac{a+(b-ap)xz}{1-pxz-qz^2}. \tag{1.3}$$

***Remark 1*** Here, and in what follows, the argument $x \in \mathbb{R}$ is *independent* of the argument $z \in \mathbb{C}$, that is, $x \neq \Re(\mathfrak{z})$.

Let $\mathcal{A}$ be the class of functions $f$ of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n = z + a_2 z^2 + a_3 z^3 + \cdots, \tag{1.4}$$

which are analytic in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}$$

and normalized under the conditions given by

$$f(0) = 0 = f'(0) - 1.$$

We denote by $\mathcal{S}$ the subclass of all functions in $\mathcal{A}$ which are univalent in $\mathbb{U}$.

**Definition 1** In order to recall the principle of subordination between analytic functions, let the functions $f$ and $g$ be analytic in $\mathbb{U}$. Then, for functions $f, g \in \mathcal{A}$, $f$ is said to be subordinate to $g$ if there exists a Schwarz function $w \in \Lambda$, where

$$\Lambda = \{w : w(0) = 0 \quad \text{and} \quad |w(z)| < 1 \quad (z \in \mathbb{U})\},$$

such that

$$f(z) = g(w(z)) \qquad (z \in \mathbb{U}).$$

We denote this subordination by

$$f \prec g \quad \text{or} \quad f(z) \prec g(z) \quad (z \in \mathbb{U}).$$

In particular, if the function $g$ is univalent in $\mathbb{U}$, the above subordination is equivalent to

$$f(0) = g(0) \qquad \text{and} \qquad f(\mathbb{U}) \subset g(\mathbb{U}).$$

We now turn to the *Koebe One-Quarter Theorem* (see Duren 1983), which ensures that the image of $\mathbb{U}$ under every function in the normalized univalent function class $\mathcal{S}$ contains a disk of radius $\frac{1}{4}$. Thus, clearly, every such univalent function has an inverse $f^{-1}$ which satisfies the following conditions:

$$f^{-1}(f(z)) = z \qquad (z \in \mathbb{U})$$

and

$$f\left(f^{-1}(w)\right) = w \qquad \left(|w| < r_0(f);\; r_0(f) \geqq \frac{1}{4}\right),$$

where

$$\begin{aligned} f^{-1}(w) &= w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots \\ &=: g(w). \end{aligned} \tag{1.5}$$

**Definition 2** A function $f \in \mathcal{A}$ is said to be *bi-univalent* in $\mathbb{U}$ if both $f$ and $f^{-1}$ are univalent in $\mathbb{U}$. We denote by $\Sigma$ the class of bi-univalent functions defined in the open unit disk $\mathbb{U}$.

For a brief historical account and for several interesting examples of functions in the class $\Sigma$, see the pioneering work on this subject by Srivastava et al. (2010), which actually revived the study of bi-univalent functions in recent years. From the work of Srivastava et al. (2010), we choose to recall the following examples of functions in the class $\Sigma$:

$$\frac{z}{1-z}, \quad -\log(1-z), \quad \frac{1}{2}\log\left(\frac{1+z}{1-z}\right),$$

and so on. However, the familiar Koebe function is not a member of the bi-univalent function class $\Sigma$. Such other common examples of functions in $S$ as

$$z - \frac{z^2}{2} \quad \text{and} \quad \frac{z}{1-z^2}$$

are also not members of $\Sigma$ (see, for details, Srivastava et al. 2010).

It may be of interest to recall that Lewin (1967) studied the class of bi-univalent functions and derived the bound 1.51 for the modulus of the second coefficient $|a_2|$. Subsequently, Brannan and Clunie (1980) conjectured that $|a_2| \leqq \sqrt{2}$ for $f \in \Sigma$. Later on, Netanyahu (1969) showed that

$$\max|a_2| = \frac{4}{3}$$

if $f \in \Sigma$. Moreover, Brannan and Taha (1985) introduced certain subclasses of the bi-univalent function class $\Sigma$ similar to the familiar subclasses $\mathcal{S}^{\star}(\alpha)$ and $\mathcal{K}(\alpha)$ of starlike and convex functions of order $\alpha$ $(0 \leqq \alpha < 1)$ in $\mathbb{U}$, respectively (see Netanyahu 1969). The classes $\mathcal{S}_{\Sigma}^{\star}(\alpha)$ and $\mathcal{K}_{\Sigma}(\alpha)$ of bi-starlike functions of order $\alpha$ in $\mathbb{U}$ and biconvex functions of order $\alpha$ in $\mathbb{U}$, corresponding to the function classes $\mathcal{S}^{\star}(\alpha)$ and $\mathcal{K}(\alpha)$, were also introduced analogously. For each of the function classes $\mathcal{S}^{\star}_{\Sigma}(\alpha)$ and $\mathcal{K}_{\Sigma}(\alpha)$, they found non-sharp estimates for the initial Taylor–Maclaurin coefficients. Recently, motivated obviously by the aforementioned pioneering work on this subject by Srivastava et al. (2010), many authors investigated the coefficient bounds for various subclasses of the bi-univalent function class $\Sigma$ (see, e.g., Srivastava and Bansal 2015; Çağlar et al. 2017; Srivastava et al. 2013a, b, 2017). However, not much is known about the bounds on the general coefficient $|a_n|$ for $n \geqq 4$. In the literature, there are only a few works determining the general coefficient bounds for $|a_n|$ for various classes of analytic and bi-univalent functions (see, e.g., Altınkaya and Yalçın 2015, 2017b; Bulut 2014; Hamidi and Jahangiri 2016; Srivastava et al. 2015, 2018). The coefficient estimate problem for each of the coefficients $|a_n|$ $(n \in \mathbb{N}\backslash\{1,2\})$ is still an open problem.

Our present investigation is motivated essentially by the fact that, in Geometric Function Theory, one can find many interesting and fruitful usages of a wide variety of special functions and special polynomials. The main purpose of this article is to make use of the Horadam polynomials $h_n(x)$, which are given by the recurrence relation (1.1) and the generating function $\Pi(x,z)$ in (1.3), in order to introduce a new subclass of the bi-univalent function class $\Sigma$. For functions belonging to this newly introduced bi-univalent function class, we derive Taylor–Maclaurin coefficient inequalities in Sect. 2. Furthermore, in Sect. 3, we consider the celebrated Fekete–Szegö problem. We also provide relevant connections of our results with those considered in earlier investigations. Finally, in the concluding section (Sect. 4), we present our remarks and observations.

**Definition 3** A function $f \in \Sigma$ is said to be in the class

$$\mathcal{W}_{\Sigma}(\mu;x) \qquad (0<\mu\leqq 1;\ z,w\in\mathbb{U})$$

if the following subordination conditions are satisfied:

$$\frac{1}{2}\left[\frac{zf'(z)}{f(z)}+\left(\frac{zf'(z)}{f(z)}\right)^{\frac{1}{\mu}}\right] \prec \Pi(x,z)+1-a \tag{1.6}$$

and

$$\frac{1}{2}\left[\frac{wg'(w)}{g(w)}+\left(\frac{wg'(w)}{g(w)}\right)^{\frac{1}{\mu}}\right] \prec \Pi(x,w)+1-a, \tag{1.7}$$

where the real constants $a$ and $b$ are as in (1.2) and the function $g$ is given by (1.5).

**Remark 2** Upon setting $\mu=1$, it is readily seen that a function $f\in\Sigma$ is in the class

$$\mathcal{W}_{\Sigma}(x) \qquad (z,w\in\mathbb{U})$$

if the following subordination conditions are satisfied:

$$\frac{zf'(z)}{f(z)} \prec \Pi(x,z)+1-a \tag{1.8}$$

and

$$\frac{wg'(w)}{g(w)} \prec \Pi(x,w)+1-a, \tag{1.9}$$

where the real constants $a$ and $b$ are as in (1.2) and the function $g$ is given by (1.5).

**Remark 3** In its special case when

$$a=1, \quad b=p=2, \quad q=-1 \quad \text{and} \quad x\mapsto t,$$

the generating function in Eq. (1.3) reduces to that of the Chebyshev polynomials $U_n(t)$ of the second kind, which is given explicitly by (see, for details, Szegö 1975)

$$U_n(t)=(n+1)\,{}_2F_1\left(-n,n+2 \quad \frac{3}{2} \quad \frac{1-t}{2}\right) = \frac{\sin(n+1)\varphi}{\sin\varphi} \qquad (t=\cos\varphi)$$

in terms of the celebrated Gauss hypergeometric function ${}_2F_1$. In this special case, the bi-univalent function class $\mathcal{W}_{\Sigma}(\mu;x)$ would become the class $\mathcal{S}_{\Sigma}(\mu,t)$, which was studied earlier by Altınkaya and Yalçın (2017a).

## 2 Inequalities for the Taylor–Maclaurin Coefficients

In this section, we propose to find the estimates on the Taylor–Maclaurin coefficients $|a_2|$ and $|a_3|$ for functions in the class $\mathcal{W}_{\Sigma}(\mu;x)$, which we introduced in Definition 3. We first state Theorem 2.

**Theorem 2** *Let the function f given by* (1.4) *be in the class* $\mathcal{W}_{\Sigma}(\mu;x)$. *Then,*

$$|a_2| \leqq \frac{2\mu|bx|\sqrt{|bx|}}{\sqrt{\left|\left[(2\mu^2+\mu+1)b-(\mu+1)^2p\right]bx^2-(\mu+1)^2qa\right|}} \tag{2.1}$$

*and*

$$|a_3| \leqq \frac{4\mu^2b^2x^2}{(\mu+1)^2}+\frac{\mu|bx|}{\mu+1}. \tag{2.2}$$

***Proof*** Let $f\in\mathcal{W}_{\Sigma}(\mu;x)$ be given by the Taylor–Maclaurin expansion (1.4). Then, by Definition 3, for some analytic functions $\Theta$ and $\Phi$ such that

$$\Theta(0)=\Phi(0)=0, \quad |\Theta(z)|<1 \quad \text{and} \quad |\Phi(w)|<1 \quad (\forall\, z,w \in \mathbb{U}),$$

we can write

$$\frac{1}{2}\left[\frac{zf'(z)}{f(z)}+\left(\frac{zf'(z)}{f(z)}\right)^{\frac{1}{\mu}}\right]=\Pi(x,\Theta(z))+1-a$$

and

$$\frac{1}{2}\left[\frac{wg'(w)}{g(w)}+\left(\frac{wg'(w)}{g(w)}\right)^{\frac{1}{\mu}}\right]=\Pi(x,\Phi(w))+1-a$$

or, equivalently,

$$\frac{1}{2}\left[\frac{zf'(z)}{f(z)}+\left(\frac{zf'(z)}{f(z)}\right)^{\frac{1}{\mu}}\right]=1+h_1(x)-a+h_2(x)\Theta(z)+h_3(x)[\Theta(z)]^2+\cdots \tag{2.3}$$

and

$$\frac{1}{2}\left[\frac{wg'(w)}{g(w)}+\left(\frac{wg'(w)}{g(w)}\right)^{\frac{1}{\mu}}\right]=1+h_1(x)-a+h_2(x)\Phi(w)+h_3(x)[\Phi(w)]^2+\cdots. \tag{2.4}$$

From these last Eqs. (2.3) and (2.4), we obtain

$$\frac{1}{2}\left[\frac{zf'(z)}{f(z)}+\left(\frac{zf'(z)}{f(z)}\right)^{\frac{1}{\mu}}\right]=1+h_2(x)\xi_1 z+\left[h_2(x)\xi_2+h_3(x)\xi_1^2\right]z^2+\cdots \tag{2.5}$$

and

$$\frac{1}{2}\left[\frac{wg'(w)}{g(w)}+\left(\frac{wg'(w)}{g(w)}\right)^{\frac{1}{\mu}}\right]=1+h_2(x)\tau_1 w+\left[h_2(x)\tau_2+h_3(x)\tau_1^2\right]w^2+\cdots. \tag{2.6}$$

It is fairly well known that if

$$|\Theta(z)|=\left|\xi_1 z+\xi_2 z^2+\xi_3 z^3+\cdots\right|<1 \qquad (z\in\mathbb{U})$$

and

$$|\Phi(w)|=\left|\tau_1 w+\tau_2 w^2+\tau_3 w^3+\cdots\right|<1 \qquad (w\in\mathbb{U}),$$

then

$$|\xi_k| \leqq 1 \quad \text{and} \quad |\tau_k| \leqq 1 \qquad (k\in\mathbb{N}).$$

Thus, upon comparing the corresponding coefficients in (2.5) and (2.6), we have

$$\frac{\mu+1}{2\mu}\,a_2=h_2(x)\xi_1, \tag{2.7}$$

$$\frac{\mu+1}{2\mu}\left(2a_3-a_2^2\right)+\frac{1-\mu}{4\mu^2}\,a_2^2=h_2(x)\xi_2+h_3(x)\xi_1^2, \tag{2.8}$$

$$-\frac{\mu+1}{2\mu}\,a_2=h_2(x)\tau_1 \tag{2.9}$$

and

$$\frac{\mu+1}{2\mu}\left(3a_2^2-2a_3\right)+\frac{1-\mu}{4\mu^2}\,a_2^2=h_2(x)\tau_2+h_3(x)\tau_1^2. \tag{2.10}$$

From Eqs. (2.7) and (2.9), we can easily see that

$$\xi_1=-\tau_1 \tag{2.11}$$

and

$$\frac{(\mu+1)^2}{2\mu^2}\,a_2^2=[h_2(x)]^2\left(\xi_1^2+\tau_1^2\right). \tag{2.12}$$

If we add (2.8) to (2.10), we get

$$\frac{2\mu^2+\mu+1}{2\mu^2}\,a_2^2=h_2(x)(\xi_2+\tau_2)+h_3(x)\left(\xi_1^2+\tau_1^2\right). \tag{2.13}$$

By using (2.12) in Eq. (2.13), we have

$$\left[\left(2\mu^2+\mu+1\right)[h_2(x)]^2-(\mu+1)^2 h_3(x)\right]a_2^2=2\mu^2[h_2(x)]^3(\xi_2+\tau_2), \tag{2.14}$$

which yields

$$|a_2| \leqq \frac{2\mu|bx|\sqrt{|bx|}}{\sqrt{\left|\left[\left(2\mu^2+\mu+1\right)b-(\mu+1)^2p\right]bx^2-(\mu+1)^2qa\right|}}.$$

Moreover, if we subtract (2.10) from (2.8), we obtain

$$\frac{2(\mu+1)}{\mu}\left(a_3-a_2^2\right)=h_2(x)(\xi_2-\tau_2)+h_3(x)\left(\xi_1^2-\tau_1^2\right). \tag{2.15}$$

In view of (2.11) and (2.12), Eq. (2.15) becomes

$$a_3=\frac{2\mu^2[h_2(x)]^2}{(\mu+1)^2}\left(\xi_1^2+\tau_1^2\right)+\frac{\mu h_2(x)}{2(\mu+1)}(\xi_2-\tau_2).$$

Finally, with the help of Eq. (1.1), we deduce that

$$|a_3| \leqq \frac{4\mu^2b^2x^2}{(\mu+1)^2}+\frac{\mu|bx|}{\mu+1}.$$

The proof of Theorem 2 is thus completed. □

In its special case when $\mu=1$, Theorem 2 leads us to Corollary 1.

**Corollary 1** *Let the function f given by* (1.4) *be in the class* $\mathcal{W}_\Sigma(x)$. *Then,*

$$|a_2| \leqq \frac{|bx|\sqrt{|bx|}}{\sqrt{|(b-p)bx^2 - qa|}}$$

*and*

$$|a_3| \leqq b^2x^2 + \frac{|bx|}{2}.$$

In light of Remark 3, Theorem 2 would yield the following known result.

**Corollary 2** (see Altınkaya and Yalçın 2017a) *Let the function f given by* (1.4) *be in the class* $\mathcal{S}_\Sigma(\mu, t)$. *Then,*

$$|a_2| \leqq \frac{4\mu t\sqrt{2t}}{\sqrt{4(\mu^2-\mu)t^2 + (\mu+1)^2}}$$

*and*

$$|a_3| \leqq \frac{16\mu^2t^2}{(\mu+1)^2} + \frac{2\mu t}{\mu+1}.$$

## 3 The Fekete–Szegö Problem for the Class $\mathcal{W}_\Sigma(\mu; x)$

The following classical Fekete–Szegö inequality, which is investigated by means of Loewner's chain method, involves the Taylor–Maclaurin coefficients of $f \in \mathcal{S}$ given by (1.4):

$$|a_3 - \vartheta a_2^2| \leqq 1 + 2\exp\left(-\frac{2\vartheta}{1-\vartheta}\right) \quad (0 \leqq \vartheta < 1). \tag{3.1}$$

In its limit as $\vartheta \to 1-$, we have an elementary inequality given by

$$|a_3 - a_2^2| \leqq 1.$$

In fact, the coefficient functional $\Psi_\vartheta(f)$, where

$$\Psi_\vartheta(f) = a_3 - \vartheta a_2^2,$$

for the normalized analytic functions $f$ in the unit disk $\mathbb{U}$ plays an important rôle in function theory. The problem of maximizing the modulus of the functional $\Psi_\vartheta(f)$ is called the Fekete–Szegö problem (see Fekete and Szegö 1933).

In this section, we derive the Fekete–Szegö inequalities for functions in the class $\mathcal{W}_\Sigma(\mu; x)$, which is introduced by Definition 3. These inequalities are asserted by Theorem 3.

**Theorem 3** *Let the function f given by* (1.4) *be in the class* $\mathcal{W}_\Sigma(\mu; x)$. *Suppose also that* $\vartheta \in \mathbb{R}$. *Then*

$$|a_3 - \vartheta a_2^2| \leqq \begin{cases} \dfrac{\mu|bx|}{\mu+1} \\ \left(|\vartheta - 1| \leqq \dfrac{1}{4\mu(\mu+1)}\left|2\mu^2+\mu+1-(\mu+1)^2\left[\dfrac{pbx^2+qa}{b^2x^2}\right]\right|\right) \\ \dfrac{4\mu^2|1-\vartheta||bx|^3}{\left|\left[(2\mu^2+\mu+1)b-(\mu+1)^2p\right]bx^2-(\mu+1)^2qa\right|} \\ \left(|\vartheta - 1| \geqq \dfrac{1}{4\mu(\mu+1)}\left|2\mu^2+\mu+1-(\mu+1)^2\left[\dfrac{pbx^2+qa}{b^2x^2}\right]\right|\right). \end{cases} \tag{3.2}$$

***Proof*** From (2.14) and (2.15), we find that

$$\begin{aligned} a_3 - \vartheta a_2^2 &= \frac{2\mu^2[h_2(x)]^3(1-\vartheta)(\xi_2+\tau_2)}{(2\mu^2+\mu+1)[h_2(x)]^2-(\mu+1)^2h_3(x)} + \frac{\mu h_2(x)(\xi_2-\tau_2)}{2(\mu+1)} \\ &= h_2(x)\left[\left(\Omega(\vartheta,x)+\frac{\mu}{2(\mu+1)}\right)\xi_2 + \left(\Omega(\vartheta,x)-\frac{\mu}{2(\mu+1)}\right)\tau_2\right], \end{aligned}$$

where

$$\Omega(\vartheta,x) = \frac{2\mu^2[h_2(x)]^2(1-\vartheta)}{(2\mu^2+\mu+1)[h_2(x)]^2-(\mu+1)^2h_3(x)}.$$

Hence, in view of (1.1), we conclude that

$$|a_3 - \vartheta a_2^2| \leqq \begin{cases} \dfrac{\mu|h_2(x)|}{\mu+1} & \left(0 \leqq |\Omega(\vartheta,x)| \leqq \dfrac{\mu}{2(\mu+1)}\right) \\ 2|h_2(x)|\cdot|\Omega(\vartheta,x)| & \left(|\Omega(\vartheta,x)| \geqq \dfrac{\mu}{2(\mu+1)}\right) \end{cases},$$

which evidently completes the proof of Theorem 3. □

An immediate consequence of Theorem 3 when $\mu = 1$ is asserted by Corollary 3.

**Corollary 3** *Let the function f given by* (1.4) *be in the class* $\mathcal{W}_\Sigma(x)$. *Suppose also that* $\vartheta \in \mathbb{R}$. *Then,*

$$|a_3 - \vartheta a_2^2| \leqq \begin{cases} \dfrac{|bx|}{2} & \left(|1-\vartheta| \leqq \dfrac{|(b-p)bx^2-qa|}{2b^2x^2}\right) \\ \dfrac{|1-\vartheta|\cdot|bx|^3}{|(b-p)bx^2-qa|} & \left(|1-\vartheta| \geqq \dfrac{|(b-p)bx^2-qa|}{2b^2x^2}\right). \end{cases} \tag{3.3}$$

In view of Remark 3, Theorem 3 can be shown to yield the following known result.

**Corollary 4** (see Altınkaya and Yalçın 2017a) *Let the function f given by* (1.4) *be in the class* $\mathcal{S}_\Sigma(\mu, t)$. *Suppose also that* $\vartheta \in \mathbb{R}$. *Then,*

$$|a_3-\vartheta a_2^2| \leqq \begin{cases} \dfrac{2\mu t}{\mu+1} & \left(|1-\vartheta| \leqq \dfrac{1}{4\mu(\mu+1)}\left|\mu^2-\mu+\dfrac{(\mu+1)^2}{4t^2}\right|\right) \\ \dfrac{32\mu^2|1-\vartheta|t^3}{4t^2(\mu^2-\mu)+(\mu+1)^2} & \left(|1-\vartheta| \geqq \dfrac{1}{4\mu(\mu+1)}\left|\mu^2-\mu+\dfrac{(\mu+1)^2}{4t^2}\right|\right). \end{cases} \tag{3.4}$$

If we set $\vartheta = 1$, we get the following corollaries.

**Corollary 5** *If the function f given by* (1.4) *is in the class* $\mathcal{W}_\Sigma(\mu;x)$, *then*

$$|a_3-a_2^2| \leqq \frac{\mu|bx|}{\mu+1}. \tag{3.5}$$

**Corollary 6** (see Altınkaya and Yalçın 2017a) *If the function f given by* (1.4) *is in the class* $\mathcal{S}_\Sigma(\mu,t)$, *then*

$$|a_3-a_2^2| \leqq \frac{2\mu t}{\mu+1}. \tag{3.6}$$

In our next section (Sect. 4), we choose to present several remarks and observations concerning our main results (Theorems 2 and 3) as well as their known or new corollaries and consequences which are stated above as Corollaries 1 and 2 and Corollaries 3 to 6, respectively.

## 4 Concluding Remarks and Observations

Our investigation in this article drew its motivation essentially by the fact that, in Geometric Function Theory, we can find many interesting and fruitful usages of a wide variety of special functions and special polynomials. The main purpose was to make use of the Horadam polynomials $h_n(x)$, which are given by the recurrence relation (1.1) and the generating function $\Pi(x,z)$ in (1.3), with a view to introducing a new subclass $\mathcal{W}_\Sigma(\mu;x)$ of the bi-univalent function class $\Sigma$ (see Definition 3). For functions belonging to this newly introduced bi-univalent function class $\mathcal{W}_\Sigma(\mu;x)$, we have derived Taylor–Maclaurin coefficient inequalities in Sect. 2 and we have considered the celebrated Fekete–Szegö problem in Sect. 3. We have also provided relevant connections of our results with those considered in earlier investigations.

The geometric properties of the function class $\mathcal{W}_\Sigma(\mu;x)$ vary according to the values assigned to the parameters involved. Nevertheless, some results for the special cases of the parameters involved could be presented as illustrative examples. If $\mu = 1$, $a = p = x = 1$, $b = 2$, and $q = 0$, then we have

$$\frac{zf'(z)}{f(z)} \prec \frac{1+z}{1-z} \qquad (z \in \mathbb{U}).$$

In this case, the function $f$ maps the open unit disk $\mathbb{U}$ onto the half-plane given by

$$\Re\left(\frac{+\mathfrak{z}}{-\mathfrak{z}}\right) > 0,$$

since the following expression:

$$\frac{zf'(z)}{f(z)}$$

takes on values in the half-plane. If, on the other hand, we restrict our considerations for a given univalent function $\mathfrak{p}(\mathfrak{z})$ in $\mathbb{U}$, we can investigate the corresponding mapping problems for other regions of the complex $z$-plane instead of the half-plane $\Re(\mathfrak{z}) > 0$. In this way, one can introduce many other subclasses of the function class $\mathcal{W}_\Sigma(\mu;x)$ which we have studied in this paper.

## References


Altınkaya Ş, Yalçın S (2015) Faber polynomial coefficient bounds for a subclass of bi-univalent functions. C R Acad Sci Paris Sér I(353):1075–1080

Altınkaya Ş, Yalçın S (2017) On the Chebyshev polynomial coefficient problem of some subclasses of bi-univalent functions. Gulf J Math 5(3):34–40

Altınkaya Ş, Yalçın S (2017) Faber polynomial coefficient estimates for certain classes of bi-univalent functions defined by using the Jackson $(p,q)$-derivative operator. J Nonlinear Sci Appl 10:3067–3074

Brannan DA, Clunie JG (eds) (1980) Aspects of contemporary complex analysis. In: Proceedings of the NATO Advanced Study Institute (University of Durham, Durham; July 1–20, 1979), Academic Press, New York and London

Brannan DA, Taha TS (1985) On some classes of bi-univalent functions, In: Mazhar SM, Hamoui A, Faour NS (eds) Mathematical analysis and its applications. Kuwait, pp 53–60, KFAS Proceedings Series, Vol. 3, Pergamon Press (Elsevier Science Limited), Oxford, 1988; see also Studia Univ. Babeş-Bolyai Math 31(2) (1986) 70–77

Bulut S (2014) Faber polynomial coefficient estimates for a comprehensive subclass of analytic bi-univalent functions. C R Acad Sci Paris Sér I(352):479–484

Çağlar M, Deniz E, Srivastava HM (2017) Second Hankel determinant for certain subclasses of bi-univalent functions. Turk J Math 41:694–706

Duren PL (1983) Univalent functions, Grundlehren der Mathematischen Wissenschaften, Band 259. Springer-Verlag, New York

Fekete M, Szegö G (1933) Eine Bemerkung Über Ungerade Schlichte Funktionen. J Lond Math Soc 89:85–89

Filipponi P, Horadam AF (1990) Derivative sequences of Fibonacci and Lucas polynomials. In: Bergum GE, Philippou AN, Horadam AF (eds) Applications of Fibonacci Numbers, vol 4, pp 99–108, Proceedings of the fourth international conference on Fibonacci numbers and their applications, Wake Forest University, Winston-Salem, North Carolina; Springer (Kluwer

Academic Publishers), Dordrecht, Boston and London, 1991. 4 (1991) 99–108
Filipponi P, Horadam AF (1993) Second derivative sequences of Fibonacci and Lucas polynomials. Fibonacci Quart. 31:194–204
Hamidi SG, Jahangiri JM (2016) Faber polynomial coefficients of bi-subordinate functions. C R Acad Sci Paris Sér I(354):365–370
Horadam AF, Mahon JM (1985) Pell and Pell–Lucas polynomials. Fibonacci Quart 23:7–20
Hörçum T, Gökçen Koçer E (2009) On some properties of Horadam polynomials. Int Math Forum 4:1243–1252
Lewin M (1967) On a coefficient problem for bi-univalent functions. Proc Am Math Soc 18:63–68
Lupaş AI (1999) A guide of Fibonacci and Lucas polynomials. Octagon Math Mag 7:2–12
Netanyahu E (1969) The minimal distance of the image boundary from the origin and the second coefficient of a univalent function in $|z| < 1$. Arch Ration Mech Anal 32:100–112
Srivastava HM, Bansal D (2015) Coefficient estimates for a subclass of analytic and bi-univalent functions. J Egypt Math Soc 23:242–246
Srivastava HM, Bulut S, Çağlar M, Yağmur N (2013) Coefficient estimates for a general subclass of analytic and bi-univalent functions. Filomat 27:831–842
Srivastava HM, Gaboury S, Ghanim F (2017) Coefficient estimates for some general subclasses of analytic and bi-univalent functions. Afrika Mat 28:693–706
Srivastava HM, Murugusundaramoorthy G, Magesh N (2013) Certain subclasses of bi-univalent functions associated with the Hohlov operator. Global J Math Anal 1(2):67–73
Srivastava HM, Mishra AK, Gochhayat P (2010) Certain subclasses of analytic and bi-univalent functions. Appl Math Lett 23:1188–1192
Srivastava HM, Sümer Eker S, Ali RM (2015) Coefficient bounds for a certain class of analytic and bi-univalent functions. Filomat 29:1839–1845
Srivastava HM, Sümer Eker S, Hamidi SG, Jahangiri JM (2018) Faber polynomial coefficient estimates for bi-univalent functions defined by the Tremblay fractional derivative operator, Bull. Iranian Math. Soc. 44: 149–157
Szegö G (1975) Orthogonal Polynomials, Fourth edition, American Mathematical Society Colloquium Publications, vol 23. American Mathematical Society, Providence, Rhode Island
Vellucci P, Bersani AM (2016) The class of Lucas-Lehmer polynomials, Rend. Mat. Appl. (Ser. 7) 37: 43–62
Wang T-T, Zhang W-P (2012) Some identities involving Fibonacci, Lucas polynomials and their applications. Bull. Math. Soc. Sci. Math. Roumanie (New Ser.) 55(103):95–103